African Diaspora Journal of Mathematics
Proceedings of Geometry and Physics V
Volume 9, Number 2, pp. 139–149 (2010)
ISSN 1539-854X
www.african-j-math.org

TRANSVERSE GEOMETRY AND GENERALIZED COMPLEX STRUCTURES

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Dedicated to the memory of Paulette Libermann

Abstract

Transversal generalized complex structures provide a framework unifying both transversely holomorphic foliations and generalized complex geometry. In this paper, we give characterizations of transversal generalized complex structures. Moreover, a natural extension of the basic Dolbeault cohomology is obtained.

AMS Subject Classification: Primary: 53Cxx Secondary: 32Qxx, 32Vxx

Keywords: Foliation; generalized complex structures; basic Dolbeault cohomology.

1 Introduction

The notion of a generalized complex structure was introduced by Hitchin [H03]. It encompasses both complex structures and symplectic structures [G04]. On the other hand, complex structures are closely related to transversely holomorphic foliations. Transversely holomorphic foliations have attracted considerable interest since the late seventies. In fact, Girbau, Haefliger and Sundararaman [GHS83] improved the deformation result of transversely holomorphic foliations stated by Duchamp and Kalka [DK79]. The existence of a versal space for deformations of transversely holomorphic foliations with a fixed underlying smooth foliation was proved by El Kacimi and Nicolau [ElKB89] in two cases, namely, the case of Hermitian foliations and that of a transversely holomorphic foliation of complex codimension one which admits a transverse projectable connection. Before that, transversely holomorphic foliations were studied by Gòmez-Mont in [G-M80]. In [G92], Girbau proves the existence of a versal space for deformations of transversely holomorphic foliations with a fixed underlying smooth foliation and a connection invariant along the leaves. Furthermore, the Bott class for transversely holomorphic foliation was studied in [A03, A00].

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In this paper, we study the concept of a transversal generalized complex structures which encompasses generalized complex structures, contact structures as well as transversely holomorphic foliations. Recall that a *transversal generalized complex structure* on a smooth $(2n+d)$-dimensional manifold M is given by a smooth foliation $\mathcal{F}$ of codimension $2n$ which is locally defined by submersions into a model manifold N, endowed with a generalized complex structure, such that the transitions maps preserve the generalized complex structure on the overlap charts [V07]. Our main purpose is to give characterization theorems for transversely generalized complex structures. The deformation theory for such geometric structures will be studied elsewhere.

The paper is organized as follows. In Sections 2, we recall the basic concepts in generalized complex geometry. In Section 3, we recall known fact about courant algebroids and their automorphisms. Section 4 is devoted to characterizations of transversal generalized complex structures. We provide various examples of transversal generalized complex manifolds in Section 5. Finally, Section 6 is devoted to the decomposition of the space of basic induced by a transversal generalized complex structure.

2 Generalized complex structures

Let N be a smooth finite-dimensional manifold. The space of local sections of the vector bundle $TN \oplus T^*N \to N$ is endowed with two natural $\mathbb{R}$-bilinear operations:

- the symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by:

$$\langle X+\alpha, Y+\beta \rangle = \frac{1}{2}\left(\alpha(Y)+\beta(X)\right), \tag{2.1}$$

- and the Courant bracket given by:

$$[\![X+\alpha, Y+\beta]\!] = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha), \tag{2.2}$$

where $X+\alpha$ and $Y+\beta$ are smooth sections of $TN \oplus T^*N$.

An *almost generalized complex structure* on the smooth manifold N is a bundle automorphism $I : TN \oplus T^*N \to TN \oplus T^*N$ such that $I^2 = -\mathrm{Id}$ and $I^* + I = 0$. It can be written in the matrix form:

$$I = \begin{pmatrix} J & \pi^\sharp \\ \theta^\flat & -J^* \end{pmatrix} \tag{2.3}$$

where $J : TN \to TN$ is a (1,1)-tensor field, π a bi-vector field on N, θ a 2-form on N and one has:

$$\iota_{\pi^\sharp\alpha}\beta = \pi(\alpha,\beta) \quad \text{and} \quad \iota_Y(\theta^\flat(X)) = \theta(X,Y),$$

for all 1-forms α, β and for all vector fields X, Y. For simplicity, we will use the notation $I = (J, \pi, \theta)$ instead of the matrix notation. The fact that $I^2 = -\mathrm{Id}$ is equivalent to the following identities:

$$J^2 + \pi^\sharp\theta^\flat = -\mathrm{Id}, \quad J\pi^\sharp = \pi^\sharp J^*, \quad \theta^\flat J = J^*\theta^\flat. \tag{2.4}$$

