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A First-Order Periodic Differential Equation at Resonance

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Abstract

We consider the existence of a periodic solution to the first-order nonlinear problem

$$\begin{aligned} &x'(t) = -a(t)x(t) + q(t, x(t)), \ \text{a.e. on } (0, T), \\ &x(0) = x(T), \end{aligned}$$

where the nonlinear term q is Carathéodory with respect to $L^1[0,T]$. The coefficient function a is such that the differential equation is non-invertible. The technique used to establish our existence result is Mahwin's coincidence degree theory.

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1 Introduction

Let $T > 0$ be fixed. We consider existence of solutions to the first-order the nonlinear periodic equation

$$\begin{aligned} &x'(t) = -a(t)x(t) + q\left(t, x(t)\right), \ a.e. \text{ on } (0, T), \\ &x(0) = x(T). \end{aligned} \tag{1.1}$$

In recent years, there have been several papers written on the existence, uniqueness, stability and positivity of solutions for periodic equations of forms similar to equation (1.1); see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14] and references therein.

In the above mentioned works, the non-linear term is assumed to be continuous in all variables. We relax this condition by assuming that q is Carathéodory with respect to $L^1[0,T]$. The map $q : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies Carathéodory conditions with respect to $L^1[0,T]$ if the following conditions hold.

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto q(t,z)$ is Lebesgue measurable.

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(ii) For almost every $t \in [0,T]$, the mapping $z \mapsto q(t,z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $\rho > 0$, there exists $\alpha_\rho \in L^1([0,T],\mathbb{R})$ such that for almost every $t \in [0,T]$ and for all z such that $|z| < \rho$, we have $|q(t,z)| \le \alpha_\rho(t)$.

Throughout the paper we assume that the function $a \in L^1[0,T]$ satisfies $e^{\int_0^T a(s)\,ds} = 1$. As such, equation (1.1) is not invertible and we say that the system is at resonance. To show the existence of a solution of (1.1) we rewrite the differential equation in the form $Lx = Nx$ and employ Mawhin's coincidence theory; see [10]. We give some concepts from coincidence theory in Section 2 that are central in our proof, as well as define the spaces and projectors P and Q employed. We state and prove our main result in Section 3.

2 Coincidence Theory

Let X and Z be normed spaces. A linear mapping $L : \text{dom } L \subset X \to Z$ is called a *Fredholm mapping* if the following two conditions hold:

(i) $\ker L$ has a finite dimension, and

(ii) $\text{Im}\, L$ is closed and has finite codimension.

If L is a Fredholm mapping, its (Fredhom) *index* is the integer, Ind L, given by Ind $L = \dim\ker L - \text{codim Im } L$.

For a Fredholm map of index zero, $L :$ dom $L \subset X \to Z$, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that

$$\text{Im } P = \ker L, \ker Q = \text{Im } L, X = \ker L \oplus \ker P, Z = \text{Im } L \oplus \text{Im } Q,$$

and the mapping

$$L|_{\text{dom } L \cap \ker P} : \text{dom } L \cap \ker P \to \text{Im } L$$

is invertible. The inverse of $L|_{\text{dom } L \cap \ker P}$ is denoted by

$$K_P : \text{Im } L \to \text{dom } L \cap \ker P.$$

The generalized inverse of L, denoted by $K_{P,Q} : Z \to \text{dom } L \cap \ker P$, is defined by $K_{P,Q} = K_P(I - Q)$.

If L is a Fredholm mapping of index zero, then for every isomorphism $J : \text{Im } Q \to \ker L$, the mapping $JQ + K_{P,Q} : Z \to \text{dom } L$ is an isomorphism and, for every $x \in \text{dom } L$,

$$(JQ + K_{P,Q})^{-1}x = (L + J^{-1}P)x.$$

Definition 2.1. Let $L : \text{dom } L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N : E \to Z$. We say that N is L-compact on E if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are compact on E. In addition, we say that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

As noted in the abstract, we formulate the periodic equation (1.1) as $Lx = Nx$, where L and N are defined below. We employ the following theorem due to Mawhin [10] to show the existence of a solution.

