Acta Math., 193 (2004), 175–191

On the rational approximations to the powers of an algebraic number: Solution of two problems of Mahler and Mendès France

by

PIETRO CORVAJA and UMBERTO ZANNIER

Università di Udine
Udine, Italy

Scuola Normale Superiore
Pisa, Italy

1. Introduction

About fifty years ago Mahler [Ma] proved that *if* $\alpha>1$ *is rational but not an integer and if* $0<l<1$*, then the fractional part of* α^n *is larger than* l^n *except for a finite set of integers* n *depending on* α *and* l. His proof used a p-adic version of Roth's theorem, as in previous work by Mahler and especially by Ridout.

At the end of that paper Mahler pointed out that the conclusion does not hold if α is a suitable algebraic number, as e.g. $\frac{1}{2}(1+\sqrt{5})$; of course, a counterexample is provided by any *Pisot* number, i.e. a real algebraic integer $\alpha>1$ all of whose conjugates different from α have absolute value less than 1 (note that rational integers larger than 1 are Pisot numbers according to our definition). Mahler also added that "It would be of some interest to know which algebraic numbers have the same property as [the rationals in the theorem]".

Now, it seems that even replacing Ridout's theorem with the modern versions of Roth's theorem, valid for several valuations and approximations in any given number field, the method of Mahler does not lead to a complete solution to his question.

One of the objects of the present paper is to answer Mahler's question completely; our methods will involve a suitable version of the Schmidt subspace theorem, which may be considered as a multi-dimensional extension of the results mentioned by Roth, Mahler and Ridout. We state at once our first theorem, where as usual we denote by $\|x\|$ the distance of the complex number x from the nearest integer in $\mathbf{Z}$, i.e.

$$\|x\| := \min\{|x-m| : m\in\mathbf{Z}\}.$$

THEOREM 1. *Let $\alpha>1$ be a real algebraic number and let $0<l<1$. Suppose that $\|\alpha^n\|<l^n$ for infinitely many natural numbers n. Then there is a positive integer d such that α^d is a Pisot number. In particular, α is an algebraic integer.*

We remark that the conclusion is best possible. For assume that for some positive integer d, $\beta:=\alpha^d$ is a Pisot number. Then for every positive multiple n of d we have $\|\alpha^n\|\ll l^n$, where l is the dth root of the maximum absolute value of the conjugates of β different from β. Here, Mahler's example with the golden ratio is typical.

Also, the conclusion is not generally true without the assumption that α is algebraic; for this see the appendix.

The present application of the subspace theorem seems different from previous ones, and occurs in Lemma 3 below. Related methods actually enable us to answer as well a question raised by Mendès France about the length of the periods of the continued fractions for α^n, where α is now a quadratic irrational; this appears as Problem 6 in [Me]. We shall prove, more generally, the following result:

THEOREM 2. *Let $\alpha>0$ be a real quadratic irrational. If α is neither the square root of a rational number, nor a unit in the ring of integers of $\mathbf{Q}(\alpha)$, then the period length of the continued fraction for α^n tends to infinity with n. If α is the square root of a rational number, the period length of the continued fraction for α^{2n+1} tends to infinity. If α is a unit, the period length of the continued fraction for α^n is bounded.*

Clearly, if α is the square root of a rational number, then the continued fraction for α^{2n} is finite,(1) so Theorem 2 gives a complete answer to the problem of Mendès France.

The main tool in the proof of both theorems is the following new lower bound for the fractional parts of S-units in algebraic number fields. We first need a definition:

Definition. We call a (complex) algebraic number α a *pseudo-Pisot* number if
(i) $|\alpha|>1$ and all its conjugates have (complex) absolute value strictly less than 1;
(ii) α has an integral trace: $\mathrm{Tr}_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha)\in\mathbf{Z}$.

Of course, pseudo-Pisot numbers are "well approximated" by their trace, hence are good candidates for having a small fractional part compared to their height. The algebraic integers among the pseudo-Pisot numbers are just the usual Pisot numbers. We shall prove the following result:

(1) Its length tends to infinity by a result of Pourchet, proved in greater generality in [CZ].

MAIN THEOREM. *Let $\Gamma\subset\overline{\mathbf{Q}}^\times$ be a finitely generated multiplicative group of algebraic numbers, let $\delta\in\overline{\mathbf{Q}}^\times$ be a non-zero algebraic number and let $\varepsilon>0$ be fixed. Then there are only finitely many pairs $(q,u)\in\mathbf{Z}\times\Gamma$ with $d=[\mathbf{Q}(u):\mathbf{Q}]$ such that $|\delta qu|>1$, δqu is not a pseudo-Pisot number and*

$$0<\|\delta qu\|<H(u)^{-\varepsilon}q^{-d-\varepsilon}. \tag{1.1}$$

Note again that, conversely, starting with a Pisot number α and taking $q=1$ and $u=\alpha^n$ for $n=1,2,...$ produces an infinite sequence of solutions to $0<\|qu\|<H(u)^{-\varepsilon}$ for a suitable $\varepsilon>0$.

The above main theorem can be viewed as a Thue–Roth inequality with "moving target", as the theorem in [CZ], where we considered quotients of power sums with integral roots instead of elements of a finitely generated multiplicative group. The main application of the theorem in [CZ] also concerned continued fractions, as for our Theorem 2.

2. Proofs

We shall use the following notation: Let K be a number field, embedded in $\mathbf{C}$ and Galois over $\mathbf{Q}$. We denote by M_K (resp. M_∞) the set of places (resp. archimedean places) of K; for each place v we denote by $|\cdot|_v$ the absolute value corresponding to v, normalized *with respect to* K; by this we mean that if $v\in M_\infty$ then there exists an automorphism $\sigma\in\mathrm{Gal}(K/\mathbf{Q})$ of K such that, for all $x\in K$,

$$|x|_v=|\sigma(x)|^{d(\sigma)/[K:\mathbf{Q}]},$$

where $d(\sigma)=1$ if $\sigma(K)=K\subset\mathbf{R}$, and $d(\sigma)=2$ otherwise (note that $d(\sigma)$ is now constant since $K/\mathbf{Q}$ is Galois). Non-archimedean absolute values are normalized accordingly, so that the product formula holds and the absolute Weil height reads

$$H(x)=\prod_{v\in M_K}\max\{1,|x|_v\}.$$

For a vector $\mathbf{x}=(x_1,...,x_n)\in K^n$ and a place $v\in M_K$ we shall denote by $\|\mathbf{x}\|_v$ the *v-norm* of $\mathbf{x}$, namely,(2)

$$\|\mathbf{x}\|_v:=\max\{|x_1|_v,...,|x_n|_v\},$$

(2) We believe that there will be no confusion with the previously recalled symbol $\|x\|$ for the distance to the nearest integer.

