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Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing

by

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Contents

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0. Introduction

We deal with the theory of iteration of forcing notions for the continuum, and prove CON($\mathfrak{a}>\mathfrak{d}$) and related results. We present it in several perspectives; so §§2 and 3 do not depend on §1; and §4 does not depend on §§1, 2 and 3. In §2 we introduce and investigate iterations which are of finite support but with non-transitive memory and linear, non-well-ordered length, and prove CON($\mathfrak{a}>\mathfrak{d}$) using a measurable. In §4 we also answer related questions ($\mathfrak{u}<\mathfrak{a}$); in §3, relying on §2 we eliminate the use of a measurable, and in §1 we rely heavily on [S8].

Very basically, the difference we use between $\mathfrak{a}$ on one hand and $\mathfrak{b}$ and $\mathfrak{d}$ on the other hand is that $\mathfrak{a}$ speaks on a set, whereas $\mathfrak{b}$ is witnessed by a sequence and $\mathfrak{d}$ by a quite directed family; it essentially deals with cofinality; so every unbounded subsequence is a witness as well, i.e. the relevant relation is transitive; when $\mathfrak{b}=\mathfrak{d}$ things are smooth, otherwise the situation is still similar. This manifests itself by using ultrapowers for some $\varkappa$-complete ultrafilter (in model-theoretic outlook), and by using "convergent sequences" (see [S1], or the existence of Av, the average, in [S3]) in §§2 and 3, respectively. The meaning of "model-theoretic outlook" is that by experience set theorists starting to hear an explanation of the forcing tend to think of an elementary embedding $\mathbf{j}:\mathbf{V}\to M$, and then the limit practically does not make sense (though of course we can translate). Note that ultrapowers by, e.g., an ultrafilter on $\varkappa$, preserve any witness for a cofinality of a linear order being $\geqslant\varkappa^+$ (or the cofinality of a $\varkappa^+$-directed partial order), as the set of old elements is cofinal and a cofinal subset of a cofinal subset is a cofinal subset. On the other hand, the ultrapower always "increases" a set of cardinality at least $\varkappa$, the completeness of the ultrafilter.

"Is $\mathfrak{a}\leqslant\mathfrak{d}$?" is one of the oldest problems and well known on cardinal invariants of the continuum (see [D]). It was mostly thought (certainly by me) that consistently $\mathfrak{a}>\mathfrak{d}$ and that the natural way to proceed is by CS-iteration $\langle\mathbf{P}_i,\underset{\sim}{\mathbf{Q}}_i:i<\omega_2\rangle$ of proper ${}^{\omega}\omega$-bounding forcing notions, starting with $\mathbf{V}\models\mathrm{GCH}$, and $|\mathbf{P}_i|=\aleph_1$ for $i<\omega_2$ and $\underset{\sim}{\mathbf{Q}}_i$ "deal" with one MAD family $\mathcal{A}_i\in\mathbf{V}^{\mathbf{P}_i}$, $\mathcal{A}_i\subseteq[\omega]^{\aleph_0}$, adding an infinite subset of ω almost disjoint to every $A\in\mathcal{A}_i$. The needed iteration theorem holds by [S4, Chapter V, §4], saying that in $\mathbf{V}^{\mathbf{P}_{\omega_2}}$, $\mathfrak{d}=\mathfrak{b}=\aleph_1$ and no cardinal is collapsed, but the single step forcing is not known to exist. This has been explained in details in [S5].

We do not proceed in this way but in a totally different direction involving making the continuum large, so we still do not know the answer to the following problem.

Problem 0.1. Is ZFC$+2^{\aleph_0}+\aleph_2+\mathfrak{a}>\mathfrak{d}$ consistent?

To clarify our idea, let D be a normal ultrafilter on $\varkappa$, a measurable cardinal, and consider a c.c.c. (countable chain condition) forcing notion $\mathbf{P}$ and

(a) a sequence $\bar{f}=\langle \underset{\sim}{f}_\alpha:\alpha<\varkappa^+\rangle$ of **P**-names such that $\Vdash_{\mathbf{P}}$ "$\langle \underset{\sim}{f}_\alpha:\alpha<\varkappa^+\rangle$ is $<^*$-increasing cofinal in ${}^\omega\omega$" (so that $\underset{\sim}{\bar{f}}$ exemplifies $\Vdash_{\mathbf{P}}$ "$\mathfrak{b}=\mathfrak{d}=\varkappa^+$"), and

(b) a sequence $\langle \underset{\sim}{A}_\alpha:\alpha<\alpha^*\rangle$ of **P**-names such that $\Vdash_{\mathbf{P}}$ "$\{\underset{\sim}{A}_\alpha:\alpha<\alpha^*\}$ is MAD, that is, $\alpha\neq\beta\Rightarrow \underset{\sim}{A}_\alpha\cap \underset{\sim}{A}_\beta$ is finite and $\underset{\sim}{A}_\alpha\in[\omega]^{\aleph_0}$".

Now $\mathbf{P}_1=\mathbf{P}^\varkappa/D$ also is a c.c.c. forcing notion by Łoś' theorem for $\mathbf{L}_{\varkappa,\varkappa}$; let $\mathbf{j}:\mathbf{P}\to\mathbf{P}_1$ be the canonical embedding; moreover, under the canonical identification we have $\mathbf{P}\prec_{\mathbf{L}_{\varkappa,\varkappa}}\mathbf{P}_1$. So also $\Vdash_{\mathbf{P}_1}$ "$\underset{\sim}{f}_\alpha\in{}^\omega\omega$", recalling that $\underset{\sim}{f}_\alpha$ actually consists of ω maximal antichains of **P** (or think of $(\mathcal{H}(\chi),\in)^\varkappa/D$, χ large enough). Similarly $\Vdash_{\mathbf{P}_1}$ "$\underset{\sim}{f}_\alpha<^*\underset{\sim}{f}_\beta$ if $\alpha<\beta<\varkappa^+$".

Now, if $\Vdash_{\mathbf{P}_1}$ "$\underset{\sim}{g}\in{}^\omega\omega$", then $\underset{\sim}{g}=\langle \underset{\sim}{g}_\varepsilon:\varepsilon<\varkappa\rangle/D$, $\Vdash_{\mathbf{P}}$ "$\underset{\sim}{g}_\varepsilon\in{}^\omega\omega$", so for some $\alpha<\varkappa^+$ we have $\Vdash_{\mathbf{P}}$ "$\underset{\sim}{g}_\varepsilon<^*\underset{\sim}{f}_\alpha$ for $\varepsilon<\varkappa$". Hence by Łoś' theorem $\Vdash_{\mathbf{P}_1}$ "$\underset{\sim}{g}<^*\underset{\sim}{f}_\alpha$" (so before the identification this means $\Vdash_{\mathbf{P}_1}$ "$\underset{\sim}{g}<^*\mathbf{j}(f_\alpha)$"), so $\langle \underset{\sim}{f}_\alpha:\alpha<\varkappa^+\rangle$ exemplifies also $\Vdash_{\mathbf{P}_1}$ "$\mathfrak{b}=\mathfrak{d}=\varkappa^+$".

On the other hand, $\langle \underset{\sim}{A}_\alpha:\alpha<\alpha^*\rangle$ cannot exemplify that $\mathfrak{a}\leqslant\varkappa^+$ in $\mathbf{V}^{\mathbf{P}_1}$ because $\alpha^*\geqslant\varkappa^+$ (as ZFC$\models\mathfrak{b}\leqslant\mathfrak{a}$), so $\langle \underset{\sim}{A}_\alpha:\alpha<\varkappa\rangle/D$ exemplifies that $\Vdash_{\mathbf{P}_1}$ "$\{\underset{\sim}{A}_\alpha:\alpha<\alpha^*\}$ is not MAD".

Our original idea here is to start with an FS-iteration $\bar{\mathbf{Q}}^0=\langle \mathbf{P}_i^0,\underset{\sim}{\mathbf{Q}}_i^0:i<\varkappa^+\rangle$ of nep c.c.c. forcing notions, $\underset{\sim}{\mathbf{Q}}_i^0$ adding a dominating real (e.g. dominating real = Hechler forcing), for $\varkappa$ a measurable cardinal, and let D be a $\varkappa$-complete uniform ultrafilter on $\varkappa$ and $\chi\gg\varkappa$. Then let $L_0=\varkappa^+$, and let $\bar{\mathbf{Q}}^1=\langle \mathbf{P}_i^1,\mathbf{Q}_i^1:i\in L_1\rangle$ be $\bar{\mathbf{Q}}^0$ as interpreted in $(\mathcal{H}(\chi),\in,<_\chi^*)^\varkappa/D$. It looks like $\bar{\mathbf{Q}}^0$ replacing $\varkappa^+$ by $(\varkappa^+)^\varkappa/D$. We look at $\mathrm{Lim}(\bar{\mathbf{Q}}^0)=\bigcup_i \mathbf{P}_i$ as a subforcing of $\mathrm{Lim}(\bar{\mathbf{Q}}^1)$ identifying $\underset{\sim}{\mathbf{Q}}_i$ with $\underset{\sim}{\mathbf{Q}}_{\mathbf{j}_0(i)}$, $\mathbf{j}_0$ being the canonical elementary embedding of $\varkappa^+$ into $(\varkappa^+)^\varkappa/D$ (no Mostowski collapse!). We continue to define $\bar{\mathbf{Q}}^n$ and then $\bar{\mathbf{Q}}^\omega$ as the following limit: for the original $i\in\varkappa^+$, we use the definition, otherwise we use direct limit ("founding father's privilege" you may say). So $\mathbf{P}^i=\mathrm{Lim}(\bar{\mathbf{Q}}^i)$ is $\lessdot$-increasing, continuous when $\mathrm{cf}(i)>\aleph_0$; so now we have a kind of iteration with non-transitive memory and a non-well-founded base. We continue $\varkappa^{++}$ times. Now in $\mathbf{V}^{\mathrm{Lim}(\bar{\mathbf{Q}}^{\varkappa^{++}})}$, the original $\varkappa^+$ generic reals exemplify $\mathfrak{b}=\mathfrak{d}=\varkappa^+$, so we know that $\mathfrak{a}\geqslant\varkappa^+$. To finish assume that $p\Vdash$ "$\{\underset{\sim}{A}_\gamma:\gamma<\varkappa^+\}\subseteq[\omega]^{\aleph_0}$ is a MAD family". Each name $\underset{\sim}{A}_\gamma$ is a "countable object" and so depends on countably many conditions, so all of them are in $\mathrm{Lim}(\bar{\mathbf{Q}}^i)$ for some $i<\varkappa^{++}$. In the next stage, $\bar{\mathbf{Q}}^{i+1}$, $\langle \underset{\sim}{A}_\gamma:\gamma<\varkappa\rangle/D$ is a name of an infinite subset of ω almost disjoint to $\underset{\sim}{A}_\beta$ for each $\beta<\varkappa^+$, a contradiction.

All this is a reasonable scheme. This is done in §1 but rely on "nep forcing" from [S8]. But another self-contained approach is in §§ 2 and 3, where the meaning of the iteration is more on the surface (and also, in §3, help to eliminate the use of large cardinals). In §4 we deal with the case of an additional cardinal invariant, $\mathfrak{u}$.

Note that just using FS-iteration on a non-well-ordered linear order L (instead of an ordinal) is impossible by a theorem of Hjorth. On non-linear orders for iterations (history and background) see [S10]. On iteration with non-transitive memory see [S6], [S7], and in particular [S7, §3]. Continuing this work J. Brendle has proved the consistency of $\mathfrak{a}=\aleph_0$ (note that in Lemma 3.5 we have assumed that $\lambda=\lambda^{\aleph_0}$ in $\mathbf{V}$, and hence $\mathrm{cf}(\lambda)>\aleph_0$ even in $\mathbf{V^P}$).

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1. On Con($\mathfrak{a}>\mathfrak{d}$)

In this section, we look at it in the context of [S8] and we use a measurable.

Definition 1.1. (1) Given sets A_l of ordinals for $l<n$, we say that $\mathcal{T}$ is an $(A_0,\dots,A_{n-1})$-*tree* if $\mathcal{T}=\bigcup_{k<\omega}\mathcal{T}_k$, where

$$\mathcal{T}_k\subseteq\{(\eta_0,\dots,\eta_l,\dots,\eta_{n-1}):\eta_l\in{}^k(A_l)\text{ for }l<n\}$$

and $\mathcal{T}$ is ordered by $\bar\eta\leqslant_{\mathcal{T}}\bar\nu\Leftrightarrow\bigwedge_{l<n}\eta_l\trianglelefteq\nu_l$, and we let $\bar\eta\upharpoonright k_1:=\langle\eta_l\upharpoonright k_1:l<n\rangle$ and demand that $\bar\eta\in\mathcal{T}_k$ & $k_1<k\Rightarrow\bar\eta\upharpoonright k_1\in\mathcal{T}_{k_1}$. We call $\mathcal{T}$ *locally countable* if $k\in[1,\omega)$ & $\bar\eta\in\mathcal{T}_k\Rightarrow$ $|\{\bar\nu\in\mathcal{T}_{k+1}:\bar\eta\leqslant_{\mathcal{T}}\bar\nu\}|\leqslant\aleph_0$. Let

$$\lim(\mathcal{T})=\{\langle\eta_l:l<n\rangle:\eta_l\in{}^\omega(A_l)\text{ for }l<k\text{ and }m<\omega\Rightarrow\langle\eta_l\upharpoonright m:l<n\rangle\in\mathcal{T}\}.$$

Lastly for $n_1\leqslant n$ we let

$$\operatorname{prj}\lim_{n_1}(\mathcal{T})=\{\langle\eta_l:l<n_1\rangle:\text{for some }\eta_{n_1},\dots,\eta_{n-1}\text{ we have }\langle\eta_l:l<n\rangle\in\lim(\mathcal{T})\};$$

and if n_1 is omitted we mean $n_1=n-1$.

(2) We let

$\mathfrak{K}=\{\bar{\mathcal{T}}$: for some sets A and B of ordinals we have
- (i) $\bar{\mathcal{T}}=(\mathcal{T}_1,\mathcal{T}_2)$;
- (ii) $\mathcal{T}_1$ is a locally countable (A,B)-tree;
- (iii) $\mathcal{T}_2$ is a locally countable (A,A,B)-tree;
- (iv) $\mathbf{Q}_{\bar{\mathcal{T}}}:=(\operatorname{prj}\lim(\mathcal{T}_1),\operatorname{prj}\lim(\mathcal{T}_2))$ is a c.c.c. forcing notion absolutely under c.c.c. forcing notions (see below)$\}$.

(2A) We say that $\mathbf{Q}_{\bar{\mathcal{T}}}$ is *c.c.c. absolutely for c.c.c. forcing* if: for c.c.c. forcing notions $\mathbf{P}\lessdot\mathbf{R}$ we have $\mathbf{P}*\mathbf{Q}_{\underset{\sim}{\bar{\mathcal{T}}}}\lessdot\mathbf{R}*\mathbf{Q}_{\underset{\sim}{\bar{\mathcal{T}}}}$ (though not necessarily $\mathbf{Q}_{\bar{\mathcal{T}}}^{\mathbf{V^P}}\lessdot\mathbf{Q}_{\bar{\mathcal{T}}}^{\mathbf{V^R}}$ in $\mathbf{V^R}$), so that membership, order, non-order, compatibility, non-compatibility, and being predense over p in the universe $\mathbf{V^P}$, are preserved in passing to $\mathbf{V^R}$; note that the predense sets belong to $\mathbf{V^P}$ (the $\mathbf{Q}_{\bar{\mathcal{T}}}$'s are snep, from [S8] with slight restriction). Similarly we define "$\mathbf{Q}_{\bar{\mathcal{T}}} \lessdot \mathbf{Q}_{\bar{\mathcal{T}}}$ absolutely under c.c.c. forcing" (compare with clause (A)(a)(iii) in Definition 2.6).

(3) For a set or class A of ordinals, $\mathfrak{K}^{\varkappa}_{A}$ is the family of $\bar{\mathcal{T}}\in\mathfrak{K}$ which are a pair of objects, the first an (A,B)-tree and the second an (A,A,B)-tree for some B such that $|\mathcal{T}_1|\leqslant\varkappa$ and $|\mathcal{T}_2|\leqslant\varkappa$. For a cardinal $\varkappa$ and a pairing function pr with inverses pr_1 and pr_2, let $\mathfrak{K}^{\varkappa}_{\mathrm{pr}_1,\gamma}=\mathfrak{K}^{\varkappa}_{\{\alpha:\mathrm{pr}_1(\alpha)=\gamma\}}$ and $\mathfrak{K}^{\varkappa}_{\mathrm{pr}_1,<\gamma}=\mathfrak{K}^{\varkappa}_{\{\alpha:\mathrm{pr}_1(\alpha)<\gamma\}}$. Let $|\bar{\mathcal{T}}|=|\mathcal{T}_1|+|\mathcal{T}_2|$.

(4) Let $\bar{\mathcal{T}},\bar{\mathcal{T}}'\in\mathfrak{K}$. We say that $\mathbf{f}$ is an *isomorphism* from $\bar{\mathcal{T}}$ onto $\bar{\mathcal{T}}'$ when $\mathbf{f}=(f_1,f_2)$ and for $m=1,2$ we have: f_m is a one-to-one function from $\mathcal{T}_m$ onto $\mathcal{T}'_m$ preserving the level (in the respective trees), preserving the relations $x=y\restriction k$, $x\neq y\restriction k$, and if $f_2((\eta_1,\eta_2,\eta_3))=(\eta'_1,\eta'_2,\eta'_3)$, $f_1((\nu_1,\nu_2))=(\nu'_1,\nu'_2)$, then $[\eta_1=\nu_1\Leftrightarrow\eta'_1=\nu'_1]$ and $[\eta_2=\nu_1\Leftrightarrow\eta'_2=\nu'_1]$.