Theorem 2.2. *Let $\Omega \subset X$ be open and bounded. Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in \big((dom\, L \setminus \ker L) \cap \partial\Omega\big) \times (0,1)$*;*

(ii) $Nx \notin Im\, L$ *for every* $x \in \ker L \cap \partial\Omega$*;*

(iii) $\deg_B \big(JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0\big) \neq 0$*, with* $Q : Z \to Z$ *a continuous projector, such that* $\ker Q = Im\, L$ *and* $J : Im\, Q \to \ker L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $dom\, L \cap \overline{\Omega}$.

Let $AC[0,T]$ denote the space of absolutely continuous functions on the interval $[0,T]$. Define $Z = L^1[0,T]$ with norm $\|\cdot\|_1$ and let

$$X = \left\{x : [0,T] \to \mathbb{R} : x \in AC[0,T] \text{ and } x' + a(t)x \in L^1[0,T]\right\}$$

with norm $\|x\| = \max_{t\in[0,T]} \left|x(t)e^{\int_0^t a(s)\,ds}\right|$. Define the mapping $L : \text{dom } L \subset X \to Z$ by

$$Lx(t) = x'(t) + a(t)x(t), \quad t \in [0,T],$$

where

$$\text{dom } L = \{x \in X : x(0) = x(T)\}.$$

Define $N : X \to Z$ by

$$Nx(t) = q(t,x(t)), \quad t \in [0,T].$$

Let $Q : Z \to Z$ be given by

$$Qg(t) = \frac{1}{T}\int_0^T g(r)e^{\int_0^r a(s)\,ds}\,dr\,e^{-\int_0^t a(s)\,ds}. \tag{2.1}$$

Note that for all $t \in [0,T]$,

$$\begin{aligned}
Q^2g(t) &= \frac{1}{T}\int_0^T Qg(r)e^{\int_0^r a(s)\,ds}\,dr\,e^{-\int_0^t a(s)\,ds} \\
&= \frac{1}{T^2}\int_0^T g(u)e^{\int_0^r a(s)\,ds}\,du \int_0^T e^{-\int_0^r a(s)\,ds}e^{\int_0^r a(s)\,ds}\,dr\,e^{-\int_0^t a(s)\,ds} \\
&= \frac{1}{T}\int_0^T g(r)e^{\int_0^r a(s)\,ds}\,dr e^{-\int_0^t a(s)\,ds} = Qg(t).
\end{aligned}$$

Hence $Q : Z \to Z$ is a continuous projector.

Lemma 2.3. *The mapping $L : dom\, L \subset X \to Z$ is a Fredholm mapping of index zero.*

Proof. Note

$$\ker L = \left\{x \in \text{dom } L : x(t) = ce^{-\int_0^t a(s)\,ds},\ c \in \mathbb{R}\right\} \cong \mathbb{R}.$$

Thus $\dim \ker L = 1$.

Let $g \in Z$ and let

$$x(t) = x(0)e^{-\int_0^t a(s)\,ds} + \int_0^t g(r)e^{-\int_r^t a(s)\,ds}\,dr.$$

Then $x'(t) = -a(t)x(t) + g(t)$ a.e. on $[0,T]$. Furthermore, suppose that g satisfies

$$\int_0^T g(r)e^{\int_0^r a(s)\,ds}\,dr = 0.$$

Then,

$$x(T) = x(0)e^{-\int_0^T a(s)\,ds} + \int_0^T g(r)e^{\int_0^r a(s)\,ds}\,dr = x(0),$$

and hence, $g \in \text{Im } L$. That is,

$$\left\{ g \in Z : \int_0^T g(r)e^{\int_0^r a(s)\,ds}\,dr = 0 \right\} \subseteq \text{Im } L. \tag{2.2}$$

Now let $g \in \text{Im } L$. Then there exists an $x \in \text{dom } L$ such that $Lx(t) = g(t)$ for a.e. $t \in [0,T]$. That is,

$$x'(t) + a(t)x(t) = g(t) \quad \text{a.e. on } [0,T].$$

It is easy to see that x satisfies

$$x(t) = x(0)e^{-\int_0^t a(s)} + e^{-\int_0^t a(s)\,ds}\int_0^t g(r)e^{\int_0^r a(s)\,ds}\,dr.$$

Since $x \in X$, then $x(0) = x(T)$ and so,

$$\int_0^T g(r)e^{\int_0^r a(s)\,ds}\,dr = 0.$$

Thus

$$\text{Im } L \subseteq \left\{ g \in Z : \int_0^T g(r)e^{\int_0^r a(s)\,ds}\,dr = 0 \right\}. \tag{2.3}$$

