THE DIOPHANTINE EQUATION $ax^3+by^3+cz^3=0$. COMPLETION OF THE TABLES

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§ 1. In a previous paper [2], I have studied the cubic curve

$$X^3 + Y^3 = A Z^3, \tag{1}$$

giving the number of generators and the basic rational solutions for nearly all positive (cube-free) integers $A \leqq 500$. The solutions were found by means of a "first descent", leading to equations of the form

$$a x^3 + b y^3 + c z^3 = 0, \quad a b c = A \tag{2}$$

$$3 a u v (u - v) + b (u^3 - 3 u^2 v + v^3) = 3 A_1 w^3, \quad A_1 (a^2 - a b + b^2) = A, \tag{3}$$

and by a "second descent" in certain cubic fields defined by these equations.

The extensive tables of [2][1] contain a few blank spaces, where no solution had been found, but where my congruence conditions of the second descent did not show insolubility. In some of these cases, the corresponding equations can be proved insoluble by the methods of CASSELS [1], showing *the insufficiency of my conditions* (§§ 2–3 below).

The remaining unsolved equations of [2] have all been solved on the electronic computer at the Institute for Advanced Study in Princeton, N. J. (§ 4; the completion of Tables 2^{a-b}, 5 and 6). Consequently, I now have *the complete solution of* (1) *for all* $A \leqq 500$.

One of my earlier conjectures concerning the equation (2) is incorrect and must be modified (§ 5; Tables 2^{c-d}).

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[1] There is a misprint in the last line of Table 3, for $p = 17$: *for* $w = 0$ *read* $w = 1$.

§ 2. It was mentioned in [2] (Ch. I, § 6) that solubility or insolubility of an equation (1) can also be decided by the methods of Cassels [1], in the purely cubic field $K(\vartheta) = K(\sqrt[3]{m}) = K(\sqrt[3]{4A})$ (which reduces to $K(\sqrt[3]{\frac{1}{2}A})$ when A is even). Since [2] was written, I have discovered that the following unsolved equations can be shown *insoluble* by these methods:

1. The equation $x^3 + 41y^3 + 46z^3 = 0$ of Table 2^a, corresponding to $A = 41 \cdot 46 = 1\,886$. — There are in fact *four* equations (2) with $abc = 1\,886$, all of which are consequently insoluble (cf. § 3 below).

2. The equation $X^3 + Y^3 = 473\,Z^3$ of Table 6, giving rise to the equation $x^3 + 11y^3 + 43z^3 = 0$ of Tables 2^{a-b}, and to an equation (3) with $a = 7$, $b = 6$, $A_1 = 11$ (Table 5). — Another example of the same kind (not within the tables of [2]) is $A = 508 = 2^2 \cdot 127$; this is the first value of $A > 500$ where my methods fail.

Class-numbers h and units η of the cubic fields used for the above exclusions are:

$$A = 1\,886,\quad m = \tfrac{1}{2}A = 943,\quad h = 15,$$

$$\eta = \frac{1}{3 \cdot 23^2}(-458\,850 + 41\,653\,\vartheta + 524\,\vartheta^2)^3$$

$$A = 473,\quad m = 4A = 1\,892,\quad h = 27,$$

$$\eta = -185\,767 - 32\,567\,\vartheta + \frac{7\,695}{2}\vartheta^2$$

$$A = 508,\quad m = \tfrac{1}{2}A = 254,\quad h = 27,\quad \eta = 19 - 3\,\vartheta.$$

In all cases, h is *odd* and m is $\not\equiv \pm 1 \pmod 9$. The two first units are not necessarily fundamental, but they are neither squares nor cubes of other units.

§ 3. For all equations of the last paragraph, my methods of the second descent fail to indicate insolubility. In the case of (3), there is only *one* (non-purely) cubic field, defined by the left hand side, to be used for each of the values $A = 473$ and $A = 508$. In the case of (2), however, there are *three* different cubic fields $K(\sqrt[3]{m})$ which might be used for exclusion, as seen from the following transformations:

$$(ax)^3 + a^2 b y^3 = -a^2 c z^3,\quad m = a^2 b$$

$$(by)^3 + b^2 c z^3 = -b^2 a x^3,\quad m = b^2 c$$

$$(cz)^3 + c^2 a x^3 = -c^2 b y^3,\quad m = c^2 a.$$

It will not lead to any new conditions if we use for instance $m = a^2 c$, since $K(\sqrt[3]{a^2 c}) = K(\sqrt[3]{c^2 a})$.

