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ANGLES AND A CLASSIFICATION OF NORMED SPACES

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ABSTRACT. We suggest a concept of generalized angles in arbitrary real normed vector spaces. We give for each real number a definition of an 'angle' by means of the shape of the unit ball. They all yield the well known Euclidean angle in the special case of real inner product spaces. With these different angles we achieve a classification of normed spaces, and we obtain a characterization of inner product spaces. Moreover we consider this construction also for a generalization of normed spaces, i.e. for spaces which may have a non-convex unit ball.

1. Introduction

In a real inner product space $(X, < .|. >)$ it is well-known that the inner product can be expressed by the norm, namely for $\vec{x}, \vec{y} \in X$ we can write
$$< \vec{x} \mid \vec{y} > = \tfrac{1}{4} \cdot (\, \|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 \,).$$
That means for $\vec{x} \neq \vec{0} \neq \vec{y}$ we have the expression
$$< \vec{x} \mid \vec{y} > = \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] .$$
Furthermore we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the usual Euclidean angle $\angle_{Euclid}(\vec{x}, \vec{y}) =$
$$\arccos \frac{< \vec{x} \mid \vec{y} >}{\|\vec{x}\| \cdot \|\vec{y}\|} = \arccos \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right) ,$$
which is defined in terms of the norm, too.

In this paper we deal with generalized real normed vector spaces. We consider vector spaces X provided with a *weight* or *functional* $\|\cdot\|$, that means we have a

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continuous map $\|\cdot\| : X \longrightarrow \mathbb{R}^+ \cup \{0\}$. We assume that the weights are *absolute homogeneous* or *balanced*, i.e. $\|r \cdot \vec{x}\| = |r| \cdot \|\vec{x}\|$ for $\vec{x} \in X, r \in \mathbb{R}$. We call such pairs $(X, \|\cdot\|)$ *balancedly weighted vector spaces*, or for short 'BW spaces'.

To avoid problems with a denominator 0 we restrict our considerations to BW spaces which are positive definite, i.e. $\|\vec{x}\| = 0$ only for $\vec{x} = \vec{0}$.

Following the lines of an inner product we define for each real number ϱ a continuous product $< . \mid . >_\varrho$ on X.

Definition 1.1. Let $\vec{x}, \vec{y}$ be two arbitrary elements of X. In the case of $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$ we set $< \vec{x} \mid \vec{y} >_\varrho := 0$, and if $\vec{x}, \vec{y} \neq \vec{0}$ (i.e. $\|\vec{x}\| \cdot \|\vec{y}\| > 0$) we define the real number

$$< \vec{x} \mid \vec{y} >_\varrho \; := \|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \cdot \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right)^\varrho .$$

It is easy to show that the product fulfils the symmetry ($< \vec{x}|\vec{y} >_\varrho \; = \; < \vec{y}|\vec{x} >_\varrho$), the positive semidefiniteness ($< \vec{x}|\vec{x} >_\varrho \geq 0$), and the homogeneity ($< r \cdot \vec{x}|\vec{y} >_\varrho \; = \; r \cdot < \vec{x}|\vec{y} >_\varrho$), for $\vec{x}, \vec{y} \in X$, $r \in \mathbb{R}$.

Let us fix a number $\varrho \in \mathbb{R}$ and a positive definite BW space $(X, \|\cdot\|)$. For two vectors $\vec{x}, \vec{y} \neq \vec{0}$ with an additional property we are able to define an 'angle' which coincides with the Euclidean angle in inner product spaces.

Definition 1.2. Let $\vec{x}, \vec{y}$ be two elements of $X \backslash \{\vec{0}\}$, and let $\vec{x}, \vec{y}$ fulfil the inequality $| < \vec{x} \mid \vec{y} >_\varrho | \; \leq \; \|\vec{x}\| \cdot \|\vec{y}\|$. We define the number from the interval $[0, \pi]$

$$\angle_\varrho(\vec{x}, \vec{y}) \; := \; \arccos \frac{< \vec{x} \mid \vec{y} >_\varrho}{\|\vec{x}\| \cdot \|\vec{y}\|} \; .$$

The number $\angle_\varrho(\vec{x}, \vec{y})$ is called the *ϱ-angle* of the pair $(\vec{x}, \vec{y})$.

We consider mainly those pairs $(X, \|\cdot\|)$ where the triple $(X, \|\cdot\|, \; < . \mid . >_\varrho)$ satisfies the *Cauchy-Schwarz-Bunjakowsky Inequality* or CSB inequality, that means for all $\vec{x}, \vec{y} \in X$ we have the inequality

$$| < \vec{x} \mid \vec{y} >_\varrho | \; \leq \; \|\vec{x}\| \cdot \|\vec{y}\|$$

for a fixed real number ϱ. In this case we get that the 'ϱ-angle' $\angle_\varrho(\vec{x}, \vec{y})$ is defined for all $\vec{x}, \vec{y} \neq \vec{0}$, and we shall express this by

' The space $(X, \|\cdot\|)$ has the angle $\angle_\varrho$ '.

This new 'angle' has seven comfortable properties (`An` 1) - (`An` 7) which are known from the Euclidean angle in inner product spaces, and for all $\varrho \in \mathbb{R}$ it corresponds to the Euclidean angle in the case that $(X, \|\cdot\|)$ already is an inner product space.

Let $(X, \|\cdot\|)$ be a real positive definite BW space. Assume that the triple $(X, \|\cdot\|, \; < . \mid . >_\varrho)$ satisfies the CSB inequality for a fixed number ϱ. Hence we are able to define the ϱ-angle $\angle_\varrho$, and we have the properties (`An` 1) - (`An` 7).

- (`An` 1) $\quad \angle_\varrho$ is a continuous map from $\left(X \backslash \{\vec{0}\} \right)^2$ into the interval $[0, \pi]$.

  For elements $\vec{x}, \vec{y} \neq \vec{0}$ it holds that
- (`An` 2) $\quad \angle_\varrho(\vec{x}, \vec{x}) = 0$,
- (`An` 3) $\quad \angle_\varrho(-\vec{x}, \vec{x}) = \pi$,

- $(\mathtt{An}\ 4)$ $\quad \angle_\varrho(\vec{x},\vec{y}) = \angle_\varrho(\vec{y},\vec{x})$,
- $(\mathtt{An}\ 5)$ $\quad$ for all $r,s>0$ we have $\angle_\varrho(r\cdot\vec{x}, s\cdot\vec{y}) = \angle_\varrho(\vec{x},\vec{y})$,
- $(\mathtt{An}\ 6)$ $\quad \angle_\varrho(-\vec{x},-\vec{y}) = \angle_\varrho(\vec{x},\vec{y})$,
- $(\mathtt{An}\ 7)$ $\quad \angle_\varrho(\vec{x},\vec{y}) + \angle_\varrho(-\vec{x},\vec{y}) = \pi$.

We define some classes of real vector spaces. Let NORM be the class of all real normed vector spaces. For all fixed real numbers ϱ let

$$\mathsf{NORM}_\varrho \;:=\; \{(X,\|\cdot\|)\in\mathsf{NORM}\mid \text{ The normed space } (X,\|\cdot\|) \text{ has the angle } \angle_\varrho\}\,.$$

We prove the statements

$$\mathsf{NORM} \;=\; \mathsf{NORM}_\varrho$$

for all real numbers ϱ from the closed interval $[-1,1]$, and also

$$\mathsf{IPspace} \;=\; \bigcap_{\varrho\in\mathbb{R}} \mathsf{NORM}_\varrho\;,$$

where $\mathsf{IPspace}$ denotes the class of all real inner product spaces. Furher, if we assume four positive real numbers $\alpha,\beta,\gamma,\delta$ such that

$$-\delta<-\gamma<-1<1<\alpha<\beta,$$

we obtain the inclusions

$$\mathsf{NORM}_{-\delta}\subset\mathsf{NORM}_{-\gamma}\subset\mathsf{NORM}\supset\mathsf{NORM}_\alpha\supset\mathsf{NORM}_\beta\,.$$

We prove the inequalities

$$\mathsf{NORM}_{-\gamma}\neq\mathsf{NORM}\neq\mathsf{NORM}_\alpha$$

and we strongly believe, but we have no proof that the demonstrated inclusions $\mathsf{NORM}_{-\delta}\subset\mathsf{NORM}_{-\gamma}$ and $\mathsf{NORM}_\alpha\supset\mathsf{NORM}_\beta$ are proper.

After that we return to the more general situation. We abandon the restriction of the triangle inequality, again we consider positive definite BW spaces $(X,\|\cdot\|)$, i.e. its weights $\|\cdot\|$ have to be positive definite and absolute homogeneous only. We say *positive definite balancedly weighted spaces* or pdBW for the class of all such pairs, and for all fixed real numbers ϱ we define the class

$$\mathsf{pdBW}_\varrho \;:=\; \{(X,\|\cdot\|)\in\mathsf{pdBW}\mid \text{ The space } (X,\|\cdot\|) \text{ has the angle } \angle_\varrho\}\,.$$

We show $\mathsf{pdBW}_{\mathbf{-1}}=\mathsf{pdBW}$. Roughly speaking this means that for the angle $\angle_\varrho$ the 'best' choice is $\varrho=-1$, since the angle $\angle_{-1}$ is defined in every element of pdBW.

For real numbers $\alpha,\beta,\gamma,\delta$ with $-\delta<-\gamma<-1<\alpha<\beta$ we get the inclusions

$$\mathsf{pdBW}_{-\delta}\subset\mathsf{pdBW}_{-\gamma}\subset\mathsf{pdBW}\supset\mathsf{pdBW}_\alpha\supset\mathsf{pdBW}_\beta\,.$$

Since $\mathsf{NORM}_{-\gamma}\neq\mathsf{NORM}$ we already know the fact $\mathsf{pdBW}_{-\gamma}\neq\mathsf{pdBW}$. Further we prove the inequality $\mathsf{pdBW}\neq\mathsf{pdBW}_\alpha$, and we conjecture that the inclusions $\mathsf{pdBW}_{-\delta}\subset\mathsf{pdBW}_{-\gamma}$ and $\mathsf{pdBW}_\alpha\supset\mathsf{pdBW}_\beta$ are proper.

To prove the above statements we define and use 'convex corners' which can occur even in normed vector spaces, and 'concave corners' which can be vectors in BW spaces which are not normed spaces. Both expressions are mathematical descriptions of a geometric shape exactly what the names associate. For instance, the well-known normed space $(\mathbb{R}^2,\|\cdot\|_1)$ with the norm $\|(x,y)\|_1=|x|+|y|$ has four convex corners at its unit sphere, they are just the corners of the generated square.

Further we introduce a function Υ,

$$\Upsilon : \mathsf{pdBW} \longrightarrow [-\infty, -1] \times [-1, +\infty], \quad \Upsilon(X, \|\cdot\|) := (\nu, \mu).$$

This function maps every real positive definite BW space $(X, \|\cdot\|)$ to a pair of extended real numbers (ν, μ), where

$$\nu := \inf\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\}, \quad \text{and}$$
$$\mu := \sup\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\} .$$

For an inner product space $(X, \|\cdot\|)$ we get immediately $\Upsilon(X, \|\cdot\|) = (-\infty, \infty)$.

If $(X, \|\cdot\|)$ is an arbitrary space from the class pdBW with $-\infty < \nu, \mu < \infty$, we show that the infimum and the supremum are attained, i.e.

$$\nu = \min\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\}, \quad \text{and}$$
$$\mu = \max\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\} .$$

Let $(X, \|\cdot\|) \in \mathsf{NORM}$. We assume that $(X, \|\cdot\|)$ has a convex corner. We prove

$$\Upsilon(X, \|\cdot\|) = (-1, 1) .$$

For instance, for the normed space $(\mathbb{R}^2, \|\cdot\|_1)$ we have $\Upsilon\left(\mathbb{R}^2, \|\cdot\|_1\right) = (-1, 1)$.

