African Diaspora Journal of Mathematics
Volume 10, Number 2, pp. 145–152 (2010)
ISSN 1539-854X
www.math-res-pub.org/adjm

ON A NEW HILBERT'S TYPE INEQUALITY

GUANGSHENG CHEN*
Department of Computer Engineering
Guangxi Modern Vocational Technology College
Hechi, 547000,China

Abstract

By introducing some parameters and estimating the weight coefficient, we establish a new Hilbert's type inequality with a best constant factor. As applications, the reverse form, some particular results and the equivalent form are considered.

AMS Subject Classification: 26D15.

Keywords: Hilbert's inequality,Hö1der's inequality,weight coefficient.

1 Introduction

If $a_n \geq 0$,$b_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then we have (see[1])

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{1/2} \tag{1.1}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{\max\{m,n\}} < 4 \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{1/2}. \tag{1.2}$$

where the constant factors π and 4 are the best possible. Inequality (1.1) is well known as Hilbert's inequality and (1.2) is named a Hilbert-type inequality. Both of them are important in analysis and its applications [2]. In 1998, Gao et al. [3,4] considered some strengthened versions of (1.1). Recently, by introducing an independent parameter λ, some extensions of (1.1) and (1.2) were given as follows [5,6]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m^\lambda + n^\lambda} < \frac{\pi}{\lambda} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 \right\}^{1/2} \tag{1.3}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{\max\{m^\lambda, n^\lambda\}} < \frac{4}{\lambda} \left\{ \sum_{n=1}^{\infty} n^{1-\lambda} a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda} b_n^2 \right\}^{1/2}. \tag{1.4}$$

*E-mail address: cgswavelets@126.com

where the constant factors π/λ and $4/\lambda$ $(0<\lambda\le 2)$ are all the best possible. A reverse Hilbert's integral inequality is given by Yang [7].

Very recently,in [8] the following extensions were given:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\max\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda} \\ < D_\lambda(a,b,c)\left\{\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}b_n^q\right\}^{\frac{1}{q}}. \tag{1.5}$$

where the constant factor $D_\lambda(a,b,c)$ (see[8,Lemma 1]) is the best possible.

In this paper,by introducing some parameters and estimating the weight coefficient, a new inequality with the best constant factor is established, which is similar to inequality (1.5) . As applications, the reverse form, some particular results and the equivalent form are considered.

2 Some Lemmas

Lemma 2.1. *If* $0<\lambda\le 2$, $a\ge 0$, $b,c>0$, *define the weight coefficient* $\omega_\lambda(a,b,c,m)$ *as*

$$\omega_\lambda(a,b,c,m)=\sum_{n=1}^{\infty}\frac{m^{\lambda/2}n^{\lambda/2-1}}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\quad (m\in \mathrm{N}) \tag{2.1}$$

Then we have the following inequality:

$$C_\lambda(a,b,c)[1-\theta_\lambda(a,b,c,m)]<\omega_\lambda(a,b,c,m)<C_\lambda(a,b,c). \tag{2.2}$$

where $C_\lambda(a,b,c)=\frac{C(a,b,c)}{\lambda}$, $\theta_\lambda(a,b,c,m):=\frac{1}{C(a,b,c)}\int_0^{m^{-\lambda}}\frac{u^{-1/2}}{(a+c)u+b}du=O(1/m^{\lambda/2})\in(0,1),(m\to\infty)$,*and the constant* $C(a,b,c)$ *is defined by*

$$C(a,b,c)=\begin{cases}\frac{2}{\sqrt{b(a+c)}}\arctan\sqrt{\frac{a+c}{b}}+\frac{2}{\sqrt{c(a+b)}}\arctan\sqrt{\frac{a+b}{c}}, & for\ a>0,b,c>0\\ \frac{\pi}{\sqrt{bc}}, & for\ a=0,b,c>0\end{cases}. \tag{2.3}$$

Proof. Setting $u=(y/m)^\lambda$, Since $0<\lambda\le 2$ and $a\ge 0,b,c>0$, we have

$$\omega_\lambda(a,b,c,m)<\int_0^\infty\frac{m^{\lambda/2}y^{\lambda/2-1}}{a\min\{m^\lambda,y^\lambda\}+bm^\lambda+cy^\lambda}dy \\ =\frac{1}{\lambda}\int_0^\infty\frac{u^{-1/2}}{a\min\{1,u\}+b+cu}du:=I. \tag{2.4}$$