Definition 2.1. An almost generalized complex structure I is *integrable* if it satisfies the torsion-free condition $\mathcal{N}_I = 0$, where

$$\mathcal{N}_I(e_1, e_2) = [\![Ie_1, Ie_2]\!] - [\![e_1, e_2]\!] - I([\![Ie_1, e_2]\!] + [\![e_1, Ie_2]\!]), \tag{2.5}$$

for all e_1, e_2 sections of the vector bundle $TN \oplus T^*N \to N$. In this case, I is simply called a generalized complex structure.

Example 2.2. (Complex structures) If J is a complex structure on M then the following tensor I defines a generalized complex structure:

$$I = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix},$$

Example 2.3. (Symplectic structures) Let ω be a symplectic form on M. Set

$$I = \begin{pmatrix} 0 & \pi^\sharp \\ \omega^\flat & 0 \end{pmatrix},$$

where $\pi = -\omega^{-1}$. Then I is a generalized complex structure.

Notice that both the Courant bracket and the symmetric $\mathbb{R}$-bilinear operation $\langle \cdot, \cdot \rangle$ can be extended to the complexified bundle $(TN \oplus T^*N) \otimes \mathbb{C}$ by linearity. By a *Dirac structure* of $(TN \oplus T^*N) \otimes \mathbb{C}$, we mean a vector sub-bundle L of $(TN \oplus T^*N) \otimes \mathbb{C}$ having complex rank $d = \dim N$ and such that, for all $e_1, e_2 \in \Gamma(L)$,

$$\langle e_1, e_2 \rangle = 0 \quad \text{and} \quad [\![e_1,\ e_2]\!] \in \Gamma(L).$$

Proposition 2.4. [G04] A generalized complex structure I on N is equivalent to a Dirac structure L of $(TN \oplus T^*N) \otimes \mathbb{C}$ such that

$$L \cap \overline{L} = \{0\}.$$

Here, L is exactly the $\sqrt{-1}$-eigenbundle of the tensor I. The proof of this theorem can be found in [G04].

3 Courant algebroids and automorphisms

Definition 3.1. [LWX97, U02] A *Courant algebroid* over a manifold N consists of a vector bundle $E \to N$ equipped with an $\mathbb{R}$-bilinear bracket $[\![\cdot, \cdot]\!]$ on $\Gamma(E)$, a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on E and a bundle morphism $\rho : E \to TN$, called the *anchor map* that satisfy:

(i) $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$

(ii) $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\rho(e_1)f)e_2$

(iii) $\rho(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$

(iv) $[\![e,e]\!] = \mathcal{D}\langle e,e\rangle$

for any $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$, where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by

$$\mathcal{D}f = \frac{1}{2}\rho^*(df).$$

We identify E with its dual E^* using the non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$.

The basic example of a Courant algebroid is $\mathbb{E}(M) = TM \oplus T^*M$ whose anchor map is the canonical projection $\rho :\ \mathbb{E}(M) \to TM$. This model is an exact Courant algebroid since the following sequence is exact

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0.$$

Let B be a 2-form on M. Recall that a *B-field transformation* of the Courant algebroid $\mathbb{E}(M)$ is a bundle automorphism denoted by $\mathcal{A}_B$ and defined as follows:

$$\mathcal{A}_B(X+\alpha) = X + (\alpha + \iota_X B),$$

for all $X \in \mathfrak{X}$ and $\alpha \in \Omega^1(M)$. Furthermore, given any diffeomorphism of $\psi : M \to M$, one can define an automorphism of $\mathbb{E}(M)$ by setting:

$$\psi_* \oplus (\psi^{-1})^* :\ (X,Y) \mapsto (\psi_* X,\ (\psi^{-1})^*\alpha).$$

Given a bundle map Φ over the diffeomorphism ψ represented by the diagram:

$$\begin{array}{ccc} TM \oplus T^*M & \xrightarrow{\Phi} & TM \oplus T^*M \\ \downarrow & & \downarrow \\ M & \xrightarrow{\psi} & M \end{array}$$

we say that Φ is an automorphism of the Courant algebroid $\mathbb{E}(M)$ over $\psi : M \to M$ if there exists a 2-form B on M such that

$$\Phi = (\psi_* \oplus (\psi^{-1})^*) \circ \mathcal{A}_B.$$

Define

$$\Phi_* I = \Phi^{-1} \circ I \circ \Phi.$$

The group of automorphisms of $\mathbb{E}(M)$ acts naturally on the space of generalized complex structures on M. Two generalized complex structures I and I' are said to be *isomorphic* is there exists an automorphism Φ such that $I' = \Phi_* I$.