In this case let $\hat{\mathbf{f}}$ be the isomorphism induced by $\mathbf{f}$ from $\mathbf{Q}_{\bar{\mathcal{T}}}$ onto $\mathbf{Q}_{\bar{\mathcal{T}}'}$.

Definition 1.2. For $\bar{\mathcal{T}}',\bar{\mathcal{T}}''\in\mathfrak{K}$ let $\bar{\mathcal{T}}'\leqslant_{\mathfrak{K}}\bar{\mathcal{T}}''$ mean:

(a) $\mathcal{T}'_l\subseteq\mathcal{T}''_l$ (as trees) for $l=1,2$;

(b) if $l\in\{1,2\}$, $\bar{\eta}\in\mathcal{T}''_l\setminus\mathcal{T}'_l$ and $\bar{\eta}\restriction k\in\mathcal{T}'_l$ then $k\leqslant 1$;

(c) $\mathbf{Q}_{\bar{\mathcal{T}}'}\lessdot\mathbf{Q}_{\bar{\mathcal{T}}''}$ (absolutely under c.c.c. forcing); note that by (a) and (b) we have:

$$x\in\mathbf{Q}_{\bar{\mathcal{T}}'}\Longrightarrow x\in\mathbf{Q}_{\bar{\mathcal{T}}''}\quad\text{and}\quad\mathbf{Q}_{\bar{\mathcal{T}}'}\vDash x\leqslant y\Longrightarrow\mathbf{Q}_{\bar{\mathcal{T}}''}\vDash x\leqslant y.$$

Remark. The definition is tailored such that the union of an increasing chain will give a forcing notion which is the union.

Claim/Definition 1.3. (0) *The relation $\leqslant_{\mathfrak{K}}$ is a partial order of $\mathfrak{K}$.*

(1) *Assume that $\langle\bar{\mathcal{T}}[i]:i<\delta\rangle$ is $\leqslant_{\mathfrak{K}}$-increasing and that $\bar{\mathcal{T}}$ is defined by $\bar{\mathcal{T}}=\bigcup_i\bar{\mathcal{T}}[i]$, that is, $\mathcal{T}_m=\bigcup_{i<\delta}\mathcal{T}_m[i]$ for $m=1,2$. Then*

(a) $i<\delta\Rightarrow\bar{\mathcal{T}}[i]\leqslant_{\mathfrak{K}}\bar{\mathcal{T}}$;

(b) $\mathbf{Q}_{\bar{\mathcal{T}}}=\bigcup_{i<\delta}\mathbf{Q}_{\bar{\mathcal{T}}[i]}$.

(2) *Assume $\bar{\mathcal{T}}',\bar{\mathcal{T}}\in\mathfrak{K}$. Then there is $\bar{\mathcal{T}}''\in\mathfrak{K}$ such that $\bar{\mathcal{T}}'\leqslant_{\mathfrak{K}}\bar{\mathcal{T}}''$ and $\mathbf{Q}_{\bar{\mathcal{T}}''}$ is isomorphic to $\mathbf{Q}_{\bar{\mathcal{T}}'}*\underset{\sim}{\mathbf{Q}}_{\bar{\mathcal{T}}}$, and this is absolute by c.c.c. forcing. Moreover, there is such an isomorphism extending the identity map from $\mathbf{Q}_{\bar{\mathcal{T}}'}$ into $\mathbf{Q}_{\bar{\mathcal{T}}''}$.*

(3) *There is $\bar{\mathcal{T}}\in\mathfrak{K}^{\aleph_0}_{\omega}$ such that $\mathbf{Q}_{\bar{\mathcal{T}}}$ is the trivial forcing.*

(4) *There is $\bar{\mathcal{T}}\in\mathfrak{K}^{\aleph_0}_{\omega}$ such that $\mathbf{Q}_{\bar{\mathcal{T}}}$ is the dominating real forcing.*

Proof. See [S8]. □

Claim 1.4. (1) *Assume that $\bar{\mathcal{T}}[\gamma]\in\mathfrak{K}_{\mathrm{pr}_1,\gamma}$ for $\gamma<\gamma(*)$. Then for each $\alpha\leqslant\gamma(*)$ there is $\bar{\mathcal{T}}\langle\alpha\rangle\in\mathfrak{K}_{\mathrm{pr}_1,<\alpha}$ such that $\mathbf{Q}_{\bar{\mathcal{T}}\langle\alpha\rangle}$ is $\mathbf{P}_\alpha$, where $\langle\mathbf{P}_\gamma,\underset{\sim}{\mathbf{Q}}_\beta:\gamma\leqslant\gamma(*),\ \beta<\gamma(*)\rangle$ is an FS-iteration, $\underset{\sim}{\mathbf{Q}}_\beta=(\mathbf{Q}_{\bar{\mathcal{T}}[\beta]})^{\mathbf{V}[\mathbf{P}_\beta]}$, $\bar{\mathcal{T}}\langle\alpha\rangle\in\mathfrak{K}_{\mathrm{pr}_1,<\alpha}$, $\bar{\mathcal{T}}\langle\alpha_1\rangle\leqslant_{\mathfrak{K}}\bar{\mathcal{T}}\langle\alpha_2\rangle$ for $\alpha_1\leqslant\alpha_2\leqslant\gamma(*)$ and $\bar{\mathcal{T}}[\gamma]\leqslant_{\mathfrak{K}}\bar{\mathcal{T}}\langle\alpha\rangle$ for $\gamma<\alpha\leqslant\gamma(*)$. We write $\bar{\mathcal{T}}\langle\alpha\rangle=\sum_{\gamma<\alpha}\bar{\mathcal{T}}[\gamma]$.*

(2) *In part* (1), *for each* $\gamma<\gamma(*)$ *there is* $\bar{\mathcal{T}}'\in\mathfrak{K}_{\mathrm{pr}_1,\gamma}$ *such that* $\bar{\mathcal{T}}'$ *and* $\bar{\mathcal{T}}$ *are isomorphic over* $\bar{\mathcal{T}}[\gamma]$. *Hence* $\mathbf{Q}_{\bar{\mathcal{T}}'}$ *and* $\mathbf{Q}_{\bar{\mathcal{T}}}$ *are isomorphic over* $\mathbf{Q}_{\bar{\mathcal{T}}[\gamma]}$.

(3) *If in addition* $\mathcal{T}[\gamma]\leqslant_{\mathfrak{K}}\mathcal{T}'[\gamma]\in\mathfrak{K}_{\mathrm{pr}_1,\gamma}$ *for* $\gamma<\gamma(*)$, *and* $\langle\mathbf{P}_\gamma,\underset{\sim}{\mathbf{Q}}'_\beta:\gamma\leqslant\gamma(*),\ \beta<\gamma(*)\rangle$ *is an FS-iteration as above with* $\mathbf{P}'_{\gamma(*)}=\mathbf{Q}_{\bar{\mathcal{T}}'}$, *then we find such a* $\bar{\mathcal{T}}'$ *with* $\bar{\mathcal{T}}\leqslant_{\mathfrak{K}}\bar{\mathcal{T}}'$.

Proof. This is straightforward. □

THEOREM 1.5. *Assume that*

(a) $\varkappa$ *is a measurable cardinal*;

(b) $\varkappa<\mu=\mathrm{cf}(\mu)<\lambda=\mathrm{cf}(\lambda)=\lambda^\varkappa$ *and* $(\forall\alpha<\mu)(|\alpha|^{\aleph_0}<\mu)$ *for simplicity.*

Then for some c.c.c. forcing notion $\mathbf{P}$ *of cardinality* λ, *in* $\mathbf{V}^{\mathbf{P}}$ *we have* $2^{\aleph_0}=\lambda$, $\mathfrak{d}=\mathfrak{b}=\mu$ *and* $\mathfrak{a}=\lambda$.

Proof. We choose by induction on $\zeta\leqslant\lambda$ the following objects satisfying the following conditions:

(a) a sequence $\langle\bar{\mathcal{T}}[\gamma,\zeta]:\gamma<\mu\rangle$;

(b) $\bar{\mathcal{T}}[\gamma,\zeta]\in\mathfrak{K}^\lambda_{\mathrm{pr}_1,\gamma}$;

(c) $\xi<\zeta\Rightarrow\bar{\mathcal{T}}[\gamma,\xi]\leqslant_{\mathfrak{K}}\bar{\mathcal{T}}[\gamma,\zeta]$;

(d) if ζ is limit then $\bar{\mathcal{T}}[\gamma,\zeta]=\bigcup_{\xi<\zeta}\bar{\mathcal{T}}[\gamma,\xi]$;

(e) if $\gamma<\mu$ and $\zeta=1$, then $\mathbf{Q}_{\bar{\mathcal{T}}}[\gamma,\zeta]$ is the dominating real forcing = Hechler forcing;

(f) if $\gamma<\mu$, $\zeta=\xi+1>1$ and ξ is even, then $\bar{\mathcal{T}}[\gamma,\zeta]$ is isomorphic to $\bar{\mathcal{T}}\langle\gamma+1,\xi\rangle$ over $\bar{\mathcal{T}}[\gamma,\xi]$, say by $\mathbf{j}_{\gamma,\xi}$, where $\bar{\mathcal{T}}\langle\gamma+1,\xi\rangle:=\sum_{\beta\leqslant\gamma}\bar{\mathcal{T}}[\beta,\xi]$; let $\hat{\mathbf{j}}_{\gamma,\xi}$ be the isomorphism induced from $\mathbf{Q}_{\bar{\mathcal{T}}\langle\gamma+1,\xi\rangle}$ onto $\mathbf{Q}_{\bar{\mathcal{T}}}[\gamma,\zeta]$ over $\mathbf{Q}_{\bar{\mathcal{T}}[\gamma,\xi]}$;

(g) if $\gamma<\mu$, $\zeta=\xi+1$ and ξ is odd, then $\bar{\mathcal{T}}[\gamma,\zeta]$ is almost isomorphic to $(\bar{\mathcal{T}}[\gamma,\xi])^\varkappa/D$ over $\bar{\mathcal{T}}_{[\gamma,\xi]}$, say that $\mathbf{j}_{\gamma,\xi}$ is an isomorphism from $(\bar{\mathcal{T}}[\gamma,\xi])^\varkappa/D$ onto $\bar{\mathcal{T}}[\gamma,\zeta]$ such that $\langle x:\varepsilon<\varkappa\rangle/D$ is mapped onto x by $\mathbf{j}_{\gamma,\xi}$.

There is no problem to carry out the induction. Let $\mathbf{P}_\zeta=\mathbf{Q}_{\bar{\mathcal{T}}\langle\mu,\zeta\rangle}$, where we have $\bar{\mathcal{T}}\langle\mu,\zeta\rangle:=\sum_{\gamma<\mu}\bar{\mathcal{T}}[\gamma,\zeta]$ for $\zeta\leqslant\lambda$, $\mathbf{P}=\mathbf{P}_\lambda$ and $\mathbf{P}_{\gamma,\zeta}=\mathbf{Q}_{\bar{\mathcal{T}}\langle\gamma,\zeta\rangle}$. Now the following holds.

$\boxtimes_1$ It is true that $|\mathbf{P}|\leqslant\lambda$. (Why? As we prove by induction on $\zeta\leqslant\lambda$ that each $\bar{\mathcal{T}}[\gamma,\zeta]$ and $\sum_{\gamma\leqslant\mu}\bar{\mathcal{T}}[\gamma,\lambda]$ has cardinality $\leqslant\lambda$. Hence for $\gamma<\mu$ we have that the forcing notion $\mathbf{Q}_{\bar{\mathcal{T}}[\gamma,\lambda]}$ in the universe $\mathbf{V}^{\mathbf{Q}_{\bar{\mathcal{T}}\langle\gamma,\lambda\rangle}}$ has cardinality $\leqslant\lambda^{\aleph_0}=\lambda$.)

$\boxtimes_2$ In $\mathbf{V}^{\mathbf{P}}$ we have $\mathfrak{b}=\mathfrak{d}=\lambda$. (Why? Let $\underset{\sim}{\eta}_\gamma$ be the $\mathbf{Q}_{\bar{\mathcal{T}}[\gamma,1]}$-name of the dominating real (see clause (e)). As $\bar{\mathcal{T}}[\gamma,1]\leqslant_{\mathfrak{K}}\bar{\mathcal{T}}[\gamma,\lambda]$, clearly $\underset{\sim}{\eta}_\gamma$ is also a $\mathbf{Q}_{\bar{\mathcal{T}}[\gamma,\lambda]}$-name of a dominating real. This is preserved by $\mathbf{P}_\gamma$, so $\Vdash_{\mathbf{P}_{\gamma+1}}$ "$\underset{\sim}{\eta}_\gamma$ dominate $({}^\omega\omega)^{\mathbf{V}[\mathbf{P}_{\gamma,\lambda}]}$". But $\langle\mathbf{P}_{\gamma,\lambda}:\gamma<\mu\rangle$ is $\lessdot$-increasing with union $\mathbf{P}$ and $\mathrm{cf}(\mu)=\mu>\aleph_0$, so $\Vdash_{\mathbf{P}}$ "$\langle\underset{\sim}{\eta}_\gamma:\gamma<\mu\rangle$ is $<^*$-increasing and dominating". The conclusion follows.)

We shall prove below that $\mathfrak{a}\geqslant\lambda$, and together this finishes the proof. (Note that it implies $2^{\aleph_0}\geqslant\lambda$, and hence as $\lambda=\lambda^{\aleph_0}$ by $\boxtimes_1$ we get $2^{\aleph_0}=\lambda$.)

$\boxtimes_3$ It is true that $\Vdash_{\mathbf{P}}$ "$\mathfrak{a}\geqslant\lambda$".

So assume that $p\Vdash$ "$\underset{\sim}{\mathcal{A}}=\{\underset{\sim}{A}_i:i<\theta\}$ is a MAD family, i.e. ($\theta\geqslant\aleph_0$ and)

(i) $\underset{\sim}{A}_i\in[\omega]^{\aleph_0}$;

(ii) $i\neq j\Rightarrow|\underset{\sim}{A}_i\cap\underset{\sim}{A}_j|<\aleph_0$;

(iii) $\underset{\sim}{\mathcal{A}}$ is maximal under (i) and (ii)".

Without loss of generality $\Vdash_{\mathbf{P}}$ "$\underset{\sim}{A}_i\in[\omega]^{\aleph_0}$".

As always $\mathfrak{a}\geqslant\mathfrak{b}$, by $\boxtimes_2$ we know that $\theta\geqslant\mu$, and toward contradiction assume $\theta<\lambda$. For each $i<\theta$ and $m<\omega$ there is a maximal antichain $\langle p_{i,m,n}:n<\omega\rangle$ of $\mathbf{P}$ and a sequence $\langle \mathbf{t}_{i,m,n}:n<\omega\rangle$ of truth values such that $p_{i,m,n}\Vdash_{\mathbf{P}}$ "$n\in\underset{\sim}{A}_i$ if and only if $\mathbf{t}_{i,m,n}$ is truth". We can find a countable $w_i\subseteq\mu$ such that $\big[\bigcup_{m,n<\omega}\mathrm{Dom}(p_{i,m,n})\subseteq w_i\big]$, $p_{i,m,n}\in\mathbf{Q}_{\Sigma\{\overline{\mathcal{T}}[\gamma,\lambda]:\gamma\in w_i\}}$; moreover, $\gamma\in\mathrm{Dom}(p_{i,m,n})\Rightarrow p_{i,m,n}(\gamma)$ is a $\mathbf{Q}_{\sum\{\overline{\mathcal{T}}[\beta,\lambda]:\beta\in\gamma\cap w_i\}}$-name. Note that $\mathbf{Q}_{\sum\{\overline{\mathcal{T}}[\beta,\lambda]:\beta\in\gamma\cap w_i,i<\theta\}}\lessdot\mathbf{Q}_{\sum\{\overline{\mathcal{T}}_\beta:\beta<\gamma\}}$, see [S8].

Clearly for some even $\zeta<\lambda$, we have $\{p_{i,m,n}:i<\theta,\ m<\omega \text{ and } n<\omega\}\subseteq\mathbf{Q}_{\sum\{\overline{\mathcal{T}}[\beta,\zeta]:\beta<\mu\}}$. Now for some stationary $S\subseteq\{\delta<\mu:\mathrm{cf}(\delta)=\varkappa\}$ and w^* we have $\delta\in S\Rightarrow w_\delta\cap\delta=w^*$ and $\alpha<\delta\in S\Rightarrow w_\alpha\subseteq\delta$. Let $\langle\delta_\varepsilon:\varepsilon<\varkappa\rangle$ be an increasing sequence of members of S, and $\delta^*=\bigcup_{\varepsilon<\varkappa}\delta_\varepsilon$. The definition of $\langle\overline{\mathcal{T}}[\gamma,\zeta+1]:\gamma<\mu\rangle$, $\langle\overline{\mathcal{T}}[\gamma,\zeta+2]:\gamma<\mu\rangle$ was made to get a name of an infinite $\underset{\sim}{A}\subseteq\omega$ almost disjoint to every $\underset{\sim}{A}_\beta$ for $\beta<\theta$ (in fact, $\big(\sum_{\gamma<\mu}\mathbf{Q}_{\overline{\mathcal{T}}[\gamma,\zeta]}\big)^{\varkappa}/D$ can be $\lessdot$-embedded into $\sum_{\gamma<\mu}\mathbf{Q}_{\overline{\mathcal{T}}[\gamma,\zeta+2]}$). $\square$

Remark. In later proofs in §2 we give more details.