The values of A in § 2 give rise to 6 insoluble equations (2) (in the abbreviated notation of [2], Ch. VII, § 4):

$$A = 1\,886 = 2 \cdot 23 \cdot 41: \{1, \ 2, \ 23 \cdot 41\}, \ \{1, \ 23, \ 2 \cdot 41\},$$
$$\{1, \ 41, \ 2 \cdot 23\}, \ \{2, \ 23, \ 41\}$$
$$A = 473 = 11 \cdot 43: \quad \{1, \ 11, \ 43\}$$
$$A = 508 = 4 \cdot 127: \quad \{1, \ 4, \ 127\}.$$

For each of these equations, I have checked in all three cubic fields that my congruence conditions of the second descent do not lead to exclusion. It seems to me very striking that *when my methods fail, they seem to fail in all the fields involved.*

§ 4. During the Spring of 1952, I had the opportunity to "code" the remaining unsolved equations of [2] for the electronic computer at the Institute for Advanced Study in Princeton, N.J., and to run the problem on the computer myself. The equations (3) were coded in this form, but "resulting equations" (cf. [2], Ch. IV) were used instead of (2). The computer scanned a certain domain for the unknowns, and halted at the first solution, or when a given limit was reached.

With the first code, the machine was unable to solve the one equation (3) corresponding to $A = 283$. I later made a separate code for this equation, utilizing special congruence conditions etc., and the problem was run successfully by Mr. Manfred Kochen at the Institute in December, 1953.[1]

The effective machine time for solving 9 equations totalled about 6 hours, corresponding to approximately 6 months of work on a desk calculator (but it took 6 weeks to prepare the problem for the computer).

The completion of Tables 2^{a-b}, 5 and 6, resulting from the Princeton solutions, appears below.

§ 5. For given $abc = A$, the number of different equations (2), possible for all moduli, is one of the numbers

$$N_A = 0, \ 1, \ 4, \ 13, \ 40, \ \ldots$$

(if trivial repetitions are avoided by the additional conditions $1 \leqq a < b < c$, $(a, b) = (a, c) = (b, c) = 1$). I conjectured in [2] (Ch. VII, § 4, 3rd conjecture) that one and

[1] I would like to express my gratitude to Mr. Kochen for his assistance, and also to Prof. von Neumann and Dr. Goldstine at the Institute, for giving me the opportunity to run my problems on the computer.

COMPLETION OF TABLE 2^a

$(x^3 + my^3 + nz^3 = 0.)$

m	n	x	y	z
17	41	149 105	-140 161	101 988
29	47	-5 646	1 705	917
11	43	Insoluble (Cassels)		
41	46			

COMPLETION OF TABLE 2^b

$(ax^3 + by^3 + cz^3 = 0, \ abc = A.)$

A	a	b	c	x	y	z
346	1	2	173	117 747	-119 635	21 799
382	1	2	191	456 771	501 542	-122 093
445	1	5	89	- 18 683	10 383	2 182
473	1	11	43	Insoluble (Cassels)		

EXTENSION OF TABLE 2^c TO $1\,000 < A \leqq 2\,500$

(Values of A with 13 possible equations $ax^3 + by^3 + cz^3 = 0$, $abc = A$, only one of which is soluble.)

A	a	b	c	x	y	z
$1230 = 2.3.5.41$	2	5	123	- 4	1	1
$1380 = 2^2.3.5.23$	3	20	23	1	1	-1
$1410 = 2.3.5.47$	1	5	282	7	- 5	1
$1518 = 2.3.11.23$	2	3	253	5	1	-1
$1590 = 2.3.5.53$	5	6	53	1	2	-1
$1650 = 2.3.5^2.11$	3	22	25	1	1	-1
$1740 = 2^2.3.5.29$	1	5	348	7	1	-1
$1770 = 2.3.5.59$	1	6	295	- 7	2	1
$1870 = 2.5.11.17$	5	17	22	1	1	-1
$1914 = 2.3.11.29$	2	11	87	-49	25	9
$2130 = 2.3.5.71$	3	10	71	- 3	1	1
$2244 = 2^2.3.11.17$	4	17	33	-23	13	7
$2460 = 2^2.3.5.41$	4	15	41	-14	9	1
$2490 = 2.3.5.83$	3	10	83	1	2	-1

TABLE 2[d] (NEW)

Values of $A \leqq 2\,500$ with 13 possible equations $ax^3+by^3+cz^3=0$, $abc=A$, all of which are soluble.