4 Transversal generalized complex structures

Let M be a smooth $(2n+d)$-dimensional manifold equipped with a foliation $\mathcal{F}$ of codimension $2n$ which is defined by a foliated cocycle $(U_i, f_i, N, \gamma_{ij})$, where $(U_i)_{i\in I}$ is an open cover of M, $f_i : U_i \to N$ submersions, and $\gamma_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ local diffeomorphisms such that $f_i = \gamma_{ij} \circ f_j$ on $U_i \cap U_j$.

Definition 4.1. [V07] A *transversal generalized complex structure* on the foliated manifold $(M,\mathcal{F})$ is given by a generalized complex structure on (N, I_N) which is preserved by the γ_{ij}, in the sense that

$$\big((\gamma_{ij})_* \oplus (\gamma^{-1})_{ij}\big)\, I_N = I_N.$$

As pointed out by I. Vaisman, one may consider a more general definition of a transversal generalized complex structure, where the model manifold N is endowed with a closed 2-form B and γ_{ij} together with the B-field transformation preserves the structure, that is the following maps Φ_{ij} preserve the generalized complex structure I_N:

$$\Phi_{ij} = \big((\gamma_{ij})_* \oplus (\gamma^{-1})_{ij}\big) \circ \mathcal{A}_B.$$

Then the collection $(f_i^* B)$ would give a Cech 1-cocycle with values in the space closed 2-forms on N.

Let $E' \to N$ be a smooth vector bundle and $f : M \to N$ a *submersion*. Recall that the *pull-back of E' at x* is defined as follows:

$$f^*(E')_x = \{(X + f^*\alpha)_x \mid (f_* X + \alpha)_{f(x)} \in E'_{f(x)}\}.$$

It determines a well-defined smooth vector bundle over M, denoted by $f^*(E')$. Similarly, one can define the pull-back of complex vector bundles.

Definition 4.2. Under the above notations, a vector bundle $E \to (M,\mathcal{F})$ is said to be an *$\mathcal{F}$-foliated bundle* if the restriction of E to any simple open subset U_i of M is a pull-back of a bundle on N, i.e. $E_{|U_i} = f_i^*(G_{U_i})$ and $\gamma_{ij}^*(G_{U_i}) = G_{U_j}$.

A subbundle E of $\mathbb{E}(M)$ (resp. $\mathbb{E}(M) \otimes \mathbb{C}$) is *integrable* if its space of sections is closed under the Courant bracket.

We have the following characterization of transversal generalized structures:

Theorem 4.3. A transversal generalized complex structure on a foliated manifold $(M,\mathcal{F})$ is determined by a maximal isotropic $\mathcal{F}$-foliated subbundle E of $(TM \oplus T^*M) \otimes \mathbb{C} \to M$ which satisfies the following properties:

(a) $E \cap \overline{E} = (T\mathcal{F} \times \{0\}) \otimes \mathbb{C}$;

(b) E is integrable,

where $T\mathcal{F}$ is the tangent bundle of the foliation. In other words, a transversal generalized complex structure on $(M,\mathcal{F})$ is given by an $\mathcal{F}$-foliated Dirac structure $E \subset (TM \oplus T^*M) \otimes \mathbb{C}$ that satisfies (a). Conversely, any $\mathcal{F}$-foliated Dirac structure E that satisfies the above condition (a) gives rise to a transversal generalized complex structure.

Proof: Consider a transversal generalized complex structure on the foliated manifold $(M,\mathcal{F})$. Let $(U_i, f_i, N, I_N, \gamma_{ij})$ be a foliated cocycle defining $\mathcal{F}$ and satisfying the conditions of Definition 4.2. Denote by G_N the $\sqrt{-1}$-eigenbundle of the associated generalized complex structure I_N. On each U_i, one gets a foliated isotropic subbundle

$$E_i = \{(X, f_i^*\alpha) \mid ((f_i)_* X, \alpha) \in G_{N_{|U_i}}\}.$$

Gluing together the E_i's, we obtain an $\mathcal{F}$ foliated vector bundle which is maximal isotropic, integrable, and satisfies condition (a).

