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BESOV–KÖTHE SPACES AND APPLICATIONS

KWOK-PUN HO

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ABSTRACT. We introduce and study the family of Besov–Köthe spaces which is a generalization of the Besov spaces, the Besov–Morrey spaces and the variable Besov spaces. As an application of the general results for the Besov–Köthe spaces, we identify a pre-dual of the Besov–Morrey space.

1. INTRODUCTION

In this paper, we aim to offer a generalization of Besov spaces. Besov spaces was introduced by Besov in [3, 4]. For the development of Besov spaces, the reader is referred to [27, 29, 36, 38]. The reader may also consult [30, Section 6.7.4.2] for a brief history of Besov spaces.

Recently, there are a substantial amount of researches considering Besov spaces associated with some generalizations of Lebesgue spaces. For instance, when the Lebesgue spaces used to defined the Besov spaces are replaced by the Morrey spaces, then we have the Besov–Morrey spaces. The study of Besov–Morrey spaces was given in [17, 22, 23, 32, 37]. Its application on the study of the solution of Navier–Stroke equation was also provided in [17, 22, 23]. The atomic, molecular and quarkonial decompositions are established in [32] and the boundedness of pseudo-differential operators on the Besov–Morrey spaces is obtained in [34]. The wavelet characterization of Besov–Morrey space is given in [33].

In [39], another family of Besov type spaces is introduced. For this family of function spaces, the Lebesgue spaces are replaced by the Lebesgue spaces with variable exponent. For the basic properties of Lebesgue space and Morrey space with variable exponent, the reader is referred to [6, 13, 19, 26]. The family of Besov type function spaces introduced in [39] is called as the variable Besov

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spaces. The smooth atomic decompositions of the variable Besov spaces are presented in [40].

From the studies of the above new function spaces, we see that there is a general approach to define and study Besov type spaces for general function spaces. In fact, a corresponding generalization of Triebel–Lizorkin spaces is obtained in [12, 14]. Roughly speaking, in [12], we find that whenever a function space satisfies the Fefferman–Stein type vector-valued inequality, then, using the terminology given in [12], the corresponding Littlewood–Paley space is well-defined and possesses some nice structures such as the atomic and molecular decompositions [14].

In this paper, we identify the condition imposed on a semi-Köthe function space X (see Definition 2.1) so that the corresponding Besov type space is well defined. The condition is given in (2.1). A large family of function spaces fulfills this condition. For instance, whenever the Hardy–Littlewood maximal operator is bounded on X, then it satisfies (2.1) (see Lemma 2.2). Moreover, if the translation operator is bounded on X, it also satisfies (2.1) (see Lemma 2.2).

As demonstrated in [12], some notions appeared in the study of Banach space are used in the Littlewood–Paley spaces such as the UMD property. Similarly, we find that the notion of associate space, the absolutely continuity of norm and the Fatou property are invoked in our study.

This paper is organized as follows. Section 2 contains the definition of the Besov–Köthe spaces and some of the background materials for the introduction of Besov–Köthe spaces. We show that the Besov–Köthe space is well-defined in Section 3. It also presents the atomic and molecular decompositions and a duality result of Besov–Köthe spaces. The general results obtained in Section 3 is applied in Section 4 to block spaces. We introduce the Besov-block space in Section 4 and show that it is a pre-dual of the Besov–Morrey spaces studied in [17, 22, 23, 39]. Finally, the proofs of Theorems 3.1, 3.2 and 3.3 are given in Section 5.

2. Besov–Köthe spaces

We begin with some notions used to study the Besov–Köthe spaces.

Let $\mathcal{M}(\mathbb{R}^n)$ and $\mathcal{P}$ denote the class of Lebesgue measurable functions and the class of polynomials on $\mathbb{R}^n$, respectively. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions and $\mathcal{S}'(\mathbb{R}^n)$ be the space of Schwartz distributions. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\hat{f}$. For any $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$.

Let $\mathcal{Q} = \{Q_{i,k} : i \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ be the set of dyadic cubes, where

$$Q_{i,k} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : k_j \leq 2^i x_i < k_j + 1,\ j = 1, \dots, n\}$$

and $k = (k_1, \dots, k_n)$. For any dyadic cube, $Q \in \mathcal{Q}$, let $l(Q) = 2^{-i}$ and $x_Q = 2^{-i}k$ denote the length of $Q = Q_{i,k}$ and the center of the cube, $Q = Q_{i,k}$, respectively. Let $\tilde{\mathcal{Q}} = \{Q \in \mathcal{Q} : \ |Q| \leq 1\}$. For any $\varphi \in \mathcal{M}(\mathbb{R}^n)$ and $Q = Q_{i,k} \in \mathcal{Q}$, write $\varphi_Q(x) = 2^{in}\varphi(2^i x - k)$.

Definition 2.1. Let μ be the Lebesgue measure on $\mathbb{R}^n$ (or the counting measure on $\mathbb{N}$) and $\mathcal{M}(\mu)$ be the set of μ-measurable functions. A mapping $\|\cdot\| : \mathcal{M}(\mu) \to$

$[0, \infty]$ is called a semi-Köthe function norm (when μ is the counting measure on $\mathbb{N}$, it is called as a semi-Köthe sequence norm) if

(1) $\|\cdot\|$ is a quasi-norm,
(2) $\|\chi_B\| < \infty$, $\forall B \in \mathbb{B}$ and
(3) $|f| \leq |g| \Rightarrow \|f\| \leq \|g\|, \quad f, g \in \mathcal{M}(\mu)$,

where χ_B is the characteristic function of $B \in \mathbb{B}$.

A quasi-Banach space $\mathbb{F} \subseteq \mathcal{M}(\mathbb{R}^n)$ is a semi-Köthe function space if $\|\cdot\|_{\mathbb{F}}$ is a complete semi-Köthe function norm and

$$\mathbb{F} = \{f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{\mathbb{F}} < \infty\}.$$

In addition, $\mathbb{F}$ is a Köthe function space if $\|\cdot\|_{\mathbb{F}}$ is a norm and $|E| < \infty$ implies $\|\chi_E\|_{\mathbb{F}} < \infty$ for any Lebesgue measurable set E.

Definition 2.1 is inspired by the definition of Köthe function space. We generalize the notion of Köthe function space to semi-Köthe function space so that our results apply to the family of block spaces introduced in Section 4 (see Proposition 4.4). The reader is referred to [20, Volume II, Definition 1.b.17] for the definition of Köthe function space. The preceding definition also generalizes the notion of Köthe function space to quasi-Banach space.

Similarly, a quasi-Banach space $\mathbb{S} \subseteq \{\{a_i\} : a_i \in \mathbb{R}, i \in \mathbb{N}\}$ is a semi-Köthe sequence space if $\|\cdot\|_{\mathbb{S}}$ is a complete semi-Köthe sequence norm and

$$\mathbb{S} = \{\{a_i\}_{i\in\mathbb{N}} : \|\{a_i\}_{i\in\mathbb{N}}\|_{\mathbb{S}} < \infty\}.$$

According to the Aoki–Rolewicz theorem [16, Theorem 1.3], there exists a $0 < p = p_{\mathbb{F}} \leq 1$ such that $\|\cdot\|_{\mathbb{F}}^p$ satisfies the triangle inequality. Therefore, on the rest of this paper, we assume that $\mathbb{F}$ is a $p_{\mathbb{F}}$-Banach space. Apparently, if $\mathbb{F}$ is a Banach space, then $p_{\mathbb{F}} = 1$.

Definition 2.2. Let $L \geq 0$. Let $\mathbb{S}$ and $\mathbb{F}$ be a semi-Köthe sequence space and a semi-Köthe function space, respectively. We call the pair $(\mathbb{F}, \mathbb{S})$ a L-regular pair if

(1) there exists a constant $C > 0$ such that for any $f \in \mathbb{F}$, $i \in \mathbb{N}$ and $l \in \mathbb{R}^n$,

$$\Big\| \sum_{k\in\mathbb{Z}^n} \inf_{w\in Q_{i,k+l}} |g(w)|\chi_{Q_{i,k}} \Big\|_{\mathbb{F}} \leq C(1 + |l|^L)\|g\|_{\mathbb{F}}. \tag{2.1}$$

(2) For any $a = (a_0, a_1, \cdots,) \in \mathbb{S}$,

$$\|(0, a_0, a_1, \cdots)\|_{\mathbb{S}} \leq C\|a\|_{\mathbb{S}} \quad \text{and} \quad \|(a_1, a_2, \cdots)\|_{\mathbb{S}} \leq C\|a\|_{\mathbb{S}}.$$

The index L in (2.1) plays a role in the order of vanishing moment conditions satisfied by the smooth molecules associated with the Besov–Köthe space, see Definition 3.3.

The reader may have a wrong impression that (2.1) is too complicate to apply. In fact, its introduction is motivated by two simple criteria. The first criterion is given by the translation on $\mathbb{R}^n$.

Lemma 2.1. *If a semi-Köthe space $\mathbb{F}$ satisfies*

$$\|T_l f\|_{\mathbb{F}} \leq C(1 + |l|^L)\|f\|_{\mathbb{F}}, \quad \forall l \in \mathbb{Z}^n \tag{2.2}$$

for some $C>0$ where $T_l f(x)=f(x+l)$, then $\mathbb{F}$ fulfills (2.1).