and by $H(\mathbf{x})$ the *projective height*

$$H(\mathbf{x}) = \prod_{v \in M_K} \max\{|x_1|_v, \ldots, |x_n|_v\}.$$

We begin by proving the main theorem. First of all notice that, by enlarging if necessary the multiplicative group Γ, we can reduce to the situation where $\Gamma \subset K^\times$ is the group of S-units:

$$\Gamma = \mathcal{O}_S^\times = \{u \in K : |u|_v = 1 \text{ for all } v \notin S\}$$

of a suitable number field K, Galois over $\mathbf{Q}$, with respect to a suitable finite set of places S, containing the archimedean ones and stable under Galois conjugation.

Our first lemma can be easily deduced from a theorem of Evertse, which in turn was obtained as an application of the already mentioned subspace theorem; we prefer to give a proof for completeness.

LEMMA 1. *Let K and S be as before, $\sigma_1, \ldots, \sigma_n$ be distinct automorphisms of K, $\lambda_1, \ldots, \lambda_n$ be non-zero elements of K, $\varepsilon > 0$ be a positive real number and $w \in S$ be a distinguished place. Let $\Xi \subset \mathcal{O}_S^\times$ be an infinite set of solutions $u \in \mathcal{O}_S^\times$ of the inequality*

$$|\lambda_1 \sigma_1(u) + \ldots + \lambda_n \sigma_n(u)|_w < \max\{|\sigma_1(u)|_w, \ldots, |\sigma_n(u)|_w\} H(u)^{-\varepsilon}.$$

Then there exists a non-trivial linear relation of the form

$$a_1 \sigma_1(u) + \ldots + a_n \sigma_n(u) = 0, \quad a_i \in K,$$

satisfied by infinitely many elements of Ξ.

Proof. Let Ξ be as in the lemma. Going to an infinite subset of Ξ we may assume that $|\sigma_1(u)|_w = \max\{|\sigma_1(u)|_w, \ldots, |\sigma_n(u)|_w\}$ for all the involved u's. Let us consider, for each $v \in S$, n linear forms $L_{v,1}, \ldots, L_{v,n}$ in n variables $\mathbf{x} = (x_1, \ldots, x_n)$ as follows: Put

$$L_{w,1}(\mathbf{x}) := \lambda_1 x_1 + \ldots + \lambda_n x_n$$

while for $(v,i) \in S \times \{1, \ldots, n\}$, with $(v,i) \neq (w,1)$, put $L_{v,i}(\mathbf{x}) = x_i$. Note that the linear forms $L_{v,1}, \ldots, L_{v,n}$ are indeed linearly independent. Now put

$$\mathbf{x} = (\sigma_1(u), \ldots, \sigma_d(u)) \in (\mathcal{O}_S^\times)^d$$

and consider the double product

$$\prod_{v \in S} \prod_{i=1}^{n} \frac{|L_{v,i}(\mathbf{x})|_v}{\|\mathbf{x}\|_v}.$$

By multiplying and dividing by $|x_1|_w = |\sigma_1(u)|_w$, and using the fact that the coordinates of $\mathbf{x}$ are S-units, we obtain the equality

$$\prod_{v\in S}\prod_{i=1}^{n}\frac{|L_{v,i}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} = |\lambda_1\sigma_1(u)+\ldots+\lambda_n\sigma_n(u)|_w\,|\sigma_1(u)|_w^{-1}H(\mathbf{x})^{-n}.$$

Since $u\in\Xi$, we have

$$|\lambda_1\sigma_1(u)+\ldots+\lambda_n\sigma_n(u)|_w\,|\sigma_1(u)|_w^{-1} < H(u)^{-\varepsilon},$$

and then

$$\prod_{v\in S}\prod_{i=1}^{n}\frac{|L_{v,i}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} < H(\mathbf{x})^{-n}H(u)^{-\varepsilon}.$$

The height of the point $\mathbf{x}=(\sigma_1(u),\ldots,\sigma_d(u))$ is easily compared with the height of u by the estimate

$$H(\mathbf{x}) \leqslant H(u)^{[K:\mathbf{Q}]}.$$

Hence the above upper bound for the double product also gives

$$\prod_{v\in S}\prod_{i=1}^{n}\frac{|L_{v,i}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} < H(\mathbf{x})^{-n-\varepsilon/[K:\mathbf{Q}]}.$$

Now, an application of the subspace theorem in the form given in e.g. [S, Chapter V, Theorem 1D′] gives the desired result. □

Our next tool is a very special case of the so-called unit-equation theorem, proved by Evertse and van der Poorten–Schlickewei. It also rests on the subspace theorem:

LEMMA 2. *Let K, S and $\sigma_1,\ldots,\sigma_n$ be as before, $\varepsilon>0$ be a real number and $a_1,\ldots,a_n$ be non-zero elements of K. Suppose that $\Xi\subset\mathcal{O}_S^\times$ is an infinite set of solutions to the equation*

$$a_1\sigma_1(u)+\ldots+a_n\sigma_n(u)=0.$$

Then there exist two distinct indices $i\neq j$, two non-zero elements $a,b\in K^\times$ and an infinite subset $\bar{\Xi}\subset\Xi$ of Ξ such that for all $u\in\bar{\Xi}$,

$$a\sigma_i(u)+b\sigma_j(u)=0.$$

For a proof of this lemma, see [S, Chapter IV].

Our last lemma is the key of the proof of the main theorem; its proof is once more based on the subspace theorem.

LEMMA 3. *Let K and S be as before, $k\subset K$ be a subfield of K of degree d over $\mathbf{Q}$, and $\delta\in K^\times$ be a non-zero element of K. Let $\varepsilon>0$ be given. Suppose that we have an infinite sequence Σ of points $(q,u)\in\mathbf{Z}\times(\mathcal{O}_S^\times\cap k)$ such that $|\delta qu|>1$, δqu is not a pseudo-Pisot number and*

$$0<\|\delta qu\|<H(u)^{-\varepsilon}q^{-d-\varepsilon}. \tag{2.1}$$

Then there exists a proper subfield $k'\subset k$, an element $\delta'\in k^\times$ and an infinite subsequence $\Sigma'\subset\Sigma$ such that for all $(q,u)\in\Sigma'$, $u/\delta'\in k'$.