At the end we consider products. For two spaces $(A, \|\cdot\|_A), (B, \|\cdot\|_B) \in \mathsf{pdBW}$ we take its Cartesian product $A \times B$, and we get a set of balanced weights $\|\cdot\|_p$ on $A \times B$, for $p > 0$. For a positive number p for each element $\left(\vec{a}, \vec{b}\right) \in A \times B$ we define the non-negative number

$$\left\|\left(\vec{a}, \vec{b}\right)\right\|_p := \sqrt[p]{\|\vec{a}\|_A^p + \|\vec{b}\|_B^p} .$$

This makes the pair $\left(A \times B, \|\cdot\|_p\right)$ to an element of the class pdBW, and with this construction we finally ask two more interesting and unanswered questions.

2. General Definitions

Let $X = (X, \tau)$ be an arbitrary real topological vector space, that means that the real vector space X is provided with a topology τ such that the addition of two vectors and the multiplication with real numbers are continuous. Further let $\|\cdot\|$ denote a *positive functional* or a *weight* on X, these notations mean that there is a continuous map $\|\cdot\|\colon\ X \longrightarrow \mathbb{R}^+ \cup \{0\}$, the non-negative real numbers $\mathbb{R}^+ \cup \{0\}$ carry the usual Euclidean topology.

We consider the following conditions.

$\widehat{(1)}$: For all $r \in \mathbb{R}$ and all $\vec{x} \in X$ we have: $\|r \cdot \vec{x}\| = |r| \cdot \|\vec{x}\|$ ('absolute homogeneity'),

$\widehat{(2)}$: $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$ ('positive definiteness'),

$\widehat{(3)}$: for $\vec{x}, \vec{y} \in X$ it holds $\|\vec{x} + \vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$ ('triangle inequality'),

$\widehat{(4)}$: for $\vec{x}, \vec{y} \in X$ it holds $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2 \cdot (\|\vec{x}\|^2 + \|\vec{y}\|^2)$

('parallelogram identity').

If $\|\cdot\|$ fulfils $\widehat{(1)}$, we call $\|\cdot\|$ a *balanced weight* on X,
if $\|\cdot\|$ fulfils $\widehat{(1)}, \widehat{(2)}, \widehat{(3)}$ the map $\|\cdot\|$ is called a *norm* on X, and
if $\|\cdot\|$ fulfil $\widehat{(1)}, \widehat{(2)}, \widehat{(3)}, \widehat{(4)}$ the map $\|\cdot\|$ generates an *inner product*.
According to this three cases we call the pair $(X, \|\cdot\|)$ a *balancedly weighted vector space* (or 'BW space'), a *normed vector space*, or an *inner product space* (or 'IP space'), respectively.

In this article we shall restrict our considerations to BW spaces which are positive definite, i.e. $\|\vec{x}\| = 0$ only for $\vec{x} = \vec{0}$, i.e. they fulfil $\widehat{(2)}$.

See also the interesting paper [11] where it has been shown that $\widehat{(1)}, \widehat{(2)}, \widehat{(4)}$ is sufficient to get $\widehat{(3)}$, and therefore to get an inner product space.

Remark 2.1. In a positive definite BW space $(X, \|\cdot\|)$ we can generate a 'distance' d by $d(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\|$. Note that generally the distance d is not a metric.

Let $< . \mid . > : \ X^2 \longrightarrow \mathbb{R}$ be a continuous map from the product space $X \times X$ into the Euclidean space $\mathbb{R}$.

We consider the following conditions.

$\overline{(1)}$: For all $r \in \mathbb{R}$ and $\vec{x}, \vec{y} \in X$ it holds $< r \cdot \vec{x} \mid \vec{y} > = \ r \cdot < \vec{x} \mid \vec{y} >$ ('homogeneity'),

$\overline{(2)}$: for all $\vec{x}, \vec{y} \in X$ it holds $< \vec{x} \mid \vec{y} > = < \vec{y} \mid \vec{x} >$ ('symmetry'),

$\overline{(3)}$: for all $\vec{x} \in X$ we have $< \vec{x} \mid \vec{x} > \ \geq \ 0$ ('positive semidefiniteness'),

$\overline{(4)}$: $< \vec{x} \mid \vec{x} > = 0$ if and only if $\vec{x} = \vec{0}$ ('definiteness'),

$\overline{(5)}$: for all $\vec{x}, \vec{y}, \vec{z} \in X$ it holds $< \vec{x} \mid \vec{y} + \vec{z} > = < \vec{x} \mid \vec{y} > + < \vec{x} \mid \vec{z} >$ ('linearity in the second component').

If $< . \mid . >$ fulfils $\overline{(1)}, \overline{(2)}, \overline{(3)}$, we call $< . \mid . >$ a *homogeneous product* on X,
if $< . \mid . >$ fulfils $\overline{(1)}, \overline{(2)}, \overline{(3)}, \overline{(4)}, \overline{(5)}$, the map $< . \mid . >$ is an *inner product* on X.
According to these cases we call the pair $(X, < . \mid . >)$ a *homogeneous product vector space*, or an *inner product space* (or IP space), respectively.

Remark 2.2. We use the term 'IP space' twice, but both definitions coincide. It is well-known that a norm is based on an inner product if and only if the parallelogram identity holds.

Let $\|\cdot\|$ denote a positive functional on a vector space X. We define two closed subsets of X.

$$\mathbf{S} := \mathbf{S}_{(X,\|\cdot\|)} := \{ \vec{x} \in X \mid \|\vec{x}\| = 1 \}, \text{ the } \textit{unit sphere} \text{ of } X,$$
$$\mathbf{B} := \mathbf{B}_{(X,\|\cdot\|)} := \{ \vec{x} \in X \mid \|\vec{x}\| \leq 1 \}, \text{ the } \textit{unit ball} \text{ of } X.$$

Assume that the real vector space X is provided with a positive functional $\|\cdot\|$ and a product $< . \mid . >$. The triple $(X, \|\cdot\|, < . \mid . >)$ satisfies the *Cauchy-Schwarz-Bunjakowsky Inequality* or CSB inequality if and only if for all $\vec{x}, \vec{y} \in X$ we have the inequality

$$| < \vec{x} \mid \vec{y} > | \ \leq \ \|\vec{x}\| \cdot \|\vec{y}\| \, .$$

Let A be an arbitrary subset of a real vector space X. Let A has the property that for arbitrary $\vec{x}, \vec{y} \in A$ and for every $0 \leq t \leq 1$ it holds $t \cdot \vec{x} + (1-t) \cdot \vec{y} \in A$. Such a set A is called *convex*. The unit ball $\mathbf{B}$ in a normed space is convex because of the triangle inequality.

A convex set A is called *strictly convex* if and only if for each number $0 < t < 1$ it holds that the linear combination $t \cdot \vec{x} + (1-t) \cdot \vec{y}$ lies in the interior of A, for all distinct vectors $\vec{x}, \vec{y} \in A$.

We call a BW space *convex* if its unit ball is convex. A positive definite convex BW space is a normed space.

For two real numbers $a < b$ the term $[a, b]$ means the closed interval of a and b, while (a, b) means the pair of two numbers or the open interval between a and b.

3. Some Balancedly Weighted Vector Spaces

We describe easy examples of positive definite balanced weights on the usual vector space $\mathbb{R}^2$. First for each $p > 0$ we introduce a known balanced weight $\| \cdot \|_p$. For $\vec{x} = (x, y) \in \mathbb{R}^2$ we define

$$\|\vec{x}\|_p := \sqrt[p]{|x|^p + |y|^p} \,,$$

and for $p = \infty$ we set $\|\vec{x}\|_\infty := \max\{|x|, |y|\}$.

The functional $\| \cdot \|_p$ is called a *Hölder weight* on $\mathbb{R}^2$. It holds that $\| \cdot \|_p$ is a norm if and only if $p \geq 1$ (the *Hölder norms*). For $p = 2$ we get the ordinary Euclidean norm on $\mathbb{R}^2$.

The next two examples are a little unusual. We construct the weight $\| \cdot \|_A$ by fixing the unit sphere $\mathbf{S}_A$,

$$\mathbf{S}_A \ := \ \left\{(x,y) \in \mathbb{R}^2 \mid \sqrt{|x|^2 + |y|^2} = 1 \ \wedge (x,y) \notin \{(1,0),(-1,0)\}\right\} \bigcup \ \{(2,0)\,,(-2,0)\}\,,$$

and extending the weight $\| \cdot \|_A$ by homogeneity.

In a similar way the weight $\| \cdot \|_B$ is constructed by fixing the unit sphere $\mathbf{S}_B$,

$$\mathbf{S}_B \ := \ \left\{(x,y) \in \mathbb{R}^2 \mid \sqrt{|x|^2 + |y|^2} = 1 \ \wedge (x,y) \notin \{(1,0),(-1,0)\}\right\} \bigcup \ \left\{\left(\frac{1}{2},0\right),\left(-\frac{1}{2},0\right)\right\},$$

and extending the weight $\| \cdot \|_B$ by homogeneity.

The pairs $(\mathbb{R}^2, \| \cdot \|_A)$ and $(\mathbb{R}^2, \| \cdot \|_B)$ are positive definite BW spaces.

4. On Angle Spaces

In the usual Euclidean plane $\mathbb{R}^2$ angles are considered for more than 2000 years. With the idea of 'metrics' and 'norms' others than the Euclidean one, the idea came to have also orthogonality and angles in generalized metric and normed spaces, respectively. The first attempt to define a concept of generalized 'angles' on metric spaces was made by Menger [6, p.749]. Since then a few ideas have been developed, see the references [2, 3, 5, 7, 8, 9, 12, 14, 15].

In this paper we focus our attention on real BW spaces as a generalization of real inner product spaces. Let $(X, < . \mid . >)$ be an IP space, and let $\| \cdot \|$ be the associated norm, $\|\vec{x}\| := \sqrt{< \vec{x} | \vec{x} >}$. The triple $(X, \| \cdot \|, < . \mid . >)$ fulfils the CSB inequality, and we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the well-known Euclidean angle $\angle_{Euclid}(\vec{x}, \vec{y}) = \arccos \frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|}$ with all its comfortable properties (An 1) - (An 7) from the introduction.

Definition 4.1. Let $(X, \|\cdot\|)$ be a real positive definite BW space. Let $\angle_X$ be a real-valued map for pairs $\vec{x}, \vec{y} \neq \vec{0}$. We call the triple $(X, \|\cdot\|, \angle_X)$ an *angle space* if and only if the seven conditions (An 1) - (An 7) are fulfilled for $\angle_X$. The map $\angle_X$ is called an *angle*.

Furthermore we write down four more properties which seem to us 'desirable', but 'not absolutely necessary'. They all are satisfied by each IP space, too.

- (An 8) For all $\vec{x}, \vec{y},\ \vec{x}+\vec{y} \in X\backslash\{\vec{0}\}$ it holds
$$\angle_X(\vec{x}, \vec{x}+\vec{y}) + \angle_X(\vec{x}+\vec{y}, \vec{y}) \ = \ \angle_X(\vec{x}, \vec{y}).$$
- (An 9) For all $\vec{x}, \vec{y},\ \vec{x}-\vec{y} \in X\backslash\{\vec{0}\}$ it holds
$$\angle_X(\vec{x}, \vec{y}) + \angle_X(-\vec{x}, \vec{y}-\vec{x}) + \angle_X(-\vec{y}, \vec{x}-\vec{y}) \ = \ \pi.$$
- (An 10) For all $\vec{x}, \vec{y},\ \vec{x}-\vec{y} \in X\backslash\{\vec{0}\}$ it holds
$$\angle_X(\vec{y}, \vec{y}-\vec{x}) + \angle_X(\vec{x}, \vec{x}-\vec{y}) \ = \ \angle_X(-\vec{x}, \vec{y}).$$
- (An 11) For any two linear independent vectors $\vec{x}, \vec{y} \in X$ there is a decreasing homeomorphism
$$\mathbb{R} \xrightarrow{\cong} (0, \pi), \quad t \mapsto \angle_X(\vec{x}, \vec{y}+t\cdot\vec{x}).$$

Remark 4.2. We mention another condition, which is more a suggestion. Assume that for all normed spaces $(X, \|\cdot\|)$ we have constructed any real valued map $\angle_X$ such that the triple $(X, \|\cdot\|, \angle_X)$ is an 'angle space' as fixed by Definition 4.1. In the special case of an IP space $(X, \|\cdot\|)$ it should hold that $\angle_X = \angle_{Euclid}$, i.e. the new angle should coincide with the Euclidean angle.

