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DOMAIN OF THE TRIPLE BAND MATRIX ON SOME MADDOX'S SPACES

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ABSTRACT. The sequence spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ were introduced and studied by Maddox [Proc. Cambridge Philos. Soc. 64 (1968), 335–340]. In the present paper, we introduce the sequence spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ of non-absolute type which are derived by the triple band matrix $B(r,s,t)$ and is proved that the spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ are paranorm isomorphic to the spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$; respectively. Besides this, the α-, β- and γ-duals of the spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ are computed and the bases of the spaces $c(B,p)$ and $c_0(B,p)$ are constructed. Finally, the matrix mappings from the sequence spaces $\lambda(B,p)$ to a given sequence space μ and from the sequence space μ to the sequence space $\lambda(B,p)$ are characterized, where $\lambda \in \{\ell_\infty, c, c_0\}$.

1. Introduction and preliminaries

By ω, we denote the space of all real valued sequences. Any vector subspace of ω is called a *sequence space*. We write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively.

A sequence space λ with a linear topology is called a *K-space* provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A K-space λ is called an *FK-space* provided λ is a complete linear metric space. An FK-space whose topology is normable is called a *BK-space*. An FK-space λ is said to have *AK property*, if

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$\phi \subset \lambda$ and $\{e^{(k)}\}$ is a basis for λ, where $e^{(k)}$ is a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$ and ϕ=span$\{e^{(k)}\}$, the set of all finitely non-zero sequences. If ϕ is dense in λ, then λ is called an *AD-space*, thus AK implies AD. For example, the spaces c_0, cs, and ℓ_p are AK-spaces, where $1 \leq p < \infty$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a *paranormed space* if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ were defined by Maddox [20] (see also Simons [26] and Nakano [24]) as follows:

$$\begin{aligned}
\ell_\infty(p) &:= \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\right\}, \\
c(p) &:= \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\right\}, \\
c_0(p) &:= \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}
\end{aligned}$$

which are the complete spaces paranormed by

$$g(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/M}.$$

We assume throughout that $p_k^{-1} + (p_k^{'})^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we write $Ax = \{(Ax)_n\}$, the *A-transform* of x, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. If $x \in \lambda$ implies that $Ax \in \mu$ then we say that A defines a *matrix mapping* from λ into μ and denote it by $A : \lambda \to \mu$. By $(\lambda : \mu)$, we mean the class of all infinite matrices A such that $A : \lambda \to \mu$.

Let us define some triangle limitation matrices which are needed in text. Define the summation matrix $S = (s_{nk})$, the difference matrix $\Delta = (\Delta_{nk}^{(1)})$ and the generalized difference matrix $B(r, s) = \{b_{nk}(r, s)\}$ by

$$s_{nk} := \begin{cases} 1 & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n, \end{cases} \quad \Delta_{nk}^{(1)} := \begin{cases} (-1)^{n-k} & , \quad n-1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

$$b_{nk}(r, s) := \begin{cases} r & , \quad k = n, \\ s & , \quad k = n-1, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$; where $r, s \in \mathbb{R} \setminus \{0\}$. The domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}, \tag{1.1}$$

which is a sequence space. If A is triangle, then one can easily observe that the normed sequence spaces λ_A and λ are norm isomorphic, i.e., $\lambda_A \cong \lambda$. If λ is a sequence space, then the continuous dual λ_A^* of the space λ_A is defined by

$$\lambda_A^* := \{f : f = g \circ A, g \in \lambda^*\}$$

Although in most cases the new sequence space λ_A generated by the limitation matrix A from a sequence space λ is the expansion or the contraction of the original space λ, it may be observed in some cases that those spaces overlap. Indeed, one can easily see that the inclusion $\lambda_S \subset \lambda$ strictly holds for $\lambda \in \{\ell_\infty, c, c_0\}$. Further, one can deduce that the inclusion $\lambda \subset \lambda_{\Delta^{(1)}}$ also strictly holds for $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$. However, if we define $\lambda := c_0 \oplus span\{z\}$ with $z = \{(-1)^k\}$, i.e., $x \in \lambda$ if and only if $x := s + ax$ for some $s \in c_0$ and some $a \in \mathbb{C}$, and consider the matrix A with the rows A_n defined by $A_n := (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$, we have $Ae = z \in \lambda$ but $Az = e \notin \lambda$ which lead us to the consequences that $z \in \lambda \backslash \lambda_A$ and $e \in \lambda_A \backslash \lambda$, where $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ is a sequence whose only non-zero term is a 1 in nth place for each $n \in \mathbb{N}$. Hence the sequence spaces λ_A and λ overlap but neither contains the other. The approach constructing a new paranormed sequence space by means of the matrix domain of a particular limitation method has recently been employed by Malkowsky [22], Choudhary and Mishra [15], Altay and Başar [2, 4, 5], Aydın and Başar [7, 8], Başar et al. [13]. $c_0(u, p)$ and $c(u, p)$ are the spaces consisting of the sequences $x = (x_k)$ such that $ux = (u_k x_k)$ in the spaces $c_0(p)$ and $c(p)$ for $u = (u_k)$ with $u_k \neq 0$ for all $k \in \mathbb{N}$, and are studied by Başarır [14]. More recently, generalized difference matrix $B(r, s) = \{b_{nk}(r, s)\}$ have been used by Kirişçi and Başar [18] to generalize the difference spaces $\ell_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$ and bv_p. Finally, the new technique for deducing certain topological properties, for example AB-, KB-, AD-properties, etc., and determining the $\beta-$ and $\gamma-$duals of the domain of a triangle matrix in a sequence space is given by Altay and Başar [5].