Note that Lemma 3 gives a finiteness result in the case $k=\mathbf{Q}$, since the rational field $\mathbf{Q}$ admits no proper subfields.

Proof. Let us suppose that the hypotheses of the lemma are satisfied, so in particular Σ is an infinite sequence of solutions of (2.1). We begin by observing that, by Roth's theorem [S, Chapter II, Theorem 2A], in any such infinite sequence u cannot be fixed; so we have $H(u)\to\infty$ in the set Σ. Let $H:=\mathrm{Gal}(K/k)\subset\mathrm{Gal}(K/\mathbf{Q})$ be the subgroup fixing k. Let $\{\sigma_1,\dots,\sigma_d\}$ $(d=[k:\mathbf{Q}])$ be a (complete) set of representatives for the left cosets of H in $\mathrm{Gal}(K/\mathbf{Q})$, containing the identity σ_1. Each automorphism $\varrho\in\mathrm{Gal}(K/\mathbf{Q})$ defines an archimedean valuation on K by the formula

$$|x|_\varrho:=|\varrho^{-1}(x)|^{d(\varrho)/[K:\mathbf{Q}]}, \tag{2.2}$$

where as usual $|\cdot|$ denotes the usual complex absolute value. Two distinct automorphisms ϱ_1 and ϱ_2 define the same valuation if and only if $\varrho_1^{-1}\circ\varrho_2$ is the complex conjugation. Let now $(q,u)\in\Sigma$ be a solution of (2.1) and let $p\in\mathbf{Z}$ be the nearest integer to δqu. Then for each $\varrho\in\mathrm{Gal}(K/\mathbf{Q})$ we have, with the notation of (2.2),

$$\|\delta qu\|^{d(\varrho)/[K:\mathbf{Q}]}=|\delta qu-p|^{d(\varrho)/[K:\mathbf{Q}]}=|\varrho(\delta)\varrho(qu)-p|_\varrho. \tag{2.3}$$

Let, for each $v\in M_\infty$, ϱ_v be an automorphism defining the valuation v according to the rule (2.2): $|x|_v:=|x|_{\varrho_v}$; then the set $\{\varrho_v:v\in M_\infty\}$ represents the left cosets of the subgroup generated by the complex conjugation in $\mathrm{Gal}(K/\mathbf{Q})$. Let S_i, for $i=1,\dots,d$, be the subset of M_∞ formed by those valuations v such that ϱ_v coincides with σ_i on k; note that $S_1\cup\ldots\cup S_d=M_\infty$. We take the product of the terms in (2.3) for ϱ running over the set $\{\varrho_v:v\in M_\infty\}$; this corresponds to taking the product over all archimedean valuations. Then we obtain

$$\prod_{v\in M_\infty}|\varrho_v(\delta)\varrho_v(qu)-p|_v=\prod_{i=1}^{d}\prod_{v\in S_i}|\varrho_v(\delta)\sigma_i(qu)-p|_v.$$

By (2.3) and the well-known formula $\sum_{v\in M_\infty} d(\varrho_v)=[K:\mathbf{Q}]$, it follows that

$$\prod_{i=1}^{d}\prod_{v\in S_i}|\varrho_v(\delta)\sigma_i(qu)-p|_v=\|\delta qu\|. \tag{2.4}$$

Now, let us define for each $v\in S$ a set of $d+1$ linearly independent linear forms in $d+1$ variables $(x_0,x_1,...,x_d)$ in the following way: for an archimedean valuation $v\in S_i$ $(i=1,...,d)$ put

$$L_{v,0}(x_0,x_1,...,x_d)=x_0-\varrho_v(\delta)x_i,$$

and for all $v\in S\setminus M_\infty$ or $0<j\leqslant d$ put

$$L_{v,j}(x_0,x_1,...,x_d)=x_j.$$

Plainly the forms $L_{v,0},...,L_{v,d}$ are independent for each $v\in S$. Finally, let $\mathbf{x}\in K^{d+1}$ be the point

$$\mathbf{x}=(p,q\sigma_1(u),...,q\sigma_d(u))\in K^{d+1}.$$

Let us estimate the double product

$$\prod_{v\in S}\prod_{j=0}^{d}\frac{|L_{v,j}(\mathbf{x})|_v}{\|\mathbf{x}\|_v}. \tag{2.5}$$

Using the fact that $L_{v,j}(\mathbf{x})=q\sigma_j(u)$ for $j\geqslant 1$ and that the $\sigma_j(u)$ are S-units, we obtain, from the product formula,

$$\prod_{v\in S}\prod_{j=1}^{d}|L_{v,j}(\mathbf{x})|_v\leqslant\prod_{v\in M_\infty}\prod_{j=1}^{d}|q|_v=|q|^d. \tag{2.6}$$

Since the coordinates of $\mathbf{x}$ are S-integers, the product of the denominators in (2.5) is $\geqslant H(\mathbf{x})^{d+1}$ ($\|\mathbf{x}\|_v\leqslant 1$ for v outside S); then in view of (2.4) and (2.6),

$$\prod_{v\in S}\prod_{j=0}^{d}\frac{|L_{v,j}(\mathbf{x})|_v}{\|\mathbf{x}\|_v}\leqslant H(\mathbf{x})^{-d-1}|q|^d\,\|\delta qu\|\leqslant H(\mathbf{x})^{-d-1}(qH(u))^{-\varepsilon},$$

the last inequality being justified by the fact that $(q,u)\in\Sigma$; hence (2.1) holds. Since $H(\mathbf{x})\leqslant|q|\cdot|p|H(u)^d$ and $|p|\leqslant|\delta qu|+1\leqslant|q|\cdot|\delta|H(u)^d+1\leqslant|q|H(u)^{2d}$ for all but finitely many pairs $(q,u)\in\Sigma$ (recall that $H(u)\to\infty$ for $(q,u)\in\Sigma$), we have $|q|^2H(u)^{3d}\geqslant H(\mathbf{x})$, so the last displayed inequality gives

$$\prod_{v\in S}\prod_{j=0}^{d}\frac{|L_{v,j}(\mathbf{x})|_v}{\|\mathbf{x}\|_v}\leqslant H(\mathbf{x})^{-d-1-\varepsilon/3d}.$$