5. An Infinite Set of Angles

Assume that a real topological vector space (X, τ) is provided with a positive definite functional $\|\cdot\|$ and a product $< . \mid . >$. We take two elements $\vec{x}, \vec{y} \in X\backslash\{\vec{0}\}$ with the property $\mid < \vec{x} \mid \vec{y} > \mid \ \leq \ \|\vec{x}\| \cdot \|\vec{y}\|$. Hence we could define an angle between these two elements, $\angle(\vec{x}, \vec{y}) := \arccos \frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|}$. If the triple $(X, \|\cdot\|, < . \mid . >)$ satisfies the CSB inequality we would be able to define for all $\vec{x}, \vec{y} \in X\backslash\{\vec{0}\}$ this angle $\angle(\vec{x}, \vec{y}) := \arccos \frac{<\vec{x} \mid \vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|} \in [0, \pi]$.

Let the pair $(X, \|\cdot\|)$ be a real BW space, i.e. the weight $\|\cdot\|$ is absolute homogeneous, or 'balanced'. Let ϱ be an arbitrary real number. In the introduction in Definition 1.1 we defined the continuous product $< . \mid . >_\varrho$ on X.

For the coming discussions it is very useful to introduce some abbreviations. For arbitrary vectors $\vec{x}, \vec{y} \neq \vec{0}$ (hence $\|\vec{x}\| \cdot \|\vec{y}\| \neq 0$) we define two non-negative real numbers $\mathbf{s}$ and $\mathbf{d}$,

$$\mathbf{s} \ := \ \mathbf{s}(\vec{x}, \vec{y}) \ := \ \left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|, \quad \text{and} \quad \mathbf{d} \ := \ \mathbf{d}(\vec{x}, \vec{y}) \ := \ \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\| ,$$

and also two real numbers $\mathbf{\Sigma}$ and $\mathbf{\Delta}$, the latter can be negative,

$$\mathbf{\Sigma} \ := \ \mathbf{\Sigma}(\vec{x}, \vec{y}) \ := \ \mathbf{s}^2 + \mathbf{d}^2 \ = \ \left\| \tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \left\| \tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 ,$$

and

$$\mathbf{\Delta} \ := \ \mathbf{\Delta}(\vec{x}, \vec{y}) \ := \ \mathbf{s}^2 - \mathbf{d}^2 \ = \ \left\| \tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 .$$

All defined four variables depend on two vectors $\vec{x}, \vec{y} \neq \vec{0}$. Since $(X, \|\cdot\|)$ is positive definite, $\mathbf{\Sigma}$ must be a positive number. Note the inequality $0 \le |\mathbf{\Delta}| \le \mathbf{\Sigma}$. With these abbreviations the formula in Definition 1.1 is shortened to

$$< \vec{x} \,|\, \vec{y} >_\varrho \;=\; \begin{cases} 0 & \text{for } \vec{x} = \vec{0} \ \text{ or } \ \vec{y} = \vec{0}\,, \\ \|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta} \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho & \text{for } \vec{x}, \vec{y} \ \neq \vec{0}\,. \end{cases}$$

Lemma 5.1. *In the case that $(X, \|\cdot\|)$ is already an* IP *space with the inner product $< . \,|\, . >_{IP}$, the product from Definition 1.1 corresponds to the inner product, i.e. for all $\vec{x}, \vec{y} \in X$ it holds the equation*

$$< \vec{x} \,|\, \vec{y} >_{IP} \;=\; < \vec{x} \,|\, \vec{y} >_\varrho \quad \textit{for all } \varrho \in \mathbb{R}.$$

Proof. Let the normed space $(X, \|\cdot\|)$ be an inner product space or 'IP space'. For two elements $\vec{x}, \vec{y} \in X$ we can express the inner product by its norms, i.e. we have

$$< \vec{x} \mid \vec{y} >_{IP} \;=\; \frac{1}{4} \cdot \left(\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 \right),$$

and by the properties of the inner product we can write for $\vec{x}, \vec{y} \ne \vec{0}$

$$\begin{aligned} < \vec{x} \mid \vec{y} >_{IP} \;&= \|\vec{x}\| \cdot \|\vec{y}\| \cdot < \frac{\vec{x}}{\|\vec{x}\|} \mid \frac{\vec{y}}{\|\vec{y}\|} >_{IP} \\ &= \|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] . \end{aligned}$$

This shows the equation $< \vec{x} \,|\, \vec{y} >_{IP} = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta}$. Further, in inner product spaces the parallelogram identity holds, that means for unit vectors $\vec{v}$ and $\vec{w}$ we have

$$\|\vec{v} + \vec{w}\|^2 + \|\vec{v} - \vec{w}\|^2 \;=\; 2 \cdot \left(\|\vec{v}\|^2 + \|\vec{w}\|^2\right) \;=\; 4\,.$$

It follows for the unit vectors $\frac{\vec{x}}{\|\vec{x}\|}$ and $\frac{\vec{y}}{\|\vec{y}\|}$

$$\frac{1}{4} \cdot \left(\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right) \;=\; \frac{1}{4} \cdot (\mathbf{s}^2 + \mathbf{d}^2) \;=\; \frac{1}{4} \cdot \mathbf{\Sigma} \;=\; 1\,,$$

and Lemma 5.1 is proven. □

Lemma 5.2. *For a positive definite* BW *space $(X, \|\cdot\|)$ the pair $(X, < . \,|\, . >_\varrho)$ is a homogeneous product vector space, with $\|\vec{x}\| = \sqrt{< \vec{x} \,|\, \vec{x} >_\varrho}$, for all $\vec{x} \in X$ and for all real numbers ϱ.*

Proof. We have $< . \,|\, . >_\varrho \colon X^2 \longrightarrow \mathbb{R}$, and the properties $\overline{(2)}$ (symmetry) and $\overline{(3)}$ (positive semidefiniteness) are rather trivial. Clearly, $\|\vec{x}\| = \sqrt{< \vec{x} \,|\, \vec{x} >_\varrho}$ for all $\vec{x} \in X$. We show $\overline{(1)}$, the homogeneity. For a real number $r > 0$ it holds $< r \cdot \vec{x} \mid \vec{y} >_\varrho = r \cdot < \vec{x} \mid \vec{y} >_\varrho$, because $(X, \|\cdot\|)$ satisfies $\widehat{(1)}$. Now we prove $< -\vec{x} \mid \vec{y} >_\varrho = - < \vec{x} \mid \vec{y} >_\varrho$. Let $\vec{x}, \vec{y} \ne \vec{0}$. Note that the factor $\mathbf{\Sigma}$ is not affected by a negative sign at $\vec{x}$ or $\vec{y}$. We have

$$- < \vec{x} \mid \vec{y} >_\varrho \;=\; -\, \tfrac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left(\left\| \tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right) \cdot \left(\tfrac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho ,$$

and as well as

$$< -\vec{x} \mid \vec{y} >_\varrho = \tfrac{1}{4} \cdot \| -\vec{x}\| \cdot \|\vec{y}\| \cdot \left(\left\| \tfrac{-\vec{x}}{\|-\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \tfrac{-\vec{x}}{\|-\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right) \cdot \left(\tfrac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho$$

$$= \tfrac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left(\left\| \tfrac{\vec{y}}{\|\vec{y}\|} - \tfrac{\vec{x}}{\|\vec{x}\|} \right\|^2 - \left\| \tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right) \cdot \left(\tfrac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho ,$$

hence $< -\vec{x} \mid \vec{y} >_\varrho = \; - < \vec{x} \mid \vec{y} >_\varrho$. Then for each number $r < 0$ it follows the equation $< r \cdot \vec{x} \mid \vec{y} >_\varrho = \; r \cdot < \vec{x} \mid \vec{y} >_\varrho$, and the homogeneity $\overline{(1)}$ is proven. □

Let ϱ be a real number. For positive definite BW spaces $(X, \|\cdot\|)$ for two elements $\vec{x}, \vec{y} \in X \backslash \{\vec{0}\}$ with the additional property $| < \vec{x} \,|\, \vec{y} >_\varrho | \;\leq\; \|\vec{x}\| \cdot \|\vec{y}\|$, or equivalently, $\left|\tfrac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\tfrac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho \leq 1$, we defined in Definition 1.2 the 'ϱ-angle' $\angle_\varrho(\vec{x}, \vec{y})$. There we had set

$$\angle_\varrho(\vec{x}, \vec{y}) \;=\; \arccos \frac{< \vec{x} \,|\, \vec{y} >_\varrho}{\|\vec{x}\| \cdot \|\vec{y}\|} \;=\; \arccos \left(\frac{1}{4} \cdot \mathbf{\Delta} \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma} \right)^\varrho \right) \;=$$

$$\arccos \left(\tfrac{1}{4} \cdot \left[\left\| \tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \cdot \left\langle \tfrac{1}{4} \cdot \left[\left\| \tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \left\| \tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right\rangle^\varrho \right) .$$

Proposition 5.3. *Let* $(X, < . \,|\, . >_{IP})$ *be an* IP *space with the inner product* $< . \,|\, . >_{IP}$ *and the generated norm* $\|\cdot\|$.

It follows that the triple $(X, \|\cdot\|, < . \,|\, . >_\varrho)$ *fulfils the* CSB *inequality, and the well-known Euclidean angle corresponds to the* ϱ*-angle, i.e. for all* $\vec{x}, \vec{y} \neq \vec{0}$ *it holds* $\angle_\varrho(\vec{x}, \vec{y}) = \angle_{Euclid}(\vec{x}, \vec{y})$ *for each real number* ϱ.

Proof. In Lemma 5.1 it was shown that $< . \,|\, . >_{IP} \;=\; < . \,|\, . >_\varrho$ holds for all real numbers ϱ. □

Lemma 5.4. *For positive definite* BW *spaces* $(X, \|\cdot\|)$ *for an arbitrary element* $\vec{x} \in X \backslash \{\vec{0}\}$, *i.e.* $\|\vec{x}\| > 0$, *the 'angles'* $\angle_\varrho(\vec{x}, \vec{x})$ *and* $\angle_\varrho(\vec{x}, -\vec{x})$ *always exist, with* $\angle_\varrho(\vec{x}, \vec{x}) = 0$ *and* $\angle_\varrho(\vec{x}, -\vec{x}) = \pi$. *That means* (An 2) *and* (An 3) *from the introduction are fulfilled, for every number* $\varrho \in \mathbb{R}$.

Proof. Trivial if we use that $\|\cdot\|$ is balanced and $\|\vec{0}\| = 0$. □

Now the reader should take a short look on (An 4) - (An 7) from the introduction to prepare the following proposition.

Proposition 5.5. *Assume a positive definite* BW *space* $(X, \|\cdot\|)$ *and two fixed vectors* $\vec{x}, \vec{y} \in X$ *such that the* ϱ*-angle* $\angle_\varrho(\vec{x}, \vec{y})$ *is defined for a fixed number* ϱ.