(i)$a,b,c>0$, we have

$$I=\frac{1}{\lambda}\int_0^\infty\frac{u^{-1/2}}{a\min\{1,u\}+b+cu}du$$
$$=\frac{1}{\lambda}[\int_0^1\frac{u^{-1/2}}{b+(a+c)u}du+\int_1^\infty\frac{u^{-1/2}}{(a+b)+cu}du]$$
$$=\frac{1}{\lambda}[\frac{2}{\sqrt{b(a+c)}}\arctan\sqrt{\frac{a+c}{b}}+\frac{2}{\sqrt{c(a+b)}}\arctan\sqrt{\frac{a+b}{c}}].$$

(ii) $a=0$,$b,c>0$ we obtain

$$I=\frac{1}{\lambda}\int_0^\infty\frac{u^{-1/2}}{b+cu}du=\frac{2}{\lambda\sqrt{bc}}[\arctan\sqrt{\frac{c}{b}u}]_0^\infty=\frac{\pi}{\lambda\sqrt{bc}}.$$

Hence $\omega_\lambda(a,b,c,m)<I=\frac{1}{\lambda}C(a,b,c)=C_\lambda(a,b,c)(m\in N)$, We have

$$\omega_\lambda(a,b,c,m)>\int_1^\infty\frac{m^{\lambda/2}y^{\lambda/2-1}}{a\min\{m^\lambda,y^\lambda\}+bm^\lambda+cy^\lambda}dy$$
$$=I-\frac{1}{\lambda}\int_0^{m^{-\lambda}}\frac{u^{-1/2}}{a\min\{1,u\}+b+cu}du=I[1-\theta_\lambda(a,b,c,m)].$$

where $I=\frac{1}{\lambda}C(a,b,c)$ and $0<\theta_\lambda(a,b,c,m):=\frac{1}{C(a,b,c)}\int_0^{m^{-\lambda}}\frac{u^{-1/2}}{(a+c)u+b}du<1$ .Since

$$0<\int_0^{m^{-\lambda}}\frac{u^{-1/2}}{(a+c)u+b}du\le\int_0^{m^{-\lambda}}\frac{u^{-1/2}}{b}du=\frac{2}{bm^{\lambda/2}}$$

then $\theta_\lambda(a,b,c,m)=O(\frac{1}{m^{\lambda/2}})$. Therefore (2.2) is valid. The lemma is proved.

Note. By the symmetry, we still have $\frac{1}{\lambda}\int_0^\infty\frac{u^{-1/2}}{a\min\{1,u\}+bu+c}du=I$ and

$$C_\lambda(a,b,c)[1-\theta_\lambda(a,b,c,n)]<\omega_\lambda(a,b,c,n)<C_\lambda(a,b,c). \tag{2.5}$$

Lemma 2.2. *Let $p>0, p\neq 1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda\le 2$, $a\ge 0,b,c>0$, and $0<\varepsilon<p\lambda/2$,setting*

$$J(\varepsilon):=\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{m^{\lambda/2-1-\varepsilon/p}n^{\lambda/2-1-\varepsilon/q}}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda} \tag{2.6}$$

then for $\varepsilon\to 0^+$,we have

$$[C_\lambda(a,b,c)-o(1)]\sum_{n=1}^\infty\frac{1}{n^{1+\varepsilon}}<J(\varepsilon)<[C_\lambda(a,b,c)+\tilde{o}(1)]\sum_{n=1}^\infty\frac{1}{n^{1+\varepsilon}}. \tag{2.7}$$

Proof. Setting $u=(x/n)^\lambda$ in the following, by the Note, we have

$$J(\varepsilon)<\sum_{n=1}^\infty\int_0^\infty\frac{x^{\lambda/2-1-\varepsilon/p}n^{\lambda/2-1-\varepsilon/q}}{a\min\{x^\lambda,n^\lambda\}+bx^\lambda+cn^\lambda}dx$$
$$=\sum_{n=1}^\infty\frac{1}{n^{1+\varepsilon}}[\frac{1}{\lambda}\int_0^\infty\frac{u^{-1/2-\varepsilon/\lambda p}}{a\min\{1,u\}+bu+c}du]=[I+\tilde{o}(1)]\sum_{n=1}^\infty\frac{1}{n^{1+\varepsilon}}$$