An application of the subspace theorem in the form given e.g. in [S, Chapter V, Theorem 1D′] implies the existence of a hyperplane containing infinitely many points $\mathbf{x}=(p,q\sigma_1(u),...,q\sigma_d(u))$. We then obtain a non-trivial linear relation of the form

$$a_0p+a_1q\sigma_1(u)+...+a_dq\sigma_d(u)=0,\quad a_i\in K, \tag{2.7}$$

satisfied by infinitely many pairs $(q,u)\in\Sigma$. Our next goal is to prove the following result:

CLAIM. *There exists such a non-trivial relation with vanishing coefficient a_0, i.e. one involving only the conjugates of u.*

Proof. Rewriting the above linear relation, if $a_0 \neq 0$, we obtain

$$p = -\frac{a_1}{a_0} q\sigma_1(u) - \dots - \frac{a_d}{a_0} q\sigma_d(u). \tag{2.8}$$

Suppose first that for some index $j \in \{2, \dots, d\}$, $\sigma_j(a_1/a_0) \neq a_j/a_0$; then by applying the automorphism σ_j to both sides of (2.8) and subtracting term-by-term from (2.8) to eliminate p, we obtain a linear relation involving only the terms $\sigma_1(u), \dots, \sigma_d(u)$. Such a relation is non-trivial since the coefficient of $\sigma_j(u)$ becomes $\sigma_j(a_1/a_0) - a_j/a_0$. Hence we have proved our claim in this case.

Therefore, we may and shall assume that $a_j/a_0 = \sigma_j(a_1/a_0)$ for all j, so in particular all coefficients a_j/a_0 are non-zero. Let us then rewrite (2.8) in a simpler form as

$$p = q(\sigma_1(\lambda)\sigma_1(u) + \dots + \sigma_d(\lambda)\sigma_d(u)) \tag{2.9}$$

with $\lambda = -a_1/a_0 \neq 0$. Suppose now that λ does not belong to k. Then there exists an automorphism $\tau \in H$ with $\tau(\lambda) \neq \lambda$. (Recall that H is the subgroup of $\mathrm{Gal}(K/\mathbf{Q})$ fixing k, and that $\{\sigma_1 = \mathrm{id}, \sigma_2, \dots, \sigma_d\}$ is a complete set of representatives of left cosets of H.) By applying the automorphism τ to both sides of (2.9) and subtracting term-by-term from (2.9) to eliminate p, we obtain the linear relation

$$(\lambda - \tau(\lambda))\sigma_1(u) + \sum_{i=2}^{d} (\sigma_i(\lambda)\sigma_i(u) - \tau \circ \sigma_i(\lambda)\tau \circ \sigma_i(u)) = 0.$$

Note that $\tau \circ \sigma_j$ coincides on k with some σ_i. Note also that since $\tau \in H$ and $\sigma_2, \dots, \sigma_d \notin H$, no $\tau \circ \sigma_j$ with $j \geqslant 2$ can belong to H. Hence the above relation can again be written as a linear combination of the $\sigma_i(u)$; in such an expression the coefficient of $\sigma_1(u)$ will remain $\lambda - \tau(\lambda)$, and will therefore be non-zero, so we obtain a non-trivial relation among the $\sigma_i(u)$, as claimed.

Therefore we may and shall suppose that $\lambda \in k$ and write (2.9) in the simpler form

$$p = q \operatorname{Tr}_{k/\mathbf{Q}}(\lambda u) = \operatorname{Tr}_{k/\mathbf{Q}}(q\lambda u). \tag{2.10}$$

After adding $-\delta qu$ to both sides in (2.9), recalling that (q, u) is a solution to (2.1) and that σ_1 is the identity, we obtain

$$|p - \delta qu| = |(\lambda - \delta)q\sigma_1(u) + q\sigma_2(\lambda)\sigma_2(u) + \dots + q\sigma_d(\lambda)\sigma_d(u)| < q^{-d-\varepsilon} H(u)^{-\varepsilon} \leqslant q^{-1} H(u)^{-\varepsilon}.$$

In particular,

$$|(\lambda-\delta)u+\sigma_2(\lambda)\sigma_2(u)+\ldots+\sigma_d(\lambda)\sigma_d(u)|<q^{-1}H(u)^{-\varepsilon}. \tag{2.11}$$

We want to apply Lemma 1. We distinguish two cases:

First case: $\lambda=\delta$ (in particular $\delta\in k$). In this case the algebraic number $q\delta u=q\lambda u$ has an integral trace. Since by assumption it is not a pseudo-Pisot number, the maximum modulus of its conjugates $|\sigma_2(q\lambda u)|, \ldots, |\sigma_d(q\lambda u)|$ is $\geqslant 1$. This yields

$$\max\{|\sigma_2(u)|, \ldots, |\sigma_d(u)|\} \geqslant q^{-1}\max\{|\sigma_2(\lambda)|, \ldots, |\sigma_d(\lambda)|\}^{-1}.$$

Hence from (2.11) we deduce that infinitely many pairs $(q,u)\in\Sigma$ satisfy the inequality of Lemma 1, where w is the archimedean place associated to the given embedding $K\hookrightarrow\mathbf{C}$, $n=d-1$, $\lambda_i=\sigma_{i+1}(\lambda)$ and with $\sigma_2, \ldots, \sigma_d$ instead of $\sigma_1, \ldots, \sigma_n$. The conclusion of Lemma 1 provides what we claimed.

Second case: $\lambda\neq\delta$. In this case the first term does appear in (2.11). Since we supposed that $|q\delta u|>1$, we have

$$\max\{|\sigma_1(u)|, \ldots, |\sigma_d(u)|\} \geqslant |u| > |\delta|^{-1}q^{-1},$$

so we can again apply Lemma 1 (with the same place w as in the first case, $n=d$ and $\lambda_1=(\lambda-\delta)$, $\lambda_2=\sigma_2(\lambda)$, ..., $\lambda_d=\sigma_d(\lambda)$) and conclude as in the first case.

