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ON SYMPLECTOMORPHISMS OF THE SYMPLECTIZATION OF A COMPACT CONTACT MANIFOLD

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Abstract

Let (N,α) be a compact contact manifold and $(N\times\mathbf{R},\, d(e^t\alpha))$ its symplectization. We show that the group G which is the identity component in the group of symplectic diffeomorphisms ϕ of $(N\times\mathbf{R},\mathbf{d}(\mathbf{e^t}\alpha))$ that cover diffeomorphisms $\underline{\phi}$ of $N\times S^1$ is simple, by showing that G is isomorphic to the kernel of the Calabi homomorphism of the associated locally conformal symplectic structure.

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1 Introduction and statement of the results

The structure of the group of compactly supported symplectic diffeomorphisms of a symplectic manifold is well understood [1], see also [2]. For instance, if (M,Ω) is a compact symplectic manifold, the commutator subgroup $[Diff_\Omega(M)_0, Diff_\Omega(M)_0]$ of the identity component $Diff_\Omega(M)_0$ in the group of all symplectic diffeomorphisms, is the kernel of a homomorphism from $Diff_\Omega(M)_0$ to a quotient of $H^1(M,\mathbf{R})$ (The Calabi homomorphism) and it is a simple group.

Unfortunately, the structure of the group of symplectic diffeomorphisms of a non compact manifold, with unrestricted supports in largely unkown. In this paper, we study the group $Diff_{\tilde{\Omega}}(N\times\mathbf{R})$ of symplectic diffeomorphisms of the symplectization $(N\times\mathbf{R},\mathbf{d}(\mathbf{e^t}\alpha))$ of a compact contact manifold (N,α). Our main result is the following

Theorem 1.1. *Let G be the subgroup of $Diff_{\tilde{\Omega}}(N\times\mathbf{R})$ consisting of elements ϕ, isotopic to the identity through isotopies ϕ_t in $Diff_{\tilde{\Omega}}(N\times\mathbf{R})$, which cover isotopies $\underline{\phi_t}$ of $N\times S^1$. Then G is a simple group.*

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Recall that a group G is said to be a simple group if it has no non-trivial normal subgroup. In particular it is equal to its commutator subgroup $[G,G]$.

For $\phi \in Diff_{\tilde{\Omega}}(N\times \mathbf{R})$, the 1-form

$$\tilde{C}(\phi) = \phi^*(e^t\alpha) - e^t\alpha$$

is closed.
Let $C(\phi)$ denotes its cohomology class in $H^1(N\times \mathbf{R},\mathbf{R}) \approx \mathbf{H^1}(\mathbf{N},\mathbf{R})$.
Let $Diff_{\tilde{\Omega}}(N\times \mathbf{R})_{\mathbf{0}}$ be the subgroup of $Diff_{\tilde{\Omega}}(N\times \mathbf{R})$ consisting of elements that are isotopic to the identity in $Diff_{\tilde{\Omega}}(N\times \mathbf{R})$.
The map $\phi \mapsto C(\phi)$, where $\phi \in Diff_{\tilde{\Omega}}(N\times \mathbf{R})_{\mathbf{0}}$ is a surjective homomorphism

$$C : Diff_{\tilde{\Omega}}(N\times \mathbf{R})_0 \to H^1(N,\mathbf{R})$$

(the Calabi homomorphism, see [1]).

Corollary 1.2. *The group G is contained in the kernel of C.*

Proof: Since G is simple, the kernel of the restriction C_0 of C to G is either the trivial group $\{id\}$ or the whole group G. But $KerC_0$ contains $[G,G] \neq \{1d\}$. Hence $KerC_0 = G$.

Theorem 1.1 follows from the study of the structure of the group of diffeomorphisms preserving a locally conformal symplectic structure. Each locally conformal symplectic manifold (M,Ω), is covered in a natural way by a symplectic manifold $(\tilde{M},\tilde{\Omega})$. We analyze the group of symplectic diffeomorphisms of $\tilde{M}$, which cover diffeomorphisms of M (Theorem 2.1). Our results will be deducted from the fact that, if (N,α) is a contact manifold, then $N\times S^1$ has a locally conformal symplectic structure and the associated symplectic manifold covering $N\times S^1$ is precisely the symplectization. We show that the group G is isomorphic to the kernel of the Calabi homomorphism for locally conformal symplectic geometry.