2. On Con($\mathfrak{a}>\mathfrak{d}$) revisited with FS, with non-transitive memory of non-well-ordered length

We first define the FSI-templates, telling us how to iterate along a linear order L; we think of having for each $t\in L$, a forcing notion $\mathbf{Q}_t$, say adding a generic $\underset{\sim}{\nu}_t$, and $\mathbf{Q}_t$ will really be $\bigcup\{\mathbf{Q}^{\mathbf{V}[\langle\underset{\sim}{\nu}_s:s\in A\rangle]}:A\in I_t\}$, where I_t is an ideal of subsets of $\{s:s<_L t\}$; so $\mathbf{Q}_t$ in the nice case is a definition. In our application this definition is constant, but we treat a more general case, so $\underset{\sim}{\mathbf{Q}}_t$ may be defined using parameters from $\mathbf{V}[\langle\underset{\sim}{\nu}_s:s\in K_t\rangle]$, K_t being a subset of $\{s:s<_L t\}$, and so the reader may consider only the case $t\in L\Rightarrow K_t=\varnothing$. In part (3), instead of distinguishing "ζ successor, ζ limit" we can consider the two cases for each ζ. The depth of L is the ordinal on which our induction rests (as order type of L is inadequate).

Definition 2.1. (1) An *FSI-template* (= finite support iteration template) $\mathfrak{t}$ is a sequence $\langle I_t:t\in L\rangle=\langle I_t^{\mathfrak{t}}:t\in L^{\mathfrak{t}}\rangle=\langle I_t[\mathfrak{t}]:t\in L[\mathfrak{t}]\rangle$ such that

(a) L is a linear order (but we may write $x\in\mathfrak{t}$ instead of $x\in L$ and $x<_{\mathfrak{t}}y$ instead of $x<_L y$);

(b) I_t is an ideal of subsets of $\{s:L\vDash s<t\}$.

We say that $\mathfrak{t}$ is *locally countable* if $t \in L^{\mathfrak{t}} \,\&\, (\forall B \in [A]^{\aleph_0})(B \in I_t) \Rightarrow A \in I_t$, we say that $\mathfrak{t}^1$ and $\mathfrak{t}^2$ are *equivalent* if $L^{\mathfrak{t}^1} = L^{\mathfrak{t}^2}$ and $t \in L^{\mathfrak{t}^1} \,\&\, |A| \leqslant \aleph_0 \Rightarrow (A \in I_t^{\mathfrak{t}^1} \equiv A \in I_t^{\mathfrak{t}^2})$, and we say that $\mathfrak{t}$ is *globally countable* if $t \in L^{\mathfrak{t}} \,\&\, A \in I_t^{\mathfrak{t}} \Rightarrow |A| \leqslant \aleph_0$.

(2) Let $\mathfrak{t}$ be an FSI-template.

(c) We say that $\overline{K} = \langle K_t : t \in L^{\mathfrak{t}} \rangle$ is a $\mathfrak{t}$-*memory choice* if

(i) $K_t \in I_t^{\mathfrak{t}}$ is countable;

(ii) $s \in K_t \Rightarrow K_s \subseteq K_t$.

(d) We say that $L \subseteq L^{\mathfrak{t}}$ is $\overline{K}$-*closed* if $t \in L \Rightarrow K_t \subseteq L$.

(e) For $\overline{K}$ a $\mathfrak{t}$-memory choice and $L \subseteq L^{\mathfrak{t}}$ which is $\overline{K}$-closed, we say that $\overline{K}' = \overline{K} \restriction L$ if $\mathrm{Dom}(\overline{K}') = L$ and K_t' is K_t for $t \in L$ (it is a $(\mathfrak{t} \restriction L)$-memory choice, see part (5)).

(3) For an FSI-template $\mathfrak{t}$, $\mathfrak{t}$-memory choice $\overline{K}$ and $\overline{K}$-closed $L \subseteq L^{\mathfrak{t}}$, we define $\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K})$, the $\mathfrak{t}$-*depth* (or $(\mathfrak{t}, \overline{K})$-depth) of L, by defining by induction on the ordinal ζ when $\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) \leqslant \zeta$.

For $\zeta = 0$: $\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) \leqslant \zeta$ when $L = \varnothing$.

For ζ a successor ordinal: $\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) \leqslant \zeta$ if and only if

(a) there is L^* such that $L^* \subseteq L$, $|L^*| \leqslant 1$, $(\forall t \in L)(\forall A \in I_t^{\mathfrak{t}})(A \cap L^* = \varnothing)$; hence $L \setminus L^*$ is $\overline{K}$-closed, $\mathrm{Dp}_{\mathfrak{t}}(L \setminus L^*, \overline{K}) < \zeta$ and for every $t \in L^*$ we have

$\boxtimes_{t,L}$ $L \setminus L^* \in I_t^{\mathfrak{t}}$ and[1] it is $\overline{K}$-closed.

For $\zeta > 0$ a limit ordinal: $\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) \leqslant \zeta$ if and only if

(b) there is a directed partial order M and a sequence $\langle L_a : a \in M \rangle$ with union L such that the sequence is increasing, i.e. $M \models a \leqslant b \Rightarrow L_a \subseteq L_b$, each L_b is $\overline{K}$-closed, $(\forall b \in M)(\zeta > \mathrm{Dp}_{\mathfrak{t}}(L_b, \overline{K}))$ and $t \in L \,\&\, A \in I_t \,\&\, A \subseteq L \Rightarrow (\exists a \in M)\, A \subseteq L_a$.

So we have $\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) = \zeta$ if and only if $\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) \geqslant \zeta \,\&\, (\forall \xi < \zeta)\, \mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) \nleqslant \xi$, and $\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) = \infty$ if and only if ($\forall$ ordinals ζ) $[\mathrm{Dp}_{\mathfrak{t}}(L, \overline{K}) \nleqslant \zeta]$.

(c) If $a_\alpha \in M$, $w_\alpha \in [L_a]^{<\aleph_0}$ and $A_{\alpha,t} \subseteq L_{a_\alpha}$ is from $I_t^{\mathfrak{t}}$ for $\alpha < \omega_1$ and $t \in w_\alpha$, then there are an unbounded $S \subseteq \omega_1$ and $b \in M$ such that $\langle \bigcup \{ A_{\alpha,t} \cup \{t\} \setminus L_b : t \in w_\alpha \} : \alpha \in S \rangle$ is a sequence of pairwise disjoint sets (we can waive this if we use only σ-linked forcing notions below (where for each of the ω parts, membership is absolute too)).

(4) We say that $\overline{K}$ is a *smooth* $\mathfrak{t}$-*memory choice* if $\mathrm{Dp}_{\mathfrak{t}}(L^{\mathfrak{t}}, \overline{K}) < \infty$ and $\overline{K}$ is a $\mathfrak{t}$-memory choice.

(5) If $\overline{K}$ is omitted we mean that it is the trivial $\overline{K}$, that is, $K_t = \varnothing$ for $t \in L^{\mathfrak{t}}$. We say that $\mathfrak{t}$ is smooth if the trivial $\overline{K}$ is a smooth $\mathfrak{t}$-memory choice. For $L \subseteq L^{\mathfrak{t}}$ let $\mathfrak{t} \restriction L = \langle I_t \cap \mathcal{P}(L) : t \in L \rangle$.

(6) Let $L_1 \leqslant_{\mathfrak{t}} L_2$ mean $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ and $t \in L_1 \,\&\, A \in I_t^{\mathfrak{t}} \Rightarrow A \cap L_2 \subseteq L_1$.

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[1] We can use less, but it seems not needed at the moment. We can go deeper to names of depth $\leqslant \varepsilon$ inductively on $\varepsilon < \omega_1$, as in [S7, §3], or in a more particular way to make the point that is used here true, and/or make $I_t^{\mathfrak{t}}$ only closed under unions (but not subsets), etc. Note that, e.g., $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})$ is well defined when $L^{\mathfrak{t}}$ is well ordered.

Definition 2.2. Let $\mathfrak{t}=\langle I_t : t\in L^{\mathfrak{t}}\rangle$ be an FS-iteration template and $\overline{K}$ a $\mathfrak{t}$-memory choice.

(1) We say that $\bar{L}$ is a $(\mathfrak{t},\overline{K})$-*representation of* L if (c) of Definition 2.1 (3) is valid and

(a) $L\subseteq L^{\mathfrak{t}}$ is $\overline{K}$-closed;

(b) $\bar{L}=\langle L_a : a\in M\rangle$;

(c) M is a directed partial order;

(d) $\bar{L}$ is increasing, that is, $a<_M b \Rightarrow L_a\subseteq L_b$;

(e) $L=\bigcup_{a\in M} L_a$;

(f) each L_a is $\overline{K}$-closed;

(g) if $t\in L$, $A\in I^{\mathfrak{t}}_t$, $A\subseteq L$ then $(\exists a\in M)(A\subseteq L_a)$.

(2) We say that L^* is a $(\mathfrak{t},\overline{K})$-**representation of* L if

(a) $L\subseteq L^{\mathfrak{t}}$ is $\overline{K}$-closed;

(b) $L^*\subseteq L$, L^* a singleton;

(c) if $t\in L$ and $A\in I^{\mathfrak{t}}_t$ then $A\cap L^*=\varnothing$ (so $L\backslash L^*\leqslant_{\mathfrak{t}} L$);

(d) if $t\in L^*$ then $L\backslash L^*\in I^{\mathfrak{t}}_t$.

CLAIM 2.3. *Let* $\mathfrak{t}$ *be an FSI-template and* $\overline{K}$ *a* $\mathfrak{t}$*-memory choice.*

(0) *The family of* $\overline{K}$*-closed sets is closed under (arbitrary) unions and intersections. Also if* $L\subseteq L^{\mathfrak{t}}$ *then* $L\cup\bigcup\{K_t : t\in L\}$ *is* $\overline{K}$*-closed.*

(1) *If* $L_2\subseteq L^{\mathfrak{t}}$ *is* $\overline{K}$*-closed and* L_1 *is an initial segment of* L_2*, then* L_1 *is* $\overline{K}$*-closed.*

(2) *If* $L_1\subseteq L_2\subseteq L^{\mathfrak{t}}$ *are* $\overline{K}$*-closed then*

(α) $\mathrm{Dp}_{\mathfrak{t}}(L_1,\overline{K})\leqslant\mathrm{Dp}_{\mathfrak{t}}(L_2,\overline{K})$; *moreover,*

(β) $(\exists t\in L_2)[L_1\in I^{\mathfrak{t}}_t]$ *implies that* $\mathrm{Dp}_{\mathfrak{t}}(L_1,\overline{K})<\mathrm{Dp}_{\mathfrak{t}}(L_2,\overline{K})$ *or both are* ∞.

(3) *If* $L_1\subseteq L_2\subseteq L^{\mathfrak{t}}$ *are* $\overline{K}$*-closed then* $\mathfrak{t}\restriction L_2$ *is an FSI-template,* L_1 *is* $(\mathfrak{t}\restriction L_2)$*-closed and* $\mathrm{Dp}_{\mathfrak{t}\restriction L_2}(L_1,\overline{K}\restriction L_2)=\mathrm{Dp}_{\mathfrak{t}}(L_1,\overline{K})$.

Proof. (0) and (1) are trivial—read the definitions.

(2) We prove by induction on the ordinal ζ that

$(*)_\zeta$ (α) if $\mathrm{Dp}_{\mathfrak{t}}(L_2,\overline{K})=\zeta$ (and L_1 and L_2 are $\overline{K}$-closed) then $\mathrm{Dp}_{\mathfrak{t}}(L_1,\overline{K})\leqslant\zeta$;

(β) if in addition $(\exists t\in L_2)(L_1\in I^{\mathfrak{t}}_t)$ then $\mathrm{Dp}_{\mathfrak{t}}(L_1,\overline{K})<\zeta$.

So assume $\mathrm{Dp}_{\mathfrak{t}}(L_2,\overline{K})=\zeta$, so that $\mathrm{Dp}_{\mathfrak{t}}(L_2,\overline{K})\not\geqslant\zeta+1$, and hence one of the following cases occurs.

First case: $\zeta=0$. This is trivial; note that clause (β) is empty.

Second case: ζ *is a successor*. It follows that L_2 has a $(\mathfrak{t},\overline{K})$-*representation L^* such that $\mathrm{Dp}_{\mathfrak{t}}(L_2\backslash L^*,\overline{K})<\zeta$; see Definition 2.2 (2).

Let $L_2^-:=L_2\backslash L^*$; if $L_1\subseteq L_2^-$ then by the induction hypothesis we have $\mathrm{Dp}_{\mathfrak{t}}(L_1,\overline{K})\leqslant\mathrm{Dp}_{\mathfrak{t}}(L_2^-,\overline{K})<\zeta$; so assume that $L_1\nsubseteq L_2^-$, and so only clause (α) is relevant. Now letting

$L_1^- = L_1 \setminus L^*$ we have [L_1^- and L_2^- are $\bar{K}$-closed] & $L_1^- \subseteq L_2^-$ & $\mathrm{Dp}_{\mathfrak{t}}(L_2^-, \bar{K}) < \zeta$, and hence $\mathrm{Dp}_{\mathfrak{t}}(L_1^-, \bar{K}) < \zeta$ by the induction hypothesis. Let $L_1^* = L_1 \cap L^*$, so that $L_1^* \subseteq L_1$, L_1 is $\bar{K}$-closed, $L_1 \setminus L_1^* = (L_2 \setminus L_2^*) \cap L_1$ is $\bar{K}$-closed, $\mathrm{Dp}_{\mathfrak{t}}(L_1 \setminus L_1^*, \bar{K}) = \mathrm{Dp}_{\mathfrak{t}}(L_1^-, \bar{K}) < \zeta$ and necessarily L_1^* has exactly one element. Also easily $t \in L_1^*$ implies $L_1^- \in I_t^{\mathfrak{t}}$, so that L_1^* is a $(\mathfrak{t}, \bar{K})$-*representation of L_1. So clearly $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leqslant \mathrm{Dp}_{\mathfrak{t}}(L_1^-, \bar{K}) + 1 \leqslant \zeta$.

Third case: *ζ is limit and $\langle L_a : a \in M \rangle$ is a $(\mathfrak{t}, \bar{K})$-representation of L_2 such that* $a \in M \Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_a, \bar{K}) < \zeta$.

Let $L_a^2 := L_a$ and $L_a^1 := L_a \cap L_1$, so that $\langle L_a^1 : a \in M \rangle$ is increasing, $\bigcup_{a \in M} L_a^1 = L_1$, each L_a^1 is $\bar{K}$-closed (as L_a^2 and L_1 are $\bar{K}$-closed, see part (0)), and $t \in L_1$ & $A \in I_t^{\mathfrak{t}}$ & $A \subseteq L_1 \Rightarrow (\exists a \in M)(A \subseteq L_a^2 \cap L_1 = L_a^1)$. Also by the induction hypothesis, $b \in M \Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_b^2, \bar{K}) < \zeta$. By the last two sentences (and Definition 2.1) we get $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leqslant \zeta$, as required in clause (α). For clause (β) we know that there is $t \in L_2$ such that $L_1 \in I_t^{\mathfrak{t}}$. Hence by clause (f) of Definition 2.2 (1) for some $b \in M$ we have $L_1 \subseteq L^b$, and we can use the induction hypothesis on ζ for L_1, L_b.

(3) This is easy. □

CLAIM 2.4. (1) *If for $l = 1, 2$ we have that $\bar{L}^l$ is a $(\mathfrak{t}, \bar{K})$-representation of L, $\bar{L}^l = \langle L_a^l : a \in M_l \rangle$ and $M = M_1 \times M_2$, then $\bar{L} = \langle L_a \cap L_b : (a, b) \in M \rangle$ is a $(\mathfrak{t}, \bar{K})$-representation of L.*

(2) *If L_l^* is a $(\mathfrak{t}, \bar{K})$-*representation of L for $l = 1, 2$ then $L_1^* = L_2^*$.*

Proof. (1) This is straightforward.

(2) This is easy, too. □

Discussion 2.5. (1) Our next aim is to define iteration for any $\bar{K}$-smooth FSI-template $\mathfrak{t}$; for this we define and prove the relevant things; of course, by induction on the depth. In the following Definition 2.6, in clause (A)(a), we avoid relying on [S8]; moreover the reader may consider only the case $K_t = \varnothing$, omit $\underset{\sim}{\eta}_t$ and have $\underset{\sim}{\mathbf{Q}}_{t, \bar{\varphi}'_t}$ be the dominating real forcing = Hechler forcing.