This finishes the proof of the claim, i.e. a relation of the kind (2.7) without the term a_0p is satisfied for all pairs (q,u) in an infinite set $\bar{\Sigma}\subset\Sigma$. □

Returning to the proof of Lemma 3, we can apply the unit theorem of Evertse and van der Poorten–Schlickewei in the form of Lemma 2, which implies that a non-trivial relation of the form $a\sigma_j(u)+b\sigma_i(u)=0$ for some $i\neq j$ and $a,b\in K^\times$ is satisfied for (q,u) in an infinite subset Σ' of $\bar{\Sigma}$. We rewrite it as

$$-\sigma_i^{-1}\Big(\frac{a}{b}\Big)(\sigma_i^{-1}\circ\sigma_j)(u)=u.$$

Then for any two solutions $(q',u'), (q'',u'')\in\Sigma'$, the element $u:=u'/u''\in k$ is fixed by the automorphism $\sigma_i^{-1}\circ\sigma_j\notin H$, and thus u belongs to the proper subfield $k':=K^{\langle H,\sigma_i^{-1}\circ\sigma_j\rangle}$ of k. In other words, if we let (q',δ') be any solution, then infinitely many solutions are of the form (q'',v) for some v of the form $v=u\delta'$ with $u\in k'$ as wanted. □

Proof of the main theorem. As we have already remarked, we can suppose that the finitely generated group $\Gamma\subset\bar{\mathbf{Q}}^\times$ is the group of S-units in a number field K, where K is

Galois over $\mathbf{Q}$ containing δ, and S is stable under Galois conjugation as in Lemmas 1, 2 and 3.

Suppose by contradiction that we have an infinite set $\Sigma\subset\mathbf{Z}\times\mathcal{O}_S^\times$ of solutions (q,u) to the inequality (2.1). Let us define by induction a sequence $\{\delta_i\}_{i=0}^{\infty}\subset K$, an infinite decreasing chain Σ_i of infinite subsets of Σ and an infinite strictly decreasing chain k_i of subfields of K with the following properties:

For each natural number $n\geqslant 0$, $\Sigma_n\subset(\mathbf{Z}\times k_n)\cap\Sigma_{n-1}$, $k_n\subset k_{n-1}$, $k_n\neq k_{n-1}$ and all but finitely many pairs $(q,u)\in\Sigma_n$ satisfy the inequalities $|\delta_0\ldots\delta_n qu|>1$ and

$$\|\delta_0\ldots\delta_n qu\|<q^{-d-\varepsilon}H(u)^{-\varepsilon/(n+1)}. \tag{2.12}$$

We shall eventually deduce a contradiction from the fact that the number field K does not admit any infinite decreasing chain of subfields. We proceed as follows: put $\delta_0=\delta$, $k_0=K$ and $\Sigma_0=\Sigma$. Suppose that we have defined δ_n, k_n and Σ_n for a natural number n. Applying Lemma 3 with $k=k_n$ and $\delta=\delta_0\ldots\delta_n$, we obtain that there exists an element $\delta_{n+1}\in k_n$, a proper subfield k_{n+1} of k_n and an infinite set $\Sigma_{n+1}\subset\Sigma_n$ such that all pairs $(q,u)\in\Sigma_{n+1}$ satisfy $u=\delta_{n+1}v$ with $v\in k_{n+1}$. Now, since for $v\in K$, $H(\delta_{n+1}v)\geqslant H(\delta_{n+1})^{-1}H(v)$, we have in particular that for almost all $v\in K$, $H(\delta_{n+1}v)\geqslant H(v)^{(n+1)/(n+2)}$; then all but finitely many such pairs satisfy

$$\|\delta_0\ldots\delta_n\delta_{n+1}qv\|<q^{-1-\varepsilon}H(v)^{-\varepsilon/(n+2)},$$

and the inductive hypothesis is fulfilled.

The contradiction is then obtained as noticed above, concluding the proof of the main theorem. □

To prove Theorem 1 we also need the following result:

LEMMA 4. *Let α be an algebraic number. Suppose that for all n in an infinite set $\Xi\subset\mathbf{N}$, there exists a positive integer $q_n\in\mathbf{Z}$ such that the sequence $\Xi\ni n\mapsto q_n$ satisfies*

$$\lim_{n\to\infty}\frac{\log q_n}{n}=0 \quad \text{and} \quad \mathrm{Tr}_{\mathbf{Q}(\alpha)/\mathbf{Q}}(q_n\alpha^n)\in\mathbf{Z}\setminus\{0\}$$

(the limit being taken for $n\in\Xi$). Then α is either the h-th root of a rational number (for some positive integer h), or an algebraic integer.

Proof. It is essentially an application of Lemma 1, so it still depends on the subspace theorem. Let us suppose that α is not an algebraic integer. Let K be the Galois closure of the extension $\mathbf{Q}(\alpha)/\mathbf{Q}$ and let h be the order of the torsion group of $K^\times$. Since Ξ is an infinite subset of $\mathbf{N}$, there exists an integer $r\in\{0,\ldots,h-1\}$ such that infinitely many elements of Ξ are of the form $n=r+hm$. Let us denote by Ξ' the infinite subset of $\mathbf{N}$ composed of those integers m such that $r+hm\in\Xi$. Let $\sigma_1,...,\sigma_d\in\mathrm{Gal}(K/\mathbf{Q})$, where $d=[\mathbf{Q}(\alpha^h):\mathbf{Q}]$, be a set of automorphisms of K giving all the embedding $\mathbf{Q}(\alpha^h)\hookrightarrow K$. In other terms, if $H\subset\mathrm{Gal}(K/\mathbf{Q})$ is the subgroup fixing $\mathbf{Q}(\alpha^h)$, the set $\{\sigma_1,...,\sigma_d\}$ is a complete set of representatives of left cosets of H in $\mathrm{Gal}(K/\mathbf{Q})$. This proves that if $d=1$ then α is an hth root of a rational number. Suppose the contrary and recall moreover that α is not an algebraic integer; we try to obtain a contradiction. Since α is not an algebraic integer, there exists a finite absolute value w of K such that $|\alpha|_w>1$. Let $H'\subset\mathrm{Gal}(K/\mathbf{Q})$ be the subgroup fixing $\mathbf{Q}(\alpha)$, so that we have the inclusions $H'\subset H\subset\mathrm{Gal}(K/\mathbf{Q})$, corresponding to the chain $\mathbf{Q}(\alpha^h)\subset\mathbf{Q}(\alpha)\subset K$.