Proof. As

$$\sum_{k\in\mathbb{Z}^n}\inf_{w\in Q_{i,k+l}}|g(w)|\chi_{Q_{i,k}}(x)\leq\sum_{k\in\mathbb{Z}^n}|g(2^{-i}l+x)|\chi_{Q_{i,k}}(x),\quad\forall i\in\mathbb{N},$$

we have

$$\Big\|\sum_{k\in\mathbb{Z}^n}\inf_{w\in Q_{i,k+l}}|g(w)|\chi_{Q_{i,k}}(x)\Big\|_{\mathbb{F}}\leq\|g(2^{-i}l+x)\|_{\mathbb{F}}\leq C(1+|l|^L)\|g(x)\|_{\mathbb{F}}.$$

□

The other criterion is expressed in term of the boundedness of the Hardy–Littlewood maximal operator.

Lemma 2.2. *If the Hardy–Littlewood maximal operator M is bounded on $\mathbb{F}$, then $\mathbb{F}$ satisfies (2.1) with $L=n$.*

Proof. Since $\mathrm{dist}(Q_{i,k},Q_{i,k+l})\leq C2^{-i}(1+|l|)$, for any $x\in Q_{i,k}$, we find that

$$\begin{aligned}&\inf_{w\in Q_{i,k+l}}|g(w)|\leq\frac{1}{2^{-in}}\int_{Q_{i,k+l}}|g(y)|dy\\&\leq C(1+|l|^n)\frac{1}{2^{-in}(1+|l|)^n}\int_{Q_{i,k+l}}|g(y)|dy\leq C(1+|l|^n)(Mg)(x).\end{aligned}$$

The above inequalities yield

$$\Big\|\sum_{k\in\mathbb{Z}^n}\inf_{w\in Q_{i,k+l}}|g(w)|\chi_{Q_{i,k}}\Big\|_{\mathbb{F}}\leq C(1+|l|^n)\|Mg\|_{\mathbb{F}}\leq C(1+|l|^n)\|g\|_{\mathbb{F}}.$$

□

We find that the above criteria guarantee that a number of well-known function spaces satisfy (2.1). The rearrangement-invariant (r.-i.) quasi-Banach space [2, 12, 25, 28], the Morrey space [12, 17, 22, 23, 32, 37] the block space [5] and the Herz space [11] satisfy (2.1). We offer some examples on non-translation invariant function spaces that fulfill (2.1). For simplicity, we consider function spaces on $\mathbb{R}$. Let $\omega:\mathbb{R}\to(0,\infty)$ be a positive Lebesgue measurable function satisfying $\omega(x+l)\leq C(|l|^L+1)\omega(x)$, $x,l\in\mathbb{R}$ for some $L>0$. It is trivial that the function space

$$\begin{aligned}L^p_\omega&=\{f\in\mathcal{M}(\mathbb{R}):\Big(\int_{\mathbb{R}}|f(x)|^p\omega(x)dx\Big)^{1/p}<\infty\},\quad 0<p<\infty\\L^\infty_\omega&=\{f\in\mathcal{M}(\mathbb{R}):\sup_{x\in\mathbb{R}}\omega(x)|f(x)|<\infty\},\end{aligned}$$

possess property (2.1). In particular, the function space of polynomial growth (or polynomial decay),

$$\mathcal{G}_L=\{f\in\mathcal{M}(\mathbb{R}):\sup_{x\in\mathbb{R}}(1+|x|)^L|f(x)|<\infty\},\ -\infty<L<0\,(0<L<\infty),$$

has property (2.1).

The variable Lebesgue spaces provide examples on the use of the boundedness of the maximal operator on the study of Besov–Köthe space. There are several criteria found so that the maximal operator is bounded on $L^{p(x)}$. The reader may referred to [6, 26] for details.

Definition 2.3. Let $\mathbb{F}$ and $\mathbb{S}$ be a semi-Köthe function space and a semi-Köthe sequence space, respectively. The quasi-Banach space $(\mathbb{F},\mathbb{S})$ consists of those sequence of Lebesgue measurable functions $\{f_i\}_{i\in\mathbb{N}}$ satisfying

$$\|\{f_i\}_{i\in\mathbb{N}}\|_{(\mathbb{F},\mathbb{S})} = \|\{\|f_i\|_{\mathbb{F}}\}_{i\in\mathbb{N}}\|_{\mathbb{S}} < \infty.$$

Definition 2.4. Let $L \geq 0$. Let $(\mathbb{F},\mathbb{S})$ be a L-regular pair. The *Besov–Köthe space* consists of those $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{\mathbb{S}}_{\mathbb{F}}(\varphi_0,\varphi)} = \|\{\varphi_i * f\}_{i\in\mathbb{N}}\|_{(\mathbb{F},\mathbb{S})} < \infty$$

where $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}\hat{\varphi}_0(\xi) \subseteq \{\xi : |\xi| \leq 1\} \quad \text{and} \quad \operatorname{supp}\hat{\varphi}(\xi) \subseteq \{\xi : 1/2 \leq |\xi| \leq 2\}, \tag{2.3}$$

$$|\hat{\varphi}_0(\xi)| > C \quad |\xi| \leq 1 \quad \text{and} \quad |\hat{\varphi}(\xi)| > C \quad 3/5 \leq |\xi| \leq 5/3 \tag{2.4}$$

for some constant $C > 0$ and $\varphi_i(x) = 2^{in}\varphi(2^i x)$, $i \in \mathbb{N}\backslash\{0\}$.

The Besov–Köthe space covers the Besov–Morrey space [17, 22, 23, 32, 33, 34, 37] and the variable Besov space [39, 40]. Moreover, the above definition also introduces some new families of Besov spaces such as Besov-Orlicz spaces, Besov-Lorentz–Karamata spaces and Besov–rearrangement-invariant quasi-Banach function spaces when $\mathbb{F}$ are Orlicz spaces [31], Lorentz–Karamata spaces [7] and rearrangement-invariant quasi-Banach function spaces [2], respectively.

3. Main Results

We present the main results in this section.

Definition 3.1. Let $(\mathbb{F},\mathbb{S})$ be a L-regular pair. A sequence $a = \{a_Q\}_{Q\in\tilde{\mathcal{Q}}}$ belongs to the *Besov–Köthe sequence space* $b^{\mathbb{S}}_{\mathbb{F}}$ if

$$\|a\|_{b^{\mathbb{S}}_{\mathbb{F}}} = \Big\|\{\sum_{k\in\mathbb{R}^n} |a_{Q_{i,k}}|\chi_{Q_{i,k}}\}_{i=0}^{\infty}\Big\|_{(\mathbb{F},\mathbb{S})} < \infty$$

where χ_Q is the characteristic function of $Q \in \tilde{\mathcal{Q}}$.

We have the following main result for Besov–Köthe space. The proofs of Theorems 3.1, 3.2 and 3.3 are presented in Section 5.

Theorem 3.1. *Let $(\mathbb{F},\mathbb{S})$ be a L-regular pair. The Besov–Köthe space $B^{\mathbb{S}}_{\mathbb{F}}$ is independent of $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ in Definition 2.4.*

Next, we show that the Besov–Köthe spaces possess the atomic and molecular decompositions.

Let $K > 0$. We say that $\{a_Q\}_{Q\in\tilde{\mathcal{Q}}}$ is a family of K-atoms if it satisfies

(1) $\operatorname{supp} a_Q \subseteq 3Q$,
(2) $|(\partial^\gamma a_Q)(x)| \leq C_\gamma |Q|^{-\frac{|\gamma|}{n}}$, $\gamma \in \mathbb{N}^n$,
(3) $\int_{\mathbb{R}^n} x^\lambda a_Q(x)dx = 0$, $|\lambda| \leq [K]$, $\lambda \in \mathbb{N}^n$, when $|Q| < 1$.

The above definition follows from a simple modification of [9, p.60] or [10, p.46].

Theorem 3.2. *Let $K > 0$. Let $(\mathbb{F}, \mathbb{S})$ be a L-regular pair. For any $f \in B^{\mathbb{S}}_{\mathbb{F}}$, we have a sequence $s = \{s_Q\}_{Q\in\mathcal{Q}} \in b^{\mathbb{S}}_{\mathbb{F}}$ and a family of K-atoms $\{a_Q\}_{Q\in\tilde{\mathcal{Q}}}$ such that*

$$f = \sum_{Q\in\tilde{\mathcal{Q}}} s_Q a_Q$$

and $\|s\|_{b^{\mathbb{S}}_{\mathbb{F}}} \le C\|f\|_{B^{\mathbb{S}}_{\mathbb{F}}}$ for some constant $C > 0$ independent of f.

For any $q \ge 1$, let $\mathbb{S}^q = \{a = \{a_i\}_{i=0}^\infty : \{|a_i|^q\}_{i=0}^\infty \in \mathbb{S}\}$ and

$$\|\{a_i\}_{i=0}^\infty\|_{\mathbb{S}^q} = \|\{|a_i|^q\}_{i=0}^\infty\|_{\mathbb{S}}^{1/q}.$$

It is obvious that $\|\cdot\|_{\mathbb{S}^q}$ is a quasi-norm. If $\mathbb{S}$ is a quasi-Banach space, $\mathbb{S}^q$ is also a quasi-Banach space. The quasi-Banach space $\mathbb{S}^q$ is called as the q-convexification of $\mathbb{S}$ or the $\frac{1}{q}$-th power of $\mathbb{S}$, see [20, Volume II, p.53-54] and [28, Section 2.2], respectively.