2 The structure of the group of diffeomorphisms covering locally conformal symplectic diffeomorphisms

A locally conformal symplectic form on a smooth manifold M is a non-degenerate 2-form Ω such that there exists a closed 1-form ω satisfying:

$$d\Omega = -\omega\wedge\Omega.$$

The 1-form ω is uniquely determined by Ω and is called the Lee form of Ω. The couple (M,Ω) is called a locally conformal symplectic (lcs, for short) manifold , see [3], [7], [11].

The group $Diff(M,\Omega)$ of automorphisms of a lcs manifold (M,Ω) consists of diffeomorphisms ϕ of M such that $\phi^*\Omega = f\Omega$ for some non-zero function f. Here we will always assume that f is a positive function. Such a diffeomorphism is said to be a locally conformal symplectic diffeomorphism.

The group of conformal symplectic diffeomorphisms of a symplectic manifold (M,σ) is defined as the group of diffeomorphisms ϕ of M such that $\phi^*\sigma = f\sigma$ for some smooth function f. If the dimension of M is at least 4, then f is a constant function (see [9], or [5]). If moreover M is compact, then $f = \pm 1$.

Let $\tilde{M}$ be the minimum regular cover of a locally conformal symplectic manifold (M, Ω) over which the Lee form ω pulls to an exact form: i.e. if $\pi : \tilde{M} \to M$ is the covering map,

$$\pi^* \omega = df = d(ln\lambda).$$

where $\lambda = e^f$. It is easy to check that

$$\tilde{\Omega} = \lambda \pi^* \Omega.$$

is a symplectic form on $\tilde{M}$.

The conformal class of $\tilde{\Omega}$ is independent of the choice of λ [4]. : Indeed, if λ' is another function such that $\pi^*\omega = d(ln\lambda')$, then $\lambda' = a\lambda$ for some constant a.

A diffeomorphism ϕ of $\tilde{M}$ is said to be fibered if there exists a diffeomorphism h of M such that $\pi \circ \phi = h \circ \pi$. We also say that ϕ covers h.

Theorem 2.1. *If a diffeomorphism ϕ of $\tilde{M}$ covers a diffeomorphism h of M, then ϕ is conformal symplectic iff h is locally conformal symplectic*

Proof: Suppose $\phi : \tilde{M} \to \tilde{M}$ is conformal symplectic, and covers $h : M \to M$. Then $\phi^*(\tilde{\Omega}) = a\tilde{\Omega}$ for some number $a \in \mathbf{R}$. We have:

$$\pi^*(h^*\Omega) = \phi^*(\pi^*\Omega) = \phi^*((1/\lambda)\tilde{\Omega}) = (\frac{1}{\lambda} \circ \phi))a\tilde{\Omega} = a(\frac{1}{\lambda} \circ \phi)\lambda\pi^*\Omega.$$

Let τ be an automorphism of the covering $\tilde{M} \to M$, then

$$\tau^*\pi^*(h^*\Omega) = (\pi \circ \tau)^*(h^*\Omega) = \pi^*(h^*\Omega)$$

$$= \tau^*[(a\frac{1}{\lambda} \circ \phi)\lambda]\tau^*\pi^*\Omega = \tau^*[(a\frac{1}{\lambda} \circ \phi)\lambda]\pi^*\Omega.$$

$$= a(\frac{1}{\lambda} \circ \phi)\lambda\pi^*\Omega.$$

Therefore $\tau^*[(a\frac{1}{\lambda} \circ \phi)\lambda] = (a\frac{1}{\lambda} \circ \phi)\lambda)$ since $\pi^*\Omega$ is non-degenerate. Hence $(a\frac{1}{\lambda} \circ \phi)\lambda) = u \circ \pi$, where u is a basic function. We thus get $\pi^*(h^*\Omega) = \pi^*(u\Omega)$. Since π is a covering map, $h^*\Omega = u\Omega$.

Conversely if $h \in Diff(M, \Omega)$, i.e. $h^*\Omega = u\Omega$ for some function u on M, and ϕ is its lift on $\tilde{M}$, then:

$$\phi^*\tilde{\Omega} = \phi^*(\lambda\pi^*\Omega) = (\lambda \circ \phi)\phi^*\pi^*\Omega = (\lambda \circ \phi)(\pi \circ \phi)^*\Omega$$

$$= (\lambda \circ \phi)(h \circ \pi)^*\Omega = (\lambda \circ \phi)\pi^*h^*\Omega = (\lambda \circ \phi)\pi^*(u\Omega) = (\frac{\lambda \circ \phi}{\lambda} u \circ \pi)\tilde{\Omega}.$$

We just proved that if $h \in Diff(M, \Omega)$, $(h^*\Omega = u\Omega)$ is covered by ϕ, then $\phi^*(\tilde{\Omega}) = a\tilde{\Omega}$ where a is the constant $a = (\frac{\lambda \circ \phi}{\lambda} u \circ \pi)$.