For each $i=1,...,d$, let $T_i\subset\mathrm{Gal}(K/\mathbf{Q})$ be a complete set of representatives for the set of automorphisms coinciding with σ_i in $\mathbf{Q}(\alpha^h)$, modulo H'. In other words, the elements of T_i, when restricted to $\mathbf{Q}(\alpha)$, give all the embedding of $\mathbf{Q}(\alpha)\hookrightarrow K$ whose restrictions to $\mathbf{Q}(\alpha^h)$ coincide with σ_i. Also $T_1\cup...\cup T_d$ is a complete set of embeddings of $\mathbf{Q}(\alpha)$ in K. Then we can write the trace of α^n ($n=r+hm$) as

$$\mathrm{Tr}_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha^{r+hm})=\sum_{i=1}^{d}\Big(\sum_{\tau\in T_i}\tau(\alpha^r)\Big)\sigma_i(\alpha^h)^m=\lambda_1\sigma_1(\alpha^h)^m+...+\lambda_d\sigma_d(\alpha^h)^m,$$

where $\lambda_i=\sum_{\tau\in T_i}\tau(\alpha^r)$. Note that not all the coefficients λ_i can vanish, since then the trace would also vanish. Since the trace of $q_n\alpha^n$ is integral, we have $|\mathrm{Tr}_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha^n)|_w\leqslant|q_n|_w^{-1}$; then, since $\log q_n=o(n)$, for every $\varepsilon<\log|\alpha|_w/\log H(\alpha)$ and sufficiently large $m\in\Xi'$ we have

$$|\lambda_1\sigma_1(\alpha^{hm})+...+\lambda_d\sigma_d(\alpha^{hm})|_w\leqslant|q_{r+hm}|_w^{-1}<|\alpha^{hm}|_w H(\alpha^{hm})^{-\varepsilon}.$$

Applying Lemma 1 we arrive at a non-trivial equation of the form

$$a_1\sigma_1(\alpha^h)^m+...+a_d\sigma_d(\alpha^h)^m=0$$

satisfied by infinitely many integers m. An application of the Skolem–Mahler–Lech theorem leads to the conclusion that for two indices $i\neq j$ some ratio $\sigma_i(\alpha^h)/\sigma_j(\alpha^h)=(\sigma_i(\alpha)/\sigma_j(\alpha))^h$ is a root of unity. But then it equals 1, since $\sigma_i(\alpha)/\sigma_j(\alpha)$ lies in $K^\times$ (and by assumption h is the exponent of the torsion group of $K^\times$). Then σ_i coincides with σ_j on $\mathbf{Q}(\alpha^h)$. This contradiction concludes the proof. □

Proof of Theorem 1. Let us suppose that the hypotheses of Theorem 1 are satisfied, so that $\|\alpha^n\|<l^n$ for infinitely many n. Then, either α is a dth root of an integer, and we are done, or $\|\alpha^n\|\neq0$ for all $n>1$. In this case, taking any $\varepsilon<-\log l/\log H(\alpha)$ we obtain that for infinitely many n,

$$0<\|\alpha^n\|<H(\alpha^n)^{-\varepsilon},$$

and of course the sequence α^n belongs to a finitely generated subgroup of $\bar{\mathbf{Q}}^\times$. Then our main theorem, with $q=1$ and $u=\alpha^n$, implies that infinitely many numbers α^n are pseudo-Pisot numbers, and in particular have integral trace. For large n, their trace cannot vanish since $|\alpha|>1$, while its conjugates have absolute value less than 1; then by Lemma 4, α^n is an algebraic integer, so it is a Pisot number as wanted. □

3. Proof of Theorem 2

We recall some basic facts about continued fractions. Let $\alpha>1$ be a real irrational number. We will use the notation

$$\alpha=[a_0,a_1,...],$$

where $a_0,a_1,...$ are positive integers, to denote the continued fraction for α. We also let p_h and q_h, for $h=0,1,...$, be the numerator and denominator of the truncated continued fraction $[a_0,a_1,...,a_h]$, so that, by a well-known fact, for all h we have

$$\left|\frac{p_h}{q_h}-\alpha\right| \leqslant \frac{1}{q_h^2 a_{h+1}}. \tag{3.1}$$

Also, we have the recurrence relation

$$q_{h+1}=a_{h+1}q_h+q_{h-1} \tag{3.2}$$

holding for $h\geqslant 2$. Let

$$T=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$$

be a unimodular matrix, and let $[b_0,b_1,...]$ be the continued fraction of

$$T(\alpha):=\frac{a\alpha+b}{c\alpha+d}.$$

Then there exists an integer k such that for all large h, $b_h=a_{h+k}$.

Consider now the case of a real *quadratic* irrational number $\alpha>0$, and let α' be its (algebraic) conjugate. The continued fraction of α is eventually periodic; it is purely periodic if and only if $\alpha>1$ and $-1<\alpha'<0$; we call such a quadratic irrational *reduced*. In this case the period of $-1/\alpha'$ is the "reversed" period of α. For every quadratic irrational number there exists a unimodular transformation $T\in\mathrm{GL}_2(\mathbf{Z})$ such that $T(\alpha)$ is reduced (this is equivalent to saying that the expansion of α is eventually periodic). Since the transformation $x\mapsto -1/x$ is also unimodular, it follows that in any case, α and α' have a period of the same length. We can summarize these facts as follows:

FACTS. *Let α be a real quadratic irrational and α' its conjugate. The continued fraction development of α is eventually periodic. The quadratic irrationals*

$$|\alpha|, \quad |\alpha'|, \quad \left|\frac{1}{\alpha}\right| \quad \textit{and} \quad \left|\frac{1}{\alpha'}\right|$$

have periods of the same length.

We begin by proving the easier part of Theorem 2, namely:

CLAIM. *If $\alpha>0$ is a unit in the ring of integers of $\mathbf{Q}(\alpha)$, then the period length for the continued fraction for α^n is uniformly bounded, and in fact is $\leqslant 2$ for all large n.*

Proof. Let us begin with the case of odd powers of a unit $\alpha>1$, with $\alpha'<0$, i.e. a unit of norm -1. Let us denote by $t_n=\alpha^n+(\alpha')^n=\mathrm{Tr}_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha^n)$ the trace of α^n. Then for all odd integers n, the number α^n satisfies the relation $(\alpha^n)^2-t_n\alpha^n-1=0$, i.e.

$$\alpha^n = t_n + \frac{1}{\alpha^n};$$

hence its continued fraction is simply $[\overline{t_n}]$, and has period one.