Conversely, given an $\mathcal{F}$-foliated subbundle E of $\mathbb{E}(M)\otimes\mathbb{C}$. Let $(U_i, f_i, N, \gamma_{ij})$, be a foliated cocycle defining $\mathcal{F}$. Since E is foliated, we can assume that there are vector bundles $G_i \to N$, for which $E_{|U_i} = f_i^*(G_i)$. The fact that every G_i defines a generalized complex structure J_i on N follows from the properties of E. Moreover, on each non-empty intersection $U_i \cap U_j$, one has $\gamma_{ij}^*(G_i) = G_j$. Hence, we can put together the G_i's in order to get a well-defined generalized complex structure G_N on N which is, by construction, preserved by the γ_{ij}. There follows the result.

■

Recall that the real index r of a maximal isotropic subspace L is the dimension of the intersection of L with its complex conjugate. A transversal generalized complex structure is a maximal isotropic subbundle of $(TM\oplus T^*M)\otimes C$ whose real index r is not necessarily zero, unlike the case of generalized complex structures. We have the following proposition:

Proposition 4.4. There is a one-to-one correspondence between transversal generalized complex structure on $(M,\mathcal{F})$ and generalized complex structures on the normal bundle $\nu(\mathcal{F}) = TM/T\mathcal{F}$ which are invariant along the leaves of $\mathcal{F}$.

Proof: At any point $x \in M$, there is a foliated chart $(U, x_1, \ldots, x_d, y_1, \ldots, y_{2n})$ such that the leaf through x is defined by

$$y_1 = 0, \ \ldots, \ y_{2n} = 0.$$

The transversal generalized complex structure on M determines a (1,1)-tensor J, a bivector field π and a basic 2-form θ defined on the transversal manifold given by $x_1 = 0, \ldots, x_d = 0$ that satisfy conditions 2.4. Using (J,π,θ), one can construct a generalized complex structure on the normal bundle ν which is invariant along the leaves of U. Starting from a foliated atlas, one can patch together the tensors to get a well-defined generalized complex structure on ν which is invariant along the leaves of $\mathcal{F}$.

Conversely, any generalized complex structure on ν induces a transversal generalized complex structure on a foliated chart $(U, x_1, \ldots, x_d, y_1, \ldots, y_{2n})$. Since the generalized complex structure is constant along the leaves, there is a well-defined transversal generalized complex structure on M.

■

5 Examples

Example 1: Generalized complex structures.

When the leaves of $\mathcal{F}$ are just points, one recovers the notion of a generalized complex structure on M from Theorem 4.3 since, in this case, $E\cap\overline{E} = \{0\}$. Compare also the definition of a generalized complex structure on M with Proposition 4.4.

Example 2: Trivial bundles.

Given a generalized complex manifold N and a smooth manifold M, the product manifold $M = B \times N$ has a canonical transversal generalized structure. The leaves of the associated foliations are the sets $B \times \{x\}$.

Example 3: Suspensions.

Let B be a compact manifold, $\pi_1(B)$ its fundamental group and $\widetilde{B}$ its universal covering space. Consider a compact generalized complex manifold (N, J_N) and the group $\mathrm{Diff}(N, J_N)$ of diffeomorphisms of N which preserve J_N. Given a group homomorphism

$$\rho : \pi_1(B) \to \mathrm{Diff}(N, J_N),$$

one gets an action of $\pi_1(B)$ on $\widetilde{B} \times N$ given by

$$\gamma \cdot (\tilde{b}, x) = (\gamma \cdot \tilde{b},\ \rho(\gamma) \cdot x).$$

The quotient space Q has a transversal generalized complex structure induced by that of the product $\widetilde{B} \times N$. Precisely, the leaves of Q are exactly the images under the projection map p of $\widetilde{B} \times \{x\}$, $x \in N$. They are transversal to the fibers of the fibration $Q \to B$, which can be identified with N.

Example 4: Coisotropic *A*-branes.

Coisotropic A-branes were introduced and studied by Kapustin and Orlov (see [KO03]). Recall that a coisotropic A-brane is a triple (N, L, ∇), where N is a submanifold of some symplectic manifold (M, ω), L a line bundle over N, and ∇ a unitary connection on L such that

- N is a coisotropic submanifold of M, i.e. $\omega_{|_N}$ has a constant rank and

  $$\mathrm{Orth}_\omega TN \subset TN,$$

  where $\mathrm{Orth}_\omega TN$ is given by:

  $$\mathrm{Orth}_\omega TN = \{v \in TM \mid \omega(u, v) = 0,\ \forall u \in TN\}.$$

- The curvature 2-form R of ∇ annihilates the tangent bundle $T\mathcal{F}$ to the foliation on N determined by $\ker(\omega)$.