Let X be a seminormed space. A set $Y \subset X$ is called *fundamental* if the span of Y is dense in X. The useful result on fundamental set which is an application of Hahn-Banach Theorem as follows: If Y is the subset of a seminormed space X and $f \in X'$, $f(Y) = 0$ implies $f = 0$, then Y is fundamental [28, p. 39].

Let r, s, t be non-zero reel numbers and define the triple band matrix $B(r, s, t) = \{b_{nk}(r, s, t)\}$ by

$$b_{nk}(r, s, t) := \begin{cases} r & , \quad n = k, \\ s & , \quad n = k + 1, \\ t & , \quad n = k + 2, \\ 0 & , \quad 0 \leq k < n - 2 \text{ or } k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$. The inverse matrix $B^{-1}(r, s, t) = C = \{c_{nk}(r, s, t)\}$ is given by Furkan et al. [16] as follows:

$$c_{nk}(r, s, t) := \begin{cases} \frac{1}{r} \sum_{j=0}^{n-k} \left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{n-k-j} \left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{j} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases} \tag{1.2}$$

for all $k, n \in \mathbb{N}$. We should record here that $B(r,s,0) = B(r,s)$, $B(1,-2,1) = \Delta^{(2)}$ and $B(1,-1,0) = \Delta^{(1)}$. So, the results related to the matrix domain of the triple band matrix $B(r,s,t)$ are more general and more comprehensive than the consequences on the matrix domain of $B(r,s)$, $\Delta^{(2)}$ and $\Delta^{(1)}$, and include them.

The main purpose of this paper is to introduce the paranormed sequence spaces $\lambda(B,p)$ and normed sequence spaces λ_B of non-absolute type which are the set of all sequences whose generalized $B(r,s,t)$-transforms are in the spaces $\lambda(p)$ and λ, and to compute their α-, β- and γ-duals. Here and after, by λ we denote any of the classical spaces ℓ_∞, c or c_0. Besides this, the basis of the spaces $c(B,p)$ and $c_0(B,p)$ are derived, and the concept of the pair of summability methods is defined and given an analysis about this type methods. Finally, the matrix mappings from the sequence spaces $\lambda(B,p)$ to a given sequence space μ and from the sequence space μ to the sequence spaces $\lambda(B,p)$ are characterized.

The rest of this paper is organized, as follows:

In section 2, the sequence spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ of non-absolute type are introduced and determined their α-, β- and γ-duals. This section is terminated with the results on the bases for the spaces $c(B,p)$ and $c_0(B,p)$. Section 3 is devoted to the characterization of matrix transformations on/in the new sequence spaces. As a consequence of the analysis related to the pair of summability methods, a basic theorem is given and the classes $(\lambda(B,p) : \mu)$ and $(\mu : \lambda(B,p))$ of infinite matrices are characterized, where $\lambda(B,p)$ denotes any of the spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$, and μ denotes any normed sequence space. In the final section of the paper, we note the significance of the present results in the literature about difference sequences and record some further suggestions.

2. The Paranormed Sequence spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ of Non-absolute Type

In this section, we define the sequence spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ of non-absolute type derived by the triple band matrix, and prove that these are the complete paranormed linear spaces and determine their β- and γ-duals. Furthermore, we give the bases for the spaces $c(B,p)$ and $c_0(B,p)$.

Başar and Altay [12] have recently examined the space $bs(p)$ which is formerly defined in [9] as the set of all series whose sequence of partial sums are in $\ell_\infty(p)$. Choudhary and Mishra [15] defined the sequence space $\overline{\ell(p)}$ which consists of all sequences whose S-transforms are in the space $\ell(p)$, the space of all sequences $x = (x_k) \in \omega$ such that $\sum_k |x_k|^{p_k} < \infty$. Quite recently, Altay and Başar [2] have studied the space $r^t_\infty(p)$ which consists of all sequences whose Riesz transforms are in the space $\ell_\infty(p)$. With the notation of (1.1), the spaces $bs(p)$, $\overline{\ell(p)}$ and $r^t_\infty(p)$ can be redefined by

$$bs(p) := [\ell_\infty(p)]_S, \quad \overline{\ell(p)} := [\ell(p)]_S \quad \text{and} \quad r^t_\infty(p) := [\ell_\infty(p)]_{R^t}.$$

Following Başar and Altay [12], Choudhary and Mishra [15], Altay and Başar [2], we define the sequence spaces $\lambda(B,p)$ for $\lambda \in \{\ell_\infty, c, c_0\}$ by

$$\lambda(B,p) := \{x = (x_k) \in \omega : y = (tx_{k-2} + sx_{k-1} + rx_k) \in \lambda(p)\}.$$

If $p_k = 1$ for every $k \in \mathbb{N}$, we write $\lambda(B)$ instead of $\lambda(B,p)$. If λ is any normed or paranormed sequence space then we call the matrix domain $\lambda_{B(r,s,t)}$ as the *generalized difference space of sequences*. It is natural that these spaces may also be defined with the notation of (1.1) that

$$\lambda(B,p) := \{\lambda(p)\}_{B(r,s,t)} \text{ and } \lambda(B) := \lambda_{B(r,s,t)}.$$

Define the sequence $y = (y_k)$, which will be frequently used, by the $B(r,s,t)$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k := tx_{k-2} + sx_{k-1} + rx_k \text{ for all } k \in \mathbb{N}. \tag{2.1}$$

Although Theorems 2.1, 2.2 and 2.3 below, are related to the sequence spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$, we give the proof only for one of those spaces. Since the proof can also be obtained in the similar way for the other spaces, to avoid undue repetition in the statements, we leave the detail to the reader. Now, we may begin with the following theorem which is essential in the study.