(2) We may more generally than here allow $\underset{\sim}{\eta}_t$ to be, e.g., a sequence of ordinals, and members of $\underset{\sim}{\mathbf{Q}}_{t, \varphi, \eta_t}$ be $\subseteq \mathcal{H}_{<\aleph_1}(\mathrm{Ord})$, and even K_t large but increasing L; we then need more "information" from $\underset{\sim}{\eta}_t \restriction \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}} \restriction L)$. We may change to: $\underset{\sim}{\mathbf{Q}}_t$ is a definition of nep c.c.c. forcing ([S8]) or just "Souslin c.c.c. forcing (= snep)" or just absolute enough c.c.c. forcing notion. All those cases do not make real problems (but when the parameter $\underset{\sim}{\eta}_t$ have length $\geqslant \varkappa$ it changes in the ultrapower!, i.e. $\mathbf{j}(\underset{\sim}{\eta}_t)$ has length greater than the length of $\underset{\sim}{\eta}_t$).

(3) If we restrict ourselves to σ-centered forcing notions (which is quite reasonable), maybe we can in Definition 2.1 (3)(a) omit $\boxtimes_{t,L}$ if in Definition 2.6 below in (A)(b), second case, we add that $t \in L^* \Rightarrow p \restriction (L \setminus L^*)$ forces a value to $\underset{\sim}{f}_t(p(t))$ where $\underset{\sim}{f}_t : \underset{\sim}{\mathbf{Q}}_t \to \omega$

witnessed σ-centerness and is absolute enough (or just assume that $\mathbf{Q}_t\subseteq\omega\times\mathbf{Q}'_t$ and that $f_t(p(t))$ is the first coordinate). More carefully, probably we can do this with σ-linked instead of σ-centered.

Definition/Claim 2.6. Let $\mathfrak{t}$ be an FSI-template and $\bar{K}=\langle K_t:t\in L^{\mathfrak{t}}\rangle$ be a smooth $\mathfrak{t}$-memory choice.

By induction on the ordinal ζ we shall define and prove:

(A) (Definition) For $L\subseteq L^{\mathfrak{t}}$ which is $\bar{K}$-closed of $(\mathfrak{t},\bar{K})$-depth $\leqslant\zeta$ we define:

(a) when $\bar{\mathbf{Q}}=\langle\mathbf{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}:t\in L\rangle$ is a $(\mathfrak{t},\bar{K})$-iteration of def-c.c.c. forcing notions, but we can let $\underset{\sim}{\eta}_t$ code $\bar{\varphi}_t$ so that we may omit $\bar{\varphi}_t$;

(b) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ for $\bar{\mathbf{Q}}$ as in (A)(a).

(B) (Claim) For $L_1\subseteq L_2\subseteq L^{\mathfrak{t}}$ which are $\bar{K}$-closed of $(\mathfrak{t},\bar{K})$-depth $\leqslant\zeta$ and a $(\mathfrak{t},\bar{K})$-iteration of def-c.c.c. forcing notions $\bar{\mathbf{Q}}=\langle\mathbf{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}:t\in L_2\rangle$ we prove:

(a) $\bar{\mathbf{Q}}\restriction L_1$ is a $(\mathfrak{t},\bar{K}\restriction L_1)$-iteration of def-c.c.c. forcing notions.

(b) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)\subseteq\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ as quasi-orders.

(c) If $L_1\leqslant_{\mathfrak{t}}L_2$ (see Definition 2.1 (6)) and $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$, then $p\restriction L_1\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ and $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})\vDash$ "$p\restriction L_1\leqslant p$".

(d) If $L_1\leqslant_{\mathfrak{t}}L_2$, $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ and $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)\vDash$ "$(p\restriction L_1)\leqslant q$", then $q\cup(p\restriction(L_2\setminus L_1))$ is a least upper bound of $\{p,q\}$ in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$; hence $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)\lessdot\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$.

(e) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)\lessdot\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$, i.e.(2)

(i) $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)\Rightarrow p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$;

(ii) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)\vDash p\leqslant q\Rightarrow\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})\vDash p\leqslant q$;

(iii) if $\mathcal{I}\subseteq\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ is predense in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$, then $\mathcal{I}$ is predense in $\mathrm{Lim}(\bar{\mathbf{Q}})$ (hence if $p,q\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ are incompatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ then they are incompatible in $\mathrm{Lim}(\bar{\mathbf{Q}})$).

(f) Assume that $L_0\subseteq L_2$ is $\bar{K}$-closed and $L=L_0\cap L_1$; if $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_0)$ and $q\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L)$ satisfies $(\forall r\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L))[q\leqslant r\rightarrow p$ and r are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_0)]$, then $(\forall r\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1))[q\leqslant r\rightarrow p$ and r are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_2)]$. (Explanation: this means that if q forces for $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L)}$ that $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_0)/\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L)$, then q forces for $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)}$ that $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})/\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$.)

(g) If $\langle L_a:a\in M_1\rangle$ is a $(\mathfrak{t},\bar{K})$-representation of L_1 then we have $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)=\bigcup_{a\in M_1}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L^1_a)$.

(h) If L^* is a $(\mathfrak{t},\bar{K})$-*representation of L_1, then $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ is as defined in (A)(b) of our definition below, second case, from L^*.

(i) (α) If $p_1,p_2\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ and $t\in\mathrm{Dom}(p_1)\cap\mathrm{Dom}(p_2)\Rightarrow p_1(t)=p_2(t)$, then $q=p_1\cup p_2$ (i.e. $p_1\cup(p_2\setminus\mathrm{Dom}(p_1))$) belongs to $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ and is a least upper bound of p_1,p_2;

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(2) Here we do not assume $L_1\leqslant_{\mathfrak{t}}L_2$.

(β) $p\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})$ if and only if p is a function with domain a finite subset of L_2 such that for every $t\in\mathrm{Dom}(p)$, for some $A\in I_t^{\mathfrak{t}}$, A is $\overline{K}$-closed, $K_t\subseteq A$ and $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)}$ "$p(t)\in\mathbf{Q}_{t,\underset{\sim}{\eta}_t}$" (so if $p\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})$ then for some countable $L\subseteq L_2$ we have $p\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L)$);

(γ) $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})\models p\leqslant q$ if and only if $p,q\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})$, for every $t\in\mathrm{Dom}(p)$ we have $t\in\mathrm{Dom}(q)$, and for some $\overline{K}$-closed $A\in I_t^{\mathfrak{t}}$ we have $q\restriction A\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)$ and $q\restriction A\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)}$ "$p(t)\leqslant q(t)$ in $\mathbf{Q}_{t,\underset{\sim}{\eta}_t}$ (as interpreted in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)}$ of course)".

(j) $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})$ is a c.c.c. forcing notion and $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})=\bigcup\{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L):L\in[L_2]^{\leqslant\aleph_0}\}$.

(k) $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})$ has cardinality $\leqslant|L_2|^{\aleph_0}$ (here we use the assumption that $\underset{\sim}{\eta}_t$ and members of $\underset{\sim}{\mathbf{Q}}_{t,\underset{\sim}{\eta}_t}$ are reals; see definition in (A)(a)(i) and (ii)) below).

Let us carry out the induction.

Part (A) (*Definition*). Assume $\mathrm{Dp}_{\mathfrak{t}}(L,\overline{K})\leqslant\zeta$. If $\mathrm{Dp}_{\mathfrak{t}}(L,\overline{K})<\zeta$ we have already defined $(\mathfrak{t},\overline{K})$-iteration and $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L)$, so assume $\mathrm{Dp}_{\mathfrak{t}}(L,\overline{K})=\zeta$.

Clause (A)(a). (i) $\underset{\sim}{\eta}_t$ is a $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K_t)$-name of a real (i.e. from ${}^{\omega}2$, used as a parameter) (legal as $K_t\subseteq L\ \&\ K_t\in I_t\ \&\ t\in L$, hence by Claim 2.3 (2), clause (β), we have $\mathrm{Dp}_{\mathfrak{t}}(K_t,\overline{K})<\mathrm{Dp}_{\mathfrak{t}}(K_t\cup\{t\},\overline{K})\leqslant\mathrm{Dp}_{\mathfrak{t}}(L,\overline{K})\leqslant\zeta$, and so $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L_t)$ is a well-defined forcing notion by the induction hypothesis and Claim 2.3 (2), clause (β)).

(ii) $\bar{\varphi}_t$ is a pair of formulas with the parameters $\underset{\sim}{\eta}_t$ defining in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K_t)}$ a forcing notion denoted by $\mathbf{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ whose elements are contained in $\mathcal{H}(\aleph_1)$.

(iii) In $\mathbf{V}_1=\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K_t)}$, if $\mathbf{P}'\lessdot\mathbf{P}''$ are c.c.c. forcing notions then $\mathbf{Q}=\mathbf{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ as interpreted in $\mathbf{V}_2=(\mathbf{V}_1)^{\mathbf{P}'}$ is a c.c.c. forcing notion there, and $\mathbf{P}'*\underset{\sim}{\mathbf{Q}}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$ is a $\lessdot$-subforcing of $\mathbf{P}''*\underset{\sim}{\mathbf{Q}}_{t,\bar{\varphi},\underset{\sim}{\eta}_t}$, where $\underset{\sim}{\mathbf{Q}}_{t,\bar{\varphi},\underset{\sim}{\eta}_t}$ means as interpreted in $(\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K_t)})^{\mathbf{P}'}$ or in $(\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K_t)})^{\mathbf{P}''}$ respectively (i.e. "$p\leqslant q$", "p and q incompatible" and "$\langle p_n:n<\omega\rangle$ is predense" (so the sequence is from the smaller universe) are preserved).

(iv) Assume that $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K_t)\lessdot\mathbf{P}_0\lessdot\mathbf{P}_l\lessdot\mathbf{P}_3$ are c.c.c. forcing notions for $l=1,2$, $\mathbf{P}_1\cap\mathbf{P}_2=\mathbf{P}_0$, $\mathbf{P}_3'$ is a forcing notion, $\mathbf{P}_l\lessdot\mathbf{P}_3'$ for $l=1,2$ (in fact, $\mathbf{P}_3'=\mathbf{P}_1*_{\mathbf{P}_0}\mathbf{P}_2$ is all right) and G_3*G^3 is a generic subset of $\mathbf{P}_3*\underset{\sim}{\mathbf{Q}}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$, and let $G_l*G^l=(G_3*G^3)\cap(\mathbf{P}_l*\underset{\sim}{\mathbf{Q}}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t})$. If $(p_l,\underset{\sim}{q}_l)\in G_l*G^l$ for $l=1,2$, then for some $p_l'\in\mathbf{P}_l$ satisfying $\mathbf{P}_l\models p_l\leqslant p_l'$ for $l=1,2$ we have: if $p^*\in\mathbf{P}_3$ is above p_1' and above p_2', then $p^*\Vdash_{\mathbf{P}_3}$ "$\underset{\sim}{q}_1$ and $\underset{\sim}{q}_2$ are compatible in $\underset{\sim}{\mathbf{Q}}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$".

Clause (A)(b).

First case: $\zeta=0$. This is trivial.

Second case: *ζ is a successor*. So let L^* be a $(\mathfrak{t},\overline{K})$-*representation of L. Define: $p\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})$ if and only if p is a finite function, $\mathrm{Dom}(p)\subseteq L$, $p\restriction(L\backslash L^*)\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction(L\backslash L^*))$, and if $t\in L^*\cap\mathrm{Dom}(p)$, then $p(t)$ is a $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction(L\backslash L^*))$-name of a member of $\mathbf{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t}$; and the order is $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})\vDash p\leqslant q$ if and only if

(i) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction(L\backslash L^*))\vDash$ "$(p\restriction(L\backslash L^*))\leqslant(q\restriction(L\backslash L^*))$", and

(ii) if $t\in L^*\cap\mathrm{Dom}(p)$ then $q\restriction(L\backslash L^*)\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction(L\backslash L^*))}$ "$p(t)\leqslant q(t)$".

Clearly $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ is a quasi-order. But we should prove that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ is well defined, which means that the definition does not depend on the representation. So we prove that

$\boxtimes_1$ if $\mathrm{Dp}_{\mathfrak{t}}(L,\bar{K})=\zeta$, and for $l=1,2$ we have that L_l^* is a $(\mathfrak{t},\bar{K})$-*representation of L with $\mathrm{Dp}_{\mathfrak{t}}(L\backslash L_l^*,\bar{K})<\zeta$, and $\mathbf{Q}^l$ is $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L)$ as defined by L_l^* above, then $\mathbf{Q}^1=\mathbf{Q}^2$.

This is immediate by Claim 2.4 (2) and the induction hypothesis clause (B)(h).

Third case: *ζ is limit.* There are a directed partial order M and $\bar{L}=\langle L_a:a\in M\rangle$, a $(\mathfrak{t},\bar{K})$-representation of L, such that $a\in M\Rightarrow\mathrm{Dp}_{\mathfrak{t}}(L_a,\bar{K})<\zeta$. By the induction hypothesis, $a\leqslant_M b\Rightarrow L_a\subseteq L_b$ & $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a)\subseteq\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_b)$.

We let $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L)=\bigcup_{a\in M}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a)$, so we have to prove that

$\boxtimes_2$ the choice of $\bar{L}$ is immaterial.

So we just assume that for $l=1,2$ we have that M_l is a directed partial order, $\bar{L}^l=\langle L_a^l:a\in M_l\rangle$, $L_a^l\subseteq L$, $M_l\vDash a\leqslant b\Rightarrow L_a^l\subseteq L_b^l$ and $(\forall t\in L)(\forall A\in I_t)[A\subseteq L\rightarrow(\exists a\in M_l)(A\subseteq L_a^l)$ and $\mathrm{Dp}_{\mathfrak{t}}(L_a^l,\bar{K})<\zeta]$.

We should prove that $\bigcup_{a\in M_1}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^1)$ and $\bigcup_{a\in M_2}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^2)$ are equal, as quasi-orders of course.

Now let $M:=M_1\times M_2$, with $(a_1,a_2)\leqslant(b_1,b_2)\Leftrightarrow a_1\leqslant_{M_1}b_1$ & $a_2\leqslant_{M_2}b_2$, be a directed partial order. We let $L_{(a_1,a_2)}=L_{a_1}^1\cap L_{a_2}^2$, so clearly $L_{(a_1,a_2)}\subseteq L^{\mathfrak{t}}$, $\mathrm{Dp}_{\mathfrak{t}}(L_{(a_1,a_2)},\bar{K})<\zeta$, $(a_1,a_2)\leqslant_M(b_1,b_2)\Rightarrow L_{(a_1,a_2)}\subseteq L_{(b_1,b_2)}$ and $\langle L_{(a_1,a_2)}:(a_1,a_2)\in M\rangle$ is a $(\mathfrak{t},\bar{K})$-representation of L by Claim 2.4 (1). So by transitivity of equality, it is enough to prove for $l=1,2$ that $\bigcup_{a\in M_l}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^l)$ and $\bigcup_{(a,b)\in M}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_{(a,b)})$ are equal as quasi-orders. By the symmetry in the situation, without loss of generality, $l=1$.

Now for every $a\in M_1$, $\bar{L}=\langle L_{(a,b)}:b\in M_2\rangle$ satisfies: $L_a^1\subseteq L$, $\mathrm{Dp}(L_a^1)<\zeta$, $L_{(a,b)}\subseteq L_a^1$, $L_a^1=\bigcup_{b\in M_2}L_{(a,b)}$ and $b_1\leqslant_{M_2}b_2\Rightarrow L_{(a,b_1)}\subseteq L_{(a,b_2)}$. Also we know that $(\forall t\in L)(\forall A\in I_t^{\mathfrak{t}})$ $(\exists b\in M_2)(A\subseteq L\rightarrow A\subseteq L_b^2)$, and hence $(\forall t\in L_a^1)(\forall A\in I_t^{\mathfrak{t}})(A\subseteq L_a^1\rightarrow(\exists b\in M_2)(A\subseteq L_{(a,b)}))$. Hence by the induction hypothesis for clause (B)(g) we have that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^1)$ and $\bigcup_{b\in M_2}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_{(a,b)})$ are equal as quasi-orders. As this holds for every $a\in M_1$ and M_1 is directed, we get that $\bigcup_{a\in M_1}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^1)$ and $\bigcup_{a\in M_1}\bigcup_{b\in M_2}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_{(a,b)})$ are equal as quasi-orders. But the second is equal to $\bigcup_{(a,b)\in M}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_{(a,b)})$, and so we are done.

Part (B).

First case: $\zeta=0$. This is trivial.

Second case: *ζ is a successor.* Similar to usual iteration, and easy using the definition and the induction hypothesis, except for clause (f), which we prove in detail.

Clause (f). Let p, q, L and L_0 be as in the assumption of clause (f). Let $r\in \mathrm{Lim}_{\mathfrak{t}}(\mathbf{Q}\restriction L_1)$ be above q there (and we should prove that p and r are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_2)$). Let t be the maximal member of L_2, and set $L_l^-:=L_l\setminus t$ and $L^-:=L\setminus t$. If $(t\notin L_0 \vee t\notin L_1)$ or just $t\notin \mathrm{Dom}(p)\cap\mathrm{Dom}(r)$, then by the induction hypothesis applied to L_1^-, L_2^-, L^-, L_0^-, $p\restriction L_0^-$, $q\restriction L^-$ and $r\restriction L_2^-$ we can find a common upper bound r^* of $p\restriction L_0^-$ and $r\restriction L_1^-$ in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_2^-)$, and $r^*\cup p\restriction\{t\}\cup r\restriction\{t\}$ is a common upper bound of p and r as required.