The following definition is introduced for the molecular decompositions of $B^{\mathbb{S}}_{\mathbb{F}}$.

Definition 3.2. Let $\alpha, L \ge 0$. Let $\mathbb{S}$ and $\mathbb{F}$ be semi-Köthe sequence space and semi-Köthe function space, respectively. We call the pair $(\mathbb{F}, \mathbb{S})$ a (α, L)-regular pair if $\mathbb{F}$ satisfies (2.1) and there exists a $1 \le q_{\mathbb{S}}$ such that for any $\delta_1 > \alpha$ and $\delta_2 > -\alpha$, the mappings

$$T_{\delta,1}(\{a_i\}_{i=0}^\infty) = \Big\{\sum_{j=0}^{i} 2^{(j-i)\delta_1/q_{\mathbb{S}}} a_j\Big\}_{i=0}^\infty \quad \text{and}$$

$$T_{\delta,2}(\{a_i\}_{i=0}^\infty) = \Big\{\sum_{j=i+1}^{\infty} 2^{(i-j)\delta_2/q_{\mathbb{S}}} a_j\Big\}_{i=0}^\infty$$

are bounded on $\mathbb{S}^{q_{\mathbb{S}}}$.

For example, if we consider the sequence space,

$$l^{r,\kappa} = \Big\{\{a_i\}_{i=0}^\infty : \|\{a_i\}_{i=0}^\infty\|_{l^{r,\kappa}} = \big(\sum_{i=0}^\infty (2^{i\kappa}|a_i|)^r\big)^{1/r} < \infty\Big\}, \quad 0 < r < \infty,$$

we find that $\alpha = \kappa$ and we can take $q_{l^{r,\kappa}} = 1$ when $1 \le r \le \infty$. If $0 < r < 1$, we can assign $q_{l^{r,\kappa}}$ to be any number in the interval $(1/r, \infty)$.

We state the definition of smooth molecules for $B^{\mathbb{S}}_{\mathbb{F}}$ in the followings. It is a modification of the definition of smooth molecules from [9, p.56]. Notice that the index L in (2.1) is used in the order of the vanishing moment condition imposed on the smooth molecules.

Definition 3.3. Let $\alpha, L \ge 0$ and $(\mathbb{F}, \mathbb{S})$ be a (α, L)-regular pair. A family of function $\{m_Q\}_{Q\in\tilde{\mathcal{Q}}}$ is a family of molecules for $B^{\mathbb{S}}_{\mathbb{F}}$, if there exist $\beta > |\alpha|$ and $N > \frac{n}{p_{\mathbb{F}}} + L - n$ such that

$$\int_{\mathbb{R}^n} x^\lambda m_Q(x)dx = 0, \quad |\lambda| \le \Big[\frac{n}{p_{\mathbb{F}}} + L - \alpha - n\Big], \lambda \in \mathbb{N}^n, \tag{3.1}$$

when $|Q| < 1$ and

$$\begin{aligned}
&|\partial^\gamma m_Q(x)| \le C_1 |Q|^{-\frac{|\gamma|}{n}} \frac{1}{(1 + l(Q)|x - x_Q|)^N}, \quad |\gamma| \le [\beta],\, \gamma \in \mathbb{N}^n, \\
&|\partial^\gamma m_Q(x) - \partial^\gamma m_Q(y)| \\
&\le C_2 |Q|^{-\frac{\beta}{n}} |x - y|^{\beta - [\beta]} \sup_{|z - x_Q| \le |x - y|} \frac{1}{(1 + l(Q)^{-1}|x - z|)^N}.
\end{aligned}$$

Theorem 3.3. *Let $(\mathbb{F}, \mathbb{S})$ be a (α, L)-regular pair and $q_{\mathbb{S}} p_{\mathbb{F}} \ge 1$. Let $\{m_Q\}_{Q \in \tilde{\mathcal{Q}}} \in \mathcal{M}_{\beta,N}$ be a family of molecules for $B^{\mathbb{S}}_{\mathbb{F}}$. Then, there exists a constant $C > 0$ such that for any $f = \sum_{Q \in \tilde{\mathcal{Q}}} s_Q m_Q$ with $s = \{s_Q\}_{Q \in \tilde{\mathcal{Q}}} \in b^{\mathbb{S}}_{\mathbb{F}}$, $f \in B^{\mathbb{S}}_{\mathbb{F}}$ and*

$$\|f\|_{B^{\mathbb{S}}_{\mathbb{F}}} \le C \|s\|_{b^{\mathbb{S}}_{\mathbb{F}}}.$$

If $\mathbb{F} = L^p$ and $\mathbb{S} = l^{q,\alpha}$, $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$, then the Besov–Köthe space $B^{\mathbb{S}}_{\mathbb{F}}$ is the "classical" Besov space $B^\alpha_{p,q}$. The previous molecular decomposition is a generalization of the celebrated molecular decomposition of Besov space $B^\alpha_{p,q}$ (see [8], Theorem 3.1). The criteria $q_{l^q} p_{L^p} \ge 1$ is satisfied as q_{l^q} can be taken as any number bigger than $1/q$. Furthermore, as L^p is translation-invariant, $L = 0$. Thus, the vanishing moment condition in (3.1) is precisely the conditions for the Besov space $B^\alpha_{p,q}$.

Moreover, the above results also provide the atomic and molecular decompositions for the Besov–Morrey spaces [32] and the variable Besov spaces [40]. For the Besov–Morrey spaces studied in [32], the corresponding "classical" Morrey space is translational invariant. That is, it satisfies (2.2) with $L = 0$, therefore the preceding results reestablish the decompositions given in [32].

For any semi-Köthe function space $\mathbb{F}$, let $\mathbb{F}'$ denote the linear space of *integral* of $\mathbb{F}$ (see [20, Volume II, p.29] and [21, 41]). More precisely, any linear functional on $\mathbb{F}$, L, of the form

$$L(f) = \int_{\mathbb{R}^n} f(x) g(x) dx, \quad g \in \mathcal{M},$$

is called an integral of $\mathbb{F}$. Whenever $\mathbb{F}$ is a Banach function space, $\mathbb{F}'$ is also called as the Köthe dual of $\mathbb{F}$ or the associate space of $\mathbb{F}$, see [2, Chapter 1, Section 3] and [28, p.35].

Similarly, for any semi-Köthe sequence space $\mathbb{S}$, the integral space is the collection of those linear functionals L on $\mathbb{S}$ having the representation

$$L(s) = \sum_{i=0}^{\infty} s_i l_i, \quad \{l_i\}_{i=0}^\infty \subset \mathbb{C}.$$

Note that the dual space and the integral space of a semi-Köthe space are not necessarily equal, see [2, Chapter 1, Corollary 4.3].

Definition 3.4. We say that a semi-Köthe sequence space $\mathbb{S}$ has the Fatou property if for any $s_n = \{s_{n,j}\}_{j=0}^\infty \in \mathbb{S}$ satisfying $s_{n,j} \ge 0$ $\forall n, j \in \mathbb{N}$,

$$s_n \uparrow s \;\text{ and }\; \sup_{n \in \mathbb{N}} \|s_n\|_{\mathbb{S}} < \infty \Rightarrow s \in \mathbb{S} \;\text{ and }\; \|s\|_{\mathbb{S}} = \lim_{n \to \infty} \|s_n\|_{\mathbb{S}}.$$

The reader is referred to [2, Chapter 1, Lemma 1.5, Theorem 1.6 and Theorem 2.7] or [20, Volume II, p.30] for the definition of the Fatou property on Banach function space, its application on the completeness of Banach function space and its relation with the Lorentz-Luxemburg theorem of the second associate space of Banach function space.

We define the notion of absolutely continuous for semi-Köthe sequence space (see [2], Chapter 1, Section 3 and [12, Definition 2.4]).

Definition 3.5. We say that a semi-Köthe sequence space $\mathbb{S}$ has absolutely continuous quasi-norm if $\lim_{i\to\infty} \|s_i\|_{\mathbb{S}} = 0$ for every sequence $\{s_i\}_{i\in\mathbb{N}} \subset \mathbb{S}$ satisfying $s_i \downarrow 0$.

Proposition 3.4. *Let $\mathbb{F}$ and $\mathbb{S}$ be a semi-Köthe function space and a semi-Köthe sequence space, respectively. If $\mathbb{S}^* = \mathbb{S}'$, $\mathbb{F}^* = \mathbb{F}'$ and $\mathbb{S}$ has absolutely continuous norm and the Fatou property, then*

$$(\mathbb{F}, \mathbb{S})^* = (\mathbb{F}^*, \mathbb{S}^*).$$

Proof. Let $\{g_i\}_{i=0}^\infty \in (\mathbb{F}', \mathbb{S}') = (\mathbb{F}^*, \mathbb{S}^*)$. For any $f = \{f_i\}_{i=0}^\infty \in (\mathbb{F}, \mathbb{S})$, define

$$G(f) = \sum_{i=0}^\infty \int_{\mathbb{R}^n} f_i(x) g_i(x) dx. \tag{3.2}$$

The definition of integral space yields

$$|G(f)| \leq \sum_{i=0}^\infty \|f_i\|_{\mathbb{F}} \|g_i\|_{\mathbb{F}'} \leq \|\{f_i\}_{i=0}^\infty\|_{(\mathbb{F},\mathbb{S})} \|\{g_i\}_{i=0}^\infty\|_{(\mathbb{F}',\mathbb{S}')}.$$

Therefore, $(\mathbb{F}^*, \mathbb{S}^*) \subseteq (\mathbb{F}, \mathbb{S})^*$.