Theorem 2.1. *The following statements hold:*

(a) *$\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ are the complete linear metric spaces paranormed by h, defined by*

$$h(x) := \sup_{k\in\mathbb{N}} |tx_{k-2} + sx_{k-1} + rx_k|^{p_k/M}.$$

h is a paranorm for the spaces $\ell_\infty(B,p)$ and $c(B,p)$ only in the trivial case $\inf p_k > 0$ when $\ell_\infty(B,p) = (\ell_\infty)_{B(r,s,t)}$ and $c(B,p) = c_{B(r,s,t)}$.

(b) *The sets $\lambda_{B(r,s,t)}$ are Banach spaces with the norm $\|x\|_{\lambda_{B(r,s,t)}} = \|y\|_\lambda$.*

Proof. We prove the theorem for the space $c_0(B,p)$. The linearity of $c_0(B,p)$ with respect to the coordinatewise addition and scalar multiplication of the sequences follows from the following inequalities which are satisfied for $u = (u_k)$, $x = (x_k) \in c_0(B,p)$ (see [21, p. 30])

$$\begin{aligned} &\sup_{k\in\mathbb{N}} |t(u_{k-2} + x_{k-2}) + s(u_{k-1} + x_{k-1}) + r(u_k + x_k)|^{p_k/M} \leq \\ &\leq \sup_{k\in\mathbb{N}} |tu_{k-2} + su_{k-1} + ru_k|^{p_k/M} + \sup_{k\in\mathbb{N}} |tx_{k-2} + sx_{k-1} + rx_k|^{p_k/M} \end{aligned} \tag{2.2}$$

and for any $\alpha \in \mathbb{R}$ (see [20])

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \tag{2.3}$$

It is clear that $h(\theta) = 0$ and $h(x) = h(-x)$ for all $x \in c_0(B,p)$. Again the inequalities (2.2) and (2.3) yield the subadditivity of h and

$$h(\alpha x) \leq \max\{1, |\alpha|\} h(x).$$

Let $\{x^n\}$ be any sequence of the points $c_0(B,p)$ and (α_n) also be any sequence of scalars such that $h(x^n - x) \to 0$ and $\alpha_n \to \alpha$, as $n \to \infty$, respectively. Then, since the inequality

$$h(x^n) \leq h(x) + h(x^n - x)$$

holds by subadditivity of h, $\{h(x^n)\}$ is bounded and we thus have

$$h(\alpha_n x^n - \alpha x) = \sup_{k\in\mathbb{N}} \left| t\left(\alpha_n x^{(n)}_{k-2} - \alpha x_{k-2}\right) + s\left(\alpha_n x^{(n)}_{k-1} - \alpha x_{k-1}\right) + r\left(\alpha_n x^{(n)}_{k} - \alpha x_{k}\right)\right|^{p_k/M} \leq |\alpha_n - \alpha| h(x^n) + |\alpha| h(x^n - x)$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, h is a paranorm on the space $c_0(B,p)$.

It remains to prove the completeness of the space $c_0(B,p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $c_0(B,p)$, where $x^i = \left\{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots\right\}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that $h(x^i - x^j) < \varepsilon$ for all $i,j \geq n_0(\varepsilon)$. We obtain by using definition of h for each fixed $k \in \mathbb{N}$ that

$$\left|\left\{B(r,s,t)x^i\right\}_k - \left\{B(r,s,t)x^j\right\}_k\right| \leq \sup_{k\in\mathbb{N}} \left|\left\{B(r,s,t)x^i\right\}_k - \left\{B(r,s,t)x^j\right\}_k\right|^{\frac{p_k}{M}} < \varepsilon \quad (2.4)$$

for every $i,j \geq n_0(\varepsilon)$, which leads us to the fact that

$$\{(B(r,s,t)x^0)_k, (B(r,s,t)x^1)_k, (B(r,s,t)x^2)_k, \ldots\}$$

is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $\{B(r,s,t)x^i\}_k \to \{B(r,s,t)x\}_k$ as $i \to \infty$. Using these infinitely many limits $(B(r,s,t)x)_0$, $(B(r,s,t)x)_1$, $(B(r,s,t)x)_2$, ..., we define the sequence $\{(B(r,s,t)x)_0, (B(r,s,t)x)_1, (B(r,s,t)x)_2, \ldots\}$. We have from (2.4) with $j \to \infty$ that

$$\left|\left\{B(r,s,t)x^i\right\}_k - \left\{B(r,s,t)x\right\}_k\right| \leq \varepsilon, \ (i \geq n_0(\varepsilon)) \quad (2.5)$$

for every fixed $k \in \mathbb{N}$. Since $x^i = \{x_k^{(i)}\} \in c_0(B,p)$,

$$\left|\left\{B(r,s,t)x^i\right\}_k\right|^{p_k/M} < \varepsilon$$

for all $k \in \mathbb{N}$. Therefore, we obtain by (2.5) that

$$|\{B(r,s,t)x\}_k|^{p_k/M} \leq \left|\{B(r,s,t)x\}_k - \left\{B(r,s,t)x^i\right\}_k\right|^{p_k/M} + \left|\left\{B(r,s,t)x^i\right\}_k\right|^{p_k/M} < \varepsilon$$

for all $i \in \mathbb{N}$ such that $i \geq n_0(\varepsilon)$. This shows that the sequence $B(r,s,t)x$ belongs to the space $c_0(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $c_0(B,p)$ is complete and this terminates the proof. □

Therefore, one can easily check that the absolute property does not hold on the spaces $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ that is $h(x) \neq h(|x|)$ for at least one sequence in those spaces, and this says that $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ are the sequence spaces of non-absolute type; where $|x| = (|x_k|)$.