$A=1020=2^2.3.5.17$:

a	b	c	x	y	z
1	3	340	−7	1	1
1	4	255	1	−4	1
1	5	204	−19	11	1
1	12	85	29	32	−17
1	15	68	11	−5	2
1	17	60	29	−13	6
1	20	51	−11	4	1
3	4	85	3	1	−1
3	5	68	3	−5	2
3	17	20	1	1	−1
4	5	51	7	1	−3
4	15	17	−2	1	1
5	12	17	1	1	−1

$A=1122=2.3.11.17$:

a	b	c	x	y	z
1	2	561	5	−7	1
1	3	374	1	−5	1
1	6	187	29	−16	1
1	11	102	41	−19	4
1	17	66	7	−5	3
1	22	51	5	−2	1
1	33	34	1	1	−1
2	3	187	10	−9	1
2	11	51	1	5	−3
2	17	33	2	1	−1
3	11	34	−9	1	4
3	17	22	13	17	−16
6	11	17	1	1	−1

$A=2310=2.3.5.7.11$:

a	b	c	x	y	z
1	6	385	1	4	−1
1	7	330	19	17	−5
1	15	154	−23	9	2
1	22	105	−29	8	5
1	42	55	13	−6	5
2	5	231	−23	17	1
2	33	35	1	1	−1
3	10	77	1	−2	1
3	11	70	−3	1	1
5	14	33	1	−4	3
6	7	55	2	1	−1
7	15	22	1	1	−1
10	11	21	1	1	−1

$A=2346=2.3.17.23$:

a	b	c	x	y	z
1	2	1173	−13	8	1
1	3	782	229	−159	4
1	6	391	7	2	−1
1	17	138	31	1	−6
1	23	102	−5	1	1
1	34	69	−41	11	7
1	46	51	11	1	−3
2	3	391	2	5	−1
2	17	69	47	25	−19
2	23	51	196	23	−67
3	17	46	3	−7	5
3	23	34	61	71	−64
6	17	23	1	1	−1

Addition to Table 4

4^h: Values of $A \leqq 500$, proved insoluble by the methods of Cassels:

$$A = 473 = 11.43$$

Completion of Table 5

(Non-excluded equations $3\,a\,u\,v\,(u-v) + b\,(u^3 - 3\,u^2\,v + v^3) = \frac{s}{3t} A_1\, w^3$.)

A	a	b	A_1	Case	u	v	w
283	19	6	1	I	31	982	1102
337	8	21	1	I	89	165	17
409	23	15	1	I	96	29	169
499	25	18	1	I	2	125	222
473 = 11.43	7	6	11	I	Insoluble (Cassels)		

Completion of Table 6

(Basic solutions of $X^3 + Y^3 = A\,Z^3$ for $A \leqq 500$.)

A	g	(X, Y, Z)
283	1	(20 824 888 493, −8 780 429 621, 3 090 590 958)
337	1	(53 750 671, −53 706 454, 1 043 511)
346	1	(47 189 035 813 499 932 580 169 103 856 786 964 321 592 777 067, 42 979 005 685 698 193 708 286 233 727 941 595 382 526 544 683, 8 108 695 117 451 325 702 581 978 056 293 186 703 694 064 735)
382	1	(58 477 534 119 926 126 376 218 390 196 344 577 607 972 745 895 728 749, 16 753 262 295 125 845 463 811 427 438 340 702 778 576 158 801 481 539, 8 122 054 393 485 793 893 167 719 500 929 060 093 151 854 013 194 574)
409	1	(22 015 523, 21 425 758, 3 687 411)
445	1	(362 650 186 970 550 612 016 862 044 970 863 425 187, −58 928 948 142 525 345 898 087 903 372 951 745 227, 47 432 800 292 536 072 666 333 861 784 516 450 106)
499	1	(80 968 219, 17 501 213, 10 242 414)

only one of these equations is soluble when $N_A = 13$. My assumption was based on an examination for $A \leqq 1\,000$, covering only 5 cases (Table 2^c of [2]).

I have later continued the examination of $N_A = 13$ up to $A = 2\,500$. In most cases (Table 2^c above), there is one soluble and 12 excluded equations. In four cases, however, *all 13 equations are soluble* (Table 2^d). The 3rd conjecture should consequently be *modified* to include this alternative.

In a separate paper [3], I have combined the conjectures of [2] with the new numerical results to the following generalized

Conjectures:

1. (Weaker form.) *The second descent excludes an even number of generators.*

2. (Stronger form.) *When a second descent exists, the number of generators is an even number less than what is indicated by the first descent.*

It is very striking that the stronger conjecture seems to hold (at least in certain cases) *also for the Weierstrass normal form* of a cubic curve, cf. [4].

References

[1]. J. W. S. Cassels, The rational solutions of the diophantine equation $Y^2 = X^3 - D$. *Acta Math.* 82 (1950), 243–273.

[2]. E. S. Selmer, The diophantine equation $ax^3 + by^3 + cz^3 = 0$. *Acta Math.* 85 (1951), 203–362.

[3]. ——, A conjecture concerning rational points on cubic curves. *Math. Scand.* 2 (1954), 49–54.

[4]. ——, The diophantine equation $\eta^2 = \xi^3 - D$. A note on Cassels' method.—To appear in *Math. Scand.* 3 (1955).