Next, let $L \in (\mathbb{F}, \mathbb{S})^*$. Write $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. For any $j \in \mathbb{N}$, we consider

$$L_j(h) = L(\{h\delta_{ij}\}_{i=0}^\infty), \quad \forall h \in \mathbb{F}.$$

Since $L_j \in \mathbb{F}^* = \mathbb{F}'$, we obtain a Lebesgue measurable function $g_j \in \mathbb{F}'$ such that

$$L_j(h) = \int_{\mathbb{R}^n} h(x) g_j(x) dx.$$

Therefore, for any $N \in \mathbb{N}$ and $f = \{f_j\}_{j=0}^\infty \in (\mathbb{F}, \mathbb{S})$, we obtain

$$\sum_{j=0}^N \int_{\mathbb{R}^n} |f_j(x) g_j(x)| dx \leq \|L\|_{op} \|f\|_{(\mathbb{F},\mathbb{S})}$$

where $\|L\|_{op}$ is the operator norm of L.

Let $\epsilon > 0$. Let $h_j \in \mathbb{F}$ with $\|h_j\|_{\mathbb{F}} \leq 1$, $j \in \mathbb{N}$, be chosen so that

$$\|g_j\|_{\mathbb{F}'} \leq \int_{\mathbb{R}^n} |h_j(x) g_j(x)| dx + \frac{\epsilon}{2^j}.$$

For any $s = \{s_j\}_{j=0}^\infty \in \mathbb{S}$ with $\|s\|_{\mathbb{S}} \leq 1$, we have

$$\sum_{j=0}^N s_j \|g_j\|_{\mathbb{F}'} \leq \sum_{j=0}^N \int_{\mathbb{R}^n} |s_j h_j(x) g_j(x)| dx + \epsilon \leq \|L\|_{op} + \epsilon$$

because $\|\{s_jh_j\}_{j=0}^{\infty}\|_{(\mathbb{F},\mathbb{S})}\leq 1$. Taking supreme over those $s\in\mathbb{S}$ with $\|s\|_{\mathbb{S}}\leq 1$ on the left hand side of the above inequalities, we have

$$\|\{g_0,g_1,\dots,g_N,0,\dots\}\|_{(\mathbb{F}',\mathbb{S}')}\leq \|L\|_{op}+\epsilon.$$

Applying the Fatou property, we find that $g=\{g_j\}_{j=0}^{\infty}\in(\mathbb{F}',\mathbb{S}')$.

Write $L_g(f)=\sum_{i=0}^{\infty}\int_{\mathbb{R}^n}f_j(x)g_j(x)dx$, $f=\{f_j\}_{j=0}^{\infty}\in(\mathbb{F},\mathbb{S})$. We find that L and L_g are identical on

$$\tilde{\mathbb{F}}=\{\{f_j\}_{j=0}^{\infty}\in(\mathbb{F},\mathbb{S}):f_j=0,\forall j\geq N,\text{ for some }N\in\mathbb{N}\}.$$

Furthermore, the absolute continuity of $\|\cdot\|_{\mathbb{S}}$ assures that $\tilde{\mathbb{F}}$ is dense in $\mathbb{F}$. Therefore, $L=L_g$ and, hence, we assert that $(\mathbb{F},\mathbb{S})^*\hookrightarrow(\mathbb{F}',\mathbb{S}')$. ☐

In fact, the above theorem also shows that any bounded linear functional on $(\mathbb{F},\mathbb{S})$ is of the form (3.2).

Theorem 3.5. *Let $\alpha,L\geq 0$ and $(\mathbb{F},\mathbb{S})$ be a (α,L)-regular pair and $q_{\mathbb{S}}p_{\mathbb{F}}\geq 1$. Suppose that $(\mathbb{F}^*,\mathbb{S}^*)$ are M-regular pair for some $M\geq 0$ and $q_{\mathbb{S}^*}p_{\mathbb{F}^*}\geq 1$. If $\mathbb{S}^*=\mathbb{S}'$, $\mathbb{F}^*=\mathbb{F}'$, $\mathbb{S}$ has absolutely continuous norm and the Fatou property and for any $\varphi\in\mathcal{S}(\mathbb{R}^n)$*

$$\|f*\varphi\|_{\mathbb{F}'}\leq C\|\varphi\|_{L^1}\|f\|_{\mathbb{F}'},\quad\forall f\in\mathbb{F}' \tag{3.3}$$

for some $C>0$, then we have

$$(B_{\mathbb{F}}^{\mathbb{S}})^*=B_{\mathbb{F}^*}^{\mathbb{S}^*}.$$

Proof. Let $g\in B_{\mathbb{F}^*}^{\mathbb{S}^*}$. For any $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfying (2.3)-(2.4) and $f\in B_{\mathbb{F}}^{\mathbb{S}}$, define

$$G(f)=\sum_{i=0}^{\infty}\int_{\mathbb{R}^n}(g*\varphi_i(x))(f*\varphi_{i-1}(x)+f*\varphi_i(x)+f*\varphi_{i+1}(x))dx.$$

According to the definition of integral on $\mathbb{F}$, we have

$$|G(f)|\leq\sum_{i=0}^{\infty}\|g*\varphi_i\|_{\mathbb{F}'}(\|f*\varphi_{i-1}\|_{\mathbb{F}}+\|f*\varphi_i\|_{\mathbb{F}}+\|f*\varphi_{i+1}\|_{\mathbb{F}}).$$

Similarly, using the definition of integral on $\mathbb{S}$ and Item (2) of Definition 2.2, we have

$$|G(f)|\leq C\|g\|_{(\mathbb{F}',\mathbb{S}')}\|f\|_{(\mathbb{F},\mathbb{S})}.$$

Thus, $B_{\mathbb{F}^*}^{\mathbb{S}^*}\hookrightarrow(B_{\mathbb{F}}^{\mathbb{S}})^*$.

Let $g\in(B_{\mathbb{F}}^{\mathbb{S}})^*$. Using Hahn-Banach theorem, we have a linear functional G on $(\mathbb{F},\mathbb{S})$ such that

$$g(f)=G(\{f*\varphi_i\}_{i=0}^{\infty}),\quad\forall f\in B_{\mathbb{F}}^{\mathbb{S}}$$

and $\|g\|_{op}=\|G\|_{op}$ where $\|\cdot\|_{op}$ denotes the operator norm.

Proposition 3.4 provides a family $\{g_i\}_{i=0}^{\infty}\in(\mathbb{F}',\mathbb{S}')$ such that

$$g(f)=\sum_{i=0}^{\infty}\int_{\mathbb{R}^n}g_i(x)(f*\varphi_i)(x)dx,\quad\forall f\in B_{\mathbb{F}}^{\mathbb{S}}. \tag{3.4}$$

Theorem 3.3 assures that $\mathcal{S}(\mathbb{R}^n) \subset B_{\mathbb{F}}^{\mathbb{S}}$. Hence, $g \in (B_{\mathbb{F}}^{\mathbb{S}})^* \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. For any $\psi \in \mathcal{S}(\mathbb{R}^n)$, applying (3.4) with $f = \varphi_j * \psi$, $j \in \mathbb{N}$, conditions (2.3) and (2.4) yield

$$g * \varphi_j = \sum_{i=j-1}^{j+1} g_i * \varphi_j * \varphi_i.$$

Thus, condition (3.3) guarantees that

$$\|g * \varphi_j\|_{\mathbb{F}'} \leq C(\|g_{j-1}\|_{\mathbb{F}'} + \|g_j\|_{\mathbb{F}'} + \|g_{j+1}\|_{\mathbb{F}'})$$

for all $j \in \mathbb{N}$.

Consequently, Item (2) of Definition 2.2 assures that

$$\|\{g * \varphi_i\}_{i=0}^{\infty}\|_{(\mathbb{F}',\mathbb{S}')} \leq C\|\{g_i\}_{i=0}^{\infty}\|_{(\mathbb{F}',\mathbb{S}')}.$$

That is, $(B_{\mathbb{F}}^{\mathbb{S}})^* \hookrightarrow B_{\mathbb{F}'}^{\mathbb{S}'}$. □

Notice that for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and locally integrable function f, we have

$$|f * \varphi| \leq C\|\varphi\|_{L^1} M(f)$$

for some $C > 0$. Therefore, whenever the Hardy–Littlewood maximal operator is bounded on $\mathbb{F}'$, $\mathbb{F}$ satisfies (3.3). In particular L^p, $1 \leq p < \infty$ satisfy (3.3).

4. Application: Besov-block spaces

In this section, we use the Besov–Köthe space to obtain a pre-dual of the Besov–Morrey spaces studied in [17, 22, 23, 32, 37]. We call it as the Besov-block space because this is the Besov type space associated with block space.

Even though the definition of Morrey space is well-known, for completeness, we recall it from [22, Definition 2.1].

Definition 4.1. Let $1 \leq q \leq p < \infty$, the Morrey space M_q^p is defined by

$$M_q^p = \{f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{M_q^p} = \sup_{z\in\mathbb{R}^n, r>0} r^{\frac{n}{p}-\frac{n}{q}} \|\chi_{B(z,r)} f\|_{L^q} < \infty\}.$$

Notice that a generalization of the above Morrey space with L^q replaced by Banach function space is given in [12]. In addition, the boundedness of singular integral operator, the boundedness of the Hardy–Littlewood maximal operator and the validity of the Fefferman–Stein vector-valued inequalities on this generalization of Morrey space are established in [12, Section 5].