Theorem 2.2. *The generalized difference spaces of sequences $\ell_\infty(B,p)$, $c(B,p)$ and $c_0(B,p)$ of non-absolute type are paranorm isomorphic to the spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$, respectively; where $0 < p_k \leq H < \infty$.*

Proof. We establish this for the space $\ell_\infty(B,p)$. To prove the theorem, we should show the existence of a paranorm preserving linear bijection between the spaces $\ell_\infty(B,p)$ and $\ell_\infty(p)$ for $1 \leq p_k \leq H < \infty$. With the notation of (2.1), define the transformation T from $\ell_\infty(B,p)$ to $\ell_\infty(p)$ by $x \mapsto y = Tx$. The linearity of

T is trivial. Further, it is obvious that $x=\theta$ whenever $Tx=\theta$ and hence T is injective.

Let $y=(y_k)\in \ell_\infty(p)$ and define the sequence $x=(x_k)$ by

$$\begin{aligned} x_k &:= \{B^{-1}(r,s,t)y\}_k \\ &= \sum_{j=1}^{k}\frac{1}{r}\sum_{v=0}^{j-1}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{k-v-1}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v} y_{k-j+1} \end{aligned}$$

for all $k\in\mathbb{N}$. Then, we have

$$\begin{aligned} \{B(r,s,t)x\}_k &= tx_{k-2}+sx_{k-1}+rx_k \\ &= t\sum_{j=1}^{k-2}\frac{1}{r}\sum_{k=0}^{j-1}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-v-1}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v} y_{k-j-1} \\ &\quad +s\sum_{j=1}^{k-1}\frac{1}{r}\sum_{k=0}^{j-1}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-v-1}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v} y_{k-j} \\ &\quad +r\sum_{j=1}^{k}\frac{1}{r}\sum_{v=0}^{j-1}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-v-1}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v} y_{k-j+1} \\ &= y_k \end{aligned}$$

for all $k\in\mathbb{N}$, which leads us to the fact that

$$h(x)=\sup_{k\in\mathbb{N}}|tx_{k-2}+sx_{k-1}+rx_k|^{p_k/M}=\sup_{k\in\mathbb{N}}\left|y_k\right|^{p_k/M}=g(y)<\infty.$$

Thus, we deduce that $x\in\ell_\infty(B,p)$ and consequently T is surjective and is paranorm preserving. Hence, T is a linear bijection and this says us that the spaces $\ell_\infty(B,p)$ and $\ell_\infty(p)$ are paranorm isomorphic, as desired. □

Theorem 2.3. *Suppose that* $|-s+\sqrt{s^2-4tr}|>|2r|$. *Then, the sequence space* $c_0(B)$ *has AD property.*

Proof. Suppose that $f\in[c_0(B)]'$. Then, $f(x)=g(Ax)$ for some $g\in c_0'=\ell_1$. Since c_0 has AK property and $c_0'\cong\ell_1$,

$$f(x)=\sum_j a_j\{B(r,s,t)x\}_j$$

for some $a=(a_j)\in\ell_1$ and the inclusion $\phi\subset c_0(B)$ holds. For any $f\in[c_0(B)]'$ and $e^{(k)}\in\phi$, we have

$$f(e^{(k)})=\sum_j a_j\{B(r,s,t)e^{(k)}\}_j=\{B'(r,s,t)a\}_k;\ (k\in\mathbb{N}),$$

where $B'(r,s,t)$ denotes the transpose of the matrix $B(r,s,t)$. Hence, from Hahn-Banach theorem, ϕ is dense in $c_0(B)$ if and only if $B'(r,s,t)a=\theta$ for $a\in\ell_1$ implies $a=\theta$. Under the condition $|-s+\sqrt{s^2-4tr}|>|2r|$ since the null space of the operator $B'(r,s,t)$ on w is $\{\theta\}$, $c_0(B)$ has AD property. □

For the sequence spaces λ and μ, the set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu)=\{z=(z_k)\in\omega : xz=(x_kz_k)\in\mu \text{ for all } x=(x_k)\in\lambda\} \tag{2.6}$$

is called the *multiplier space* of λ and μ. One can easily observe for a sequence space ν with $\lambda\supset\nu$ and $\nu\supset\mu$ that the inclusions

$$S(\lambda,\mu)\subset S(\nu,\mu) \text{ and } S(\lambda,\mu)\subset S(\lambda,\nu)$$

hold. With the notation of (2.6), the α-, β- and γ-*duals* of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\lambda^\alpha:=S(\lambda,\ell_1) \quad \lambda^\beta:=S(\lambda,cs) \quad \text{and} \quad \lambda^\gamma:=S(\lambda,bs).$$

It is also known that the f-*dual* λ^f of a sequence space λ is $\lambda^f:=\big\{\{f(e^k)\}:f\in\lambda'\big\}$.

Lemma 2.4. [28, pp. 106, 108] Let λ be an FK-space which contains ϕ. Then, the following statements hold:

(i) $\lambda^\beta\subseteq\lambda^\gamma$.
(ii) If λ has AD then $\lambda^\beta=\lambda^\gamma$.
(iii) λ has AD iff $\lambda^f=\lambda'$.

Now, as an easy consequence of Theorem 2.3 and Lemma 2.4 (iii), we have

Corollary 2.5. *The f-dual and the continuous dual of the space $c_0(B)$ is the set*

$$\left\{a=(a_k)\in\omega : \left(\sum_{j=k}^{\infty}\sum_{v=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-v}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v}\frac{a_j}{r}\right)_{k\in\mathbb{N}}\in\ell_1\right\}.$$

Now, we can give the theorems determining the α-, β- and γ-duals of the sequence space $\ell_\infty(B,p)$. In proving them, we apply the technique used for the spaces of single sequences by Altay and Başar [2, 3, 4, 5]. Since the $\alpha-$, $\beta-$ and $\gamma-$duals of the sequence spaces $c(B,p)$ and $c_0(B,p)$ are determined in the similar way, we omit the detail.

We quote some lemmas which are needed in proving our theorems.