From (2.2) and (2.3) we have that

$$\text{Im } L = \left\{ g \in Z : \int_0^T g(r)e^{\int_0^r a(s)\,ds}\,dr = 0 \right\}.$$

The projector defined by (2.1) is continuous and linear. Also,

$$\ker Q = \left\{ g \in Z : \int_0^T g(r)e^{\int_0^r a(s)\,ds}dr = 0 \right\} = \text{Im } L.$$

Since $Q(g - Qg) = Qg - Q^2 g = 0$ for all $g \in Z$, then $g - Qg \in \ker Q = \text{Im } L$. Hence $Z = \text{Im } L + \text{Im } Q$. Let $g \in \text{Im } L \cap \text{Im } Q$. Since $g \in \text{Im } Q$, then $g = Qg$ and since $g \in \text{Im } L = \ker Q$, then $Qg = 0$. Consequently, $g \equiv 0$. We have $\text{Im } L \cap \text{Im } Q = \{0\}$ and so, $Z = \text{Im } L \oplus \text{Im } Q$. Hence, $\dim \ker L = 1 = \dim \text{Im } Q = \text{codim Im } L$. Since L is linear, then L is a Fredholm map of index 0 and the proof is complete.

We need to define the second projector P. Let $P : X \to X$ be given by

$$Px(t) = x(0)e^{-\int_0^t a(s)\,ds}. \tag{2.4}$$

Since $Px(0) = x(0)$ then it follows trivially that $P^2x(t) = Px(t)$, $t \in [0,T]$. Note that $\ker P = \{x \in X : x(0) = 0\}$ and that $\mathrm{Im}\ P = \ker L$. Since $\ker P = \{x \in X : x(0) = 0\}$, an argument similar to the one showing $Z = \mathrm{Im}\ L \oplus \mathrm{Im}\ Q$, implies that $X = \ker P \oplus \ker L$.

Define $K_P : \mathrm{Im}\ L \subset Z \to \mathrm{dom}\ L \cap \ker P$ by

$$K_P g(t) = \int_0^t g(r)e^{\int_0^r a(s)\,ds}\,dr\,e^{-\int_0^t a(s)\,ds}.$$

Then

$$\begin{aligned}
\|K_P g\| &= \max_{t\in[0,T]} \left| \int_0^t g(r)e^{\int_0^r a(s)\,ds}\,dr\,e^{-\int_0^t a(s)\,ds}\,e^{\int_0^t a(s)\,ds} \right| \\
&\le \max_{t\in[0,T]} \int_0^t \left| g(r)e^{\int_0^r a(s)\,ds} \right| dr \\
&\le \|g\|T.
\end{aligned} \tag{2.5}$$

Note that, if $x \in \mathrm{dom}\ L \cap \ker P$ then $K_P Lx(t) = x(t)$, and if $g \in \mathrm{Im}\ L$ then $LK_P g(t) = g(t)$. Consequently, $K_P = (L|_{\mathrm{dom}\ L \ker P})^{-1}$.

Consider the map $QN : X \to Z$ defined by

$$QNx(t) = \frac{1}{T}\int_0^T q(r,x(r))e^{\int_0^r a(s)\,ds}\,dr\,e^{-\int_0^t a(s)\,ds}, \quad t \in [0,T].$$

We define the generalized inverse of L by

$$\begin{aligned}
K_{P,Q}Nx(t) &= \int_0^t (Nx(r) - QNx(r))\,e^{\int_0^r a(s)\,ds}dr\,e^{-\int_0^t a(s)\,ds} \\
&= \int_0^t q(r,x(r))e^{\int_0^r a(s)\,ds}\,dr\,e^{-\int_0^t a(s)\,ds} \\
&\quad -\frac{t}{T}\int_0^T q(\tau,x(\tau))e^{\int_0^\tau a(s)\,ds}\,d\tau\,e^{-\int_0^t a(s)\,ds}.
\end{aligned}$$

We end this section by showing that N is L-completely continuous. To do so, we first define the quantity

$$M = \max_{t\in[0,T]} e^{-\int_0^t a(s)\,ds}.$$

Lemma 2.4. *The mapping $N : X \to Z$ given by $Nu(t) = q(t,u(t))$ is L-completely continuous.*