So assume that $t\in\mathrm{Dom}(p)\cap\mathrm{Dom}(r)\subseteq L_0\cap L_1$, and let $\mathbf{P}:=\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L^-)$ and $\mathbf{P}_l:=\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_l^-)$ for $l=0,1,2$. Let $\mathbf{P}_2'$ be $\mathbf{P}_0 *_{\mathbf{P}} \mathbf{P}_1$, and let $G_2 * G^2$ be a generic subset of $\mathbf{P}'_2 * \underset{\sim}{\mathbf{Q}}_{t,\bar{\varphi}_t,\eta_t}$ to which p and r belong. Now we get p' and r' by applying clause (A)(a)(iv) for t with

$$\mathbf{P},\ \mathbf{P}_0,\ \mathbf{P}_1,\ \mathbf{P}_2,\ \mathbf{P}_2',\ (p\restriction L_0^-, p(t)),\ (r\restriction L_1^-, r(t)),\ p_1' \text{ and } p_2'$$

here standing for

$$\mathbf{P}_0,\ \mathbf{P}_1,\ \mathbf{P}_2,\ \mathbf{P}_3,\ \mathbf{P}_3',\ \underset{\sim}{q}_1,\ \underset{\sim}{q}_2,\ p' \text{ and } r'$$

there, respectively.

By the induction hypothesis in $\mathbf{P}_2$ for the conditions p' and r' we can find a common upper bound p^*. So clearly $p^*\Vdash_{\mathbf{P}_2}$ "$p(t)$ and $q(t)$ are compatible inside $\underset{\sim}{\mathbf{Q}}_{t,\bar{\varphi}_t,\eta_t}$", and we can finish.

Third case: *ζ is limit*. Let $\langle L_a^2 : a\in M\rangle$ be a $(\mathfrak{t},\bar{K})$-representation of L_2 with $a\in M\Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_a,\bar{K})<\zeta$, and let $L_a^1=L_1\cap L_a$.

Clause (B)(a). This is trivial.

Clause (B)(b). Clearly $\mathrm{Dp}_{\mathfrak{t}}(L_1,\bar{K})\leqslant\zeta$ by Claim 2.3 (2)(α). Hence $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ is well defined by (A)(b), which we have already above, i.e. $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})=\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_2)=\bigcup_{a\in M_2}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^2)$ as quasi-orders.

Clearly $\langle L_a^1=L_1\cap L_a^2 : a\in M\rangle$ is a $(\mathfrak{t},\bar{K})$-representation of L_1. Hence by the induction hypothesis (if $\mathrm{Dp}_{\mathfrak{t}}(L_1,\bar{K})<\zeta$) or by the uniqueness proved in (A)(b) (if $\mathrm{Dp}_{\mathfrak{t}}(L_1,\bar{K})=\zeta$), we know that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)=\bigcup_{a\in M}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^1)$ as quasi-orders, and by the induction hypothesis for (B)(b) we know that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^1)\subseteq\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^2)$ as quasi-orders (for $a\in M$), and we can easily finish.

Clause (B)(c), (d). Use the proof of clause (B)(b) noting that $L_a^1\leqslant_{\mathfrak{t}} L_a^2$, and so we can use the induction hypothesis (i.e. if $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_2)$, as M is directed there is $a\in M$ such that $\mathrm{Dom}(p)\subseteq L_a^2$; now $a\leqslant_M b\Rightarrow p\restriction L_b^1=p\restriction L_a^1$, and we can finish easily).

Clause (B)(e). The statements (i) and (ii) hold by clause (b). The statement (iii) holds: let $\mathcal{I}$ be a predense subset of $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ and let $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$, so that for some

$a\in M$ we have $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^2)$. By the induction hypothesis, applying clause (B)(e) to L_a^1 and L_a^2, we have $\mathrm{Lim}_{\mathfrak{t}}(\mathbf{Q}\restriction L_a^1)\lessdot\mathrm{Lim}_{\mathfrak{t}}(\mathbf{Q}\restriction L_a^2)$. Hence as $p\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^1)$ clearly there is $q\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_a^1)$ such that p is compatible with r in $\mathrm{Lim}_{\mathfrak{t}}(\mathbf{Q}\restriction L_a^2)$ whenever $\mathrm{Lim}_{\mathfrak{t}}(\mathbf{Q}\restriction L_a^1)\vDash$ "$q\leqslant r$". Now by the assumption on "$\mathcal{I}\subseteq\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ is dense", as $q\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ (by clause (B)(b)) we can find $q_0\in\mathcal{I}$ and q_1 such that $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)\vDash q_0\leqslant q_1\ \&\ q\leqslant q_1$, and so for some $b\in M$ we have $q,q_0,q_1\in L_b^1$ and $a\leqslant_M b$ (as M is directed). Now we consider $p,q,L_a^1,L_a^2,L_b^1,L_b^2$ and apply clause (B)(f).

Clause (B)(f). This is easy to check using clause (f) for the L_a^2's, which holds by the induction hypothesis.

Clause (B)(g). Let $M_2:=M$ (and recall M_1 that is from clause (B)(g)). For each $a_1\in M_1$, clearly $\mathrm{Dp}_{\mathfrak{t}}(L_a,\bar{K})\leqslant\zeta$ as $L_a\subseteq L_2$, $\langle L_{a_1}\cap L_{a_2}^2:a_2\in M_2\rangle$ is a $(\mathfrak{t},\bar{K})$-representation of L_a and $\mathrm{Dp}_{\mathfrak{t}}(L_{a_1}\cap L_{a_2}^2,\bar{K})\leqslant\mathrm{Dp}_{\mathfrak{t}}(L_{a_2}^2,\bar{K})<\zeta$. Hence by (A)(b) we know that

$$\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_{a_1})=\bigcup_{a_2\in M_2}\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction(L_{a_1}\cap L_{a_2}^2)).$$

The rest should be clear.

Clause (B)(h). This is easy.

Clause (B)(i). This is easy.

Clause (B)(j). So let $p_\alpha\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ for $\alpha<\omega_1$; let $w_\alpha=\mathrm{Dom}(p_\alpha)$, and without loss of generality assume that $\langle w_\alpha:\alpha<\omega_1\rangle$ is a Δ-system with heart w. So for some $a\in M$ we have $w\subseteq L_a^2$. For each α, for some $a_\alpha\in M$, we have $a\leqslant_M a_\alpha$ and $p_\alpha\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_{a_\alpha}^2)$.

We can choose a countable set $A_{\alpha,t}\in I_t^{\mathfrak{t}}$ such that $A_{\alpha,t}\subseteq L_{a_\sigma}$ for $t\in w_\alpha$ and $p_\alpha(t)$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_{a_\alpha})$-name. So by clause (c) of Definition 2.1 there are an unbounded $S\subseteq\omega_1$ and $b\in M$ such that $\langle\bigcup\{A_{\alpha,t}\cup\{t\}\setminus L_b:t\in w_\alpha\}:\alpha\in S\rangle$ is a sequence of pairwise disjoint sets. Without loss of generality, $\alpha\in S\Rightarrow b\subseteq a_\alpha$. By clause (B)(e), for each $\alpha\in S$ there is $p_\alpha^*\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_b)$ such that if $p_\alpha^*\leqslant q\in\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_b)$ then p_α and q are compatible. By clause (B)(j) of the induction hypothesis, for some $\alpha<\beta$ from S the conditions p_α^* and p_β^* are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_b)$. Let q exemplify this, and by clause (B)(f) we can finish easily.

Clause (k). This is easy. □

CLAIM 2.7. (1) *Assume that*

(a) $\mathfrak{t}$ *is an FSI-template,* $\mathrm{Dp}_{\mathfrak{t}}(L,\bar{K})<\infty$, *i.e.* $\bar{K}$ *is a smooth* $\mathfrak{t}$*-memory choice;*

(b) $\bar{\mathbf{Q}}=\langle\mathbf{Q}_{t,\eta_t}:t\in L\rangle$ *is a* $(\mathfrak{t},\bar{K})$*-iteration of def-c.c.c. forcing notions;*

(c_1) $L_1,L_2\subseteq L$, $L_1<L_2$ (*i.e.* $(\forall t_1\in L_1)(\forall t_2\in L_2)(L^{\mathfrak{t}}\vDash t_1<t_2)$) *and* $t\in L_2\Rightarrow L_1\in I_t^{\mathfrak{t}}$, *or at least* $t\in L_2\ \&\ L'\subseteq L_1\ \&\ |L'|\leqslant\aleph_0\Rightarrow L'\in I_t^{\mathfrak{t}}$ *and* $L=L_1\cup L_2$.

Then

(α) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ *is actually a definition of a forcing (in fact a c.c.c. one), and so meaningful in bigger universes; moreover, for extensions* $\mathbf{V}_1 \subseteq \mathbf{V}_2$ *of* $\mathbf{V}=\mathbf{V}_0$ *(with the same ordinals of course), we get* $[\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})]^{\mathbf{V}_1} \subseteq [\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})]^{\mathbf{V}_2}$,(3) *and every maximal antichain* $\mathcal{I}$ *of* $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ *from* $\mathbf{V}_1$ *is a maximal antichain of* $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ *(in* $\mathbf{V}_2$*);*

(β) $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ *is in fact the composition* $\mathbf{Q}_1 * \underset{\sim}{\mathbf{Q}}_2$*, where* $\mathbf{Q}_1 = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_1)$ *and* $\mathbf{Q}_2 = [\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_2)]^{\mathbf{V}[\underset{\sim}{G}_{\mathbf{Q}_1}]}$.

(2) *Assume clauses* (a) *and* (b) *of part* (1), *and*

(c_2) *suppose that* L *has a last element* t^* *and let* $L^- = L\setminus\{t^*\}$.

Then for any $G^- \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L^-)$ *generic over* $\mathbf{V}$*, letting* $\eta_{t^*} = \underset{\sim}{\eta}_{t^*}[G^-]$ *in* $\mathbf{V}[G^-]$ *we have: the forcing notion* $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})/G^-$ *is equivalent to* $\bigcup\{\mathbf{Q}_{t^*,\eta_{t^*}}^{\mathbf{V}[G_A^-]} : A \in I_{t^*}^{\mathfrak{t}}$ *is* $\bar{K}$*-closed*$\}$*, where* $G_A^- := G^- \cap \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction A)$ *and* $\eta_{t_1^*} = \underset{\sim}{\eta}_{t^*}[G^-]$.

(3) *Assume clauses* (a) *and* (b) *of part* (1), *and*

(c_3) *suppose that* $\langle L_i : i<\delta\rangle$ *is an increasing continuous sequence of initial segments of* L *with union* L*, and that* δ *is a limit ordinal.*

Then $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ *is* $\bigcup_{i<\delta} \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_i)$*; moreover,* $\langle \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction L_i) : i<\delta\rangle$ *is* $\lessdot$*-increasing continuous.*

(4) *Assume that* $\mathfrak{t}^1$ *and* $\mathfrak{t}^2$ *are FSI-templates, that* $L^{\mathfrak{t}^1} = L^{\mathfrak{t}^2}$*, call it* L*, that for every* $t \in L$*,* $I_t^{\mathfrak{t}^1} \cap [L]^{\leqslant\aleph_0} = I_t^{\mathfrak{t}^2} \cap [L]^{\leqslant\aleph_0}$*, that* $\bar{K}$ *is a smooth* $\mathfrak{t}^l$*-memory choice, and that* $\bar{\mathbf{Q}} = \langle \mathbf{Q}_{t,\bar{\varphi}_t,\eta_t} : t\in L\rangle$ *is a* $(\mathfrak{t}^l, \bar{K})$*-iteration of def-c.c.c. forcing notions for* $l=1,2$*. Then* $\mathrm{Lim}_{\mathfrak{t}^1}(\bar{\mathbf{Q}}) = \mathrm{Lim}_{\mathfrak{t}^2}(\bar{\mathbf{Q}})$.

Proof. This is straightforward (or read [S8]). □

We now give sufficient conditions for: "if we force by $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$ from Definition 2.6, then some cardinal invariants are small or equal/bigger than some μ". The necessity of such a claim in our framework is obvious; we deal with two-place relations only as this is the case in the popular cardinal invariants, in particular those we deal with.

CLAIM 2.8. *Assume that* $\mathfrak{t}$ *is a smooth FSI-template, that* $\bar{K} = \langle K_t : t\in L^{\mathfrak{t}}\rangle$ *and* $\bar{\mathbf{Q}} = \langle \underset{\sim}{\mathbf{Q}}_{t,\eta_t} : t \in L^{\mathfrak{t}}\rangle$ *are as in Definition* 2.6 *and that* $\mathbf{P} = \mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$.

(1) *Assume that*

(a) R *is a Borel*(4) *two-place relation on* ${}^{\omega}\omega$ *(we shall use* $<^*$*);*

(b) $L^* \subseteq L^{\mathfrak{t}}$;

(c) *for every countable* $\bar{K}$*-closed* $A \subseteq L^{\mathfrak{t}}$*, for some* $t\in L^*$*, we have* $A \in I_t^{\mathfrak{t}}$;

(d) *for* $t \in L^*$ *and* $\bar{K}$*-closed* $A \in I_t^{\mathfrak{t}}$ *which includes* K_t*, in* $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction A)}$ *we have* $\Vdash_{\underset{\sim}{\mathbf{Q}}_{t,\eta_t}}$ "$\underset{\sim}{\nu}_t \in {}^{\omega}\omega$ *is an* R*-cover of the old reals, i.e.* $\varrho \in ({}^{\omega}\omega)^{\mathbf{V}[\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction A)]} \Rightarrow \varrho R \underset{\sim}{\nu}_t$"*, where* $\underset{\sim}{\nu}_t$

(3) Of course possibly $L_1 = \varnothing$.

(4) Here and below just enough absoluteness is enough, of course.

is a name in the forcing $\underset{\sim}{\mathbf{Q}}_{t,\eta_t}$, *i.e. all this is in* $(\mathbf{Q}_{t,\eta_t[G]})^{\mathbf{V}[G]}$, $\underset{\sim}{G}$ *the generic subset of* $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)$, *not depending on* A *(usually* $\underset{\sim}{\nu}_t$ *is the generic real of* $\underset{\sim}{\mathbf{Q}}_{t,\eta_t}$, *hence* $\underset{\sim}{\mathbf{Q}}_{t,\eta_t}$ *is interpreted in the universe* $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)}$, *and so* $\underset{\sim}{\eta}_t$ *is determined by the generic; normally we assume this absolutely).*

Then $\Vdash_{\mathbf{P}}$ "$(\forall\varrho\in{}^{\omega}\omega)(\exists t\in L^*)(\varrho R\underset{\sim}{\nu}_t)$, *i.e.* $\{\underset{\sim}{\nu}_t: t\in L^*\}$ *is an* R*-cover, which, if* $R={<^*}$, *means* $\mathfrak{d}\leqslant|L^*|$".

(1A) *If we weaken assumption* (d) *to "for some* $\underset{\sim}{\nu}_t$ *a* $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K'_t)$*-name", where* $K'_t=K_t$, *or we use* $\overline{K}'=\langle K'_t: t\in L^*\rangle$, $K'_t\subseteq L^{\mathfrak{t}}_{\leqslant t}$, *then we get* $\Vdash_{\mathbf{P}}$ "$\{\underset{\sim}{\nu}_t: t\in L^*\}$ *is an* R*-cover". If we weaken assumption* (d) *to* $\Vdash_{\mathbf{P}}$ "*for every* $\varrho\in{}^{\omega}\omega$, *for some* $t\in L^{\mathfrak{t}}$ *and* $\nu\in \mathbf{V}({}^{\omega}\omega)^{\mathbf{V}[\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K'_t)]}$, *we have* $\varrho R\nu$", *then*

$$\Vdash_{\mathbf{P}} \text{“}(\forall\varrho\in{}^{\omega}\omega)(\exists t\in L^*)(\exists\nu\in\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K'_t)})[\varrho R\nu]\text{”}.$$

This implies that in $\mathbf{V}^{\mathbf{P}}$, *if* $R={<^*}$ *then* $\mathfrak{d}\leqslant\sum_{t\in L^{\mathfrak{t}}}|K'_t|$; *we could use a sequence* $\overline{K}'$ *indexed by other sets.*

(2) *Assume that*

(a) R *is a Borel two-place relation on* ${}^{\omega}\omega$ *(we shall use* $<^*$*);*

(b) μ *is a cardinality;*

(c) *if* $L^*\subseteq L^{\mathfrak{t}}$ *and* $|L^*|<\mu$, *then for some* $t\in L^{\mathfrak{t}}$ *and* $\overline{K}$*-closed* $L^{**}\supseteq L^*$ *we have* $L^{**}\in I^{\mathfrak{t}}_t$, *and in* $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L^{**})}$, $\Vdash_{\mathbf{Q}_{t,\eta_t}}$ "*some* $\nu\in{}^{\omega}\omega$ *is an* R*-cover of the old reals" (usually* $\underset{\sim}{\nu}$ *is the generic real of* $\mathbf{Q}_{t,\underset{\sim}{\nu}_1}$; *this we assume absolutely).*

Then $\Vdash_{\mathbf{P}}$ "$(\forall X\in[{}^{\omega}\omega]^{<\mu})(\exists\nu\in{}^{\omega}\omega)(\bigwedge_{\varrho\in X}\varrho R\nu)$" *(so for* $R={<^*}$ *this means* $\mathfrak{b}\geqslant\mu$*).*

(3) *Assume that*

(a) R *is a Borel two-place relation*(5) *on* ${}^{\omega}\omega$ *(we use* $R=\{(\varrho,\nu):\varrho,\nu\in{}^{\omega}2$ *and* $\varrho^{-1}\{1\}$, $\nu^{-1}\{1\}$ *are infinite with finite intersection*$\}$*);*

(b) $\varkappa$ *and* θ *are cardinals, and* $\varkappa\leqslant\theta\leqslant\lambda$;

(c) *if* $t_{i,n}\in L^{\mathfrak{t}}$ *for* $i<i(*)$, $n<\omega$, $\varkappa\leqslant i(*)<\theta$, *and each* $\{t_{i,n}: n<\omega\}$ *is* $\overline{K}$*-closed, then we can find* $t_n\in L^{\mathfrak{t}}$ *for* $n<\omega$ *such that* $\{t_n: n<\omega\}\subseteq L^{\mathfrak{t}}$ *is* $\overline{K}$*-closed and*

$(*)$ *for every* $i<i(*)$, *for some* $j<\varkappa$, $j\neq i$, *and the mapping* $t_{i,n}\mapsto t_{i,n}$, $t_{j,n}\mapsto t_n$ *is a partial isomorphism of* $(\mathfrak{t},\overline{\mathbf{Q}})$ *(see Definition* 2.9 *below).*

Then in $\mathbf{V}^{\mathbf{P}}$ *we have:*

$\boxtimes^R_{\theta,\varkappa}$ *if* $\varrho_i,\nu_i\in{}^{\omega}\omega$ *for* $i<i(*)$, $\varkappa\leqslant i(*)<\theta$ *and* $i\neq j\Rightarrow\nu_i R\varrho_j$, *then we can find* $\varrho\in{}^{\omega}\omega$ *such that* $i<i(*)\Rightarrow\nu_i R\varrho$.