Lemma 2.6. [17, Theorem 5.1.3 with $q_n=1$] $A=(a_{nk})\in(\ell_\infty(p):\ell_1)$ if and only if

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}a_{nk}B^{1/p_k}\right|<\infty \text{ for all integers } B>1.$$

Lemma 2.7. [19, Theorem 3] Let $p_k>0$ for every k. Then $A=(a_{nk})\in(\ell_\infty(p):\ell_\infty)$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|B^{1/p_k}<\infty \text{ for all integers } B>1. \tag{2.7}$$

Lemma 2.8. [19, Corollary for Theorem 3] Let $p_k>0$ for every k. Then $A=(a_{nk})\in(\ell_\infty(p):c)$ if and only if (2.7) holds and $\lim_{n\to\infty}a_{nk}=\alpha_k$ for each fixed $k\in\mathbb{N}$.

Theorem 2.9. *Define the matrix $D = (d_{nk})$ by means of a sequence $a = (a_n) \in \omega$ whose n^{th} row D_n is the product of the n^{th} row C_n of the matrix C given by (1.2) and the sequence a, that is to say that $D_n = C_n a$ for all $n \in \mathbb{N}$. Then,*

$$[\ell_\infty(B,p)]^\alpha := \left\{ a = (a_n) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} d_{nk} B^{1/p_k} \right| < \infty \text{ for all integers } B > 1 \right\}.$$

Proof. Let $a = (a_n) \in \omega$. Then, we have by (2.1) that

$$\begin{aligned} a_n x_n &= a_n (Cy)_n \qquad (2.8) \\ &= \sum_{k=0}^{n} \frac{a_n}{r} \left[\sum_{j=0}^{k} \left(\frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{k-j} \left(\frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^{j} \right] y_k \\ &= \sum_{k=0}^{n} d_{nk}(r,s,t) y_k = (Dy)_n \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, one can see from (2.8) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_n) \in \ell_\infty(B,p)$ if and only if $Dy \in \ell_1$ whenever $y = (y_n) \in \ell_\infty(p)$. This gives that $a = (a_n) \in [\ell_\infty(B,p)]^\alpha$ if and only if $D \in (\ell_\infty(p) : \ell_1)$. Thus, Lemma 2.6 leads to the desired result. □

Theorem 2.10. *Define the matrix $G = \{g_{nk}(r,s,t)\}$ by means of a sequence $a = (a_n) \in \omega$ and the matrix C given by (1.2) as*

$$g_{nk}(r,s,t) := \begin{cases} \sum_{j=k}^{n} c_{jk}\, a_j & , \quad 0 \le k \le n, \\ 0 & , \quad k > n, \end{cases} \qquad (2.9)$$

for all $k, n \in \mathbb{N}$. Then, $[\ell_\infty(B,p)]^\beta = \{a = (a_k) \in \omega : G \in (\ell_\infty(p) : c)\}$.

Proof. Take any $a = (a_k) \in \omega$ and consider the equality obtained with (2.1) that

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \sum_{j=k}^{n} \frac{1}{r} \sum_{v=0}^{j-k} \left(\frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{j-k-v} \left(\frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^{v} a_j y_k \\ &= \sum_{k=0}^{n} g_{nk}(r,s,t) y_k = (Gy)_n \quad \text{for all} \quad n \in \mathbb{N}. \qquad (2.10) \end{aligned}$$

Thus, we deduce from (2.10) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \ell_\infty(B,p)$ if and only if $Gy \in c$ whenever $y = (y_k) \in \ell_\infty(p)$, where $G = \{g_{nk}(r,s,t)\}$ is defined by (2.9). Therefore, we obtain that $a = (a_k) \in [\ell_\infty(B,p)]^\beta$ if and only if $G = (g_{nk}) \in (\ell_\infty(p) : c)$. Thus, Lemma 2.8 yields the result. □

Theorem 2.11. $[\ell_\infty(B,p)]^\gamma := \{a = (a_k) \in \omega : G \in (\ell_\infty(p) : \ell_\infty)\}$, *where the matrix $G = \{g_{nk}(r,s,t)\}$ is defined by (2.9).*

Proof. This is obtained in the similar way to the proof of Theorem 2.10. So, we omit the detail. □

If a sequence space λ paranormed by h_1 contains a sequence (b_k) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_k) such that

$$\lim_{n\to\infty} h_1\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0$$

then (b_n) is called a *Schauder basis* (or briefly *basis*) for λ. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$. Now, we may give the sequence of the points of the spaces $c_0(B,p)$ and $c(B,p)$ which form the Schauder bases for those spaces. Because of the isomorphism T, defined in the proof of Theorem 2.2, between the sequence spaces $c_0(B,p)$ and $c_0(p)$, $c(B,p)$ and $c(p)$ is onto, the inverse image of the bases of the spaces $c_0(p)$ and $c(p)$ are the bases of our new spaces $c_0(B,p)$ and $c(B,p)$, respectively. Therefore, we have:

Theorem 2.12. *Let $\alpha_k = \{B(r,s,t)x\}_k$ and $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$. Define the sequence $z = (z_n)$ and $b^{(k)}(r,s,t) = \{b^{(k)}(r,s,t)\}_{n\in\mathbb{N}}$ for every fixed $k \in \mathbb{N}$ by*

$$z_n := \sum_{k=0}^{n} c_{nk} \quad \text{and} \quad b_n^{(k)}(r,s,t) := \begin{cases} 0 & , \quad n < k, \\ c_{nk} & , \quad n \ge k. \end{cases}$$

Then, the following statements hold:

(a) *The sequence $\{b^{(k)}(r,s,t)\}_{k\in\mathbb{N}}$ is a basis for the space $c_0(B,p)$ and any x in $c_0(B,p)$ has a unique representation of the form $x = \sum_k \alpha_k b^{(k)}$.*