In this case the following ϱ*-angles are also defined, and it holds*

- *(a)* $\angle_\varrho(\vec{x}, \vec{y}) = \angle_\varrho(\vec{y}, \vec{x})$,
- *(b)* $\angle_\varrho(r \cdot \vec{x}, s \cdot \vec{y}) = \angle_\varrho(\vec{x}, \vec{y})$ *for all positive real numbers* r, s,
- *(c)* $\angle_\varrho(-\vec{x}, -\vec{y}) = \angle_\varrho(\vec{x}, \vec{y})$,
- *(d)* $\angle_\varrho(\vec{x}, \vec{y}) + \angle_\varrho(-\vec{x}, \vec{y}) = \pi$.

Proof. Easy. We defined $\angle_\varrho(\vec{x}, \vec{y}) \;=\; \arccos \frac{<\vec{x} \,|\, \vec{y}>_\varrho}{\|\vec{x}\| \cdot \|\vec{y}\|}$, and in Lemma 5.2 we proved that the space $(X, < . \,|\, . >_\varrho)$ is a homogeneous product vector space. We have that (a) is true since $< . \,|\, . >_\varrho$ is symmetrical. We have (b) and (c) because the product is homogeneous, i.e. $< r \cdot \vec{x} \,|\, s \cdot \vec{y} >_\varrho \;=\; r \cdot s \cdot < \vec{x} \,|\, \vec{y} >_\varrho$ for all

$r, s \in \mathbb{R}$, as well as $\|r \cdot \vec{y}\| = |r| \cdot \|\vec{y}\|$ for real numbers r. And (d) follows because $\arccos(v) + \arccos(-v) = \pi$ holds for all v from the interval $[-1, 1]$. $\square$

We consider three special cases, let us take ϱ from the set $\{1, 0, -1\}$. For two vectors $\vec{x}, \vec{y} \neq \vec{0}$ of a positive definite BW space we assume $| < \vec{x} | \vec{y} >_1 | \leq \|\vec{x}\| \cdot \|\vec{y}\|$ or $| < \vec{x} | \vec{y} >_0 | \leq \|\vec{x}\| \cdot \|\vec{y}\|$ or $| < \vec{x} | \vec{y} >_{-1} | \leq \|\vec{x}\| \cdot \|\vec{y}\|$, respectively. In Definition 1.2 we defined the 'ϱ-angle' $\angle_\varrho$, including the cases $\angle_1, \angle_0, \angle_{-1}$. We have

$$\angle_1(\vec{x}, \vec{y}) = \arccos\left(\frac{1}{16} \cdot \left[\left\|\frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|}\right\|^4 - \left\|\frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|}\right\|^4 \right]\right),$$

$$\angle_0(\vec{x}, \vec{y}) = \arccos\left(\frac{1}{4} \cdot \left[\left\|\frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|}\right\|^2 - \left\|\frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|}\right\|^2 \right]\right),$$

$$\angle_{-1}(\vec{x}, \vec{y}) = \arccos\left(\frac{\left\|\frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|}\right\|^2 - \left\|\frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|}\right\|^2}{\left\|\frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|}\right\|^2 + \left\|\frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|}\right\|^2}\right) = \arccos\left(\frac{\mathbf{s}^2 - \mathbf{d}^2}{\mathbf{s}^2 + \mathbf{d}^2}\right).$$

Remark 5.6. The angle $\angle_0$ reflects the fact that the cosine of an inner angle in a rhombus with the side length 1 can be expressed as the fourth part of the difference of the squares of the two diagonals, while $\angle_{-1}$ means that the cosine of an inner angle in a rhombus with the side length 1 is the difference of the squares of the two diagonals divided by its sum.

Proposition 5.7. *Assume a real normed space* $(X, \|\cdot\|)$, *let* $\vec{x}, \vec{y} \in X\backslash\{\vec{0}\}$. *Let the number* ϱ *be from the set* $\{1, 0, -1\}$.

(a) *The triple* $(X, \|\cdot\|, < . | . >_\varrho)$ *fulfils the* CSB *inequality, i.e. the 'ϱ- angle'* $\angle_\varrho(\vec{x}, \vec{y})$ *is defined for all* $\vec{x}, \vec{y} \neq \vec{0}$.

(b) *The triple* $(X, \|\cdot\|, \angle_\varrho)$ *fulfils all seven demands* (`An` 1) - (`An` 7)*, i.e. it is an 'angle space' as it has been defined in Definition 4.1.*

(c) *The triple* $(X, \|\cdot\|, \angle_\varrho)$ *generally does not fulfil* (`An` 8), (`An` 9), (`An` 10).

(d) *In the special case of* $\varrho = 0$ *the triple* $(X, \|\cdot\|, \angle_0)$ *fulfils* (`An` 11).

As we mentioned in the introduction, for the first property (a) we say that 'the normed vector space $(X, \|\cdot\|)$ has the angle $\angle_\varrho$', for $\varrho \in \{1, 0, -1\}$.

Proof. (a) We show the CSB inequality for $\varrho = 1$. Since $(X, \|\cdot\|)$ is a normed vector space, because of the triangle inequality and $\left\|\frac{\vec{x}}{\|\vec{x}\|}\right\| = 1$ we get

$$|< \vec{x} | \vec{y} >_1| = \left|\tfrac{1}{16} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\|\tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|}\right\|^4 - \left\|\tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|}\right\|^4 \right]\right| \leq$$

$$\tfrac{1}{16} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \max\left\{\left\|\tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|}\right\|^4, \left\|\tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|}\right\|^4\right\} \leq \tfrac{1}{16} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot 2^4 = \|\vec{x}\| \cdot \|\vec{y}\|.$$

The same way works with $\varrho = 0$, and it is obvious that the CSB inequality holds for $\varrho = -1$.

(b) (`An` 1) is fulfilled because the map $\angle_\varrho : \left(X\backslash\{\vec{0}\}\right)^2 \longrightarrow [0, \pi]$ is continuous. The demands (`An` 2) and (`An` 3) are shown in Lemma 5.4. For (`An` 4) - (`An` 7) see Proposition 5.5.

(*c*) Recall the pairs $(\mathbb{R}^2, \|\cdot\|_p)$ with the 'Hölder weights' $\|\cdot\|_p$, $p > 0$, $\|(x_1,x_2)\|_p = \sqrt[p]{|x_1|^p + |x_2|^p}$. Let us take, for instance, $p = 1$, because it is easy to calculate with it. Let $\vec{x} := (1,0)$, $\vec{y} := (0,1)$, both are unit vectors in the normed space $(\mathbb{R}^2, \|\cdot\|_1)$. We choose $\varrho := 0$. We have

$$\begin{aligned}
\angle_0(\vec{x},\vec{y}) &= \arccos\left(\frac{1}{4}\cdot\left[\ \left\|\frac{\vec{x}}{\|\vec{x}\|_1}+\frac{\vec{y}}{\|\vec{y}\|_1}\right\|_1^2 - \left\|\frac{\vec{x}}{\|\vec{x}\|_1}-\frac{\vec{y}}{\|\vec{y}\|_1}\right\|_1^2\ \right]\right)\\
&= \arccos\left(\frac{1}{4}\cdot\left[\ \|(1,0)+(0,1)\|_1^2 - \|(1,0)-(0,1)\|_1^2\ \right]\right)\\
&= \arccos\left(\frac{1}{4}\cdot[\ 4 - 4\ ]\right) = \arccos(0) = \frac{\pi}{2} = 90\,\mathrm{deg},\\
\angle_0(\vec{x},\vec{x}+\vec{y}) &= \arccos\left(\frac{1}{4}\cdot\left[\ \left\|(1,0)+\frac{1}{2}\cdot(1,1)\right\|_1^2 - \left\|(1,0)-\frac{1}{2}\cdot(1,1)\right\|_1^2\ \right]\right)\\
&= \arccos\left(\frac{1}{4}\cdot\left[\ (2)^2 - (1)^2\ \right]\right) = \arccos\left(\frac{3}{4}\right) \approx 41.41\,\mathrm{deg}\,.
\end{aligned}$$

With similar calculations we get $\angle_0(\vec{x}+\vec{y},\vec{y}) = \arccos\left(\frac{3}{4}\right)$, and this contradicts (An 8).
The property (An 9) means that the sum of the inner angles of a triangle is π. We can use the same example of the normed space $(\mathbb{R}^2, \|\cdot\|_1)$ and the same unit vectors $(1,0)$ and $(0,1)$ to find counterexamples for (An 9) and (An 10).
(*d*) This is the main content of [13] on 'arXiv'.
The proof of the proposition is complete. □

Remark 5.8. Note that one ϱ-angle was considered first by Pavle M. Miličić, see the references [7, 8, 9], where he dealt with the case $\varrho = 1$. He named his angle as the 'g-angle'. In the recent article [10] it is shown that the different definitions of the angle $\angle_1$ and the 'g-angle' are equivalent at least in *quasi-inner product spaces*. Following an idea in [12], the case $\varrho = 0$ was introduced by the author in [13]. There it has been called the 'Thy-angle'. In [10] some properties of the g-angle and the Thy-angle are compared.

6. On Classes and Corners

We define some classes of real BW spaces and real normed spaces.

Definition 6.1. Let pdBW be the class of all real positive definite BW spaces.
Let NORM be the class of all real normed vector spaces.
Let IPspace be the class of all real inner product spaces (or IP spaces).
For a fixed real number ϱ let

$$\mathsf{NORM}_\varrho := \{(X,\|\cdot\|) \in \mathsf{NORM} \mid \text{ The normed space } (X,\|\cdot\|) \text{ has the angle } \angle_\varrho\},$$
$$\mathsf{pdBW}_\varrho := \{(X,\|\cdot\|) \in \mathsf{pdBW} \mid (X,\|\cdot\|) \text{ has the angle } \angle_\varrho\}.$$

We have IPspace $\subset$ NORM $\subset$ pdBW $\subset$ BW spaces and $\mathsf{NORM}_\varrho \subset \mathsf{pdBW}_\varrho$, of course.

Proposition 6.2. *Let ϱ be a fixed real number. For every element $(X, \|\cdot\|)$ of pdBW_ϱ it holds that the triple $(X, \|\cdot\|, \angle_\varrho)$ is an angle space as it has been defined in Definition 4.1.*

Proof. By definition of the class pdBW_ϱ the space $(X, \|\cdot\|)$ has the angle $\angle_\varrho$, i.e. for each pair $\vec{x}, \vec{y} \neq \vec{0}$ the angle $\angle_\varrho(\vec{x}, \vec{y})$ exists. Please see Lemma 5.4 and Proposition 5.5. There we find that the six properties (`An` 2) - (`An` 7) from the introduction are fulfilled. (`An` 1) is trivial. $\square$

Proposition 6.3. *It holds*

$$\mathsf{pdBW} = \mathsf{pdBW}_{-\mathbf{1}} \quad \textit{and} \quad \mathsf{NORM} = \mathsf{NORM}_{-\mathbf{1}} = \mathsf{NORM}_{\mathbf{0}} = \mathsf{NORM}_{\mathbf{1}}.$$

Proof. We had defined $\angle_{-1}(\vec{x}, \vec{y}) = \arccos\left(\frac{\mathbf{s}^2 - \mathbf{d}^2}{\mathbf{s}^2 + \mathbf{d}^2}\right)$. Therefore for all $\vec{x}, \vec{y} \neq \vec{0}$ this angle exists always. For the second claim see Proposition 5.7. $\square$

Theorem 6.4. *Let $\alpha, \beta, \gamma, \delta$ be four real numbers, α and β may be negative, with $-\delta < -\gamma < -1 < \alpha < \beta$. There are inclusions*

$$\mathsf{pdBW}_{-\delta} \subset \mathsf{pdBW}_{-\gamma} \subset \mathsf{pdBW} \supset \mathsf{pdBW}_{\alpha} \supset \mathsf{pdBW}_{\beta}\ .$$

Proof. First we consider $-1 < \alpha < \beta$. Let $(X, \|\cdot\|) \in \mathsf{pdBW}_\beta$. By Definition 6.1, for each pair of two vectors $\vec{x}, \vec{y} \neq \vec{0}$ the angle $\angle_\beta(\vec{x}, \vec{y})$ is defined. By Definition 1.2, this means that the triple $(X, \|\cdot\|, <\,.\mid.\,>_\beta)$ fulfils the CSB inequality, i.e. for any pair $\vec{x}, \vec{y} \neq \vec{0}$ of vectors we have the inequality

$$|<\vec{x} \mid \vec{y}>_\beta| \leq \|\vec{x}\| \cdot \|\vec{y}\|\ , \quad \text{i.e.} \quad \left|\ \|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta} \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\beta\ \right| \leq \|\vec{x}\| \cdot \|\vec{y}\|\ ,$$

$$\text{or equivalently} \quad \left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\beta \leq 1\ .$$

To prove that the angle $\angle_\alpha(\vec{x}, \vec{y})$ exists we have to show the corresponding inequality

$$\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\alpha \leq 1\,.$$

We distinguish two cases. In the first case of $\frac{1}{4} \cdot \mathbf{\Sigma} \geq 1$ we have for all real numbers $\kappa \leq \beta$

$$\left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\kappa \leq \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\beta.$$

Since $\alpha < \beta$ it follows

$$0 \leq \left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\alpha \leq \left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\beta \leq 1,$$

and the angle $\angle_\alpha(\vec{x}, \vec{y})$ exists.