We shall now consider units α^n of norm 1, so including also even powers of units of norm -1. Suppose that $\alpha>1$, with $0<\alpha'<1$, is such a unit, so that $\alpha'=\alpha^{-1}$. Denoting again by t_n the trace of α^n, we see that the integral part of α^n is t_n-1, so we put $a_0(n)=t_n-1$. We have

$$(\alpha^n-a_0(n))^{-1}=(1-(\alpha')^n)^{-1}=\frac{1-\alpha^n}{(1-(\alpha')^n)(1-\alpha^n)}=\frac{\alpha^n-1}{t_n-2}.$$

For sufficiently large n, its integral part is 1. So put $a_1(n)=1$ (at least for sufficiently large n). Then

$$\left(\frac{\alpha^n-1}{t_n-2}-1\right)^{-1}=\left(\frac{\alpha^n-1-t_n+2}{t_n-2}\right)^{-1}=\frac{t_n-2}{1-(\alpha')^n}.$$

Again for sufficiently large n, the integral part of the above number is t_n-2, so put $a_2(n)=t_n-2$. Now

$$\frac{t_n-2}{1-(\alpha')^n}-(t_n-2)=\frac{t_n-2-t_n+t_n(\alpha')^n+2-2(\alpha')^n}{1-(\alpha')^n}=\frac{1+(\alpha')^{2n}-2(\alpha')^n}{1-(\alpha')^n}=1-(\alpha')^n,$$

where we used the equation satisfied by $(\alpha')^n$ to simplify the numerator. Then

$$\left(\frac{t_n-2}{1-(\alpha')^n}-a_2(n)\right)^{-1}=(1-(\alpha')^n)^{-1}=\frac{\alpha^n-1}{t_n-2},$$

and the algorithm ends giving the continued fraction expansion, valid for sufficiently large n,

$$\alpha^n=[a_0(n),\overline{a_1(n),a_2(n)}\,]=[t_n-1,\overline{1,t_n-2}\,]. \qquad \square$$

We now prove our last lemma:

LEMMA 5. *Let $\alpha>1$ be a real quadratic number, not the square root of a rational number. Let $a_0(n), a_1(n), \dots$ be the partial quotients of the continued fraction for α^n. Then either α is a Pisot number, or for every i with $i\neq 0$,*

$$\lim_{n\to\infty} \frac{\log a_i(n)}{n} = 0. \tag{3.3}$$

If $\alpha>1$ is the square root of a rational number, then (3.3) *holds provided the limit is taken over odd integers n.*

Proof. We argue by contradiction. Let $h\in\mathbf{N}$, $h>0$, be the minimum index i such that (3.3) does not hold. Then for a positive real δ and all n in an infinite set $\Xi\subset\mathbf{N}$,

$$a_h(n) > e^{\delta n}. \tag{3.4}$$

Here it is meant that Ξ contains only odd integers if α is the square root of a rational number.

On the other hand, since (3.3) holds for all $i=1,\dots,h-1$, we have, in view of the recurrence relation (3.2), that the denominators $q_{h-1}(n)$ of

$$\frac{p_{h-1}(n)}{q_{h-1}(n)} = [a_0(n), a_1(n), \dots, a_{h-1}(n)]$$

satisfy

$$\lim_{n\to\infty} \frac{\log q_{h-1}(n)}{n} = 0. \tag{3.5}$$

Put $\varepsilon=\delta/2\log H(\alpha)$. By the vanishing of the above limit we have, in particular,

$$q_{h-1}(n)^{1+\varepsilon} \leqslant e^{\delta n/2}$$

for sufficiently large $n\in\Xi$. In view of (3.1) we have, for all such n,

$$\left|\alpha^n - \frac{p_{h-1}(n)}{q_{h-1}(n)}\right| < \frac{1}{q_{h-1}(n)^2 e^{\delta n}} < \frac{1}{q_{h-1}(n)^{3+\varepsilon} e^{\delta n/2}},$$

which can be rewritten as

$$\|q_{h-1}(n)\alpha^n\| < q_{h-1}(n)^{-2-\varepsilon} H(\alpha^n)^{-\varepsilon}.$$

An application of our main theorem, with $u=\alpha^n$ (for $n\in\Xi$), gives the conclusion that for large n the algebraic number $q_{h-1}(n)\alpha^n$ is pseudo-Pisot. Now, taking into account (3.5), our Lemma 4 implies that α is an algebraic integer (it cannot have a vanishing trace, since then it would be the square root of a rational number, which we excluded). This in turn implies that α^n is a Pisot number. Since α is a quadratic irrational and α^n is not a rational, α itself is a Pisot number. □

We can now finish the proof of Theorem 2. Let α be a real quadratic irrational number; let us first treat the case when α is not the square root of a rational number. Suppose that for all positive integers n in an infinite set $\Xi \subset \mathbf{N}$, the period of the continued fraction of α^n has the same length r. We would like to prove that α is a unit.

Let us first show that, by the above-mentioned facts, we can reduce to the case where

$$\alpha > 1 \quad \text{and} \quad \alpha > |\alpha'|. \tag{3.6}$$

In fact, after replacing, if necessary, α by $\pm 1/\alpha$ we can suppose that $\alpha>1$. Observe now that if $\alpha=-\alpha'$ then α is the square root of a rational number, which we have excluded. Thus $|\alpha|\neq|\alpha'|$ (since we cannot have $\alpha=\alpha'$). If now $\alpha>|\alpha'|$, we are done; otherwise $|\alpha'|>\alpha>1$; in this case we replace α by $\pm\alpha'$, and obtain (3.6) as wanted.