(b) *The set $\{z, b^{(k)}(r,s,t)\}$ is a basis for the space $c(B,p)$ and any x in $c(B,p)$ has a unique representation of the form $x = lz + \sum_k [\alpha_k - l] b^{(k)}(r,s,t)$, where $l = \lim_{k\to\infty}\{B(r,s,t)x\}_k$.*

3. Some Matrix Mappings Related to the Sequence Spaces $\lambda(B,p)$

In this section, subsequent to defining the pair of summability methods such that one of them applied to the sequences in the space $\lambda(B,p)$ and the other one applied to the sequences in the space $\lambda(p)$, we give a basic theorem as a consequence of an analysis related to this type summability methods. The reader may refer to Başar [10] and Başar [11, pp. 41–45 and Chapter 4]. Finally, we characterize the classes $(\lambda(B,p) : \mu)$ and $(\mu : \lambda(B,p))$ of infinite matrices and derive the characterization of some other classes from them, where μ is any given sequence space. We use the convention that any term with negative subscript is equal to zero.

Let us suppose that the infinite matrices $E = (e_{nk})$ and $F = (f_{nk})$ map the sequences $x = (x_k)$ and $y = (y_k)$ which are connected with the relation (2.1) to the sequences $u = (u_n)$ and $v = (v_n)$, respectively, i.e.,

$$u_n := \left(Ex\right)_n = \sum_k e_{nk} x_k \quad \text{for each fixed} \quad n \in \mathbb{N}, \tag{3.1}$$

$$v_n := \left(Fy\right)_n = \sum_k f_{nk} y_k \quad \text{for each fixed} \quad n \in \mathbb{N}. \tag{3.2}$$

One can immediately deduce here that the method F is applied to the $B(r,s,t)$-transform of the sequence $x=(x_k)$ while the method E is directly applied to the terms of the sequence $x=(x_k)$. So, the methods E and F are essentially different.

Let us assume that the matrix product $FB(r,s,t)$ exists in general. We shall say in this situation that the methods E and F in (3.1) and (3.2) are the *pair of summability methods*, shortly PSM, if u_n becomes v_n (or v_n becomes u_n) under the application of the formal summation by parts. This leads us to the fact that $FB(r,s,t)$ exists and is equal to E and $[FB(r,s,t)]x = F[B(r,s,t)x]$ formally holds, if one side exists. This statement is equivalent to the relation

$$\begin{aligned} e_{nk} &:= rf_{nk}+sf_{n,k+1}+tf_{n,k+2} \text{ or equivalently} \\ f_{nk} &:= \sum_{j=k}^{\infty}\sum_{v=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-v}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{v}\frac{e_{nj}}{r}, \end{aligned} \tag{3.3}$$

for all $k,n\in\mathbb{N}$.

Now, we can give a short analysis on the PSM. One can see that v_n reduces to u_n as follows:

$$\begin{aligned} v_n &= \sum_k f_{nk}y_k = \sum_k f_{nk}(rx_k+sx_{k-1}+tx_{k-2}) \\ &= \sum_k (rf_{nk}+sf_{n,k+1}+tf_{n,k+2})\,x_k \\ &= \sum_k e_{nk}x_k = u_n. \end{aligned}$$

The partial sums of the series on the right of (3.1) and (3.2) are connected with the relation

$$\sum_{k=0}^{m} e_{nk}x_k = \sum_{k=0}^{m-1}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{n-j}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{j}\frac{e_{nk}}{r}y_k + \frac{me_{nm}}{r}y_m \tag{3.4}$$

for every fixed $m,n\in\mathbb{N}$. Hence if, for a given $n\in\mathbb{N}$, one of the series on the right of (3.1) and (3.2) converges then the other converges if and only if

$$\lim_{m\to\infty}\frac{me_{nm}}{r}y_m = z_n \tag{3.5}$$

for every fixed $n\in\mathbb{N}$. If (3.5) holds then we have from (3.4) by letting $m\to\infty$ that $u_n=v_n+z_n$ for all $n\in\mathbb{N}$. Hence, if (y_n) is summable by one of the methods E and F then it is summable by the other one if and only if (3.5) holds and

$$\lim_{n\to\infty} z_n = \alpha. \tag{3.6}$$

Hence the limits of (u_n) and (v_n) differ by α. Therefore the E- and F-limits of any sequence summable by one of them agree if and only if F summability implies that (3.6) holds with $\alpha=0$. The similar result holds with E and F interchanged. It follows by the validity of (3.6) with $\alpha\neq 0$ that the methods E and F are inconsistent, and conversely.

The analysis given above, leads us to the following basic theorem related to the matrix mappings on the sequence space $\lambda(B,p)$:

Theorem 3.1. *Let the matrices $E=(e_{nk})$ and $F=(f_{nk})$ are connected with the relation (3.3) and μ be any given sequence space. Then, $E\in(\lambda(B,p):\mu)$ if and only if $F\in(\lambda(p):\mu)$ and*

$$F^{(n)}\in(\lambda(p):c), \tag{3.7}$$

for every fixed $n\in\mathbb{N}$, where $F^{(n)}=\left(f_{mk}^{(n)}\right)$ with

$$f_{mk}^{(n)}:=\begin{cases}\frac{1}{r}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{n-j}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{j}e_{nk} & , \quad k<m\\ \dfrac{me_{nm}}{r} & , \quad k=m\\ 0 & , \quad k>m\end{cases}$$

for all $k,m\in\mathbb{N}$.

Proof. Suppose that E and F be PSM, that is to say that (3.3) holds, μ be any given sequence space and take into account that the spaces $\lambda(B,p)$ and $\lambda(p)$ are paranorm isomorphic.