Let $Diff_{\tilde{\Omega}}(\tilde{M})_C$ be the group of conformal symplectic of $\tilde{M}$ (a diffeomorphism ϕ of $\tilde{M}$ belongs to this group if $\phi^*\tilde{\Omega} = a\tilde{\Omega}$ for some positive number a).

The group $Diff_{\tilde{\Omega}}(\tilde{M})$ of symplectic diffeomorphisms is the kernel of the homomorphism:

$$d : Diff_{\tilde{\Omega}}(\tilde{M})_C \to \mathbf{R}^+$$

sending ϕ to $a \in \mathbf{R}^+$ when $\phi^*\tilde{\Omega} = a\tilde{\Omega}$.

We consider the subgroups $Diff_{\tilde{\Omega}}(\tilde{M})^F_C$, resp. $Diff_{\tilde{\Omega}}(\tilde{M})^F$ of $Diff_{\tilde{\Omega}}(\tilde{M})_C$, resp. of $Diff_{\tilde{\Omega}}(\tilde{M})$ consisting of fibered elements.

Finally, let G_C, resp G be the subgroups of $Diff_{\tilde{\Omega}}(\tilde{M})^F_C$, resp. $Diff_{\tilde{\Omega}}(\tilde{M})^F$ consisting of elements that are isotopic to the identity through these respective groups. We denote by $Diff(M,\Omega)_0$ the identity component in the group $Diff(M,\Omega)$, endowed with the C^∞ topology.

By Theorem 2.1, we have a homomorphism $\rho : G_C \to Diff(M,\Omega)_0$. This homomorphism is surjective: indeed, any diffeomorphism isotopic to the identity lifts to a diffeomorphism of the covering space $\tilde{M}$. See for instance [6]. By Theorem 2.1, that lifting must be a conformal symplectic diffeomorphism.

Let A be the group of automorphisms of the covering $\pi : \tilde{M} \to M$. For any $\tau \in A$, $(\lambda \circ \tau)/\lambda$ is a constant c_τ independent of λ and the map $\tau \mapsto c_\tau$ is a group homomorphism [5]

$$c : A \to \mathbf{R}^+$$

Let us denote by $\Delta \subset \mathbf{R}^+$ the image of c and by $K \subset A$ its kernel.

For $\tau \in A$, we have:

$$\tau^*\tilde{\Omega} = \tau^*(\lambda\pi^*\Omega) = (\lambda \circ \tau)\tau^*\pi^*\Omega = (\lambda \circ \tau)\pi^*\Omega = ((\lambda \circ \tau)/\lambda)(\lambda\pi^*\Omega) = c_\tau\tilde{\Omega}.$$

This shows that

$$Ker\rho = A.$$

Each element $h \in Diff(M,\Omega)_0$ lifts to an element $\phi \in G_C$ and two different liftings differ by an element of A. Hence the mapping $h \mapsto d(\phi)$ is a well defined map

$$L^* : Diff(M,\Omega)_0 \to \mathbf{R}/\Delta.$$

It is a homomorphism since a lift of $\phi\psi$ differs from the product of their lifts by an element of A.

Let $L(M,\Omega)$ be the Lie algebra of locally conformal symplectic vector fields. These are of vector fields X such that $L_X\Omega = \mu_X\Omega$ for some function μ_X on M. Here L_X stands for the Lie derivative in the direction X.

Let Ω be a lcs form with Lee form ω on a manifold M. One verifies that for all $X \in L(M,\Omega)$, the function

$$l(X) = \omega(X) + \mu_X$$

is a constant, and that the map

$$l : L(M,\Omega) \to \mathbf{R}; \qquad X \mapsto l(X)$$

is a Lie algebra homomorphism, called the extended Lee homomorphism [1], see also [3], [5].

We need now to recall the definition of the Lichnerowicz cohomology [7]. This is the cohomology of the complex of differential forms $\Lambda(M)$ on a smooth manifold with the de Rham differential replaced by d_ω, $d_\omega\theta = d\theta + \omega\wedge\theta$, where ω is a closed 1-form on M. We denote this cohomology by : $H^*_\omega(M)$.