$$
\begin{aligned}
J(\varepsilon) &> \sum_{n=1}^{\infty}\int_1^{\infty}\frac{x^{\lambda/2-1-\varepsilon/p}n^{\lambda/2-1-\varepsilon/q}}{a\min\{x^\lambda,n^\lambda\}+bx^\lambda+cn^\lambda}dx\\
&=\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}[\frac{1}{\lambda}\int_{n^{-\lambda}}^{\infty}\frac{u^{-1/2-\varepsilon/\lambda p}}{a\min\{1,u\}+bu+c}du]\\
&>\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}[\frac{1}{\lambda}\int_{n^{-\lambda}}^{\infty}\frac{u^{-1/2-\varepsilon/\lambda p}}{a\min\{1,u\}+bu+c}du]-\frac{1}{\lambda}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{n^{-\lambda}}\frac{u^{-1/2-\varepsilon/\lambda p}}{c}du\\
&=\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[I+\tilde{o}(1)-\frac{1}{(\frac{\lambda}{2}-\frac{\varepsilon}{p})c}\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{\lambda}{2}-\frac{\varepsilon}{p}}}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}\right]\\
&=\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}(I-o(1))(\varepsilon\to 0^+).
\end{aligned}
$$

Hence (2.7) is valid, and the lemma is proved.

3 Main results

Theorem 3.1. *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda\le 2$, $a\ge 0, b, c>0$, $a_n, b_n\ge 0$, such that $0<\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}a_n^p<\infty$, $0<\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q<\infty$, then the following inequalities hold and are equivalent*

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\\
&<C_\lambda(a,b,c)\{\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q\}^{\frac{1}{q}}
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
&\sum_{n=1}^{\infty}n^{p\lambda/2-1}[\sum_{m=1}^{\infty}\frac{a_m}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}]^p\\
&<[C_\lambda(a,b,c)]^p\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}a_n^p.
\end{aligned}
\tag{3.2}
$$

where the constant factor $C_\lambda(a,b,c)$ and $[C_\lambda(a,b,c)]^p$ are the best possible. In particular, (i)for $\lambda=a=b=c=1$,we have $C_1(1,1,1)=2\sqrt{2}\arctan\sqrt{2}$ and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\min\{m,n\}+m+n}<2\sqrt{2}\arctan\sqrt{2}\{\sum_{n=1}^{\infty}n^{p/2-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q/2-1}b_n^q\}^{\frac{1}{q}} \tag{3.3}$$

$$\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}\sum_{m=1}^{\infty}\frac{a_m b_n}{\min\{m,n\}+m+n}<[2\sqrt{2}\arctan\sqrt{2}]^p\sum_{n=1}^{\infty}n^{p/2-1}a_n^p. \tag{3.4}$$

(ii)for $a=0,\lambda=b=c=1$,we have $C_1(0,1,1)=\pi$ and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\pi\{\sum_{n=1}^{\infty}n^{p/2-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q/2-1}b_n^q\}^{\frac{1}{q}} \tag{3.5}$$

$$\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\pi^p\sum_{n=1}^{\infty}n^{p/2-1}a_n^p. \tag{3.6}$$

Proof. By Hölders inequality with weight [9] and in view of (2.1), we find

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}$$
$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\frac{n^{(\lambda/2-1)/p}}{m^{(\lambda/2-1)/q}}\frac{m^{(\lambda/2-1)/q}}{n^{(\lambda/2-1)/p}}$$
$$\leq\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m^p m^{\lambda/2}n^{\lambda/2-1}m^{p(1-\lambda/2)-1}}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{b_n^q m^{\lambda/2-1}n^{\lambda/2}n^{q(1-\lambda/2)-1}}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\}^{\frac{1}{q}}$$
$$=\{\sum_{m=1}^{\infty}\omega_\lambda(a,b,c,m)m^{p(1-\lambda/2)-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\omega_\lambda(a,b,c,n)n^{q(1-\lambda/2)-1}b_n^q\}^{\frac{1}{q}}.$$

Then by (2.2) and (2.5), we have (3.1).

For $0<\varepsilon<\frac{p\lambda}{2}$, setting $\widetilde{a}_m=m^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}$,$\widetilde{b}_n=n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}$ $(m,n\in N)$,by (2.6), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}=J(\varepsilon).$$

If there exists a constant $0<K\leq C_\lambda(a,b,c)$ such that (3.1) is still valid if we replace $C_\lambda(a,b,c)$ by K, then in particular, by (2.7) we find

$$[C_\lambda(a,b,c)-o(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}<J(\varepsilon)<K\{\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}\widetilde{a}_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}\widetilde{b}_n^q\}^{\frac{1}{q}}=K\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}.$$

It follows that $C_\lambda(a,b,c)-o(1)<K$, and then $C_\lambda(a,b,c)\leq K(\varepsilon\to 0^+)$. Hence $K=C_\lambda(a,b,c)$ is the best constant factor of (3.1).