Let $E\in(\lambda(B,p):\mu)$ and take any $y\in\lambda(p)$. Then, $FB(r,s,t)$ exists and $(e_{nk})_{k\in\mathbb{N}}\in[\lambda(B,p)]^{\beta}$ which yields that (3.7) is necessary and $(f_{nk})_{k\in\mathbb{N}}\in\lambda^{\beta}(p)$ for each $n\in\mathbb{N}$. Hence, Fy exists for each $y\in\lambda(p)$ and thus by letting $m\to\infty$ in the equality (3.4), we have by (3.3) that $Ex=Fy$ which leads us to the consequence $F\in(\lambda(p):\mu)$.

Conversely, let $F\in(\lambda(p):\mu)$ and (3.7) holds, and take any $x\in\lambda(B,p)$. Then, we have $(f_{nk})_{k\in\mathbb{N}}\in\lambda^{\beta}(p)$ which gives together with (3.7) that $(e_{nk})_{k\in\mathbb{N}}\in[\lambda(B,p)]^{\beta}$ for each $n\in\mathbb{N}$. Hence, Ex exists. Therefore, we obtain from the equality (3.4) as $m\to\infty$ that $Fy=Ex$ and this shows that $E\in(\lambda(B,p):\mu)$.

This completes the proof. □

By changing the roles of the spaces $\lambda(B,p)$ and $\lambda(p)$ with μ, we have

Theorem 3.2. *Suppose that the elements of the infinite matrices $A=(a_{nk})$ and $B=(b_{nk})$ are connected with the relation*

$$b_{nk}:=ta_{n-2,k}+sa_{n-1,k}+ra_{nk}$$

for all $k,n\in\mathbb{N}$ and μ be any given sequence space. Then, $A\in(\mu:\lambda(B,p))$ if and only if $B\in(\mu:\lambda(p))$.

Proof. Let $z=(z_k)\in\mu$ and consider the following equality

$$\sum_{k=0}^{m}b_{nk}z_k=\sum_{k=0}^{m}(ra_{nk}+sa_{n-1,k}+ta_{n-2,k})z_k \text{ for all } m,n\in\mathbb{N},$$

which yields as $m\to\infty$ that $(Bz)_n=[B(r,s,t)(Az)]_n$. Therefore, one can immediately observe from here that $Az\in\lambda(B,p)$ whenever $z\in\mu$ if and only if $Bz\in\lambda(p)$ whenever $z\in\mu$.

This step concludes the proof. □

It is clear that Theorem 3.1 and Theorem 3.2 have several consequences depending on the choice of the sequence spaces λ, μ and the sequence $p = (p_k)$. Therefore by Theorem 3.1 and Theorem 3.2, the necessary and sufficient conditions for $(\lambda(B, p) : \mu)$ and $(\mu : \lambda(B, p))$ may be derived by replacing the entries of E and A by those of the entries of $F = EB^{-1}(r, s, t)$ and $B = B(r, s, t)A$, respectively; where the necessary and sufficient conditions on the matrices F and B are read from the concerning results in the existing literature. We may give the following lemma due to Grosse-Erdmann [17]:

Lemma 3.3. *The necessary and sufficient conditions for* $A = (a_{nk}) \in (\lambda : \mu)$ *when* $\lambda \in \{\ell_\infty(p), c(p), c_0(p), \ell(p)\}$ *and* $\mu \in \{\ell_\infty(p), c(p), c_0(p)\}$ *can be read from the following table:*

From / *To*	$\ell_\infty(p)$	$c(p)$	$c_0(p)$	$\ell(p)$
$\ell_\infty(q)$	**1.**	**2.**	**3.**	**4.**
$c(q)$	**5.**	**6.**	**7.**	**8.**
$c_0(q)$	**9.**	**10.**	**11.**	**12.**

Table 1: Characterization of matrix transformations between some Maddox' sequence spaces,

where

1. (3.8) $\forall M \in \mathbb{N},\ \sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}|\, M^{1/p_k} \right)^{q_n} < \infty$

2. (3.9) $\exists M \ni\ \sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| M^{-1/p_k} \right)^{q_n} < \infty$

(3.10) $\sup_{n\in\mathbb{N}} \left| \sum_k a_{nk} \right|^{q_n} < \infty$

3. (3.9)

4. (3.11) $\exists L \ni\ \sup_{n\in\mathbb{N}} \sup_{k\in K_1} \left| a_{nk} L^{-1/q_n} \right|^{p_k} < \infty$

(3.12) $\exists L \ni\ \sup_{n\in\mathbb{N}} \sum_{k\in K_2} \left| a_{nk} L^{-1/q_n} \right|^{p_k'} < \infty$

5. (3.13) $\forall M \in \mathbb{N},\ \sup_{n\in\mathbb{N}} \sum_k |a_{nk}|\, M^{1/p_k} < \infty$

(3.14) $\exists (\alpha_k) \in \omega \ni\ \forall M \in \mathbb{N},\ \lim_{n\to\infty} \left(\sum_k |a_{nk} - \alpha_k|\, M^{1/p_k} \right)^{q_n} = 0$

6. (3.15) $\exists M \ni\ \sup_{n\in\mathbb{N}} \sum_k |a_{nk}| M^{-1/p_k} < \infty$

(3.16) $\exists \alpha \in \mathbb{C} \ni\ \lim_{n\to\infty} \left| \sum_k a_{nk} - \alpha \right|^{q_n} = 0$

(3.17) $\exists M$ *and* $(\alpha_k) \in \omega \ni \forall L,\ \sup_{n\in\mathbb{N}} \sum_k |a_{nk} - \alpha_k|\, L^{1/q_n} M^{-1/p_k} < \infty$