For the second case we assume $\frac{1}{4} \cdot \mathbf{\Sigma} < 1$. That means for any positive exponent κ the inequality $\left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\kappa < 1$. Now note $0 \leq \left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \leq \frac{1}{4} \cdot \mathbf{\Sigma} < 1$. In the subcase of a positive α it follows the inequality $\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\alpha < 1$, and the angle $\angle_\alpha(\vec{x}, \vec{y})$ exists.

If α is from the interval $[-1,0]$, i.e. $-\alpha\in[0,1]$, we can write the inequality

$$1\geq\frac{|\mathbf{\Delta}|}{\mathbf{\Sigma}}=\frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\frac{1}{4}\cdot\mathbf{\Sigma}}\geq\frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\alpha}}\geq\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\;.$$

Again we get the desired inequality $\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\alpha}\leq 1$, and the angle $\angle_{\alpha}(\vec{x},\vec{y})$ exists. We get that $(X,\|\cdot\|)$ is an element of pdBW_{α}, too.

We look at $-\delta<-\gamma<-1$. We have $1<\gamma<\delta$.
Let $(X,\|\cdot\|)\in\mathsf{pdBW}_{-\delta}$, and take two vectors $\vec{x},\vec{y}\in X$, $\vec{x},\vec{y}\neq\vec{0}$. The angle $\angle_{-\delta}(\vec{x},\vec{y})$ exists. Hence we have the inequality $\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\delta}\leq 1$.
As above we distinguish two cases. The first case is $\frac{1}{4}\cdot\mathbf{\Sigma}\geq 1$. We have

$$1\geq\frac{|\mathbf{\Delta}|}{\mathbf{\Sigma}}=\frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\frac{1}{4}\cdot\mathbf{\Sigma}}\geq\frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\gamma}}\geq\frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\delta}}=\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\delta}\;.$$

We get the inequality $\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\gamma}\leq 1$. It follows that the angle $\angle_{-\gamma}(\vec{x},\vec{y})$ exists.
The second case is $\frac{1}{4}\cdot\mathbf{\Sigma}<1$. We get

$$0\leq\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\delta}\leq\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\gamma}\leq\frac{1}{4}\cdot\mathbf{\Sigma}\;,\quad\text{hence}\quad\frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\gamma}}\leq\frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\delta}}\leq 1,$$

and the angle $\angle_{-\gamma}(\vec{x},\vec{y})$ exists. This proves $(X,\|\cdot\|)\in\mathsf{pdBW}_{-\gamma}$, and Theorem 6.4 is shown. □

Corollary 6.5.
It holds $\mathsf{NORM}=\mathsf{NORM}_{\varrho}$ *for all real numbers* ϱ *from the closed interval* $[-1,1]$.

Proof. See both Proposition 6.3 and Theorem 6.4. □

Corollary 6.6. *Let us take four positive numbers* $\alpha,\beta,\gamma,\delta$ *with*
$$-\delta<-\gamma<-1<1<\alpha<\beta.$$
There are inclusions $\mathsf{NORM}_{-\delta}\subset\mathsf{NORM}_{-\gamma}\subset\mathsf{NORM}\supset\mathsf{NORM}_{\alpha}\supset\mathsf{NORM}_{\beta}$.

Proof. This follows directly from Theorem 6.4. □

Theorem 6.7.
$$\mathsf{IPspace}\;=\;\bigcap_{\varrho\in\mathbb{R}}\;\mathsf{NORM}_{\varrho}$$

Proof. " $\subset$ ": This is trivial with Proposition 5.3.
" $\supset$ ": This is not trivial, but easy. We show that a real normed space $(X,\|\cdot\|)$ which in not an inner product space is not an element of NORM_{ϱ} for at least one real number ϱ.
Let $(X,\|\cdot\|)$ be a real normed space which in not an inner product space. Hence it must exist a two dimensional subspace U of X such that its unit sphere $\mathbf{S}\cap\mathsf{U}$ is not an ellipse. Hence there are two unit vectors $\vec{v},\vec{w}\in\mathbf{S}\cap\mathsf{U}$ such that the parallelegram identity is not fulfilled, i.e. it holds

$$\|\vec{v}+\vec{w}\|^{2}+\|\vec{v}-\vec{w}\|^{2}\neq 4=2\cdot\left[\|\vec{v}\|^{2}+\|\vec{w}\|^{2}\right]\;.$$

(Case A): First we assume $\|\vec{v}+\vec{w}\| \neq \|\vec{v}-\vec{w}\|$, i.e. $\mathbf{\Delta} := \|\vec{v}+\vec{w}\|^2 - \|\vec{v}-\vec{w}\|^2 \neq 0$. In the case of $\mathbf{\Sigma} := \|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 > 4$, i.e. $\frac{1}{4}\cdot\mathbf{\Sigma} > 1$, we can choose a very big number β such that

$$\left| \; \|\vec{v}\|\cdot\|\vec{w}\|\cdot\frac{1}{4}\cdot\mathbf{\Delta}\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\beta} \; \right| > \|\vec{v}\|\cdot\|\vec{w}\| = 1 \ ,$$

and if $\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 < 4$, i.e. $\frac{1}{4}\cdot\mathbf{\Sigma} < 1$, we can find a big γ such that

$$\left| \; \|\vec{v}\|\cdot\|\vec{w}\|\cdot\frac{1}{4}\cdot\mathbf{\Delta}\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\gamma} \; \right| > \|\vec{v}\|\cdot\|\vec{w}\| = 1 \ .$$

We get that the angle $\angle_{\beta}(\vec{v},\vec{w})$ or $\angle_{-\gamma}(\vec{v},\vec{w})$, respectively, does not exist.

(Case B): If we have $\|\vec{v}+\vec{w}\| = \|\vec{v}-\vec{w}\|$, hence $\mathbf{\Delta} = 0$, we have to replace $\vec{w}$ by another unit vector $\widetilde{w}$. Note that $\{\vec{v},\vec{w}\}$ is a linear independent set, since $\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 \neq 4$. We regard a continuous map $\underline{E} : \mathbb{R} \longrightarrow (-1,+1)$, we define

$$\underline{E}(t) \ := \ \frac{1}{4}\cdot\left[\ \left\|\vec{v}+\frac{\vec{w}+t\cdot\vec{v}}{\|\vec{w}+t\cdot\vec{v}\|}\right\|^2 \ - \ \left\|\vec{v}-\frac{\vec{w}+t\cdot\vec{v}}{\|\vec{w}+t\cdot\vec{v}\|}\right\|^2\ \right] \ .$$

For $t=0$ we get $\underline{E}(0) = \frac{1}{4}\cdot\left[\|\vec{v}+\vec{w}\|^2 \ - \ \|\vec{v}-\vec{w}\|^2\right] = \frac{1}{4}\cdot\mathbf{\Delta} = 0$. In the paper [13, Theorem 1] it is proven that the map $\underline{E}$ yields a homeomorphism from $\mathbb{R}$ onto the open interval $(-1,1)$. Hence we can replace the factor $t=0$ by any $\widetilde{t} \neq 0$ such that

$$\underline{E}(\widetilde{t}) \ = \ \frac{1}{4}\cdot\left[\ \left\|\vec{v}+\frac{\vec{w}+\widetilde{t}\cdot\vec{v}}{\|\vec{w}+\widetilde{t}\cdot\vec{v}\|}\right\|^2 \ - \ \left\|\vec{v}-\frac{\vec{w}+\widetilde{t}\cdot\vec{v}}{\|\vec{w}+\widetilde{t}\cdot\vec{v}\|}\right\|^2\ \right] \ \neq \ 0 \ .$$

For each $\widetilde{t}$ we abbreviate the unit vector

$$\widetilde{w} := \frac{\vec{w}+\widetilde{t}\cdot\vec{v}}{\|\vec{w}+\widetilde{t}\cdot\vec{v}\|} \ ,$$

and since $\underline{E}$ is a homeomorphism we can choose a very small $\widetilde{t} \neq 0$ such that still holds $\|\vec{v}+\widetilde{w}\|^2 + \|\vec{v}-\widetilde{w}\|^2 \neq 4$, but $\mathbf{\Delta} := \mathbf{\Delta}(\vec{v},\widetilde{w}) \neq 0$. At this point we can continue as in (Case A).

In both cases (Case A) and (Case B) it follows that $(X,\|\cdot\|)$ is not an element of the classes NORM_{β} or $\mathsf{NORM}_{-\gamma}$, respectively. Now the proof of Theorem 6.7 is finished. □

We define a function Υ which maps every real positive definite BW space to a pair of extended numbers (ν,μ),

$$\Upsilon : \mathsf{pdBW} \longrightarrow [-\infty,-1]\times[-1,+\infty].$$

Definition 6.8. Let $(X, \|\cdot\|)$ be a positive definite balancedly weighted vector space. We define

$$\begin{aligned} \nu &:= \inf\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\}, \\ \mu &:= \sup\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\}, \\ \Upsilon(X, \|\cdot\|) &:= (\nu, \mu). \end{aligned}$$

With Proposition 6.3 we get that ν is from the interval $[-\infty, -1]$ and μ is from the interval $[-1, +\infty]$. If $(X, \|\cdot\|)$ is even a normed vector space we have $\mu \in [+1, +\infty]$. If $(X, \|\cdot\|)$ is even an inner product space it follows from Theorem 6.7 the identity $\Upsilon(X, \|\cdot\|) = (-\infty, +\infty)$.

Proposition 6.9. *Let $(X, \|\cdot\|) \in \mathsf{pdBW}$, i.e. $(X, \|\cdot\|)$ is a positive definite balancedly weighted vector space. Let $\Upsilon(X, \|\cdot\|) = (\nu, \mu)$, and we assume $\nu \neq -\infty$ and $\mu \neq \infty$, i.e. ν and μ are real numbers.*

In this case the infimum and the supremum will be attained, i.e. it holds

$$\nu = \min\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \textit{ has the angle } \angle_\varrho\},$$
$$\mu = \max\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \textit{ has the angle } \angle_\varrho\}.$$

Proof. We show the first claim $\nu = \min\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\}$.

Let us suppose the opposite. We assume two unit vectors $\vec{v}, \vec{w} \in X$ such that the angle $\angle_\nu(\vec{v}, \vec{w})$ does not exist in $(X, \|\cdot\|)$. Hence there is a positive real number ε with

$$1 \;<\; 1+\varepsilon \;=\; |<\vec{v} \mid \vec{w}>_\nu| \;=\; \frac{1}{4} \cdot |\mathbf{\Delta}(\vec{v}, \vec{w})| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}(\vec{v}, \vec{w})\right)^\nu . \tag{6.1}$$

Since $\mathsf{pdBW} = \mathsf{pdBW}_{-\mathbf{1}}$ we have $|<\vec{v} \mid \vec{w}>_{-1}| \leq 1$, and it must be $\nu < -1$. Because $\nu < -1$ and $0 \leq |\mathbf{\Delta}| \leq \mathbf{\Sigma}$ it has to be $\mathbf{\Sigma} < 4$.