Proof. This is straightforward, but being requested we give details:

(1) Let $\underset{\sim}{\varrho}$ be a $\mathbf{P}$-name of a member of $({}^{\omega}\omega)^{\mathbf{V}^{\mathbf{P}}}$, so that as $\mathbf{P}$ satisfies the c.c.c. (see Definition 2.6 (B)(j)), for each n there is a maximal antichain $\{p_{n,i}: i<i_n\}$ such

(5) So R is defined in $\mathbf{V}$; if R is from $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K)}$ we need a partial isomorphism (see below) of $(\mathfrak{t},\overline{\mathbf{Q}})$ extending id_K.

that $p_{n,i}$ forces a value of $\underset{\sim}{\varrho}(n)$ and, of course, i_n is countable. Let $M=\{a: a$ is a countable $\overline{K}$-closed subset of $L^{\mathfrak{t}}\}$. Then obviously M is closed under countable unions and $\bigcup\{a: a\in M\}=L^{\mathfrak{t}}$. Let $L_a=a$ for $a\in M$, so that by Definition 2.6 (B)(i)(β) we have $p\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})\Leftrightarrow\bigcup\{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L_a): a\in M\}$ but $\mathbf{P}=\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})$. Hence for $n<\omega$ and $i<i_n$, for some $a_{n,i}\in M$ we have $p_{n,i}\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L_a)$. But M is $\aleph_1$-directed, so for some $a\in M$ we have $\{a_{n,i}: n<\omega$ and $i<i_n\}\subseteq\{c: c<_M a\}$. Also by Definition 2.6 (B)(e) we know that $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L_a)\lessdot\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})=\mathbf{P}$, so $\underset{\sim}{\varrho}$ is a $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L_a)$-name. Now by assumption (c) of what we are proving, as $L_a\subseteq L$ is countable, we can find $t\in L^*\subseteq L^{\mathfrak{t}}$ such that $L_a\in I^{\mathfrak{t}}_t$. Also we know that $K_t\in I^{\mathfrak{t}}_t$ (see Definition 2.1 (2)(c)), hence $A:=K_t\cup L_a$ belongs to $I^{\mathfrak{t}}_t$ and is $\overline{K}$-closed; and easily also $B=A\cup\{t\}$ is $\overline{K}$-closed.

Then $A\subseteq B\subseteq L^{\mathfrak{t}}$ are $\overline{K}$-closed, so as above $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)\lessdot\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction B)\lessdot\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}})=\mathbf{P}$ and $\underset{\sim}{\varrho}$ is a $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)$-name (hence also a $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction B)$-name) of a member of ${}^{\omega}\omega$.

Now by assumption (d), in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)}$ we have $\Vdash_{\mathbf{Q}_{t,\eta_t}}$ "$\underset{\sim}{\varrho}R\underset{\sim}{\nu}_t$", and therefore by Claim 2.7 (2) we know that $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction B)=\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction A)*\underset{\sim}{\mathbf{Q}}_{t,\eta_t}$. So together $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(B)}$ "$\underset{\sim}{\varrho}R\underset{\sim}{\nu}_t$", and hence by the previous sentence and obvious absoluteness we have $\Vdash_{\mathbf{P}}$ "$\underset{\sim}{\varrho}R\underset{\sim}{\nu}_t$". So as $\underset{\sim}{\varrho}$ was any $\mathbf{P}$-name of a member of $({}^{\omega}\omega)^{\mathbf{V}^{\mathbf{P}}}$, we are done.

(1A) The proof is the same as above.

(2) Assume $p\Vdash_{\mathbf{P}}$ "$\underset{\sim}{X}\subseteq{}^{\omega}\omega$ has cardinality $<\mu$". As we can increase p without loss of generality, for some $\theta<\mu$ we have $p\Vdash_{\mathbf{P}}$ "$|\underset{\sim}{X}|=\theta$", so we can find a sequence $\langle\underset{\sim}{\varrho}_\alpha:\alpha<\theta\rangle$ of $\mathbf{P}$-names of members of $({}^{\omega}\omega)^{\mathbf{V}^{\mathbf{P}}}$ such that $p\Vdash_{\mathbf{P}}$ "$\underset{\sim}{X}=\{\underset{\sim}{\varrho}_\alpha:\alpha<\theta\}$". Let $\{p_{\alpha,n,i}: i<i_{\alpha,n}\}$ be a maximal antichain of $\mathbf{P}$, with $p_{\alpha,n,i}$ forcing a value to $\underset{\sim}{\varrho}_\alpha(n)$ and $i_{\alpha,n}$ countable.

Define $M=\{a\subseteq L^{\mathfrak{t}}: a$ is countable and $\overline{K}$-closed$\}$, so that for each $\alpha<\theta$, $n<\omega$, $i<i_{\alpha,n}$, for some $a_{\alpha,n,i}\in M$ we have $p_{\alpha,n,i}\in\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L_{a_{\alpha,n,i}})$. Then for some $\overline{K}$-closed $L^{**}\subseteq L^{\mathfrak{t}}$ and $t\in L^{\mathfrak{t}}$ we have $L^{**}\in I^{\mathfrak{t}}_t$ and $a_{\alpha,n,i}\subseteq L^{**}$ for $\alpha<\theta$, $n<\omega$ and $i<i_{\alpha,n}$. We now continue as in part (1).

(3) Assume $i(*)\in[\varkappa,\theta)$ and $\Vdash_{\mathbf{P}}$ "$\underset{\sim}{\nu}_i,\underset{\sim}{\varrho}_i\in{}^{\omega}\omega$ and $i\neq j\Rightarrow\underset{\sim}{\nu}_i R\underset{\sim}{\varrho}_j$". So as above we can find a countable $\overline{K}$-closed $K^*_i\subseteq L^{\mathfrak{t}}$ such that $\underset{\sim}{\nu}_i$ and $\underset{\sim}{\varrho}_i$ are $\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction K^*_i)$-names; without loss of generality, $K^*_i\neq\varnothing$ and even $|K^*_i|=\aleph_0$; this is impossible only if $L^{\mathfrak{t}}$ is finite, and then all is trivial. Let $\langle t_{i,n}: n<\omega\rangle$ be a list of the members of K^*_i with no repetitions. Let $f_{i,j}$ be the mapping from K^*_j to K^*_i defined by $f_{i,j}(t_{j,n})=t_{i,n}$.

We define the two-place relations E_1 and E_2 on $i(*)$ and on $i(*)\times i(*)$ respectively by

- iE_1j if and only if $f_{i,j}$ is a partial isomorphism of $(\mathfrak{t},\overline{\mathbf{Q}})$ such that $\hat{f}_{i,j}$ (see claim (B) of Definition 2.9 below) maps $(\underset{\sim}{\varrho}_j,\underset{\sim}{\nu}_j)$ to $(\underset{\sim}{\varrho}_i,\underset{\sim}{\nu}_i)$;
- $(i_1,i_2)E_2(j_1,j_2)$ if and only if $i_1E_1j_1$, $i_2E_2j_2$ and $f_{i_1,j_1}\cup f_{i_2,j_2}$ is a partial isomorphism of $(\mathfrak{t},\overline{\mathbf{Q}})$.

We easily have that

$\otimes$ (i) E_1 and E_2 are equivalence relations over their domain;

(ii) $f_{j,i}=f_{i,j}^{-1}$.

As $|i(*)/E_1|<\mathrm{cf}(\varkappa)$ (by clause (c) of the assumption) and we can replace $i(*)$ by $i(*)+\varkappa$, without loss of generality, $i<\varkappa \Rightarrow 0 E_1 i$. Now we apply assumption (c), and get $\langle t_n : n<\omega\rangle$. By $(*)$ of clause (c) and clause (A)(b) of Definition 2.9 below, for any i and j clearly $K_i^* \cup K_j^*$ and $K_i^* \cup \{t_n : n<\omega\}$ are $\bar{K}$-closed (see the definition below). For any $i<i(*)$ let $j_i<\varkappa$ be as in $(*)$ of clause (c), which means that $j_i \neq i$ and the following mapping g_i is a partial isomorphism of $(\mathfrak{t}, \bar{\mathbf{Q}})$: $\mathrm{Dom}(g_i)=\{t_{i,n}, t_{j_i,n} : n<\omega\}$, $g_i(t_{i,n})=t_{i,n}$ and $g_i(t_{j,n})=t_n$.

Let $\underset{\sim}{\nu}$ and $\underset{\sim}{\varrho}$ be $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}} \restriction K^*)$-names such that for some, equivalently any, i, $\hat{g}_i$ maps $\underset{\sim}{\nu}_{j_i}, \underset{\sim}{\varrho}_{j_i}$ to $\underset{\sim}{\nu}, \underset{\sim}{\varrho}$, respectively (this is all right as for any i_1 and i_2 we have $j_{i_1} E_1 j_{i_2}$ because $j_{i_1} E_1 j_{i_2}$ and hence $g_{i_2} \circ f_{j_{i_2}}, j_{i_1} = g_{i_1} \restriction K^*_{j_{i_1}}$). Now for any $i<i(*)$, as $j_i \neq i$, we know that $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}} \restriction (K_i^* \cup K_{j_i}^*))}$ "$\underset{\sim}{\nu}_i R \underset{\sim}{\varrho}_{j_i}$", so applying g_i we have $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(K_i^* \cup K^*)}$ "$\underset{\sim}{\nu}_i R \underset{\sim}{\varrho}$". Thus we have proved $\boxtimes^R_{\theta,\varkappa}$ and Claim 2.8. □

In Definition 2.9 below we note that isomorphisms (or embeddings) of $\mathfrak{t}$'s tend to induce isomorphisms (or embeddings) of $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}})$, and we deal (in Definitions 2.10 and 2.11) with some natural operation. In Definition 2.9 we could use two $\mathfrak{t}$'s, but this can trivially be reduced to one.

Definition/Claim 2.9. Assume that $\mathfrak{t}$, $\bar{K}$ and $\bar{\mathbf{Q}}=\langle \underset{\sim}{\mathbf{Q}}_{t,\eta_t} : t\in L^{\mathfrak{t}}\rangle$ are as in Definition 2.6. By induction on ζ we define and prove the following:(6)

(A) (Definition) We say that f is a *partial isomorphism* of $(\mathfrak{t}, \bar{\mathbf{Q}})$ of depth $\leqslant\zeta$ if (omitting ζ means for some ordinal ζ; writing $\mathfrak{t}$ instead of $(\mathfrak{t}, \bar{\mathbf{Q}})$ means that we assume $\underset{\sim}{\mathbf{Q}}_{t,\eta_t}=\mathbf{Q}$, i.e. constant, $K_t=\varnothing$ for every $t\in L^{\mathfrak{t}}$, and that we may say "$\mathfrak{t}$-partial isomorphism")

(a) f is a partial one-to-one function from $L^{\mathfrak{t}}$ to $L^{\mathfrak{t}}$;

(b) $\mathrm{Dom}(f)$ and $\mathrm{Rang}(f)$ are $(\mathfrak{t}, \bar{K})$-closed sets of depth $\leqslant\zeta$;

(c) for $t\in\mathrm{Dom}(f)$ and $A\subseteq\mathrm{Dom}(f)$ we have $A\in I_t^{\mathfrak{t}} \Leftrightarrow f''(A)\in I^{\mathfrak{t}}_{f(t)}$;

(d) for $t\in\mathrm{Dom}(f)$, we have that f maps K_t onto $K_{f(t)}$, and $f\restriction K_t$ maps $\underset{\sim}{\eta}_t$ to $\underset{\sim}{\eta}_{f(t)}$; more exactly, the isomorphism $\hat{f}$ which f induces from $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction K_t)$ onto $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction K_{f(t)})$ does this.

(B) (Claim) (a) f induces naturally an isomorphism, which we call $\hat{f}$, from $\mathrm{Lim}(\bar{\mathbf{Q}}\restriction\mathrm{Dom}(f))$ onto $\mathrm{Lim}_{\mathfrak{t}}(\bar{\mathbf{Q}}\restriction\mathrm{Rang}(f))$.

Proof. This is straightforward. □

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(6) If $K_t=\varnothing$ and all $\underset{\sim}{\mathbf{Q}}_{t,\eta}$ have the same definition of forcing notion as in our main case, we can separate the definition and claim.

Definition 2.10. (1) We say that $\mathfrak{t}=\mathfrak{t}^1+\mathfrak{t}^2$ if

(a) $L^{\mathfrak{t}}=L^{\mathfrak{t}}+L^{\mathfrak{t}^2}$ (as linear orders);

(b) for $t\in L^{\mathfrak{t}^1}$, $I_t^{\mathfrak{t}^1}=I_t^{\mathfrak{t}}$;

(c) for $t\in L^{\mathfrak{t}^2}$, $I_t^{\mathfrak{t}^2}=\{A\subseteq L^{\mathfrak{t}}:A\cap L^{\mathfrak{t}^2}\in I_t^{\mathfrak{t}^2}\}$.

So $\mathfrak{t}^1+\mathfrak{t}^2$ is well defined if $\mathfrak{t}^1$ and $\mathfrak{t}^2$ are disjoint, i.e. $L^{\mathfrak{t}^1}\cap L^{\mathfrak{t}^2}=\varnothing$.

(2) We say that $\mathfrak{t}^1\leqslant_{\mathrm{wk}}\mathfrak{t}^2$ if and only if

(a) $L^{\mathfrak{t}^1}\subseteq L^{\mathfrak{t}^2}$ (as linear orders) and $t\in L^{\mathfrak{t}^1}\Rightarrow I_t^{\mathfrak{t}^1}\subseteq I_t^{\mathfrak{t}^2}$;

(b) for every countable(7) $A\subseteq L^{\mathfrak{t}^1}$ and $t\in L^{\mathfrak{t}^1}$ we have $A\in I_t^{\mathfrak{t}^1}\Leftrightarrow A\in I_t^{\mathfrak{t}^2}$.

(3) If $\langle\mathfrak{t}^\zeta:\zeta<\xi\rangle$ is $\leqslant_{\mathrm{wk}}$-increasing, with ξ a limit ordinal, we define $\mathfrak{t}^\xi:=\bigcup_{\zeta<\xi}\mathfrak{t}^\zeta$ by

$$L^{\mathfrak{t}^\xi}=\bigcup_{\zeta<\xi}L^{\mathfrak{t}^\zeta}\quad\text{(as linear orders)},$$
$$I_t^{\mathfrak{t}^\xi}=\bigcup\{I_t^{\mathfrak{t}^\zeta}:\zeta<\xi\text{ and }t\in L^{\mathfrak{t}_\zeta}\}.$$

Clearly $\zeta<\xi\Rightarrow\mathfrak{t}^\zeta\leqslant_{\mathrm{wk}}\mathfrak{t}^\xi$. Such a $\mathfrak{t}^\xi$ is the *limit* of $\langle\mathfrak{t}^\zeta:\zeta<\xi\rangle$; now a $\leqslant_{\mathrm{wk}}$-increasing sequence $\langle\mathfrak{t}^\zeta:\zeta<\xi\rangle$ is *continuous* if for every limit ordinal $\delta<\xi$ we have $\mathfrak{t}^\delta=\bigcup_{\zeta<\delta}\mathfrak{t}^\zeta$.

(4) If $\langle\mathfrak{t}^\zeta:\zeta<\xi\rangle$ are pairwise disjoint (i.e. $\zeta\neq\varepsilon\Rightarrow L^{\mathfrak{t}^\zeta}\cap L^{\mathfrak{t}^\varepsilon}=\varnothing$), we define $\sum_{\zeta<\xi}\mathfrak{t}^\zeta$ by induction on ξ naturally: for $\xi=1$ it is $\mathfrak{t}^0$, for ξ limit it is $\bigcup_{\zeta<\xi}\big(\sum_{\zeta<\varepsilon}\mathfrak{t}^\zeta\big)$, and for $\xi=\varepsilon+1$ it is $\big(\sum_{\zeta<\varepsilon}\mathfrak{t}^\zeta\big)+\mathfrak{t}^\varepsilon$; so $\xi_1\leqslant\xi_2\Rightarrow\sum_{\zeta<\xi_1}\mathfrak{t}^\zeta\leqslant_{\mathrm{wk}}\sum_{\zeta<\xi_2}\mathfrak{t}^\zeta$ (even an initial segment).