- The basic skew-symmetric 2-form σ induced by $\omega_{|_N}$ and the 2-form R determine a (1,1)-tensor $J = \sigma^{-1}R$ which is a complex structure on the normal bundle $\mathcal{N} = TN/T\mathcal{F}$.

Therefore, every coisotropic A-brane naturally carries a transversely holomorphic foliation. Moreover, $R + i\sigma$ defines a holomorphic symplectic form on the normal bundle. Clearly, transversely holomorphic foliations are special cases of transverse generalized complex structures where the transverse model N is a complex manifold.

Example 5: Generalized contact structures.

Generalized contact structures are odd-dimensional analogues of generalized complex structures. They include contact structure and normal almost contact structures. They were studied in [IW, PW]. We refer to these works for more details. A generalized contact structure on a smooth $(2n+1)$-dimensional manifold M is called regular if the foliation on M given by the maximum integral curves of its Reeb vector field is a regular foliation. Any regular generalized contact manifold is a circle bundle over a generalized complex manifold. In particular, any regular generalized contact structure defines a transverse generalized complex structure.

6 Basic cohomology

Definition 6.1. A function f on a foliated manifold $(M,\mathcal{F})$ is said to be *basic* if f is constant along the leaves of $\mathcal{F}$. Moreover, a k-form ω is basic if $\iota_X\omega = 0$ and $\mathcal{L}_X\omega = 0$ for all vector field X tangent to $\mathcal{F}$. A vector field X is projectable if $[X,Y]$ is tangent to $\mathcal{F}$ for any vector field Y tangent to $\mathcal{F}$. We denote by $\mathfrak{X}_b(M)$ the space of projectable vector fields and Ω^1_b the space of basic 1-forms on M. We say that a smooth section $e = X+\alpha$ of $\mathbb{E}(M)\otimes\mathbb{C}\to M$ is *basic* if α is basic and X projectable.

We denote $\mathcal{A}^0_b$ and $\mathcal{A}^1_b$ the space of basic functions on M and basic sections of $\mathbb{E}(M)\otimes\mathbb{C}$, respectively. In fact, the space $\mathcal{A}^1_b$ is a $\mathcal{A}^0_b$-module. Observe that Properties (i) and (iii) in Definition 3.1 imply

$$[\mathcal{A}^1_b,\mathcal{A}^1_b]\subset\mathcal{A}^1_b.$$

The basic Clifford algebra acts naturally on basic k-differential forms as follows:

$$(X+\alpha)\cdot\beta = \iota_X\beta + \alpha\wedge\beta,$$

for any basic section $X+\alpha\in\mathcal{A}^1(M)$ and for any $\beta\in\Omega^1_b(M)$.

Notice that the notion of a transversal generalized complex structure can be formulated using pure spinors. Recall that given a nonzero spinor $\rho\in\Omega^*(M)$, its nullspace is

$$E_\rho = \{(X+\alpha)\in\mathbb{E}(M)\otimes\mathbb{C} \mid (X+\alpha)\cdot\rho = 0\}.$$

The nullspace E_ρ is always isotropic since

$$0 = (X+\alpha)\cdot((X+\alpha)\cdot\rho) = \langle X+\alpha,\, X+\alpha\rangle\rho,$$

for every $X+\alpha\in\Gamma(E_\rho)$.

Definition 6.2. A spinor ρ is called a *pure spinor* if its nullspace E_ρ is maximal isotropic.

We know that there is a one-to-one correspondence between pure spinors and maximal isotropic subbundles of $(TM\oplus T^*M)\otimes\mathbb{C}$. Therefore, a transversal generalized complex structure on $(M,\mathcal{F})$ is given by a basic pure spinor ρ for which $E_\rho\cap\overline{E}_\rho$ can be completely determined by the tangent bundle $T\mathcal{F}$ of the foliation $\mathcal{F}$. Consider the morphism

$$\phi : \Lambda^{n+d-k}\overline{E}\to\Lambda^*T^*M\otimes\mathbb{C}$$

defined by

$$\phi(s) = s \cdot \rho, \quad \forall s \in \Lambda^{n+d-k}\overline{E}, \quad \forall k = -n, \ldots, n+d.$$

where $m = 2n+d$ is the dimension of M. Set

$$U^k = \phi(\Lambda^{n+d-k}\overline{E}),$$

and denote by $\Gamma(U_b^k)$ the subspace of basic forms of $\Gamma(U^k)$. One gets the following:

Remark 6.3. A transversal generalized complex structure I on $(M, \mathcal{F})$ gives rise to a $\mathbb{Z}$-grading of the basic differential forms:

$$\Omega_b^\bullet(M) = \Gamma(U_b^{-n}) \oplus \cdots \oplus \Gamma(U_b^{n+d}).$$

Theorem 6.4. *Given a transversely generalized complex structure on $(M, \mathcal{F})$, the basic de Rham operator d_b maps $\Gamma(U_b^k)$ into $\Gamma(U_b^{k-1}) \oplus \Gamma(U_b^{k+1})$. It induces two operators via the natural projections:*

$$\partial_b : \Gamma(U_b^k) \to \Gamma(U_b^{k-1}) \quad \text{and} \quad \overline{\partial}_b : \Gamma(U_b^k) \to \Gamma(U_b^{k+1}).$$

The fact that the basic de Rham operator sends $\Gamma(U_b^k)$ into $\Gamma(U_b^{k-1}) \oplus \Gamma(U_b^{k+1})$ comes from the integrability condition of the transversal generalized complex structure. The proof is similar to that of generalized complex structures given in [G04]. It is left to the reader. One has:

$$\overline{\partial}_b^2 = \partial_b \circ \overline{\partial}_b + \overline{\partial}_b \circ \partial_b = \partial_b^2 = 0.$$

Example.

Now, we will focus on the case transversely symplectic structures coming from contact forms. Given a contact 1-form η on a $(2n+1)$-dimensional manifold M, the foliation $\mathcal{F}$ is generated by its corresponding Reeb vector field ξ. Recall that a 1-form η on M is said to be a contact structure if $\eta \wedge (d\eta)^n \neq 0$ at every point. The classical Darboux theorem states that, around every point of M, there are canonical coordinates $(t, p_1, \ldots, p_n, q_1, \ldots, q_n)$ such that

$$\eta = dt - \sum_{i=1}^{n} p_i dq_i.$$

Consider the isomorphism of $C^\infty(M)$-module $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ defined by

$$\flat(X) = \iota_X d\eta + \eta(X)\eta,$$

for any vector field X. Define the bivector field π by setting:

$$\pi(\alpha, \beta) = -d\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta)).$$

In the canonical coordinates $(t, p_1, \ldots, p_n, q_1, \ldots, q_n)$, one has:

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \left(\frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial t} \right).$$

The transversal generalized contact structure on M is determined by the complex bundle:

$$E \equiv \{X - \sqrt{-1}\, \iota_X d\eta \mid X \in TM \otimes \mathbb{C}\}$$

which corresponds to the pure spinor $\rho = e^{id\eta}$ and

$$\Gamma(U_b^{n+d-k}) = \{e^{id\eta}e^{\frac{\pi}{2i}}\alpha \mid \alpha \in \Omega_b^k(M)\},$$

where for any complex bivector B, one has

$$e^B\alpha = \alpha + \iota_B\alpha + \frac{1}{2}\iota_B^2\alpha + \ldots.$$

One has a similar formula for 2-forms:

$$e^\omega\alpha = \alpha + \omega\wedge\alpha + \frac{1}{2}\omega\wedge\omega\wedge\alpha + \cdots,$$

for any differential form α. Now, we consider the Koszul-Brylinski operator [CLM96]:

$$\delta_\pi : \Omega_b^k(M) \to \Omega_b^{k-1}(M)$$

given by

$$\delta_\pi = [\iota_\pi, d_b] = \iota_\pi \circ d_b - d_b \circ \iota_\pi.$$

Similarly to the symplectic case [C05], one has

$$-2i\partial\left(e^{id\eta}e^{\frac{\pi}{2i}}\alpha\right) = e^{id\eta}e^{\frac{\pi}{2i}}(\delta_\pi\alpha) \qquad \overline{\partial}\left(e^{id\eta}e^{\frac{\pi}{2i}}\alpha\right) = e^{id\eta}e^{\frac{\pi}{2i}}(d_b\alpha),$$

for any basic form α.

Aknowledgement. I would like to thank Yat Sun Poon and Izu Vaisman for helpful comments.

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