The following Morrey type space is introduced in [12, Definition 5.1].

Definition 4.2. Let X be a r.-i. Banach function space on $\mathbb{R}^n$ and $\omega(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. A $f \in \mathcal{M}(\mathbb{R}^n)$ belongs to $\mathcal{M}_\omega^X$ if it satisfies

$$\|f\|_{\mathcal{M}_\omega^X} = \sup_{x_0\in\mathbb{R}^n, r>0} \frac{1}{\omega(x_0,r)} \|\chi_{B(x_0,r)} f\|_X < \infty$$

where $\chi_{B(x,r)}$ denotes the characteristic function of $B(x,r)$.

In particular, $M^p_q = \mathcal{M}^{L^q}_\omega$ where $\omega(x,r) = r^{\frac{n}{q}-\frac{n}{p}}$ for all $x \in \mathbb{R}^n$.

Using [12, Theorem 5.5], we have the following fundamental result of $\mathcal{M}^X_\omega$. To formulate the following proposition for r.-i. Banach function space X, we use the notion of Boyd's indices p_X, q_X. For the definition of Boyd's indices, the reader is referred to [12, Definition 4.2] and [20, Volume II, Definition 2.b.1].

Theorem 4.1. *Let X be a r.-i. Banach function space on $\mathbb{R}^n$ with Boyd's indices satisfying $1 < p_X \le q_X < \infty$. If ω satisfies*

(1) *for some $0 \le \lambda < \frac{n}{q_X}$, $\omega(x, 2^j r) \le C 2^{j\lambda}\omega(x,r)$ for any $x \in \mathbb{R}^n$, $j \in \mathbb{N}$ and $r > 0$,*
(2) *there exists $C_1 > 0$ such that $\omega(x,r) \ge C_1$, for all $r \ge 1$, $x \in \mathbb{R}^n$ and*
(3) *there exists $C_2 > 0$ such that*

$$C_2^{-1} \le \omega(x,t)/\omega(x,r) \le C_2, \quad 0 < r \le t < 2r,$$

then $\mathcal{M}^X_\omega$ is a Köthe function space.

Proof. It is obvious that $\mathcal{M}^X_\omega$ is a Banach lattice with respect to the ordering $f \le g$ of Lebesgue measurable functions. It remains to show that $\chi_E \in \mathcal{M}^X_\omega$ whenever E is a Lebesgue measurable set with $|E| < \infty$.

As $E \cap B(x_0, 1), x_0 \in \mathbb{R}^n$, is a bounded Lebesgue set, the proof of [12, Theorem 5.5] guarantees that

$$\sup_{x_0 \in \mathbb{R}^n, 0 < r \le 1} \frac{1}{\omega(x_0, r)} \|\chi_{B(x_0,r)}\chi_E\|_X < \infty. \tag{4.1}$$

Next, as X is a Banach function space, $\|\chi_E\|_X < \infty$ (see [2, Chapter 1, Definition 1.1 (P4)]). Item (2) assures that $\omega(x_0, r) \ge C_1$, for all $r \ge 1$ and $x_0 \in \mathbb{R}^n$. Therefore, (4.1) and the assertion,

$$\sup_{x_0 \in \mathbb{R}^n, 1 \le r} \frac{1}{\omega(x_0, r)} \|\chi_{B(x_0,r)}\chi_E\|_X < C\|\chi_E\|_X < \infty,$$

yield $\chi_E \in \mathcal{M}^X_\omega$. □

We generalize the definition of block space introduced in [5].

Definition 4.3. Let X be a semi-Köthe function space on $\mathbb{R}^n$ and $\omega(x,r)$: $\mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. A $b \in \mathcal{M}(\mathbb{R}^n)$ is a (ω, X)-block if it is supported in a ball $B(x_0, r)$, $x_0 \in \mathbb{R}^n$, $r > 0$, and

$$\|b\|_X \le \frac{1}{\omega(x_0, r)}.$$

Define $\mathfrak{B}_{\omega,X}$ by

$$\mathfrak{B}_{\omega,X} = \Big\{ \sum_{k=1}^\infty \lambda_k b_k \ : \ \sum_{k=1}^\infty |\lambda_k| < \infty \text{ and } b_k \text{ is a } (\omega, X)\text{-block} \Big\}.$$

The space $\mathfrak{B}_{\omega,X}$ is endowed with the norm

$$\|f\|_{\mathfrak{B}_{\omega,X}} = \inf \Big\{ \sum_{k=1}^\infty |\lambda_k| \text{ such that } f = \sum_{k=1}^\infty \lambda_k b_k \Big\}.$$

Notice that the terminology "block space" is used in [24, 35] to represent another family of function spaces.

For any $1 < p \le q \le \infty$ and $\omega(x,r) = r^{\frac{n}{q}-\frac{n}{p}}$, $\forall x \in \mathbb{R}^n$, write $\mathfrak{B}^p_q = \mathfrak{B}_{\omega,L^q}$. In [5, Theorem 1], we have

$$(\mathfrak{B}^p_q)^* = M^{p'}_{q'} \tag{4.2}$$

where $\frac{1}{q}+\frac{1}{q'} = 1$ and $\frac{1}{p}+\frac{1}{p'} = 1$. Furthermore, we have the subsequent result which shows that the family of block space is an extension of the family of Lebesgue spaces.

Proposition 4.2. *Let X be a Banach function space on $\mathbb{R}^n$. If $\omega \equiv 1$, then*

$$\mathfrak{B}_{\omega,X} = X.$$

Proof. In view of the definition of block space, for any $f \in \mathfrak{B}_{\omega,X}$, $f = \sum_{k=1}^{\infty} \lambda_k b_k$, there exist a family of (ω, X)-blocks $\{b_k\}_{k=1}^{\infty}$ and a sequence of scalars $\{\lambda_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} |\lambda_k| \le 2\|f\|_{\mathfrak{B}_{\omega,X}}$. Thus,

$$\|f\|_X \le \sum_{k=1}^{\infty} |\lambda_k| \|b_k\|_X \le \sum_{k=1}^{\infty} |\lambda_k| \le 2\|f\|_{\mathfrak{B}_{\omega,X}}. \tag{4.3}$$

For the reverse embedding, we see that for any $f \in X$ and any $R > r > 0$,

$$\frac{1}{\|f\chi_{B(0,R)\setminus B(0,r)}\|_X} f\chi_{B(0,R)\setminus B(0,r)}, \quad \frac{1}{\|f\chi_{B(0,R)}\|_X} f\chi_{B(0,R)}$$

are (ω, X)-blocks. That is, we have

$$\|f\chi_{B(0,R)\setminus B(0,r)}\|_{\mathfrak{B}_{\omega,X}} \le \|f\chi_{B(0,R)\setminus B(0,r)}\|_X \tag{4.4}$$

$$\|f\chi_{B(0,R)}\|_{\mathfrak{B}_{\omega,X}} \le \|f\chi_{B(0,R)}\|_X. \tag{4.5}$$

As $\{f\chi_{B(0,2^j)}\}_{j\in\mathbb{N}}$ is a Cauchy sequence in X with limit function f, (4.3) and (4.4) assure that $\{f\chi_{B(0,2^j)}\}_{j\in\mathbb{N}}$ is also a Cauchy sequence in $\mathfrak{B}_{\omega,X}$ that also converges to f. Therefore, (4.3) and (4.5) ensure that $\mathfrak{B}_{\omega,X} = X$. □

In particular, we have $\mathfrak{B}^p_p = L^p$, $1 \le p \le \infty$. We now present and prove an extension of (4.2).

Theorem 4.3. *Let X and ω satisfy the conditions in Theorem 4.1. If $X^* = X'$, then*

$$\mathfrak{B}'_{\omega,X} = (\mathfrak{B}_{\omega,X})^* = \mathcal{M}^{X^*}_{\omega}.$$

Proof. Let $f \in \mathcal{M}^{X^*}_{\omega}$ and b be a (ω, X)-block supported in $B(x_0, r)$. We find that

$$\left|\int_{\mathbb{R}^n} f(x)b(x)dx\right| \le \|\chi_{B(x_0,r)} f\|_{X'} \|\chi_{B(x_0,r)} b\|_X \le \frac{1}{\omega(x_0,r)} \|\chi_{B(x_0,r)} f\|_{X^*}$$

because $X^* = X'$.

Thus, if $g = \sum_{k\in\mathbb{N}} \lambda_k b_k \in \mathfrak{B}_{\omega,X}$, we have

$$\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \le \sum_{k\in\mathbb{N}} |\lambda_k| \left|\int_{\mathbb{R}^n} f(x)b_k(x)dx\right| \le \|g\|_{\mathfrak{B}_{\omega,X}} \|f\|_{\mathcal{M}^{X^*}_{\omega}}.$$

That is, $\mathcal{M}^{X^*}_{\omega} \hookrightarrow (\mathfrak{B}_{\omega,X})^*$.

For the reverse direction, we first notice that for any $h \in X$ and $B(x_0, r) \in \mathbb{B}$,

$$\frac{\chi_{B(x_0,r)}h}{\|\chi_{B(x_0,r)}h\|_X \omega(x_0, r)} \tag{4.6}$$

is a (ω, X)-block. In particular, $\|\chi_{B(x_0,r)}h\|_{\mathfrak{B}_{\omega,X}} \le \|\chi_{B(x_0,r)}h\|_X \omega(x_0, r)$.