(5) We can replace $\mathfrak{t}^\zeta$ in (1)–(4) above by $(\mathfrak{t}^\zeta,\overline{K}^\zeta)$.

CLAIM 2.11. *Let* $\mathfrak{t}$ *be an FSI-template.*

(1) *If* $L^{\mathfrak{t}}=\varnothing$ *or just finite, and* $\{s:L^{\mathfrak{t}}\models s<t\}\in I_t^{\mathfrak{t}}$, *then* $\mathfrak{t}$ *is smooth.*

(2) *If* $\mathfrak{t}^1$ *and* $\mathfrak{t}^2$ *are disjoint FSI-templates, then* $\mathfrak{t}^1+\mathfrak{t}^2$ *is an FSI-template and* $l\in\{1,2\}\Rightarrow\mathfrak{t}^l\leqslant_{\mathrm{wk}}\mathfrak{t}^1+\mathfrak{t}^2$.

(3) *If* $\mathfrak{t}^1$ *and* $\mathfrak{t}^2$ *are disjoint smooth FSI-templates, then* $\mathfrak{t}^1+\mathfrak{t}^2$ *is a smooth FSI-template; moreover,* $\mathrm{Dp}_{\mathfrak{t}}(L^{\mathfrak{t}})\leqslant\mathrm{Dp}_{\mathfrak{t}^1}(L^{\mathfrak{t}^1})+\mathrm{Dp}_{\mathfrak{t}^2}(L^{\mathfrak{t}^2})$ *and* $\mathrm{Dp}_{\mathfrak{t}}(L^{\mathfrak{t}^l})=\mathrm{Dp}_{\mathfrak{t}^l}(L^{\mathfrak{t}^l})$.

(4) *If* $\langle\mathfrak{t}^\zeta:\zeta<\xi\rangle$ *is a* $\leqslant_{\mathrm{wk}}$*-increasing sequence (see Definition* 2.10(2)) *of FSI-templates and* ξ *is a limit ordinal, then* $\mathfrak{t}^\xi:=\bigcup_{\zeta<\xi}\mathfrak{t}^\zeta$ *is an FSI-template and* $\zeta<\xi\Rightarrow\mathfrak{t}^\zeta\leqslant_{\mathrm{wk}}\mathfrak{t}^\xi$.

(5) *If* $\langle\mathfrak{t}^\zeta:\zeta<\xi\rangle$ *is an increasing continuous sequence (see Definition* 2.10(3)) *of smooth FSI-templates and* ξ *is a limit ordinal, then* $\mathfrak{t}^\xi:=\bigcup_{\zeta<\xi}\mathfrak{t}^\zeta$ *is a smooth FSI-template,* $\zeta<\xi\Rightarrow\mathfrak{t}^\zeta\leqslant_{\mathrm{wk}}\mathfrak{t}^\xi$ *and* $\mathrm{Dp}_{\mathfrak{t}^\xi}(L^{\mathfrak{t}_\xi})\leqslant\sum_{\zeta<\xi}\mathrm{Dp}_{\mathfrak{t}^\zeta}(L^{\mathfrak{t}_\zeta})$.

(6) *If* $\langle\mathfrak{t}^\zeta:\zeta<\xi\rangle$ *is a sequence of pairwise disjoint (smooth) FSI-templates, then* $\sum_{\zeta<\xi}\mathfrak{t}^\zeta$ *is a (smooth) FSI-template and* $\big\langle\sum_{\zeta<\varepsilon}\mathfrak{t}^\zeta:\varepsilon\leqslant\zeta\big\rangle$ *is increasing continuous.*

(7) *We can expand* $\mathfrak{t}^\zeta$ *by* $\overline{K}^\zeta$.

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(7) We may restrict ourselves to FSI-templates $\mathfrak{t}$ of a globally countable, i.e. such that $A\in I_t^{\mathfrak{t}}$ & $t\in L^{\mathfrak{t}}\Rightarrow|A|\leqslant\aleph_0$, or a locally countable, with no loss. We use this restriction as in Definition 2.13. If $A_i\subseteq A_j\in I_t^{\mathfrak{t}}$ for $i<j<\varkappa$, then in $\mathfrak{t}^*=\mathfrak{t}^\varkappa/D$, $\bigcup\{j^*_{D,\mathfrak{t}}(A_i):i<\varkappa\}\in I^{\mathfrak{t}^*}_{j_{D,\mathfrak{t}}(t)}$ even if $\bigcup\{A_i:i<\varkappa\}\notin I_t^{\mathfrak{t}}$.

(8) *We can restrict ourselves to locally countable* $\mathfrak{t}$*'s (so that the sums are locally countable if the summands are) or to globally countable ones.*

(9) *Assume that* $\mathbf{J}$ *is a linear order,* $\mathfrak{t}_x$ *is a smooth template for every* $x\in\mathbf{J}$ *and* $\langle L^{\mathfrak{t}_x}: x\in\mathbf{J}\rangle$ *are pairwise disjoint (for notational simplicity). We define* $\mathfrak{t}$ *by* $L^{\mathfrak{t}}=\sum_{x\in\mathbf{J}} L^{\mathfrak{t}_x}$ *(so that* $L^{\mathfrak{t}}\models s<t$ *if and only if* $(\exists x,y)(s\in L^{\mathfrak{t}_x}\wedge t\in L^{\mathfrak{t}_y}\wedge x<_{\mathbf{J}} y)\vee(\exists x\in\mathbf{J})(L^{\mathfrak{t}_x}\models s<t)$*) and*

$$I_t^{\mathfrak{t}}=\{A\subseteq L^{\mathfrak{t}}:(\forall s\in A)(s<_{L^{\mathfrak{t}}} t) \text{ and letting } t\in\mathfrak{t}^x \text{ we have } A\cap L^{\mathfrak{t}_x}\in I_t^{\mathfrak{t}_x} \text{ and } \{y: y<_{\mathbf{J}} x,\ A\cap L^{\mathfrak{t}_y}\neq\varnothing\} \text{ is finite}\}.$$

Then $\mathfrak{t}$ *is a smooth template (can be expanded by* $\bar{K}$*'s) (this will be used in §3).*

Proof. This is easy. For example, part (3) is proved by induction on $\mathrm{Dp}_{\mathfrak{t}}(L^{\mathfrak{t}})$, part (6) by induction on ξ, and in part (7) let M be $[\mathbf{J}]^{<\aleph_0}$ ordered by inclusion and $L_{\{x(1),\dots,x(n)\}}=\bigcup\{L^{\mathfrak{t}_{x(l)}}: l=1,\dots,n\}$ for any $x(1),\dots,x(n)\in\mathbf{J}$. □

Discussion 2.12. To prove our desired result $\mathrm{CON}(\mathfrak{a}>\mathfrak{d})$ we need to construct an FSI-template $\mathfrak{t}$ of the right form. Now we do it by using a measurable cardinal. The point is that if we are given $\langle\langle t_{i,n}: n<\omega\rangle: i<i(*)\rangle$, $L^{\mathfrak{t}}$, $i(*)\geqslant\varkappa$, and D is a normal ultrafilter on $\varkappa$, then in $\mathfrak{t}^{\varkappa}/D$ the ω-sequence $\langle\langle t_{i,n}: i<\varkappa\rangle/D: n<\omega\rangle$ is as required in Claim 2.8 (3)(c), considering $\mathfrak{t}^{\varkappa}/D$ as an extension of $\mathfrak{t}$.

Definition 2.13. For a template $\mathfrak{t}$ and a $(2^{\aleph_0})^+$-complete ultrafilter D on $\varkappa$, we define $\mathfrak{t}^*:=\mathfrak{t}^{\varkappa}/D$, $\mathbf{j}_{D,\mathfrak{t}}$ and $\mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t})$ as follows:

(a) We define $\mathfrak{t}^*$ by

$$L^{\mathfrak{t}^*}=(L^{\mathfrak{t}})^{\varkappa}/D \quad \text{as a linear order,}$$

and if $t^*=\langle t_i: i<\varkappa\rangle/D$, where $t_i\in L^{\mathfrak{t}}$, then we let $I_{t^*}^{\mathfrak{t}^*}=\{A:\text{we can find } A_i\in I_{t_i}^{\mathfrak{t}} \text{ for } i<\varkappa$ such that $A\subseteq\prod_{i<\varkappa} A_i/D\}$.

(b) We then let $\mathbf{j}_{D,\mathfrak{t}}$ be the canonical embedding of $\mathfrak{t}$ into $\mathfrak{t}^{\varkappa}/D$, i.e. $\mathbf{j}_{D,\mathfrak{t}}(t)=\langle t: i<\varkappa\rangle/D$ for every $t\in L^{\mathfrak{t}}$.

(c) Let finally $\mathfrak{t}'=\mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t})$ be defined by $L^{\mathfrak{t}'}=L^{\mathfrak{t}^*}\restriction\{\mathbf{j}_{D,\mathfrak{t}}(s): s\in L^{\mathfrak{t}}\}$ and $I_{\mathbf{j}_{D,\mathfrak{t}}(s)}^{\mathfrak{t}'}=\{\{\mathbf{j}_{D,\mathfrak{t}}(t): t\in A\}: A\in I_s^{\mathfrak{t}}\}$.

(We can deal with $\bar{K}$ if D is $(\bigcup_{t\in L^{\mathfrak{t}}}|K_t|^+)$-complete, which holds here as each K_t is countable, and we can also deal with $\bar{\mathbf{Q}}$ if we have less than $\mathrm{com}(D)$ kinds of $\bar{\varphi}_t$ (letting $\underset{\sim}{\eta}_i$ vary), which holds here too.)

CLAIM 2.14. *In Definition* 2.13,

(1) $\mathfrak{t}^{\varkappa}/D$ *is also an FSI-template,* $\mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t})\leqslant_{\mathrm{wk}}\mathfrak{t}^{\varkappa}/D$ *and* $\mathbf{j}_{D,\mathfrak{t}}$ *is an isomorphism from* $\mathfrak{t}$ *onto* $\mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t})$;

(2) *if* $\mathfrak{t}$ *is a smooth FSI-template, then* $\mathfrak{t}^\varkappa/D$ *is a smooth FSI-template*;

(3) *moreover,* $\mathrm{Dp}_{\mathfrak{t}^\varkappa/D}(L^{\mathfrak{t}^\varkappa/D})\leqslant(\mathrm{DP}_\mathfrak{t}(L^\mathfrak{t}))^\varkappa/D$.

Proof. This is straightforward. □

Now Definition 2.15 and Observation 2.16 below are used only in the short proof of Conclusion 2.17, depending on §1, so you may ignore them.

Definition 2.15. Fix $\aleph_0<\varkappa<\mu=\mathrm{cf}(\mu)<\lambda=\mathrm{cf}(\lambda)=\lambda^\varkappa$ and D, a $\varkappa$-complete (or just $(2^{\aleph_0})^+$-complete) uniform ultrafilter on $\varkappa$. We define, by induction on $\zeta\leqslant\lambda$, a smooth FSI-template $\mathfrak{t}_{\gamma,\zeta}$ for $\gamma<\mu$ such that

(a) $\mathfrak{t}_{\gamma,\zeta}$ is an FSI-template;

(b) if $\gamma_1\neq\gamma_2$ then $\mathfrak{t}_{\gamma_1,\zeta}$ and $\mathfrak{t}_{\gamma_2,\zeta}$ are disjoint, i.e. $L^{\mathfrak{t}_{\gamma_1,\zeta}}\cap L^{\mathfrak{t}_{\gamma_2,\zeta}}=\varnothing$;

(c) for $\xi<\zeta$ we have $\mathfrak{t}_{\gamma,\xi}\leqslant_{\mathrm{wk}}\mathfrak{t}_{\gamma,\zeta}$;

(d) if ζ is limit then $\mathfrak{t}_{\gamma,\zeta}=\bigcup_{\xi<\zeta}\mathfrak{t}_{\gamma,\xi}$, see Definition 2.10 (3) and Claim 2.11 (6);

(e) if $\zeta=\xi+1$ and ξ is even, then there is an isomorphism $\mathbf{j}_{\gamma,\zeta}$ from $\sum_{\beta\leqslant\gamma}\mathfrak{t}_{\beta,\xi}$ onto $\mathfrak{t}_{\gamma,\zeta}$ which is the identity over $\mathfrak{t}_{\gamma,\xi}$;

(f) if $\zeta=\xi+1$ and ξ is odd, then there is an isomorphism $\mathbf{j}_{\gamma,\zeta}$ from $(\mathfrak{t}_{\gamma,\xi})^\varkappa/D$ onto $\mathfrak{t}_{\gamma,\zeta}$ which extends the inverse of $\mathbf{j}_{D,\mathfrak{t}_{\gamma,\xi}}$.

OBSERVATION 2.16. *Definition* 2.15 *is legitimate.*

Proof. This follows by the previous claims. □

CONCLUSION 2.17. *Assume that* $\varkappa$ *is measurable and* $\varkappa<\mu=\mathrm{cf}(\mu)<\lambda=\mathrm{cf}(\lambda)=\lambda^\varkappa$. *Then for some c.c.c. forcing notion* $\mathbf{P}$ *of cardinality* λ, *in* $\mathbf{V}^\mathbf{P}$ *we have* $\mathfrak{a}=\lambda$ *and* $\mathfrak{b}=\mathfrak{d}=\mu$.

Short proof (depending on §1). Let $\mathfrak{t}_{\gamma,\zeta}$ (for $\gamma<\mu$ and $\zeta\leqslant\lambda$) be as in Definition 2.15. Let $\mathfrak{t}=\sum_{\gamma<\mu}\mathfrak{t}_{\gamma,\lambda}$, $\overline{K}=\langle K_t:t\in L^\mathfrak{t}\rangle$, $K_t=\varnothing$ and $\bar{\mathbf{Q}}=\langle\underset{\sim}{\mathbf{Q}}_t:t\in L^\mathfrak{t}\rangle$ with $\underset{\sim}{\mathbf{Q}}_t$ being constantly the dominating real forcing (= Hechler forcing). Lastly let $\mathbf{P}=\mathrm{Lim}_\mathfrak{t}(\bar{\mathbf{Q}})$.

The rest is as at the end of §1. □

Alternative presentation of the proof of Conclusion 2.17, *self-contained, not depending on Definition* 2.15 *and Observation* 2.16. We define an FSI-template $\mathfrak{t}^\zeta$ for $\zeta\leqslant\lambda$ by induction on ζ.

Case 1: $\zeta=0$. Let $\mathfrak{t}^\zeta$ be defined by

$$L^{\mathfrak{t}^\zeta}=\mu,\quad I_\alpha^{\mathfrak{t}^\zeta}=\{A:A\subseteq\alpha\}.$$

Case 2: $\zeta=\xi+1$. We choose $\mathfrak{t}^\zeta$ such that there is an isomorphism $\mathbf{j}_\zeta$ from $L^{\mathfrak{t}^\zeta}$ onto $(L^{\mathfrak{t}^\xi})^\varkappa/D$ satisfying that $\mathbf{j}_\zeta\restriction L^{\mathfrak{t}^\xi}$ is the canonical embedding $\mathbf{j}_{D,\mathfrak{t}^\xi}$, and if $x\in L^{\mathfrak{t}^\zeta}$ and $\mathbf{j}_{\zeta(x)}=$

$\langle x_\varepsilon:\varepsilon<\varkappa\rangle/D\in(L^{\mathfrak{t}^\xi})^\varkappa/D$, then $A\in I_x^{\mathfrak{t}^\zeta}$ if and only if for some $\bar{A}=\langle A_\varepsilon:\varepsilon<\varkappa\rangle$ we have $A_\varepsilon\in I_{x_\varepsilon}^{\mathfrak{t}^\xi}$ and $\{\mathbf{j}_\zeta(y):y\in A\}\subseteq\{\langle y_\varepsilon:\varepsilon<\varkappa\rangle/D:\{\varepsilon<\varkappa:y_\varepsilon\in A_\varepsilon\}\in D\}$.

Case 3: ζ *is limit.* We choose $\mathfrak{t}^\zeta$ from

$$L^{\mathfrak{t}^\zeta}=\bigcup_{\xi<\zeta}L^{\mathfrak{t}^\xi}\quad\text{as linear orders.}$$

Subcase 3A. If $x\in L^{\mathfrak{t}^0}$ then $I_x^{\mathfrak{t}^\zeta}=\{A:A\subseteq\{s:L^{\mathfrak{t}^\zeta}\vDash\text{“}s<x\text{”}\}\}$.

Subcase 3B. If $x\notin L^{\mathfrak{t}^0}$ but $x\in L^{\mathfrak{t}^\zeta}$, then[8]

$I_x^{\mathfrak{t}^\zeta}=\{A$: for some $\xi<\zeta$ we have $x\in L^{\mathfrak{t}^\xi}$, and if $y=\min\{y\in L^{\mathfrak{t}^0}:L^{\mathfrak{t}^\zeta}\vDash\text{“}x<y\text{”}\}$, which is in $L^{\mathfrak{t}^0}$ (and is always well defined, see clause (b) of (*) below), then $A\setminus\{t\in L^{\mathfrak{t}^\zeta}:L^{\mathfrak{t}^\zeta}\vDash\text{“}t<z\text{”}$ for some z such that $L^{\mathfrak{t}^0}\vDash\text{“}z<y\text{”}\}$ belongs to $I_x^{\mathfrak{t}^\xi}$ (and hence is a subset of $L^{\mathfrak{t}^\xi}$)$\}$.