Let's show that (3.1) and (3.2) are equivalent.First,assume that inequality (3.1) is valid. setting

$$b_n=n^{\lambda p/2-1}\left\{\sum_{m=1}^{\infty}\frac{a_m}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\right\}^{p-1}$$

then by (3.1) we get

$$\begin{aligned}\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q&=\sum_{n=1}^{\infty}n^{p\lambda/2-1}[\sum_{m=1}^{\infty}\frac{a_m}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}]^p\\&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\\&\leq C_\lambda(a,b,c)\left\{\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q\right\}^{1/q}.\end{aligned}\tag{3.7}$$

Hence,we have

$$\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q\leq[C_\lambda(a,b,c)]^p\sum_{n=1}^{\infty}n^{p(1-\lambda/2)-1}a_n^p.\tag{3.8}$$

Thus, by (3.1),both (3.7) and (3.8) keep the form of strict inequalities, then we have (3.2).

By Holder's inequality,we find

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\\
&=\sum_{n=1}^{\infty}\left\{n^{\lambda/2-1/p}\sum_{m=1}^{\infty}\frac{a_m}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}]^p\right\}n^{1/p-\lambda/2}b_n \qquad (3.9)\\
&\leq\left\{\sum_{n=1}^{\infty}n^{p\lambda/2-1}\left\{\sum_{n=1}^{\infty}\frac{a_m}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\right\}^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q\right\}^{1/q}.
\end{aligned}$$

Therefore, by (3.2) we have (3.1), and inequality (3.1) and (3.2) are equivalent. If the constant factor in (3.2) is not the best possible, then by (3.9) can get a contradiction that the constant factor in (3.1) is not the best possible. The theorem is proved.

Theorem 3.2. *If* $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda\leq 2$,$a\geq 0,b,c>0$,$a_n,b_n\geq 0$, *such that* $0<\sum\limits_{n=1}^{\infty}n^{p(1-\lambda/2)-1}a_n^p<\infty$,$0<\sum\limits_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q<\infty$ *, then the following inequalities hold and are equivalent*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}\\
&>C_\lambda(a,b,c)\{\sum_{n=1}^{\infty}[1-\theta_\lambda(a,b,c,n)]n^{p(1-\lambda/2)-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q(1-\lambda/2)-1}b_n^q\}^{\frac{1}{q}}
\end{aligned} \qquad (3.10)$$

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{p\lambda/2-1}[\sum_{m=1}^{\infty}\frac{a_m}{a\min\{m^\lambda,n^\lambda\}+bm^\lambda+cn^\lambda}]^p\\
&>[C_\lambda(a,b,c)]^p\sum_{n=1}^{\infty}[1-\theta_\lambda(a,b,c,n)]n^{p(1-\lambda/2)-1}a_n^p.
\end{aligned} \qquad (3.11)$$

where $\theta_\lambda(a,b,c,m):=\frac{1}{C(a,b,c)}\int_0^{m^{-\lambda}}\frac{u^{-1/2}}{(a+c)u+b}du=O(1/m^{\lambda/2})\in(0,1)$.*(*$m\to\infty$*),and the constant factor* $C_\lambda(a,b,c)$ *and* $[C_\lambda(a,b,c)]^p$*are the best possible. In particular,*
(i)for $\lambda=a=b=c=1$,*we have* $C_1(1,1,1)=2\sqrt{2}\arctan\sqrt{2}$ *and*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\min\{m,n\}+m+n}\\
&>2\sqrt{2}\arctan\sqrt{2}\{\sum_{n=1}^{\infty}[1-\frac{\arctan(\sqrt{2}/n^{1/2})}{2\arctan\sqrt{2}}]n^{p/2-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q/2-1}b_n^q\}^{\frac{1}{q}}
\end{aligned} \qquad (3.12)$$

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\frac{p}{2}-1}\sum_{m=1}^{\infty}\frac{a_m b_n}{\min\{m,n\}+m+n}\\
&>[2\sqrt{2}\arctan\sqrt{2}]^p\sum_{n=1}^{\infty}[1-\frac{\arctan(\sqrt{2}/n^{1/2})}{2\arctan\sqrt{2}}]n^{p/2-1}a_n^p.
\end{aligned} \qquad (3.13)$$