For any $L \in (\mathfrak{B}_{\omega,X})^*$, by using Hahn–Banach theorem, the linear functional defined by $l(g) = L(\chi_{B(,0,r)}g)$ belongs to X^*. In view of $X^* = X'$, we have a $f_r \in X'$ such that

$$l(g) = \int_{\mathbb{R}^n} f_r(x)g(x)dx, \quad \forall g \in X.$$

In addition, without loss of generality, we can assume that $\mathrm{supp} f_r \subseteq B(0, r)$. Note that for any $r, s > 0$,

$$\int_B f_r(x)dx = l(\chi_B) = \int_B f_s(x)dx$$

for any $B \in \mathbb{B}$ with $B \subseteq B(0, r) \cap B(0, s)$. Hence, $f_r = f_s$ almost everywhere on $B(0, r) \cap B(0, s)$. Therefore, there is an unique Lebesgue measurable function f such that $f(x) = f_r(x)$ when $x \in B(0, r)$.

Next, we show that $f \in \mathcal{M}_\omega^{X^*}$. For any $x_0 \in \mathbb{R}^n$ and $r > 0$, let $s > 0$ be selected such that $B(x_0, r) \subseteq B(0, s)$. As the function given in (4.6) is a (ω, X)-block, we have

$$\begin{aligned}
&\frac{1}{\omega(x_0, r)}\|\chi_{B(x_0,r)}f\|_{X^*} = \frac{1}{\omega(x_0, r)} \sup_{\|h\|_X=1} \left| \int_{B(x_0,r)} f(x)h(x)dx \right| \\
&= \sup_{\|h\|_X=1} \left| \int_{B(0,s)} f_s(x) \frac{\chi_{B(x_0,r)}(x)h(x)}{\omega(x_0, r)} dx \right| \\
&\le \|L\|_{(\mathfrak{B}_{\omega,X})^*} \sup_{\|h\|_X=1} \left\| \frac{h\chi_{B(x_0,r)}}{\omega(x_0, r)} \right\|_{\mathfrak{B}_{\omega,X}} = \|L\|_{(\mathfrak{B}_{\omega,X})^*}.
\end{aligned}$$

As the functionals $L_f(g) = \int_{\mathbb{R}^n} f(x)g(x)dx$ and L are identical on the set of (ω, X)-block and the set of finite linear combinations of (ω, X)-block is dense in $\mathfrak{B}_{\omega,X}$, we conclude that $L_f = L$ and $(\mathfrak{B}_{\omega,X})^* = \mathcal{M}_\omega^{X^*}$. Moreover, as for any $L \in (\mathfrak{B}_{\omega,X})^*$, it can be represented as L_f for some Lebesgue measurable function $f \in \mathcal{M}_\omega^{X^*}$. Thus, $\mathfrak{B}'_{\omega,X} = (\mathfrak{B}_{\omega,X})^*$. $\square$

The above theorem identifies a pre-dual of the Morrey space $\mathcal{M}_\omega^X$. It extends the duality result in [5, Theorem 1] for the "classical" Morrey space. Moreover, notice that there are some researches consider the boundedness of the Calderón–Zygmund operators on the pre-dual of some Morrey type spaces, see [1, 18].

The above duality result shows the following fundamental result for the block spaces.

Proposition 4.4. *Let X and ω satisfy the conditions in Theorem 4.1. If $X^* = X'$, then $\mathfrak{B}_{\omega,X}$ is a semi-Köthe function space and $\|\cdot\|_{\mathfrak{B}_{\omega,X}}$ is a norm.*

Proof. It follows from the definition that $\chi_B \in \mathfrak{B}_{\omega,X}$ when $B \in \mathbb{B}$ and $\|\cdot\|_{\mathfrak{B}_{\omega,X}}$ is a norm. It remains to show that $\mathfrak{B}_{\omega,X}$ is a lattice. That is, it fulfills item (3) of Definition 2.1.

Let $|f| \leq |g|$ with $g \in \mathfrak{B}_{\omega,X}$ where f, g are Lebesgue measurable functions. From the proof of Theorem 4.3, we have $\mathfrak{B}'_{\omega,X} = \mathfrak{B}^*_{\omega,X}$. For any $h \in \mathcal{M}^X_\omega$, we find that

$$\left|\int f(x)h(x)dx\right| = \left|\int |f(x)|(sgnf(x))h(x)dx\right| \leq \int |f(x)||h(x)|dx$$
$$\leq \int |g(x)||h(x)|dx \leq \|g\|_{\mathfrak{B}_{\omega,X}}\|h\|_{\mathcal{M}^{X^*}_\omega}$$

where we use the $(\mathcal{M}^{X^*}_\omega, \mathfrak{B}_{\omega,X})$ duality and the fact that $\mathcal{M}^{X^*}_\omega$ is a lattice.

Taking supreme over those $h \in \mathcal{M}^{X^*}_\omega$ with $\|h\|_{\mathcal{M}^{X^*}_\omega} \leq 1$ on both sides of the above inequalities, we obtain $\|f\|_{\mathfrak{B}_{\omega,X}} \leq \|g\|_{\mathfrak{B}_{\omega,X}}$. □

In order to apply the duality results in the previous section, it remains to show that the block space $\mathfrak{B}_{\omega,X}$ fulfills Item (1) of Definition 2.2. In case $\omega(x, r)$ is independent of $x \in \mathbb{R}^n$, then the block space satisfies (2.2) with $L = 0$. For the general case, we need the following theorem.

The subsequent result has its own independent interest because it extends the boundedness of Hardy–Littlewood maximal operator to block space.

Theorem 4.5. *Let X be a r.-i. Banach function space where the Hardy–Littlewood maximal operator is bounded on X. If there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, ω satisfies*

$$\sum_{j=0}^{\infty} 2^{-jn}r^{-n}\phi_{X'}(r^n)\phi_X(2^{(j+1)n}r^n)\omega(x, 2^{j+1}r) < C\omega(x, r), \tag{4.7}$$

where ϕ_X and $\phi_{X'}$ are the fundamental functions of X and X', respectively (see [2, Chapter 2, Definition 5.1]*), then the Hardy–Littlewood maximal operator is bounded on $\mathfrak{B}_{\omega,X}$.*

Proof. Let b be a (ω, X)-block with support $B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$, $r > 0$. Write $B_k = B(x_0, 2^{k+1}r)$, $k \in \mathbb{N}$. Let $m_k = \chi_{B_{k+1}\setminus B_k}M(b)$, $k \in \mathbb{N}\setminus\{0\}$ and $m_0 = \chi_{B_0}M(b)$. We have $M(b) = \sum_{k=0}^\infty m_k$ and supp$m_k \subseteq B_{k+1}\setminus B_k$. As the Hardy–Littlewood maximal operator M is bounded on X, we have

$$\|m_0\|_X \leq C\|M(b)\|_X \leq \frac{C}{\omega(x_0, r)}$$

for some constant $C > 0$ independent of x_0 and r. That is, m_0 is a constant-multiple of a (ω, X)-block.

Furthermore, according to the definition of Hardy–Littlewood maximal operator, we have

$$\chi_{B_{k+1}\setminus B_k}|M(b)| \leq \frac{1}{2^{kn}r^n}\int_{B(x,r)} |b(x)|dx \leq \frac{1}{2^{kn}r^n}\|b\|_X\|\chi_{B(x,r)}\|_{X'}$$

where we use the Hölder inequality for X in the last inequality.

Consequently,

$$\begin{aligned}\|m_k\|_X &\le \frac{\|\chi_{B_{k+1}\setminus B_k}\|_X}{2^{kn}r^n}\|b\|_X\|\chi_{B(x,r)}\|_{X'}\\ &\le 2^{-kn}r^{-n}\phi_X(2^{(k+1)n}r^n)\phi_{X'}(r^n)\frac{\omega(x_0,2^{k+1}r)}{\omega(x_0,r)}\frac{1}{\omega(x_0,2^{k+1}r)}.\end{aligned}$$

Write $m_k = \sigma_k b_k$ where

$$\sigma_k = 2^{-kn}r^{-n}\phi_X(2^{(k+1)n}r^n)\phi_{X'}(r^n)\frac{\omega(x_0,2^{k+1}r)}{\omega(x_0,r)}.$$

Then, b_k is a (ω, X)-block. Moreover, inequality (4.7) yields $\sum_{k=0}^{\infty}\sigma_k < C$ for some $C > 0$. So, $M(b) \in \mathfrak{B}_{\omega,X}$ and there exists a constant $C_0 > 0$ so that for any (ω, X)-block b,

$$\|M(b)\|_{\mathfrak{B}_{\omega,X}} < C_0.$$

Finally, let $f \in \mathfrak{B}_{\omega,X}$. In view of the definition of block space, there exist a family of (ω, X)-blocks $\{c_k\}_{k=1}^{\infty}$ and a sequence $\Lambda = \{\lambda_k\}_{k=1}^{\infty} \in l^1$ such that $f = \sum_{k=1}^{\infty}\lambda_k c_k$ with $\|\Lambda\|_{l^1} \le 2\|f\|_{\mathfrak{B}_{\omega,X}}$. Therefore, we find that

$$\begin{aligned}\|M(f)\|_{\mathfrak{B}_{\omega,X}} &\le \sum_{k=1}^{\infty}|\lambda_k|\|M(c_k)\|_{\mathfrak{B}_{\omega,X}}\\ &\le C_0\sum_{k=1}^{\infty}|\lambda_k|\|c_k\|_{\mathfrak{B}_{\omega,X}} \le 2C_0\|f\|_{\mathfrak{B}_{\omega,X}}.\end{aligned}$$

□

Condition (4.7) is also used in [12, 15] to study the Fefferman–Stein vector-valued inequality on Morrey spaces associated with r.-i. Banach functions spaces and weighted Morrey spaces. In addition, (4.7) is fulfilled if the Boyd indices of X satisfy $1 < p_X \le q_X < \infty$ and ω satisfies

$$\omega(x, 2^j r) \le C2^{j\lambda}\omega(x, r), \quad \forall x \in \mathbb{R}^n,\ r > 0 \text{ and } j \in \mathbb{N},$$

for some $0 \le \lambda < n/q_X$, see [12, Corollary 5.3].