If (M,Ω) is a locally conformal symplectic form with Lee form ω, the equation $d\Omega = -\omega\wedge\Omega$ says that the 2-form Ω is d_ω closed, and hence defines a class $[\Omega]\in H^2_\omega(M)$.

Proposition 2.2. *Let Ω be a lcs form with Lee form ω on a smooth manifold M. The extended Lee homomorphism is surjective iff the Lichnerowicz cohomology class $[\Omega]\in H^2_\omega(M)$ is zero, i.e. iff Ω is d_ω - exact.*

Proposition 2.2 is essentially due to Guedira-Lichnerowicz [7] and Vaisman [11]. Its proof can be found in several places [4], [5], [8].

Let ϕ_t be a smooth family of locally conformal symplectic diffeomorphisms with $\phi_0 = id_M$, and let X_t be the family of vector fields defined by:

$$X_t(\phi_t(x)) = \frac{d}{dt}(\phi_t(x)).$$

Then X_t is a family of locally conformal symplectic vector fields : there exists a smooth family of functions μ_{X_t} such that $L_{X_t}\Omega = \mu_{X_t}\Omega$.

The mapping:

$$\phi_t \mapsto \int_0^1 l(X_t))dt$$

induces a well defined homomorphism $\tilde{L}$ from the universal covering $U(Diff(M,\Omega)_0)$ of $Diff(M,\Omega)_0$ to $\mathbf{R}$, and therefore induces a homomorphism

$$L : Diff(M,\Omega)_0 \to \mathbf{R}/\Gamma$$

where $\Gamma\subset\mathbf{R}$ is the image by $\tilde{L}$ of the fundamental group of $Diff(M,\Omega)_0$.

This integration of the extended Lee homomorphism $l : L(M,\Omega)\to\mathbf{R}$ was considered in [8].

Another integration of the extended Lee homomorphism was constructed in [4], [5]. It is shown there that the subgroups Δ and Γ of $\mathbf{R}$ below are the same and that the homomorphisms L^* and L above coincide.

We will need the following result of Haller and Rybicki [8]:

Theorem 2.3. *Let (M,Ω) be a compact lcs manifold with $[\Omega] = 0 \in H^2_\omega(M)$, where ω is the Lee form of Ω, then*

1. $KerL = [Diff(M,\Omega)_0, Diff(M,\Omega)_0]$.

2. There is a surjective homomorphism S from $KerL$ to a quotient of $H^1_\omega(M)$ whose kernel is a simple group.

The homomorphism S is an analogue of the Calabi homomorphism [1], and the theorem above is a generalization to locally conformal symplectic manifolds of the results on symplectic manifolds in [1]. The definition of the homomorphism S is recalled in the appendix.

As a consequence of these constructions and results, we have the following

Theorem 2.4. *Let (M,Ω) be a compact lcs manifold with Lee form ω and such that $[\Omega] = 0 \in H^2_\omega(M)$. Then:*

1. d and L^* are surjective.

2. We have the following exact sequence:

$$\{1\} \longrightarrow K \longrightarrow G \longrightarrow KerL^* \longrightarrow \{1\}$$

3. $KerL^* \approx [Diff(M,\Omega)_0, Diff(M,\Omega)_0]$.

Proof

Let θ be a 1-form such that $\Omega = d_\omega\theta$ and let X be defined by $i_X\Omega = \theta$. Then $X \in L(M,\omega)$ and $l(X) = 1$. Hence L is surjective. The horizontal lift $\tilde{X}$ of X to $\tilde{M}$ is a complete vector field, and if h is its time 1 flow, then $d(h) = 1$. Hence the mapping d is surjective.

Since L is equal to L^*, point 3 is just a part of Haller-Rybicki theorem.

Let $h, g \in Diff(M,\Omega)_0$ and their lifts ϕ, ψ on $\tilde{M}$. Let $a, b \in \mathbf{R}$ such that $\phi^*\tilde{\Omega} = a\tilde{\Omega}$, $\psi^*\tilde{\Omega} = b\tilde{\Omega}$. Then the commutator $hgh^{-1}g^{-1}$ lifts to $\phi\psi\phi^{-1}\psi^{-1}$,and $(\phi\psi\phi^{-1}\psi^{-1})^*\tilde{\Omega} = b^{-1}a^{-1}ba\tilde{\Omega} = \tilde{\Omega}$. Hence all of $KerL^*$ lifts to G since $KerL \approx [Diff(M,\omega)_0, Diff(M,\Omega)_0]$. This finishes the proof that the sequence 2 is exact.