Proof. Let $E \subset X$ be a bounded set and let ρ be such that $\|x\| \le \rho$ for all $x \in E$. Since q satisfies Carathéodory conditions, there exists an $\alpha_\rho \in L^1[0,T]$ such that for a.e. $t \in [0,T]$ and for all z such that $|z| < \rho$ we have $|q(t,z)| \le \alpha_\rho(t)$. Then,

$$\begin{aligned}
|QNx(t)| &\le \frac{1}{T}\int_0^T |q(r,x(r))|e^{\int_0^r a(s)\,ds}\,dr\,e^{-\int_0^t a(s)\,ds} \\
&\le \frac{M}{MT}\int_0^T \alpha_\rho(r)\,dr \\
&\le \frac{1}{T}\|\alpha_\rho\|_1.
\end{aligned}$$

Hence, $QN(E)$ is uniformly bounded.

It is clear that the functions QNx are equicontinuous on E. By the Arzelà-Ascoli Theorem, $QN(E)$ is relatively compact. Furthermore, it can be shown that $K_{P,Q}N(E)$ is relatively compact. As such, the mapping $N : X \to Z$ is L-completely continuous and the proof is complete.

3 Main Result

In this section we state and prove our main result. We will assume that the following conditions hold.

(H_1) There exists a constant $c_1 > 0$ such that for all $x \in \text{dom } L \setminus \ker L$ satisfying $|x(t)| > c_1, t \in [0,T]$, we have

$$QNx(t) \neq 0.$$

(H_2) There exist $\beta, \delta \in L^1[0,T]$, such that for all $x \in \mathbb{R}$ and for all $t \in [0,T]$,

$$|q(t,x)| \leq \beta(t)|x| + \delta(t).$$

(H_3) There exists a constant $B > 0$ such that for all $c_2 \in \mathbb{R}$ with $|c_2| > B$, either

$$c_2 \int_0^T q\left(r, c_2 e^{-\int_0^r a(s)\,ds}\right) e^{\int_0^r a(s)\,ds}\,dr < 0$$

or

$$c_2 \int_0^T q\left(r, c_2 e^{-\int_0^r a(s)\,ds}\right) e^{\int_0^r a(s)\,ds}\,dr > 0.$$

Theorem 3.1. *Assume that conditions (H_1) - (H_3) hold. Then the nonlinear periodic problem* (2.2) *has at least one solution provided that* $\|\beta\| < \frac{1}{(1+M)T}$.

Proof. Let $Q : Z \to Z$ and $P : X \to X$ be defined as in (2.1) and (2.4), respectively. We first construct a bounded open set Ω that satisfies Theorem 2.2. With this goal in mind, we define the set Ω_1 by

$$\Omega_1 = \{x \in \text{dom } L \setminus \ker L : Lx = \mu Nx \text{ for some } \mu \in (0,1)\}.$$

Let $x \in \Omega_1$ and write x as $x = Px + (I-P)x$. Then

$$\|x\| \leq \|Px\| + \|(I-P)x\|. \tag{3.1}$$

Since $x \in \Omega_1$ then $(I-P)x \in \text{dom } L \cap \ker P = \text{Im } K_P$. Note that $Nx = \frac{1}{\mu}Lx \in \text{Im } L, \mu \in (0,1)$. We obtain from the inequality (2.5) that

$$\|(I-P)x\| = \|K_P L(I-P)x\| \leq \|L(I-P)x\|T = \|Lx\|T < \|Nx\|T. \tag{3.2}$$

From (H_2) we have that $\|Nx\| \leq \|\beta\|\|x\| + \|\delta\|$, and so by (3.1) and (3.2), we obtain,

$$\|x\| < \|Px\| + \|\beta\|\|x\|T + \|\delta\|T. \tag{3.3}$$

Now, $Px(t) = x(0)e^{-\int_0^t a(s)\,ds}$. So,

$$\|Px\| = |x(0)|. \tag{3.4}$$

Since $x \in \Omega_1$ and $\ker Q = \operatorname{Im} L$, then

$$QNx(t) = 0, \quad \text{for all } t \in [0,T].$$

By (H_1) there exists $t_0 \in [0,T]$ such that $|x(t_0)| < c_1$. Also, since

$$x'(t) + a(t)x(t) = q(t,x(t))$$

then,

$$x(0) = x(t_0)e^{\int_0^{t_0} a(s)\,ds} - \int_0^{t_0} q(r,x(r))e^{\int_0^r a(s)\,ds}\,dr.$$