The above result and Lemma 2.2 assure that whenever ω satisfies (4.7), $(\mathfrak{B}_{\omega,X}, l^q)$, $0 < q < \infty$, is an (α, L)-regular pair for some $\alpha, L > 0$.

Theorem 4.6. *Let $1 \le r < \infty$. Suppose that X and ω satisfy the conditions in Theorem 4.1. If $X^* = X'$, then*

$$(\mathbb{B}^{l^r}_{\mathfrak{B}_{\omega,X}})^* = \mathbb{B}^{l^{r'}}_{\mathcal{M}^{X^*}_{\omega}}.$$

Proof. According to [12, Corollary 5.3], if ω satisfies condition (1) of Theorem 4.1, then it fulfills (4.7).

Furthermore, in [12, Theorem 5.5], we find that whenever X satisfies the conditions in Theorem 4.1, then the Hardy–Littlewood maximal operator is bounded on $\mathcal{M}^{X^*}_{\omega}$. Thus, Theorems 3.5, 4.1 and 4.3 provide the desired result. □

In particular, we have $(\mathbb{B}^{l^r}_{\mathfrak{B}^p_q})^* = \mathbb{B}^{l^{r'}}_{M^{p'}_{q'}}$, $1 \leq r < \infty$ and $1 < p \leq q \leq \infty$. This gives a pre-dual of the Besov–Morrey space studied in [17, 22, 23, 32, 37].

Being the dual space of a normed space, the Besov–Morrey space has some remarkable features. For instance, the Alaoglu's Theorem guarantees the following result.

Corollary 4.7. *Let $1 < r \leq \infty$ and $1 \leq q \leq p < \infty$. The unit ball of the Besov–Morrey space $\mathbb{B}^{l^r}_{M^p_q}$ is weak-star compact.*

Furthermore, according to the Krein–Smulian Theorem, we have the subsequent corollary.

Corollary 4.8. *For any $1 < r \leq \infty$ and $1 \leq q \leq p < \infty$, if A is a convex set of $\mathbb{B}^{l^r}_{M^p_q}$ such that $A \cap \{f \in \mathbb{B}^{l^r}_{M^p_q} : \|f\|_{\mathbb{B}^{l^r}_{M^p_q}} < R\}$ is weak-star closed for every $R > 0$, then A is weak-star closed.*

5. Proofs of Theorems 3.1, 3.2 and 3.3

Lemma 5.1. *Let $i \in \mathbb{N}$. Let $\mathbb{F}$ be a semi-Köthe function space satisfying (2.1) for some $L \geq 0$. Suppose that $g \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $\operatorname{supp} \hat{g} \subseteq \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{i+1}\}$, then,*

$$\|\sum_{k \in \mathbb{Z}^n} \sup_{z \in Q_{i,k}} |g(z)| \chi_{Q_{i,k}}\|_{\mathbb{F}} \leq C\|g\|_{\mathbb{F}}$$

for some constant $C > 0$ independent of g.

Proof. Applying the Paley-Wiener theorem and following the proof of Lemma 2.4 of [8], we find that for any $M > 0$, there exists a constant $C > 0$ such that for any $y \in Q_{i,k}$, $i \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$

$$\sup_{z \in Q_{i,k}} |g(z)| \chi_{Q_{i,k}}(x) \leq C \sum_{l \in \mathbb{Z}^n} |g(2^{-i}l + y)| \chi_{Q_{i,k}}(x)(1 + |l|)^{-M}.$$

As $y \in Q_{i,k}$ is arbitrary, we have

$$\sup_{z \in Q_{i,k}} |g(z)| \chi_{Q_{i,k}}(x) \leq C \sum_{l \in \mathbb{Z}^n} \inf_{y \in Q_{i,k}} |g(2^{-i}l + y)| \chi_{Q_{i,k}}(x)(1 + |l|)^{-M}.$$

Taking summation with respect to k on both sides and interchanging the summations on the right hand side, we obtain that

$$\sum_{k \in \mathbb{Z}^n} \sup_{z \in Q_{i,k}} |g(z)| \chi_{Q_{i,k}}(x) \leq C \sum_{l \in \mathbb{Z}^n} \sum_{k \in \mathbb{Z}^n} \inf_{w \in Q_{i,k+l}} |g(w)| \chi_{Q_{i,k}}(x)(1 + |l|)^{-M}.$$

As $\mathbb{F}$ is a $p_{\mathbb{F}}$-Banach space, we have

$$\begin{aligned}
&\|\sum_{k \in \mathbb{Z}^n} \sup_{z \in Q_{i,k}} |g(z)| \chi_{Q_{i,k}}\|_{\mathbb{F}}^{p_{\mathbb{F}}} \\
&\leq C^{p_{\mathbb{F}}} \sum_{l \in \mathbb{Z}^n} \left\| \sum_{k \in \mathbb{Z}^n} \inf_{w \in Q_{i,k+l}} |g(w)| \chi_{Q_{i,k}} \right\|_{\mathbb{F}}^{p_{\mathbb{F}}} (1 + |l|)^{-Mp_{\mathbb{F}}}.
\end{aligned}$$

By (2.1), if $M > L + \frac{n}{p_{\mathbb{F}}}$, we have

$$\Big\| \sum_{k\in\mathbb{Z}^n} \sup_{z\in Q_{i,k}} |g(z)|\chi_{Q_{i,k}}\Big\|_{\mathbb{F}}^{p_{\mathbb{F}}} \le C^{p_{\mathbb{F}}} \sum_{l\in\mathbb{Z}^n} \|g\|_{\mathbb{F}}^{p_{\mathbb{F}}} |l|^{Lp_{\mathbb{F}}}(1+|l|)^{-Mp_{\mathbb{F}}}.$$

Hence,

$$\Big\| \sum_{k\in\mathbb{Z}^n} \sup_{z\in Q_{i,k}} |g(z)|\chi_{Q_{i,k}}\Big\|_{\mathbb{F}} < C\|g\|_{\mathbb{F}}.$$

□

Lemma 5.2. *Let $i \in \mathbb{N}$. Let $\mathbb{F}$ be a semi-Köthe function space satisfying (2.1) for some $L \ge 0$. Suppose that $F(x) = \sum_{k\in\mathbb{Z}^n} s_{Q_{i,k}} f_{Q_{i,k}}(x)$ where*

$$|f_{Q_{i,k}}(x)| \le (1 + 2^i|x - 2^{-i}k|)^{-M}$$

for some $M > L + \frac{n}{p_{\mathbb{F}}}$. Then, there exists a constant $C > 0$ such that

$$\|F\|_{\mathbb{F}} \le C\Big\| \sum_{k\in\mathbb{Z}^n} |s_{Q_{i,k}}|\chi_{Q_{i,k}}\Big\|_{\mathbb{F}}.$$

Proof. We have a constant so that for any fixed $k \in \mathbb{Z}^n$,

$$(1 + 2^i|x - 2^{-i}k|)^{-M} \le C \sum_{l\in\mathbb{Z}^n} \chi_{Q_{i,l+k}}(x)(1 + |l|)^{-M}.$$

Therefore,

$$|F(x)| \le C \sum_{l\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n} |s_{Q_{i,k}}|\chi_{Q_{i,l+k}}(x)(1 + |l|)^{-M}.$$

We write $g(x) = \sum_{k\in\mathbb{Z}^n} |s_{Q_{i,k}}|\chi_{Q_{i,k}}(x)$. So, we have

$$\inf_{w\in Q_{i,k+l}} |g(w)| = g(x) = s_{Q_{i,k+l}}, \quad \forall x \in Q_{i,k+l}$$

and

$$|F(x)| \le C \sum_{l\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}^n} \inf_{w\in Q_{i,k-l}} |g(w)|\chi_{Q_{i,k}}(x)(1 + |l|)^{-M}.$$

Inequalities (2.1) and $M > L + \frac{n}{p_{\mathbb{F}}}$ ensure that

$$\|F\|_{\mathbb{F}} \le C\Big(\sum_{l\in\mathbb{Z}^n} \|g\|_{\mathbb{F}}^{p_{\mathbb{F}}} |l|^{Lp_{\mathbb{F}}}(1 + |l|)^{-Mp_{\mathbb{F}}}\Big)^{1/p_{\mathbb{F}}} < C\|g\|_{\mathbb{F}}.$$

□

The following lemma is a simple modification of [8, Lemma 3.3].