We make the exponent ν 'less negative'. By the continuity of the terms in Equation (6.1) we find two positive numbers $\overline{\eta}, \overline{\lambda}$ with the properties

$$\nu < \nu + \overline{\eta} < -1 \quad \text{and} \quad 0 < \overline{\lambda} < \varepsilon \quad \text{such that}$$

$$|<\vec{v} \mid \vec{w}>_{\nu+\overline{\eta}}| \;=\; \frac{1}{4} \cdot |\mathbf{\Delta}| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^{\nu+\overline{\eta}} \;=\; 1 + \overline{\lambda} \;<\; 1 + \varepsilon\,.$$

By the continuity of the product $<.\,|\,.>_\varrho$ we can choose the positive numbers $\overline{\eta}$ and $\overline{\lambda}$ such that we have for all η with $0 \leq \eta \leq \overline{\eta}$ the inequality

$$1 \;<\; 1 + \overline{\lambda} \;\leq\; |<\vec{v} \mid \vec{w}>_{\nu+\eta}|\,.$$

We get for all η from the interval $[\,0, \overline{\eta}\,]$ that the angle $\angle_{\nu+\eta}(\vec{v}, \vec{w})$ does not exist in $(X, \|\cdot\|)$. This contradicts the definition of ν as an infimum. This confirms the first statement of the proposition. $\square$

For the next proposition we need the term of a 'strictly curved space'.

Definition 6.10. A BW space $(X, \|\cdot\|)$ is called *strictly convex* if and only if the interior of the line segment $\{t \cdot \vec{u} + (1-t) \cdot \vec{v} \mid 0 \leq t \leq 1\}$ lies in the interior of the unit ball of $(X, \|\cdot\|)$ for each pair of distinct unit vectors $\vec{u}, \vec{v} \in X$, $\vec{u} \neq \vec{v}$. That means it holds

$$\|t \cdot \vec{u} + (1-t) \cdot \vec{v}\| < 1 \text{ for } 0 < t < 1.$$

We call a BW space $(X, \|\cdot\|)$ *strictly curved* if and only if for each pair of distinct unit vectors $\vec{u}, \vec{v} \in X,\ \vec{u} \neq \vec{v}$, the line segment $\{t \cdot \vec{u} + (1-t) \cdot \vec{v} \mid 0 \leq t \leq 1\}$ contains at least one element which is not a unit vector, i.e. there is a number $0 < \widehat{t} < 1$ with the inequality

$$\left\|\widehat{t} \cdot \vec{u} + (1-\widehat{t}) \cdot \vec{v}\right\| \neq 1.$$

Note that in normed spaces both definitions are equivalent. Further, a positive definite BW space which is strictly convex is a normed space, and it is strictly curved.

Further, a BW space $(X, \|\cdot\|)$ which is not strictly curved must contain a piece of a straight line which is completely in the unit sphere of $(X, \|\cdot\|)$. As examples we can take the two Hölder norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$. The unit spheres of both spaces have the shape of a square.

The Hölder weights $\|\cdot\|_p$ on $\mathbb{R}^2$ with $0 < p < 1$ yield examples of BW spaces which are strictly curved, but not strictly convex.

Proposition 6.11. *Let $(X, \|\cdot\|) \in \mathsf{pdBW}$ with $\Upsilon(X, \|\cdot\|) = (\nu, \mu)$. If the space $(X, \|\cdot\|)$ is not strictly curved it holds the estimate $-1 \leq \mu \leq 1$.*

Proof. Let us consider a BW space $(X, \|\cdot\|)$ which is not strictly curved. As we said above it contains a line segment which is completely in the unit sphere. One way to describe this fact in formulas is that we have two unit vectors $\vec{a}, \vec{b} \in X$ and a real number $0 < \mathsf{z} < 1$ such that

$$\|\vec{a} + t \cdot \vec{b}\| = 1 \quad \text{holds for all} \quad t \in [-\mathsf{z}, \mathsf{z}]\ .$$

Now we show that for each exponent $\varrho > 1$ we can find two unit vectors $\vec{x}, \vec{y}$ with the property $\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho > 1$. This would mean that the ϱ-angle $\angle_\varrho(\vec{x}, \vec{y})$ does not exist.

Let us take the unit vectors $\vec{x} := \vec{a} + t \cdot \vec{b}$ and $\vec{y} := \vec{a} - t \cdot \vec{b}$ for $0 \leq t \leq \mathsf{z}$. With $\mathbf{\Delta} = \mathbf{\Delta}(\vec{x}, \vec{y})$ and $\mathbf{\Sigma} = \mathbf{\Sigma}(\vec{x}, \vec{y})$ for some t we consider the desired inequality $\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho > 1$, i.e.

$$\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho = \left|\frac{1}{4} \cdot \left(\mathbf{s}^2 - \mathbf{d}^2\right)\right| \cdot \left(\frac{1}{4} \cdot \left(\mathbf{s}^2 + \mathbf{d}^2\right)\right)^\varrho = \left(1 - t^2\right) \cdot \left(1 + t^2\right)^\varrho > 1\ . \tag{6.2}$$

This is equivalent to

$$\varrho > -\frac{\log\left(1 - t^2\right)}{\log\left(1 + t^2\right)}\ .$$

The right hand side is greater than 1 for all $0 < t \leq \mathsf{z}$. By the rules of L'Hospital we get the limit

$$\lim_{t \searrow 0} \left(-\frac{\log\left(1 - t^2\right)}{\log\left(1 + t^2\right)}\right) = 1\ .$$

This means that for all $\varrho > 1$ we can find a suitable t such that Inequality (6.2) is fulfilled. Hence, for each $\varrho > 1$, we are able to find a pair of unit vectors

$\vec{x} = \vec{a} + t \cdot \vec{b}$ and $\vec{y} = \vec{a} - t \cdot \vec{b}$ such that the ϱ-angle $\angle_\varrho(\vec{x}, \vec{y})$ does not exist. Proposition 6.11 is proven. □

Now we introduce the concept of a 'convex corner'. The word 'convex' seems to be superfluous in normed spaces. But later we define also something that we shall call 'concave corner'. These can occur in BW spaces which have a non-convex unit ball. This justifies the adjective 'convex'.

Definition 6.12. Let the pair $(X, \|\cdot\|)$ be a BW space, let $\widehat{y} \in X$. The vector $\widehat{y}$ is called a *convex corner* if and only if there are another vector $\overline{x} \in X$ and two real numbers $m_- < m_+$ such that for each $\delta \in [0, 1]$ there is a pair of unit vectors, more precisely we have

$$\|\delta \cdot \overline{x} + (1 + \delta \cdot m_-) \cdot \widehat{y}\| \;=\; 1 \;=\; \| - \delta \cdot \overline{x} + (1 - \delta \cdot m_+) \cdot \widehat{y}\| \;.$$

Remark 6.13. A convex corner is only the mathematical description of something what everybody already has in his mind. We can imagine it as an intersection of two straight lines of unit vectors which meet with an Euclidean angle of less than 180 degrees.

Note that from the definition follows $\|\widehat{y}\| = 1$ and $\{\widehat{y}, \overline{x}\}$ is linear independent. Further note that a space with a convex corner is not strictly curved.

As examples we can take the Hölder weights $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$. Both spaces have four convex corners, e.g. $(\mathbb{R}^2, \|\cdot\|_1)$ has one at $(0, 1)$, and $(\mathbb{R}^2, \|\cdot\|_\infty)$ has one at $(1, 1)$. They are just the corners of the corresponding unit spheres, i.e. the corners of the squares.

Proposition 6.14. *Let $(X, \|\cdot\|)$ be a positive definite balancedly weighted vector space which has a convex corner. Let $\Upsilon(X, \|\cdot\|) = (\nu, \mu)$. It holds $\nu = -1$.*

Proof. We assume in the proposition a convex corner $\widehat{y} \in X$ and another element $\overline{x} \in X$ and two real numbers $m_- < m_+$ with the properties of Definition 6.12. We get with Proposition 6.3 the inequality $\nu \leq -1$. Let us fix a number $\varrho > 1$, hence $-\varrho < -1$. We want to find two unit vectors $\widetilde{v}, \widetilde{w} \in X$ with $| < \widetilde{v} | \widetilde{w} >_{-\varrho} | \; > 1$. This would mean that the angle $\angle_{-\varrho}(\widetilde{v}, \widetilde{w})$ does not exist.

For each $\delta \in [0, 1]$ we define the pair of unit vectors $\vec{v}, \vec{w}$,

$$\vec{v} := \delta \cdot \overline{x} + (1 + \delta \cdot m_-) \cdot \widehat{y} \quad \text{ and } \quad \vec{w} := -\delta \cdot \overline{x} + (1 - \delta \cdot m_+) \cdot \widehat{y} \;.$$

We use the abbreviations $\mathbf{\Delta} = \mathbf{\Delta}(\vec{v}, \vec{w}) = \mathbf{s}^2 - \mathbf{d}^2$ and $\mathbf{\Sigma} = \mathbf{\Sigma}(\vec{v}, \vec{w}) = \mathbf{s}^2 + \mathbf{d}^2$ as usual, and we compute

$$\begin{aligned}
< \vec{v} \,|\, \vec{w} >_{-\varrho} &= \; < \delta \cdot \overline{x} + (1 + \delta \cdot m_-) \cdot \widehat{y} \;\; | \; - \delta \cdot \overline{x} + (1 - \delta \cdot m_+) \cdot \widehat{y} >_{-\varrho} \\
&= \|\vec{v}\| \cdot \|\vec{w}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta} \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma} \right)^{-\varrho} \\
&= 1 \cdot 1 \cdot \frac{\frac{1}{4} \cdot \mathbf{\Delta}}{\left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^{\varrho}} \;=\; \frac{\frac{1}{4} \cdot (\mathbf{s}^2 - \mathbf{d}^2)}{\left[\frac{1}{4} \cdot (\mathbf{s}^2 + \mathbf{d}^2)\right]^{\varrho}} \\
&= \frac{\frac{1}{4} \cdot (\| \, [2 + \delta \cdot (m_- - m_+)] \cdot \widehat{y}\|^2 - \|2 \cdot \delta \cdot \overline{x} + \delta \cdot (m_- + m_+) \cdot \widehat{y}\|^2)}{\left[\frac{1}{4} \cdot (\| \, [2 + \delta \cdot (m_- - m_+)] \cdot \widehat{y}\|^2 + \|2 \cdot \delta \cdot \overline{x} + \delta \cdot (m_- + m_+) \cdot \widehat{y}\|^2)\right]^{\varrho}} \\
&= \frac{\frac{1}{4} \cdot \left([2 + \delta \cdot (m_- - m_+)]^2 \cdot \|\widehat{y}\|^2 \; - \; \delta^2 \cdot \|2 \cdot \overline{x} + (m_- + m_+) \cdot \widehat{y}\|^2 \right)}{\left[\frac{1}{4} \cdot \left([2 + \delta \cdot (m_- - m_+)]^2 \cdot \|\widehat{y}\|^2 \; + \; \delta^2 \cdot \|2 \cdot \overline{x} + (m_- + m_+) \cdot \widehat{y}\|^2 \right)\right]^{\varrho}} \; .
\end{aligned}$$

$$\text{Hence} \quad < \vec{v} \,|\, \vec{w} >_{-\varrho} = \frac{\frac{1}{4} \cdot \mathbf{\Delta}}{\left[\frac{1}{4} \cdot \mathbf{\Sigma}\right]^{\varrho}} \;=\; \frac{1 + \delta \cdot (m_- - m_+) + \frac{1}{4} \cdot \delta^2 \cdot \mathsf{K}_-}{\left[1 + \delta \cdot (m_- - m_+) + \frac{1}{4} \cdot \delta^2 \cdot \mathsf{K}_+\right]^{\varrho}} \; , \tag{6.3}$$

if we define two real constants K_- , K_+ by setting

$\mathsf{K}_- := (m_- - m_+)^2 - \|2 \cdot \overline{x} + (m_- + m_+) \cdot \widehat{y}\|^2$, $\mathsf{K}_+ := (m_- - m_+)^2 + \|2 \cdot \overline{x} + (m_- + m_+) \cdot \widehat{y}\|^2$.