So, from now on, let us suppose that (3.6) holds for α. Then for all $n \geqslant n_0(\alpha)$ we have $\alpha^n > (\alpha')^n+2$, so there exists an integer k_n such that

$$\alpha^n - k_n > 1 \quad \text{and} \quad -1 < (\alpha^n - k_n)' < 0,$$

and so $\alpha^n - k_n$ is reduced. Then we have for all $n \in \Xi$,

$$\alpha^n - k_n = [\,\overline{a_0(n), a_1(n), ..., a_{r-1}(n)}\,],$$

where $a_0(n), ..., a_{r-1}(n)$ are positive integers and the period $(a_0(n), ..., a_{r-1}(n))$ is the same as the one for α^n (still for n in the infinite set Ξ). Suppose first that α is not a Pisot number. We would like to derive a contradiction. Our Lemma 5 implies that all partial quotients $a_i(n)$ satisfy (3.3), including $a_0(n)=a_r(n)$. The algebraic numbers $\alpha^n - k_n$ and $(\alpha')^n - k_n$ satisfy, for $n \in \Xi$, the algebraic equation

$$x = [a_0(n); a_1(n), ..., a_{r-1}(n), x] = \frac{p_{r-1}(n)x + p_{r-2}(n)}{q_{r-1}(n)x + q_{r-2}(n)},$$

which can also be written as

$$q_{r-1}(n)x^2 + (q_{r-2}(n) - p_{r-1}(n))x - p_{r-2}(n) = 0.$$

In view of (3.5), the coefficients of this equation have logarithms that are $o(n)$, whence the logarithmic heights of $x=\alpha^n - k_n$ and $x'=(\alpha')^n - k_n$ are $o(n)$ for $n \in \Xi$. But then we would have

$$\log|\alpha^n - (\alpha')^n| = o(n),$$

which is clearly impossible.

We have then proved that α is a Pisot number, so it is an algebraic integer and satisfies $|\alpha'|<1$. But now the quadratic irrational $\pm 1/\alpha'$ also satisfies (3.6), so the same reasoning implies that $1/\alpha'$, and hence $1/\alpha$, is also an integer. This proves that α is a unit as wanted.

The method is exactly the same in the second case, when $\alpha=\sqrt{a/b}$ is the square root of a positive rational number a/b. Here too, by replacing, if necessary, α with $1/\alpha$, we reduce to the case $\alpha>1$. In this case the pre-period has length one, so $(\alpha^n-k_n)^{-1}$ is reduced, where k_n is the integral part of α^n. Under the hypothesis that the period length of α^n remains bounded for an infinite set of odd integers, we can still apply our Lemma 5 and conclude as before.

Appendix

In this appendix we show that the word "algebraic" cannot be removed from the statement of Theorem 1. A little more precisely, we prove the following statement:

There exists a real number $\alpha>1$ such that $\|\alpha^n\|\leqslant 2^{-n}$ for infinitely many positive integers n, but α^d is not a Pisot number for any positive integer d.

Our assertion will follow from a simple construction which we are going to explain. For a set I of positive real numbers and a positive real t, we put $I^t:=\{x^t : x\in I\}$.

We start by choosing an arbitrary sequence $\{\beta_n\}_{n\geqslant 1}$ of real numbers in the interval $[0,\frac{1}{2}]$; we shall then define inductively integers $b_0, b_1, \ldots$ and closed intervals $I_0, I_1, \ldots$ of positive length and contained in $[2,\infty)$. We set $I_0=[2,3]$ and $b_0=1$, and having constructed I_n and b_n we continue as follows. Let b_{n+1} be a positive integer divisible by $n+1$ and such that $I_n^{b_{n+1}}$ contains an interval of length larger than 2; such an integer will certainly exist, since by induction every element of I_n is $\geqslant 2$ and since I_n has positive length.

Put $B_n=b_0 \ldots b_n$. Then we choose I_{n+1} to be any subinterval of $I_n^{b_{n+1}}$ of the shape

$$I_{n+1}=[q+\beta_{n+1}, q+\beta_{n+1}+2^{-B_{n+1}}],$$

where $q=q_n$ is an integer. This will be possible since $I_n^{b_{n+1}}$ has length larger than 2. Clearly $q\geqslant 2$, so I_{n+1} has the required properties.

For $n\in\mathbf{N}$, let now J_n denote the closed interval I_n^{1/B_n}. Since $I_{n+1}\subset I_n^{b_{n+1}}$, we have $J_{n+1}\subset J_n$ for all n, and $J_0=I_0=[2,3]$. Hence there exists $\alpha\in\bigcap_{n=0}^{\infty} J_n$ such that $\alpha\geqslant 2$.

Note that $\alpha^{B_n}\in I_n$; hence we conclude that *for $n\geqslant 1$ the fractional part $\{\alpha^{B_n}\}$ of α^{B_n} lies between β_n and $\beta_n+2^{-B_n}$.*

To obtain our original assertion, we apply this construction by choosing $\beta_n=0$ for even integers $n\geqslant 1$, and $\beta_n=\frac{1}{3}$, say, for odd n. We then have in the first place

$$\|\alpha^{B_n}\| \leqslant \{\alpha^{B_n}\} \leqslant 2^{-B_n}$$

for all even integers $n\geqslant 1$. This yields the first required property of α, since $B_n\to\infty$.

Further, suppose that α^d were a Pisot number for some positive integer d. Then for some fixed positive $\lambda<1$ we would have $\|\alpha^{dm}\|\ll\lambda^m$ for every integer $m>1$ (by a well-known argument, used also in the previous proofs of this paper). However, this is not possible, because for odd integers $n>d$ we have that B_n is divisible by d (recall that b_n is divisible by n for $n>0$), and the fractional part of α^{B_n} is between $\frac{1}{3}$ and $\frac{1}{3}+2^{-B_n}$, whence the norm $\|\alpha^{B_n}\|$ is $\geqslant\frac{1}{6}$.

This concludes the argument. It should be noted that the construction may easily be sharpened; also, Theorem 1 obviously implies that any α likewise obtained is necessarily transcendental.

References

[CZ] Corvaja, P. & Zannier, U., On the length of the continued fraction for the ratio of two power sums. To appear in *J. Théor. Nombres Bordeaux.*

[Ma] Mahler, K., On the fractional parts of the powers of a rational number, II. *Mathematika*, 4 (1957), 122–124.

[Me] Mendès France, M., Remarks and problems on finite and periodic continued fractions. *Enseign. Math.*, 39 (1993), 249–257.

[S] Schmidt, W. M., *Diophantine Approximations and Diophantine Equations.* Lecture Notes in Math., 1467. Springer, Berlin, 1991.

Pietro Corvaja
Dipartimento di Matematica e Informatica
Università di Udine
Via delle Scienze
IT-33100 Udine
Italy
corvaja@dimi.uniud.it

Umberto Zannier
Scuola Normale Superiore
Piazza dei Cavalieri, 7
IT-56100 Pisa
Italy
u.zannier@sns.it

Received April 13, 2004