3 The symplectization of a contact manifold

Let α be a contact form on a smooth manifold N. Let p_1, p_2 be the projections from $M = N \times S^1$ to the factors N, S^1. If μ is the canonical 1-form on S^1 such that $\int_{S^1}\mu = 1$, then $\Omega = d\theta + \omega \wedge \theta$, where $\theta = p_1^*\alpha, \omega = p_2^*\mu$, is a lcs form on $M = N \times S^1$.

The hypothesis of Theorem 3 are satisfied for $M = N \times S^1$, where N is a compact contact manifold and $\Omega = d_\omega\theta$ as above.

The minimum cover $\tilde{M}$ is $N \times \mathbf{R}$, the projection $\pi : N \times \mathbf{R} \to \mathbf{N} \times \mathbf{S^1}$ is the standard projection : $\pi(x,t) = (x, e^{2\pi it})$, and $\pi^*\omega = dt$, $\lambda = e^t$. We have: $\tilde{\Omega} = \lambda\pi^*\Omega = e^t(d\alpha + dt \wedge \alpha) = d(e^t\alpha)$. Hence $(\tilde{M}, \tilde{\Omega})$ is the symplectization $(N \times \mathbf{R}, \mathbf{d}(\mathbf{e^t}\alpha))$.

Here A consists of maps $\gamma_n(x,t) = (x, n+t)$, for all $n \in \mathbf{Z}$. We have $\gamma_n^*\tilde{\Omega} = d(\gamma_n^*(e^t\alpha)) = d(e^{(t+n)}\alpha = e^n\tilde{\Omega}$. Hence $\gamma_n \in Kerc = K$ iff $n = 0$, i.e. $Kerc = \{id\}$. This and Theorem 2.1 (2) show that

$$G = Diff_{\tilde{\Omega}}(N \times \mathbf{R})_0^F \approx KerL$$

The last step is to show that $KerL$ is a simple group. The Calabi homomrphism S takes $KerL$ to a quotient of $H^1_\omega(N \times S^1)$, as one can see in the appendix. But we know that:

$$H^*_\omega(N \times S^1) \approx 0$$

Indeed, take an exact 1-form σ on N and consider $\omega' = \omega + p_1^*\sigma$. Then $H^*_\omega(N \times S^1) \approx H^*_{\omega'}(N \times S^1)$ since ω and ω' are cohomologous. By the Kunneth formula for the Lichnerowicz cohomology, $H^i_{\omega'}(N \times S^1) \approx \oplus(H^j_\mu(S^1) \otimes H^{i-j}_\sigma(N)$. But is known that $H^j_\mu(S^1) = 0$ for all j [7], [8], [3]. Therefore $H^*_{\omega''}(N \times S^1) \approx H^*_\omega(N \times S^1) = \{0\}$.

Hence, $KerS = KerL$ is a simple group. This ends the proof of Theorem 1.1.

Appendix

For completeness, we recall briefly the Calabi homomorphism in lcs geometry[8]: an element $\tilde{\phi}$ of the universal covering of $KerL$ can be represented by an isotopy $\phi_t \in Diff(M, \Omega)$ with tangent vector fields $X_t \in Kerl$. Recall that X_t is defined by : $X_t(\phi_t(x)) = \frac{d}{dt}(\phi_t(x))$. This implies that $d_\omega(i(X_t)\Omega) = 0$, since

$$d_\omega(i(X_t)\Omega) = d(i(X_t)\Omega) + \omega \wedge (i(X_t)\Omega) =$$

$$L_{X_t}\Omega - i(X_t)(-\omega \wedge \Omega) + \omega \wedge (i(X_t)\Omega)$$

$$= (\mu_{X_t} + \omega(X_t))\Omega = l(X_t)\Omega = 0.$$

One shows that

$$[\int_0^1 (i(X_t)\Omega)dt] \in H^1_\omega(M)$$

depends only on $\tilde{\phi}$, and that the correspondence

$$\tilde{\phi} \mapsto [\int_0^1 (i(X_t)\Omega)dt]$$

is a surjective homomorphism from the universal cover of $KerL$ to $H^1_\omega(M)$. This defines a surjective homomorphism $S : KerL \rightarrow H^1_\omega(M)/\Lambda$, where Λ is the image of the fundamental group of $KerL$

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