We obtain that,

$$\begin{aligned} |x(0)| &\le c_1 e^{\int_0^{t_0} a(s)\,ds} + \int_0^{t_0} |q(r,x(r))|e^{\int_0^r a(s)\,ds}\,dr \\ &\le c_1 M + \|Nx\|MT \\ &\le c_1 M + \|\beta\|\|x\|MT + \|\delta\|MT. \end{aligned} \tag{3.5}$$

From (3.2), (3.4), and (3.5) we get that

$$\|x\| \le c_1 M + \|\beta\|\|x\|MT + \|\delta\|MT + \|\delta\|T + \|\beta\|\|x\|T.$$

That is,

$$\|x\| \le \frac{c_1 + \|\delta\|T(1+M)}{1 - \|\beta\|T(1+M)}.$$

Since $\|\beta\| < \frac{1}{(1+M)T}$, the set Ω_1 is bounded.

Define

$$\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$$

and let $x \in \Omega_2$. Since $x \in \ker L$, then there exists a constant c such that

$$x(t) = ce^{-\int_0^t a(s)\,ds}.$$

Since $Nu \in \operatorname{Im} L = \ker Q$, then

$$\int_0^T q\left(r, ce^{-\int_0^r a(s)\,ds}\right) e^{\int_0^r a(s)\,ds}\,dr = 0.$$

By (H_3), we have that $|c| \le B$ and so $\|x\| = |c| \le B$. The set Ω_2 is bounded.

Before we define the set Ω_3, we must state our isomorphism, $J : \operatorname{Im} Q \to \ker L$. Let

$$J\left(ce^{-\int_0^t a(s)\,ds}\right) = ce^{-\int_0^t a(s)\,ds}.$$

If the first part of (H_3) is satisfied, then define

$$\Omega_3 = \{x \in \ker L : -\lambda J^{-1}x + (1-\lambda)QNx = 0\}.$$

Let $x \in \Omega_3$. Since $x \in \ker L$, then there exists c_2 such that

$$x(t) = c_2 e^{-\int_0^t a(s)\,ds}.$$

Assume that $|c_2| > B > 0$. Since $x \in \Omega_3$, we have

$$\lambda J^{-1} x = (1-\lambda) QNx$$

from which we obtain,

$$\lambda c_2 = (1-\lambda)\frac{1}{T}\int_0^T q\left(r, c_2 e^{-\int_0^r a(s)\,ds}\right) e^{\int_0^r a(s)\,ds}\,dr.$$

If $\lambda = 1$, then $c_2 = 0$. If $\lambda \in (0,1)$ then

$$\lambda c_2^2 = (1-\lambda)\frac{c_2}{T}\int_0^T q\left(r, c_2 e^{-\int_0^r a(s)\,ds}\right) e^{\int_0^r a(s)\,ds}\,dr < 0.$$

That is, $c_2^2 < 0$. If $\lambda = 0$, we obtain from the above equation that $c_2 = 0$. Consequently, if $\lambda \in [0,1]$ we obtain a contradiction and hence $|c_2| \leq B$. Thus, Ω_3 is bounded.

Let Ω be an open and bounded set such that $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. Then the assumptions (i) and (ii) of Theorem 2.2 are satisfied. By Lemma 2.3, $L : \operatorname{dom} L \subset X \to Z$ is a Fredholm mapping of index zero. By Lemma 2.4, the mapping $N : X \to Z$ is L-completely continuous. We only need to verify that condition (iii) of Theorem 2.2 is satisfied.

We apply the invariance under a homotopy property of the Brower degree. Let

$$H(x,\mu) = \pm\mu \mathrm{Id}\,x + (1-\mu) JQNx.$$

If $x \in \ker L \cap \partial\Omega$, then

$$\begin{aligned}
\deg_B\left(JQN|_{\ker L\cap\partial\Omega}, \Omega\cap\ker L, 0\right) &= \deg_B\left(H(\cdot,0), \Omega\cap\ker L, 0\right)\\
&= \deg_B\left(H(\cdot,1), \Omega\cap\ker L, 0\right)\\
&= \deg_B\left(\pm\,\mathrm{Id}, \Omega\cap\ker L, 0\right)\\
&\neq 0.
\end{aligned}$$

All the assumptions of Theorem 2.2 are fulfilled and the proof is complete.

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