Lemma 5.3. *Let $(\mathbb{F}, \mathbb{S})$ be a (α, L)-regular pair. If $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.3)-(2.4) and $\{m_Q\}_{Q\in\tilde{\mathcal{Q}}}$ is a family of molecules for $B^{\mathbb{S}}_{\mathbb{F}}$, then*

(1) *if $i \ge j$ where $|Q| = 2^{-jn}$, then*

$$|(\varphi_i * m_Q)(x)| \le C2^{-(i-j)\beta}(1 + 2^j|x - x_Q|)^{-N}.$$

(2) *if $j > i$ where $|Q| = 2^{-jn}$, then for any $0 < \delta < \frac{n}{p_{\mathbb{F}}} + L - \alpha - [\frac{n}{p_{\mathbb{F}}} + L - \alpha]$,*

$$|(\varphi_i * m_Q)(x)| \le C2^{-(j-i)(\frac{n}{p_{\mathbb{F}}}+L-\alpha+\delta)}(1 + 2^i|x - x_Q|)^{-N}.$$

Proof of Theorem 3.1:
Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (2.3)-(2.4). To prove the theorem, we are only required to show that there exits a constant such that $\|\cdot\|_{B^{\mathbb{S}}_{\mathbb{F}}(\psi_0,\psi)} \leq C\|\cdot\|_{B^{\mathbb{S}}_{\mathbb{F}}(\varphi_0,\varphi)}$. As $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (2.3)-(2.4), we can construct a pair of functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.3)-(2.4) and the Calderón reproducing formula

$$\theta_0(\xi)\overline{\varphi_0(\xi)} + \sum_{i=1}^{\infty} \theta(2^{-i}\xi)\overline{\varphi(2^{-i}\xi)} = 1, \quad \xi \in \mathbb{R}^n. \tag{5.1}$$

The Littlewood–Paley analysis and the Shannon sampling theorem conclude that

$$f(x) = \sum_{Q \in \tilde{\mathcal{Q}}} a_Q \theta_Q(x), \quad \forall f \in B^{\mathbb{S}}_{\mathbb{F}}(\psi_0, \psi) \tag{5.2}$$

where $a_{Q_{i,k}} = (\varphi_i * f)(2^{-i}k)$. Moreover, by Lemma 5.1, we obtain

$$\begin{aligned} \|\{a_Q\}_{Q\in\tilde{\mathcal{Q}}}\|_{b^{\mathbb{S}}_{\mathbb{F}}} &= \big\|\{\sum_{k\in\mathbb{Z}^n} |a_{Q_{i,k}}|\chi_{Q_{i,k}}\}_{i=0}^{\infty}\big\|_{(\mathbb{F},\mathbb{S})} \\ &\leq C\|\{f * \varphi_i\}_{i=0}^{\infty}\|_{(\mathbb{F},\mathbb{S})} = C\|f\|_{B^{\mathbb{S}}_{\mathbb{F}}(\varphi_0,\varphi)}, \quad \forall f \in B^{\mathbb{S}}_{\mathbb{F}}(\varphi_0,\varphi). \end{aligned} \tag{5.3}$$

Next, we consider $\psi_i * f$, $i \geq 0$. Form (2.3) and (5.2), for any $M > 0$, we have

$$\begin{aligned} |(f * \psi_i)(x)| &= \sum_{j=i-1}^{i+1} \sum_{k\in\mathbb{Z}^n} |a_{Q_{j,k}}||(\theta_{Q_{j,k}} * \psi_i)(x)| \\ &\leq C \sum_{j=i-1}^{i+1} \sum_{k\in\mathbb{Z}^n} |a_{Q_{j,k}}|(1 + 2^j|x - x_{Q_{j,k}}|)^{-M}. \end{aligned}$$

Lemma 5.2 assures that

$$\|f * \psi_i\|_{\mathbb{F}} \leq C \sum_{j=i-1}^{i+1} \|\sum_{k\in\mathbb{Z}^n} |a_{Q_{j,k}}|\chi_{Q_{j,k}}\|_{\mathbb{F}}. \tag{5.4}$$

Condition (2) of Definition 2.2 and inequality (5.3) show that for any $f \in B^{\mathbb{S}}_{\mathbb{F}}(\psi_0, \psi)$,

$$\|f\|_{B^{\mathbb{S}}_{\mathbb{F}}(\psi_0,\psi)} \leq C\|\{\sum_{k\in\mathbb{Z}^n} |a_{Q_{j,k}}|\chi_{Q_{j,k}}\}_{j=0}^{\infty}\|_{(\mathbb{F},\mathbb{S})} \leq C\|f\|_{B^{\mathbb{S}}_{\mathbb{F}}(\varphi_0,\varphi)}.$$

Moreover, with the independent of the functions φ_0 and φ on the definition of $B^{\mathbb{S}}_{\mathbb{F}}$, inequalities (5.3) establish the boundedness of the φ-transform because $(\varphi_i * f)(2^{-i}k) = \langle f, \varphi_Q\rangle$. Similarly, identity (5.2) with θ_i replaced by ψ_i, $i \in \mathbb{N}$ and the first inequality in (5.4) assure the boundedness of the ψ-transform. $\square$

Proof of Theorem 3.2:
By some simple modifications of the argument in Theorem 2.6 of [8], we have

$\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\mathrm{supp}\theta_0, \mathrm{supp}\theta \subseteq \{x : |x| \leq 1\}$ and $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.3)-(2.4) such that they fulfill (5.1). The Calderón reproducing formula guarantees that for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$f(x) = \sum_{i=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \int_{Q_{i,k}} \theta_i(x-y)(\varphi_i * f)(y)dy.$$

For $Q = Q_{i,k} \in \tilde{\mathcal{Q}}$, we define $s_Q = \sup_{y\in Q} |(\varphi_i * f)(y)|$ and, for $s_Q \neq 0$,

$$a_Q(x) = \frac{1}{s_Q} \int_{Q_{i,k}} \theta_i(x-y)(\varphi_i * f)(y)dy.$$

We have $f = \sum_{Q\in\tilde{\mathcal{Q}}} s_Q a_Q$. The estimate, $\|\{s_Q\}_{Q\in\tilde{\mathcal{Q}}}\|_{b^{\mathbb{S}}_{\mathbb{F}}} \leq C\|f\|_{B^{\mathbb{S}}_{\mathbb{F}}}$, follows from Lemma 5.1. $\square$

Proof of Theorem 3.3:
As $\|\cdot\|^r_{\mathbb{F}}$ satisfies the triangle inequality provided that $r \leq p_{\mathbb{F}}$, without loss of generality, we can assume that $q_{\mathbb{S}} = 1/p_{\mathbb{F}}$. Pick an $\epsilon > 0$ such that $N > L + \frac{n}{p_{\mathbb{F}}} + \epsilon$ and $\delta > \epsilon$ (the δ is given in Definition 5.3). Applying Lemma 5.3, we have

$$\begin{aligned}|(\psi_i * f)(x)| &\leq \sum_{j=0}^{i} \sum_{k\in\mathbb{Z}^n} |s_{Q_{j,k}}||(\psi_i * m_{Q_{j,k}})(x)| + \sum_{j=i+1}^{\infty} \sum_{k\in\mathbb{Z}^n} |s_{Q_{j,k}}||(\psi_i * m_{Q_{j,k}})(x)| \\ &\leq C\big(\sum_{j=0}^{i} 2^{-(i-j)\beta} \sum_{k\in\mathbb{Z}^n} |s_{Q_{j,k}}|(1+2^j|x-x_{Q_{j,k}}|)^{-N} \\ &+ \sum_{j=i+1}^{\infty} 2^{-(j-i)(-\alpha+\delta-\epsilon)} \sum_{k\in\mathbb{Z}^n} |s_{Q_{j,k}}|(1+2^j|x-x_{Q_{j,k}}|)^{-L-\frac{n}{p_{\mathbb{F}}}-\epsilon}\big).\end{aligned}$$

Using the fact that $\mathbb{F}$ is a $p_{\mathbb{F}}$-Banach space, Lemma 5.2 assures that

$$\begin{aligned}\|\psi_i * f\|_{\mathbb{F}} \leq & C\big(\sum_{j=0}^{i} 2^{-p_{\mathbb{F}}(i-j)\beta} \|\sum_{k\in\mathbb{Z}^n} |s_{Q_{j,k}}|\chi_{Q_{j,k}}\|^{p_{\mathbb{F}}}_{\mathbb{F}} \\ & + \sum_{j=i+1}^{\infty} 2^{p_{\mathbb{F}}(i-j)(-\alpha+\delta-\epsilon)} \|\sum_{k\in\mathbb{Z}^n} |s_{Q_{j,k}}|\chi_{Q_{j,k}}\|^{p_{\mathbb{F}}}_{\mathbb{F}}\big)^{1/p_{\mathbb{F}}}.\end{aligned}$$

In view of the definition of (α, L)-regular, we assert that

$$\|f\|_{B^{\mathbb{S}}_{\mathbb{F}}} = \|\{\psi_i * f\}_{i=0}^{\infty}\|_{(\mathbb{F},\mathbb{S})} \leq C\big\|\{\sum_{k\in\mathbb{Z}^n} |s_{Q_{j,k}}|\chi_{Q_{j,k}}\}_{i=0}^{\infty}\big\|_{(\mathbb{F},\mathbb{S})} = C\|s\|_{b^{\mathbb{S}}_{\mathbb{F}}}$$

because $-\alpha + \delta - \epsilon > -\alpha$. $\square$

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Department of Mathematics and Information Technology, The Hong Kong Institute of Education, 10 Lo Ping Road, Tai Po, Hong Kong, China

E-mail address: vkpho@ied.edu.hk