The above chain of identities holds for all $\delta \in [0, 1]$. For a shorter display we abbreviate the parts of the fraction by

$$\begin{aligned}
\mathsf{T} \;&:= \frac{1}{4} \cdot \mathbf{\Delta} \;=\; 1 + \delta \cdot (m_- - m_+) + \frac{1}{4} \cdot \delta^2 \cdot \mathsf{K}_- \; , \\
\mathsf{B} \;&:= \frac{1}{4} \cdot \mathbf{\Sigma} \;=\; 1 + \delta \cdot (m_- - m_+) + \frac{1}{4} \cdot \delta^2 \cdot \mathsf{K}_+ \; .
\end{aligned}$$

Since $\mathsf{K}_- < \mathsf{K}_+$ and $m_- - m_+ < 0$ we can find a positive number s with $0 < \mathsf{s} < 1$ such that for all positive δ with $0 < \delta \leq \mathsf{s}$ we have the inequality

$$0 < \mathsf{T} < \mathsf{B} < 1, \quad \text{i.e.} \quad 1 \; < \frac{\log(\mathsf{T})}{\log(\mathsf{B})} \; . \tag{6.4}$$

Our aim is to find vectors $\vec{v}, \vec{w}$ such that the product $< \vec{v} | \vec{w} >_{-\varrho}$ is greater than 1. With Equation (6.3) this is equivalent to

$$\begin{aligned}
< \vec{v} \,|\, \vec{w} >_{-\varrho} \; = \; \frac{\mathsf{T}}{[\mathsf{B}]^{\varrho}} \; > \; 1 \; &\Longleftrightarrow \; \log(\mathsf{T}) \; > \; \varrho \cdot \log(\mathsf{B}) \\
&\Longleftrightarrow \; \frac{\log(\mathsf{T})}{\log(\mathsf{B})} \; < \; \varrho \; , \quad \text{since} \log(\mathsf{B}) \text{ is negative, see (6.4).}
\end{aligned}$$

By the rules of L'Hospital we get the limit

$$\lim_{\delta \searrow 0} \left(\frac{\log(\mathsf{T})}{\log(\mathsf{B})} \right) \; = \; 1 \, .$$

Since $\varrho > 1$ it follows with Inequality (6.4) that we can find a very small $\widetilde{\delta}$ with $0 < \widetilde{\delta} < \mathsf{s}$ such that

$$1 \;<\; \frac{\log(\mathsf{T})}{\log(\mathsf{B})} \;<\; \varrho$$

is fulfilled. That means with the definition of

$$\widetilde{v} := \widetilde{\delta} \cdot \overline{x} + (1 + \widetilde{\delta} \cdot m_-) \cdot \widehat{y} \quad \text{ and } \quad \widetilde{w} := -\widetilde{\delta} \cdot \overline{x} + (1 - \widetilde{\delta} \cdot m_+) \cdot \widehat{y}$$

we get the desired inequality $< \widetilde{v} | \widetilde{w} >_{-\varrho} > 1$. Hence the $-\varrho$-angle $\angle_{-\varrho}(\widetilde{v}, \widetilde{w})$ does not exist. Since the variable $-\varrho$ is an arbitrary number less than -1, Proposition 6.14 is proven. □

Corollary 6.15. *Let $(X, \|\cdot\|) \in \mathsf{NORM}$. Further we assume that $(X, \|\cdot\|)$ has a convex corner. It follows*

$$\Upsilon(X, \|\cdot\|) = (-1, 1).$$

Proof. This is a direct consequence of Corollary 6.5 and of the propositions 6.11 and 6.14. □

Corollary 6.16. *For the Hölder weights $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$ it holds*

$$\Upsilon(\mathbb{R}^2, \|\cdot\|_1) \;=\; (-1, 1) \;=\; \Upsilon(\mathbb{R}^2, \|\cdot\|_\infty).$$

Now we introduce a corresponding definition of 'concave corners'. Note that they can not occur in normed spaces. In a normed space the triangle inequality $\widehat{(3)}$ holds, as a consequence its unit ball is 'everywhere' convex.

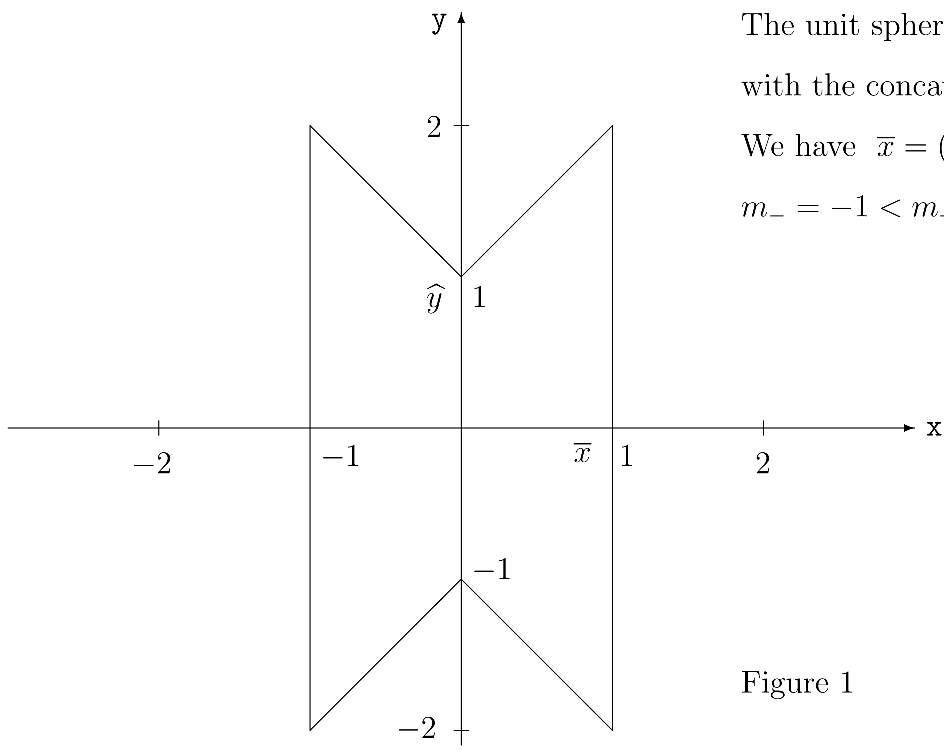
Definition 6.17. Let the pair $(X, \|\cdot\|)$ be a BW space, let $\widehat{y} \in X$. The element $\widehat{y}$ is called a *concave corner* if and only if there is an $\overline{x} \in X$, and there are two real numbers $m_- < m_+$ such that there is a pair of unit vectors for each $\delta \in [0, 1]$, more precisely we have

$$\|\delta \cdot \overline{x} + (1 + \delta \cdot m_+) \cdot \widehat{y}\| \;=\; 1 \;=\; \| - \delta \cdot \overline{x} + (1 - \delta \cdot m_-) \cdot \widehat{y}\| \;.$$

Remark 6.18. Note that from the definition follows $\|\widehat{y}\| = 1$ and that $\{\widehat{y}, \overline{x}\}$ is linear independent. Further note that a space $(X, \|\cdot\|)$ with a concave corner contains a line segment which is completely in its unit sphere, i.e. $(X, \|\cdot\|)$ is not strictly curved.

As the name suggests, a space with a concave corner has an unit ball which is not convex.

We get a set of balanced weights on $\mathbb{R}^2$ if for each $r \geq 0$ we define a weight $\|\cdot\|_{\mathtt{hexagon},r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$, we fix the unit sphere $\mathbf{S}$ of $(\mathbb{R}^2, \|\cdot\|_{\mathtt{hexagon},r})$ with the polygon through the six points $\{(0,1), (1,r), (1,-r), (0,-1), (-1,-r), (-1,r)\}$ and returning to $(0,1)$, and then we extend $\|\cdot\|_{\mathtt{hexagon},r}$ by homogeneity. (See Figure 1 with the example for $r = 2$.)



The unit sphere of $(\mathbb{R}^2, \|\cdot\|_{\texttt{hexagon},2})$ with the concave corner $\widehat{y} = (0,1)$. We have $\overline{x} = (1,0)$, and $m_- = -1 < m_+ = +1$.

Figure 1

Note that the balanced weights $\|\cdot\|_{\texttt{hexagon},0}$ and $\|\cdot\|_{\texttt{hexagon},1}$ on $\mathbb{R}^2$ coincide with the Hölder weights $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. Further, the pairs $(\mathbb{R}^2, \|\cdot\|_{\texttt{hexagon},r})$ are normed spaces if and only if $0 \le r \le 1$.

Lemma 6.19. *For all $r > 1$ the space $(\mathbb{R}^2, \|\cdot\|_{\texttt{hexagon},r})$ has a concave corner at $\widehat{y} = (0,1)$, with $\overline{x} = (1,0)$, $m_- = 1-r < 0 < m_+ = r-1$.*

Proof. Follow Definition 6.17 of a concave corner. □

In the next proposition we consider the special angle $\angle_{\mathbf{0}}$.

Proposition 6.20. *Let the pair $(X, \|\cdot\|)$ be a positive definite* BW *space, let $\widehat{y} \in X$ be a concave corner.*

It follows that the triple $(X, \|\cdot\|, <.\,|\,.>_0)$ does not fulfil the CSB *inequality, i.e. it holds $(X, \|\cdot\|) \notin \mathsf{pdBW}_{\mathbf{0}}$. Therefore with $\Upsilon(X, \|\cdot\|) = (\nu, \mu)$ it holds the estimate $-1 \le \mu < 0$.*

Proof. We use the vectors $\widehat{y}$, $\overline{x}$ from the above definition of a concave corner, and for all $\delta \in [0,1]$ we take the unit vectors $\vec{v} := \delta \cdot \overline{x} + (1 + \delta \cdot m_+) \cdot \widehat{y}$ and $\vec{w} := -\delta \cdot \overline{x} + (1 - \delta \cdot m_-) \cdot \widehat{y}$. We compute the value of $< \vec{v} \,|\, \vec{w} >_0$. Please follow now in [13] the proof of Proposition 4, p.18. There we have $<\vec{v}\,|\,\vec{w}>_0 \;=$

$$1+\delta\cdot(m_+-m_-)+\frac{1}{4}\cdot\delta^2\cdot\Big[(m_+-m_-)^2\;-\;\|\,2\cdot\overline{x}+(m_++m_-)\cdot\widehat{y}\,\|^2\Big]\;.$$

This result holds for all $\delta\in[0,1]$. Hence, because of $m_+-m_->0$, there is a positive but very small $\widetilde{\delta}$ such that for the two unit vectors

$$\widetilde{v}:=\widetilde{\delta}\cdot\overline{x}+(1+\widetilde{\delta}\cdot m_+)\cdot\widehat{y}\quad\text{ and }\quad\widetilde{w}:=-\widetilde{\delta}\cdot\overline{x}+(1-\widetilde{\delta}\cdot m_-)\cdot\widehat{y}$$

we get $<\widetilde{v}|\widetilde{w}>_0\;>1$, i.e. the CSB inequality is not satisfied and the angle $\angle_{\mathbf{0}}(\widetilde{v},\widetilde{w})$ does not exist. It follows $(X,\|\cdot\|)\notin\mathsf{pdBW}_{\mathbf{0}}$. □

Corollary 6.21. *For all $r>1$ the space $(\mathbb{R}^2,\|\cdot\|_{\mathtt{hexagon},r},<.\,|\,.>_0)$ does not fulfil the CSB inequality. Hence, there are vectors $\vec{v},\vec{w}\neq\vec{0}$ such that the angle $\angle_{\mathbf{0}}(\vec{v},\vec{w})$ is not defined. Hence, for $r>1$ it means that $(\mathbb{R}^2,\|\cdot\|_{\mathtt{hexagon},r})$ is no element of $\mathsf{pdBW}_{\mathbf{0}}$.*

Proposition 6.22. *Let α,β be two real numbers with $\alpha<-1<\beta$. It holds that the inclusions*

$$\mathsf{pdBW}_\alpha\subset\mathsf{pdBW}\supset\mathsf{pdBW}_\beta$$

are proper.