(ii)for $a=0,\lambda=b=c=1$,*we have* $C_1(0,1,1)=\pi$ *and*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}>\pi\{\sum_{n=1}^{\infty}[1-\frac{2\arctan n^{-1/2}}{\pi}]n^{p/2-1}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}n^{q/2-1}b_n^q\}^{\frac{1}{q}} \qquad (3.14)$$

$$\sum_{n=1}^{\infty} n^{\frac{p}{2}-1} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} > \pi^p \sum_{n=1}^{\infty} [1 - \frac{2\arctan n^{-1/2}}{\pi}] n^{p/2-1} a_n^p. \tag{3.15}$$

Proof. By the reverse Hölders inequality with weight [9] and in view of (2.1), we find

$$\begin{aligned} H &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{a\min\{m^\lambda, n^\lambda\} + bm^\lambda + cn^\lambda} \\ &\geq \{\sum_{m=1}^{\infty} \omega_\lambda(a,b,c,m) m^{p(1-\lambda/2)-1} a_m^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} \omega_\lambda(a,b,c,n) n^{q(1-\lambda/2)-1} b_n^q\}^{\frac{1}{q}}. \end{aligned}$$

Then by (2.2) and (2.5), in view of $q < 0$, we have (3.10).

For $0 < \varepsilon < \frac{p\lambda}{2}$, setting $\widetilde{a}_m = m^{\frac{\lambda}{2}-1-\frac{\varepsilon}{p}}, \widetilde{b}_n = n^{\frac{\lambda}{2}-1-\frac{\varepsilon}{q}}$ $(m,n \in N)$, if there exists a constant $k \geq C_\lambda(a,b,c)$ such that (3.10) is still valid if we replace $C_\lambda(a,b,c)$ by k, then in particular, by (2.6) and (2.7) we find

$$\begin{aligned} &[C_\lambda(a,b,c) + \tilde{o}(1)] \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} > J(\varepsilon) \\ &> k\{\sum_{n=1}^{\infty} [1 - \theta_\lambda(a,b,c,n)] n^{p(1-\lambda/2)-1} \widetilde{a}_n^p\}^{\frac{1}{p}} \{\sum_{n=1}^{\infty} n^{q(1-\lambda/2)-1} \widetilde{b}_n^q\}^{\frac{1}{q}} \\ &= k\left\{\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} - \sum_{n=1}^{\infty} \left[O(\frac{1}{n^{\lambda/2}}) \frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p} \left\{\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}\right\}^{1/q} \\ &= k \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left\{1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}\right)^{-1} \sum_{n=1}^{\infty} \left[O(\frac{1}{n^{\lambda/2}}) \frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}. \end{aligned}$$

It follows that

$$C_\lambda(a,b,c) + \tilde{o}(1) > k\left\{1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}\right)^{-1} \sum_{n=1}^{\infty} \left[O(\frac{1}{n^{\lambda/2}}) \frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}$$

and then $C_\lambda(a,b,c) \geq k(\varepsilon \to 0^+)$. Hence $k = C_\lambda(a,b,c)$ is the best constant factor of (3.10). The proof of (3.12) is similar to that of (3.2), we omit it. The theorem is proved.

References

[1] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*. Cambridge University Press, Cambridge 1952.

[2] D.S. Mitrinovic, J.E. Pecaric and A.M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*. Kluwer Academic Publishers, Boston 1991.

[3] M. Gao and B. Yang, On the extended Hilberts inequality.*Proc. Amer. Math. Soc.* **126** **(3)** (1998) 751-759.

[4] B. Yang and M. Gao, On a best value of HardyCHilberts inequality. *Adv. Math.* **26 (2)** (1997) 159-164.

[5] B. Yang, On an extension of HardyCHilberts inequality.*Chinese Ann. Math.* **23A (2)** (2002) 247-254.

[6] B. Yang, On a generalization of the Hilberts inequality and its applications.*Chinese J. Eng. Math.* **21 (5)** (2004) 821-824.

[7] B. Yang, A reverse HardyCHilberts inequality.*J. Jilin Univ. Sci.* **42 (4)** (2004) 489-493.

[8] B. Yang, On a relation between Hilberts inequality and a Hilbert-type inequality.*Applied Mathematics Letters.* **21** (2008) 483-488.

[9] J. Kuang, *Applied Inequalities*, Shangdong Science and Technology Press, Jinan, China, 2004.