(3.18) $\exists (\alpha_k) \in \omega \ni\ \forall k \in \mathbb{N},\ \lim_{n\to\infty} |a_{nk} - \alpha_k|^{q_n} = 0$

7. (3.15), (3.17), (3.18)

8. (3.18),

(3.19) $\sup_{n\in\mathbb{N}} \sup_{k\in K_1} |a_{nk}|^{p_k} < \infty$

(3.20) $\exists M \ni\ \sup_{n\in\mathbb{N}} \sum_{k\in K_2} |a_{nk} M^{-1}|^{p_k'} < \infty$

(3.21) $\exists (\alpha_k) \in \omega \ni\ \forall L,\ \sup_{n\in\mathbb{N}} \sup_{k\in K_1} \left(|a_{nk} - \alpha_k|\, L^{1/q_n} \right)^{p_k} < \infty$

(3.22) $\exists M$ *and* $(\alpha_k) \in \omega \ni \forall L,\ \sup_{n\in\mathbb{N}} \sum_{k\in K_2} \left(|a_{nk} - \alpha_k|\, L^{1/q_n} M^{-1} \right)^{p_k'} < \infty$

9. (3.23) $\forall M \in \mathbb{N},\ \lim_{n\to\infty} \left(\sum_k |a_{nk}| M^{1/p_k} \right)^{q_n} = 0$

10. (3.24) $\forall k \in \mathbb{N},\ \lim_{n\to\infty} |a_{nk}|^{q_n} = 0$

(3.25) $\exists M \ni\ \forall L,\ \sup_{n\in\mathbb{N}} \sum_k |a_{nk}|\, L^{1/q_n} M^{-1/p_k} < \infty$

(3.26) $\lim_{n\to\infty} \left| \sum_k a_{nk} \right|^{q_n} = 0$

11. (3.24), (3.25)

12. (3.24),

(3.27) $\forall L,\ \sup_{n\in\mathbb{N}} \sup_{k\in K_1} \left| a_{nk} L^{1/q_n} \right|^{p_k} < \infty$

(3.28) $\exists M \ni\ \forall L,\ \sup_{n\in\mathbb{N}} \sum_{k\in K_2} \left| a_{nk} L^{1/q_n} M^{-1} \right|^{p_k'} < \infty$

Now, by combining Theorem 3.2 and Lemma 3.3, as an easy consequence we have the following:

Corollary 3.4. *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $B = (b_{nk})$ as in Theorem 3.2. Then,*

(i) $A \in (\ell_\infty(p) : \ell_\infty(B, q))$ *if and only if (3.8) holds with* b_{nk} *instead of* a_{nk}.
(ii) $A \in (c(p) : \ell_\infty(B, q))$ *if and only if (3.9) and (3.10) hold with* b_{nk} *instead of* a_{nk}.
(iii) $A \in (c_0(p) : \ell_\infty(B, q))$ *if and only if (3.9) holds with* b_{nk} *instead of* a_{nk}.
(iv) $A \in (\ell(p) : \ell_\infty(B, q))$ *if and only if (3.11) and (3.12) hold with* b_{nk} *instead of* a_{nk}.
(v) $A \in (\ell_\infty(p) : c(B, q))$ *if and only if (3.13) and (3.14) hold with* b_{nk} *instead of* a_{nk}.
(vi) $A \in (c(p) : c(B, q))$ *if and only if (3.15), (3.16), (3.17) and (3.18) hold with* b_{nk} *instead of* a_{nk}.
(vii) $A \in (c_0(p) : c(B, q))$ *if and only if (3.15), (3.17) and (3.18) hold with* b_{nk} *instead of* a_{nk}.
(viii) $A \in (\ell(p) : c(B, q))$ *if and only if (3.18), (3.19), (3.20), (3.21) and (3.22) hold with* b_{nk} *instead of* a_{nk}.
(ix) $A \in (\ell_\infty(p) : c_0(B, q))$ *if and only if (3.23) holds with* b_{nk} *instead of* a_{nk}.
(x) $A \in (c(p) : c_0(B, q))$ *if and only if (3.24), (3.25) and (3.26) hold with* b_{nk} *instead of* a_{nk}.
(xi) $A \in (c_0(p) : c_0(B, q))$ *if and only if (3.24) and (3.25) hold with* b_{nk} *instead of* a_{nk}.
(xii) $A \in (\ell(p) : c_0(B, q))$ *if and only if (3.24), (3.27) and (3.28) hold with* b_{nk} *instead of* a_{nk}.

4. Conclusion

Quite recently, as a nice generalization of Kirişçi and Başar [18], Sönmez [27] has examined the domain of the triple band matrix $B(r, s, t)$ in the classical spaces ℓ_∞, c, c_0 and ℓ_p. Although the domain of some particular matrices in certain normed sequence spaces were studied by several researchers, the domain of matrices in some paranormed sequence spaces were not sufficiently worked except Altay and Başar [2, 3, 4], Aydın and Başar [7, 8], Aydın and Altay [6]. So, there is many number of open problems and to work on the domain of some infinite matrices in the paranormed sequence spaces is meaningful. In the cases $B(r, s, 0) \equiv B(r, s)$ and $p_k = 1$ for all $k \in \mathbb{N}$, since $\ell_\infty(B, p) \equiv \widehat{\ell}_\infty(p)$, $c(B, p) \equiv \widehat{c}(p)$, $c_0(B, p) \equiv \widehat{c}_0(p)$ and $\ell_\infty(B, e) \equiv \ell_\infty(B)$, $c(B, e) \equiv c(B)$, $c_0(B, e) \equiv c_0(B)$ our results are much more general than the corresponding results of Aydın and Altay [6], and Sönmez [27], respectively.

Finally, we should note from now on that our next paper will be devoted to the investigation of the domain of the triple band matrix $B(r, s, t)$ in the Maddox's space $\ell(p)$.

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