Proof. From Proposition 6.14 we know $\mathsf{pdBW}_\alpha\neq\mathsf{pdBW}$. For instance, the space $\mathbb{R}^2$ with the Hölder norm $\|\cdot\|_\infty$ has convex corners, hence it follows that the pair $(\mathbb{R}^2,\|\cdot\|_\infty)$ is no element of pdBW_α, but $(\mathbb{R}^2,\|\cdot\|_\infty)\in\mathsf{pdBW}_{-\mathbf{1}}$.

Now we consider $-1<\beta$. Let us take the spaces $(\mathbb{R}^2,\|\cdot\|_{\mathtt{hexagon},r})$ which are defined above. The balanced weight $\|\cdot\|_{\mathtt{hexagon},r}$ is not a norm if $r>1$, since it has a concave corner at $(0,1)$. We take the unit vectors $\vec{v}:=(1,r)$ and $\vec{w}:=(-1,r)$. We compute $<\vec{v}\,|\,\vec{w}>_{-\varrho}$ for an arbitrary positive number ϱ, i.e. $-\varrho<0$. We get

$$\begin{aligned}<\vec{v}\,|\,\vec{w}>_{-\varrho}\;&=\;\|\vec{v}\|\cdot\|\vec{w}\|\cdot\frac{1}{4}\cdot\mathbf{\Delta}\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\varrho}\\&=\;1\cdot1\cdot\frac{1}{4}\cdot\big(\mathbf{s}^2-\mathbf{d}^2\big)\cdot\left(\frac{1}{4}\cdot\big(\mathbf{s}^2+\mathbf{d}^2\big)\right)^{-\varrho}\\&=\;\frac{1}{4}\cdot\big[(2\cdot r)^2-2^2\big]\cdot\left(\frac{1}{4}\cdot\big[(2\cdot r)^2+2^2\big]\right)^{-\varrho}\\&=\;(r^2-1)\cdot(r^2+1)^{-\varrho}\;=\;\frac{r^2-1}{(r^2+1)^\varrho}\;.\end{aligned}$$

We assume the inequality $<\vec{v}\,|\,\vec{w}>_{-\varrho}\;>1$. This is equivalent to

$$(r^2-1)>(r^2+1)^\varrho\;,$$

and also to

$$\frac{\log(r^2-1)}{\log(r^2+1)}>\varrho\;. \tag{6.5}$$

By the rules of L'Hospital we can calculate the limit of the last term, and we get

$$\lim_{r\to\infty}\;\frac{\log(r^2-1)}{\log(r^2+1)}\;=\;1\;.$$

Hence for all $0 < \varrho < 1$, i.e. $-1 < -\varrho < 0$, we find a big number R such that Inequality (6.5) is fulfilled with $r := R$. This means $< \vec{v} \mid \vec{w} >_{-\varrho} \; > 1$, i.e. the angle $\angle_{-\varrho}(\vec{v}, \vec{w})$ does not exist in $(\mathbb{R}^2, \|\cdot\|_{\mathtt{hexagon},R})$. We get that the space $(\mathbb{R}^2, \|\cdot\|_{\mathtt{hexagon},R})$ is not an element of $\mathsf{pdBW}_{-\varrho}$.

Since the pair $(\mathbb{R}^2, \|\cdot\|_{\mathtt{hexagon},R})$ is an element of $\mathsf{pdBW}_{-\mathbf{1}}$, Proposition 6.22 is proven. □

Proposition 6.23. *Let α, β be two positive numbers with $-\alpha < -1 < 1 < \beta$. It holds that there are two proper inclusions*

$$\mathsf{NORM}_{-\alpha} \subset \mathsf{NORM} \supset \mathsf{NORM}_{\beta} \; .$$

Proof. This follows directly from Proposition 6.11 and Proposition 6.14. □

For instance, $\mathbb{R}^2$ with the Hölder norm $\|\cdot\|_\infty$ has convex corners, hence it is not strictly curved. That means that the space $(\mathbb{R}^2, \|\cdot\|_\infty)$ neither is an element of $\mathsf{NORM}_{-\alpha}$ nor an element of NORM_{β}.

7. Some Conjectures

We formulate two open questions.

Conjecture 7.1. Let us take four positive real numbers $\alpha, \beta, \gamma, \delta$ with

$$-\delta < -\gamma < -1 < 1 < \alpha < \beta.$$

From Corollary 6.6 we know

$$\mathsf{NORM}_{-\delta} \subset \mathsf{NORM}_{-\gamma} \subset \mathsf{NORM} \supset \mathsf{NORM}_{\alpha} \supset \mathsf{NORM}_{\beta} \; ,$$

and from Proposition 6.23 we have $\mathsf{NORM}_{-\gamma} \neq \mathsf{NORM} \neq \mathsf{NORM}_{\alpha}$. We are convinced that in fact all four inclusions are proper, and we believe that all five classes are different.

Conjecture 7.2. Let us assume four real numbers $\alpha, \beta, \gamma, \delta$, with

$$-\delta < -\gamma < -1 < \alpha < \beta.$$

We already know from Proposition 6.22 the inequalities $\mathsf{pdBW}_{-\gamma} \neq \mathsf{pdBW} \neq \mathsf{pdBW}_{\alpha}$. From Theorem 6.4 we have the following inclusions

$$\mathsf{pdBW}_{-\delta} \subset \mathsf{pdBW}_{-\gamma} \subset \mathsf{pdBW} \supset \mathsf{pdBW}_{\alpha} \supset \mathsf{pdBW}_{\beta} \; ,$$

and we believe that in fact all inclusions are proper and that all five classes are different, too.

In Section 3 for each $p > 0$ we introduced a balanced weight on $\mathbb{R}^2$, the 'Hölder weight'. The pairs $(\mathbb{R}^2, \|\cdot\|_p)$ may be a supply of suitable examples to prove or disprove the above conjectures.

Now we say something about finite products of BW spaces, and we ask interesting questions. We just have mentioned the Hölder weights. The method we used there can be generalized to construct products. Note that we restrict our description to products with have only two factors. But this can be extended to a finite number of factors very easily.

Assume two real vector spaces A, B provided with the balanced weights $\|\cdot\|_A$ and $\|\cdot\|_B$. That means we have two BW spaces, both spaces are not necessarily positive definite. Let p be any element from the extended real numbers, i.e.

$p \in \mathbb{R} \cup \{-\infty, +\infty\}$. If $A \times B$ denotes the usual cartesian product of the vector spaces A and B, we define a balanced weight $\|\cdot\|_p$ for $A \times B$. If p is a positive real number, for an element $\left(\vec{a}, \vec{b}\right) \in A \times B$ we define (corresponding to the definition in the third section) the real numbers

$$\left\|\left(\vec{a}, \vec{b}\right)\right\|_p := \sqrt[p]{\|\vec{a}\|_A^p + \|\vec{b}\|_B^p} \qquad \text{for the positive number } p, \text{ and}$$

$$\left\|\left(\vec{a}, \vec{b}\right)\right\|_{-p} := \begin{cases} \sqrt[-p]{\|\vec{a}\|_A^{-p} + \|\vec{b}\|_B^{-p}} & \text{if} \quad \|\vec{a}\|_A \cdot \|\vec{b}\|_B \neq 0 \\ 0 & \text{if} \quad \|\vec{a}\|_A \cdot \|\vec{b}\|_B = 0 \,. \end{cases}$$

To make the definition complete we set $\left\|\left(\vec{a}, \vec{b}\right)\right\|_0 := 0$ and

$$\left\|\left(\vec{a}, \vec{b}\right)\right\|_\infty := \max\left\{\|\vec{a}\|_A, \|\vec{b}\|_B\right\}, \ \left\|\left(\vec{a}, \vec{b}\right)\right\|_{-\infty} := \min\left\{\|\vec{a}\|_A, \|\vec{b}\|_B\right\} .$$

It is easy to verify some properties of $\|\cdot\|_p$. For instance, the weight $\|\cdot\|_p$ is positive definite if and only if $p > 0$ and both $\|\cdot\|_A$ and $\|\cdot\|_B$ are positive definite. Further, the pair $(A \times B, \|\cdot\|_p)$ is a normed space if and only if $p \geq 1$ and both $\|\cdot\|_A$ and $\|\cdot\|_B$ are norms. Further, the pair $(A \times B, \|\cdot\|_p)$ is an inner product space if and only if $p = 2$ and both $\|\cdot\|_A$ and $\|\cdot\|_B$ are inner products.

The next conjecture deals with a more intricate problem.

Conjecture 7.3. We take four real vector spaces provided with a positive definite balanced weight, i.e. we have $(A, \|\cdot\|_A), (B, \|\cdot\|_B), (C, \|\cdot\|_C), (D, \|\cdot\|_D) \in \mathsf{pdBW}$. Let us assume the identities

$$\Upsilon\left(A, \|\cdot\|_A\right) = \Upsilon(C, \|\cdot\|_C) \quad \text{and} \quad \Upsilon(B, \|\cdot\|_B) = \Upsilon(D, \|\cdot\|_D).$$

We conjecture that the identity

$$\Upsilon(A \times B, \|\cdot\|_p) \ = \ \Upsilon(C \times D, \|\cdot\|_p) \quad \text{holds for an arbitrary } \ p > 0.$$

At the end we try to find an 'algebraic structure' on the class pdBW_α, for a fixed number α. For two elements $(A, \|\cdot\|_A), (B, \|\cdot\|_B)$ of pdBW_α we look for a weight $\|\cdot\|_{A\times B}$ on $A \times B$ such that the pair $(A \times B, \|\cdot\|_{A\times B})$ is an element of pdBW_α, too. Before we formulate the conjecture we consider an example.

Take two copies of the real numbers $\mathbb{R}$ provided with the Euclidean norm $|\cdot|$. The pair $(\mathbb{R}, |\cdot|)$ is an inner product space and hence an element of pdBW_ϱ for all real numbers ϱ, see Theorem 6.7. We take the Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and we provide it with a Hölder weight $\|\cdot\|_p$. But the pair $(\mathbb{R}^2, \|\cdot\|_p)$ is an inner product space only for $p = 2$. Hence it is an element of the classes pdBW_ϱ for each ϱ only for $p = 2$. This example leads to a natural question.

Conjecture 7.4. Let α be a fixed real number. Let $(A, \|\cdot\|_A), (B, \|\cdot\|_B)$ be two elements of pdBW_α, i.e. both have the angle $\angle_\alpha$. We consider the product vector space $A \times B$. We provide $A \times B$ with the positive definite weight $\|\cdot\|_2$.

We ask whether the BW space $(A \times B, \|\cdot\|_2)$ has the angle $\angle_\alpha$, too.

Remark 7.5. In our paper we abstained from discussing properties of orthogonality, i.e. $< \vec{x} \,|\, \vec{y} >_\varrho = 0$, since this is the known 'Singer Orthogonality'. There are some papers about this topic, e.g. [1] or [12].

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