We now prove by induction on $\zeta\leqslant\lambda$ that

(*) (a) $\mathfrak{t}^\zeta$ is an FSI-template;

(b) $L^{\mathfrak{t}^0}$ is an unbounded subset of $L^{\mathfrak{t}^\zeta}$;

(c) $\mathfrak{t}^\zeta$ is smooth;

(d) $\mathfrak{t}^\xi\leqslant_{\mathrm{wk}}\mathfrak{t}^\zeta$ for $\xi<\zeta$;

(e) if $x\in L^{\mathfrak{t}^\zeta}$ then $\{z:$ for some $y\in L^{\mathfrak{t}^0}$ we have $L^{\mathfrak{t}^\zeta}\vDash\text{“}z\leqslant y<x\text{”}\}\in I_x^{\mathfrak{t}^\zeta}$;

(f) $L^{\mathfrak{t}^\zeta}$ has cardinality $\leqslant(\mu+|\zeta|)^\varkappa$;

(g) $\langle\mathfrak{t}^\varepsilon:\varepsilon\leqslant\zeta\rangle$ is $\leqslant_{\mathrm{wk}}$-increasing continuous;

(h) we have $\mathfrak{t}^\zeta=\sum_{\gamma<\mu}\mathfrak{s}^{\gamma,\zeta}$, where $\mathfrak{s}^{\gamma,\zeta}=\mathfrak{t}^\zeta\restriction\{x\in L^{\mathfrak{t}^\zeta}:L^{\mathfrak{t}^\zeta}\vDash\text{“}x<\gamma\text{”}\}$.

(Why? This is easy. For example, why do clauses (a) and (c) hold? For $\zeta=0$ by Claim 2.11 (1) and (6). For $\zeta=\xi+1$ by Claim 2.14 (2). For ζ limit, for any $t\in L^{\mathfrak{t}^0}$ clearly $\mathfrak{s}^{\gamma,\zeta}$ is the union of the increasing continuous sequence $\langle\mathfrak{s}^{\gamma,\varepsilon}:\varepsilon<\zeta\rangle$, and is hence a smooth FSI-template by clause (h) and Claim 2.11 (5). Now also $\mathfrak{t}^\zeta$ is a smooth FSI-template by Claim 2.11 (6). Of course, we let $\overline{K}^\zeta=\langle K_t^\zeta:t\in L^{\mathfrak{t}^\zeta}\rangle$, $K_t^\zeta=\varnothing$ and $\mathbf{Q}_t$ be a dominating real forcing.)

Lastly let for $\zeta\leqslant\lambda$, $\mathbf{P}_\zeta=\mathrm{Lim}_{\mathfrak{t}}(\overline{\mathbf{Q}}\restriction L^{\mathfrak{t}^\zeta})$. Now

(α) $\mathbf{P}_\lambda$ is a c.c.c. forcing notion of cardinality $\leqslant\lambda^{\aleph_0}$, and hence $\mathbf{V}^{\mathbf{P}_\lambda}\vDash2^{\aleph_0}\leqslant\lambda$ by Definition 2.6 (B)(j) as $\lambda=\lambda^\varkappa$;

(β) in $\mathbf{V}^{\mathbf{P}_\lambda}$ we have $\mathfrak{d}\leqslant\mu$, by Claim 2.8 (1) applied with $R=<^*$ and $L^*=L^{\mathfrak{t}^0}$ using (*)(b) and (e);

(γ) in $\mathbf{V}^{\mathbf{P}_\lambda}$ we have $\mathfrak{b}\geqslant\mu$ by Claim 2.8 (2) applied with $R=<^*$;

(δ) $\mathfrak{b}=\mathfrak{d}=\mu$ and $\mathfrak{a}\geqslant\mu$ by (β) and (γ), as it is well known that $\mathfrak{b}\leqslant\mathfrak{d}$ and $\mathfrak{b}\leqslant\mathfrak{a}$.

(8) So members of $L^{\mathfrak{t}^0}$ have the “veteranity privilege”, i.e. “founding father's right”; i.e. members t of $L^{\mathfrak{t}^0}$ have the maximal $I_t^{\mathfrak{t}^\zeta}$.

But in order to sort out the value of $\mathfrak{a}$, we intend to use Claim 2.8 (3) with θ there chosen as λ here.

But why does the demand (c) from Claim 2.8 (3) holds? Assume that $i(*)\in[\varkappa,\lambda)$ and $t_{i,n}\in L^{\mathfrak{t}^\lambda}$, for $i<i(*)$ and $n<\omega$, are given. As λ is regular and $>i(*)$, necessarily for some $\xi<\lambda$ we have $\{t_{i,n}:i<i(*) \text{ and } n<\omega\}\subseteq L^{\mathfrak{t}^\xi}$. Now let $t_n\in L^{\mathfrak{t}^{\xi+1}}$ be such that $\mathbf{j}_{\xi+1}(t_n)=\langle t_{i,n}:i<\varkappa\rangle/D$; easily $\langle t_n:n<\omega\rangle$ is as required (note that the number of isomorphism types of ω-sequences $\langle t_n:n<\omega\rangle$ in $\mathfrak{t}$ is trivially $\leqslant\beth_2$).(9) So

(ε) in $\mathbf{V}^{\mathbf{P}_\lambda}$ we have $\mathfrak{a}\geqslant\varkappa\Rightarrow\mathfrak{a}\geqslant\lambda$ by Claim 2.8 (3).

Thus we are done. □

3. Eliminating the measurable

Without a measurable cardinal our problem is to verify condition (c) in Claim 2.8 (3). Toward this it is helpful to show that for some $\aleph_1$-complete filter D on $\varkappa$, for any $i(*)\in[\varkappa,\lambda)$ and $t_{i,n}\in L^{\mathfrak{t}}$, for $i<i(*)$ and $n<\omega$, we have: for some $B\in D^+$, for every $j<i(*)$ some $A\in D$ satisfies that for any $i_0,i_1\in A\cap B$, the mapping $t_{j,n}\mapsto t_{j,n}$, $t_{i_0,n}\mapsto t_{i_1,n}$ is a partial isomorphism of $\mathfrak{t}$. So D behaves as an $\aleph_1$-complete ultrafilter for our purpose.

(If you know enough model theory, this is the problem of finding convergent sequences, see [S1, Chapter II]. For stable first-order T with $\varkappa=\varkappa_r(T)$, any indiscernible sequence (equivalently set) $\langle\bar a_\alpha:\alpha<\alpha^*\rangle$ of cardinality $\geqslant\varkappa$ is convergent. Why? As for any $\bar{\mathbf{b}}\in{}^{\varkappa>}\mathfrak{C}$, for all but $<\varkappa$ ordinals $\alpha<\alpha^*$, $\bar{\mathbf{b}}{}^\frown\bar a_\alpha$ has a fixed type so that average is definable. In [S1, Chapter II], we deal with it in general (harder to prove existence, which we do there under the relevant assumptions).)

Lemma 3.1. *Assume $2^{\aleph_0}<\mu=\mathrm{cf}(\mu)<\lambda=\mathrm{cf}(\lambda)=\lambda^{\aleph_0}$. Then for some $\mathbf{P}$ we have:*

(a) *$\mathbf{P}$ is a c.c.c. forcing notion of cardinality λ;*

(b) *in $\mathbf{V}^{\mathbf{P}}$, $\mathfrak{b}=\mathfrak{d}=\mu$ and $\mathfrak{a}=2^{\aleph_0}=\lambda$.*

Proof. We rely on Definition 2.6 and Claim 2.8. Let L_0^+ be a linear order isomorphic to λ, let L_0^- be a linear order anti-isomorphic to λ (and $L_0^-\cap L_0^+=\varnothing$) and let $L_0=L_0^-+L_0^+$.

Let $\mathbf{J}$ be the following linear order:

(a) its set of elements is ${}^{\omega>}(L_0)$;

(b) $\eta<_{\mathbf{J}}\nu$ if and only if for some $n<\omega$ we have $\eta\restriction n=\nu\restriction n$ and $\lg(\eta)=n$ & $\nu(n)\in L_0^+$ or $\lg(\nu)=n$ & $\eta(n)\in L_0^-$, or we have $\lg(\eta)>n$ & $\lg(\nu)>n$ & $L_0\models\eta(n)<\nu(n)$.

(See more on such orders in [L], [S2, Appendix] and [S9, XIII, §2], but we are self-contained.)

(9) In fact, it is $\leqslant 2^{\aleph_0}$ by the construction, but this is irrelevant here.

Note that

$\boxdot_1$ every interval of $\mathbf{J}$, as well as $\mathbf{J}$ itself, has cardinality λ;

$\boxdot_1^+$ if $\aleph_0<\theta=\mathrm{cf}(\theta)<\lambda$ or $\theta=1$ or $\theta=0$ and $\langle t_i:i<\theta\rangle$ is a strictly decreasing sequence in $\mathbf{J}$, then $\mathbf{J}\restriction\{y\in\mathbf{J}:(\forall i<\theta)(y<_{\mathbf{J}} t_i)\}$ has cofinality λ if non-empty;

$\boxdot_1^-$ the inverse of $\mathbf{J}$ satisfies $\boxdot_1^+$ and is moreover isomorphic to $\mathbf{J}$;

$\boxdot_2$ if $\theta=\mathrm{cf}(\theta)>\aleph_0$ and $s_\alpha,t_\alpha\in\mathbf{J}$ for $\alpha<\theta$, then we can find a function $f:\theta\to\theta$ which is regressive and a club E of θ such that: if $\alpha_l<\beta_l$ are from E for $l=1,2$ and $f(\alpha_1)=f(\beta_1)=f(\alpha_2)=f(\beta_2)$, then $t_{\alpha_1}<_{\mathbf{J}} s_{\beta_1}\Leftrightarrow t_{\alpha_2}<_{\mathbf{J}} s_{\beta_2}$ and $t_{\alpha_1}=s_{\beta_1}\Leftrightarrow t_{\alpha_2}=s_{\beta_2}$ (we can add $t_{\alpha_1}<_{\mathbf{J}} t_{\beta_1}\Leftrightarrow t_{\alpha_2}<_{\mathbf{J}} t_{\beta_2}$, etc., but this can be deduced using the above several times).

We now define by induction on $\zeta<\mu$ FSI-templates $\mathfrak{t}_\zeta$ such that

$(*)^1_\zeta$ the set of members of $L^{\mathfrak{t}_\zeta}$ is a set of finite sequences starting with ζ, hence disjoint to $\mathfrak{t}_\varepsilon$ for $\varepsilon<\zeta$; for $x\in L^{\mathfrak{t}_\zeta}$ let $\xi(x)=\zeta$.

Defining $\mathfrak{t}_\zeta$. *Case* 1: $\zeta=0$, ζ *a successor or* $\mathrm{cf}(\zeta)=\aleph_0$. Let $L^{\mathfrak{t}_\zeta}=\{\langle\zeta\rangle\}$ and $I^{\mathfrak{t}_\zeta}_{\langle\zeta\rangle}=\{\varnothing\}$.

Case 2: $\mathrm{cf}(\zeta)>\aleph_0$. Let $h_\zeta:\mathbf{J}\to\zeta$ be a function such that $\varepsilon<\zeta\Rightarrow h_\zeta^{-1}\{\varepsilon\}$ is a dense subset of $\mathbf{J}$. The set of elements of $\mathfrak{t}_\zeta$ is

$$\{\langle\zeta\rangle\}\cup\Big\{\langle\zeta\rangle^\frown\langle\eta\rangle^\frown x:\eta\in\mathbf{J}\text{ and }x\in\bigcup_{\varepsilon\leqslant h_\zeta(\eta)}L^{\mathfrak{t}_\varepsilon}\Big\}.$$

The order $<_{\mathfrak{t}_\zeta}$ is defined by:

- $\langle\zeta\rangle$ is maximal;
- $\langle\zeta\rangle^\frown\langle\eta_1\rangle^\frown x_1<_{\mathfrak{t}_\zeta}\langle\zeta\rangle^\frown\langle\eta_2\rangle^\frown x_2$ if and only if

$$\eta_1<_{\mathbf{J}}\eta_2\vee(\eta_1=\eta_2\ \&\ \xi(x_1)<\xi(x_2))\vee(\eta_1=\eta_2\ \&\ \xi(x_1)=\xi(x_2)\ \&\ x_1<_{\mathfrak{t}_{\xi(x_1)}}x_2).$$

Lastly, for $y\in\mathfrak{t}_\zeta$ we define the ideal $I=I_y^{\mathfrak{t}_\zeta}$:

(α) if $y=\langle\zeta\rangle$ then $I=\{Y:Y\subseteq L^{\mathfrak{t}_\zeta}\setminus\{\langle\zeta\rangle\}\}$;

(β) if $y=\langle\zeta\rangle^\frown\langle\nu\rangle^\frown x$ then I is the family of sets Y satisfying the following conditions:

(i) $Y\subseteq L^{\mathfrak{t}_\zeta}$;

(ii) $(\forall z\in Y)(z<_{\mathfrak{t}_\zeta}y)$;

(iii) for each $\eta\in\mathbf{J}$ and $\xi<h_\zeta(\eta)$ we have

$$\{z:\langle\zeta\rangle^\frown\langle\eta\rangle^\frown z\in Y,\ \xi(z)=\xi\text{ and }z\neq\langle\xi\rangle\}\in I^{\mathfrak{t}_\xi}_{\langle\xi\rangle};$$

(iv) the set $\{\eta\in\mathbf{J}:(\exists x)(\langle\zeta\rangle^\frown\langle\eta\rangle^\frown x\in Y)\}$ is finite.

Why is $\mathfrak{t}_\zeta$ really an FSI-template? We prove, of course, by induction on ζ that

$(*)^2_\zeta$ (i) $L^{\mathfrak{t}_\zeta}$ is a linear order;

(ii) $I_t^{\mathfrak{t}_\zeta}$ is an ideal of subsets of $\{s\in I_t^{\mathfrak{t}_\zeta}:s<t\}$;

(iii) $\mathfrak{t}_\zeta$ is an FSI-template;

(iv) $\mathfrak{t}_\zeta$ is disjoint to $\mathfrak{t}_\varepsilon$ for $\varepsilon<\zeta$.

(Why? By Claim 2.11 and looking at the definitions.)

Next we prove, by induction on ζ, that $\mathfrak{t}_\zeta$ is a smooth FSI-template. Assume that $\mathfrak{t}_\xi$ is a smooth FSI-template for all $\xi<\zeta$.

$(*)^3_\zeta$ For $\eta\in\mathbf{J}$ and $\varepsilon\leqslant h_\zeta(\eta)+1$, we have that $\mathfrak{t}_\zeta\restriction\{\langle\zeta\rangle\hat{}\langle\eta\rangle\hat{}\varrho:\varrho\in\bigcup_{\xi<\varepsilon}\mathfrak{t}_\xi\}$ is a smooth FSI-template.

(Why? We prove this by induction on ε: for $\varepsilon=0$ by Claim 2.11 (1); for ε successor by Claim 2.11 (3); for ε limit by Claim 2.11 (5) and (6).)

$(*)^4_\zeta$ For $Z\subseteq\mathbf{J}$ we have that $\mathfrak{t}_\zeta\restriction(\bigcup_{\eta\in Z}\{\langle\zeta\rangle\hat{}\langle\eta\rangle\hat{}\varrho:\varrho\in\bigcup_{\xi<h_\zeta(\eta)}\mathfrak{t}_\xi\})$ is a smooth FSI-template.

(Why? By induction on $|Z|$: for $|Z|=0$ and $|Z|=n+1$ by Claim 2.11 (3); for $|Z|\geqslant\aleph_0$ by Claim 2.11 (5).)

$(*)^5_\zeta$ $\mathfrak{t}_\zeta\restriction(L^{\mathfrak{t}_\zeta}\setminus\{\langle\zeta\rangle\})$ is a smooth FSI-template.

(Why? By $(*)^4_\zeta$ for $Z=\mathbf{J}$.)

$(*)^6_\zeta$ $\mathfrak{t}_\zeta$ is a smooth FSI-template.

(Why? By Claim 2.11 (3).)

$(*)^7_\zeta$ If $K\subseteq L^{\mathfrak{t}_\zeta}$ and $t\in L^{\mathfrak{t}_\zeta}$ then the ideal $I^{\mathfrak{t}_\zeta}_t\cap\mathcal{P}(K)$ is generated by a countable family of subsets of K.

(Why? Check by induction on ζ.)

Now for $\zeta\leqslant\mu$ let

$\boxdot_2$ $\mathfrak{s}_\zeta:=\sum_{\varepsilon<\zeta}\mathfrak{t}_\varepsilon$, i.e.

(i) the set of elements of $\mathfrak{s}_\zeta$ is $\bigcup_{\varepsilon<\zeta}L^{\mathfrak{t}_\varepsilon}$;

(ii) for $x,y\in\mathfrak{s}_\zeta$ we have $x<_{\mathfrak{s}_\zeta}y$ if and only if $\xi(x)<\xi(y)\vee(\xi(x)=\xi(y)\ \&\ x<_{\mathfrak{t}_{\xi(x)}}y)$;

(iii) $I^{\mathfrak{s}_\zeta}_y=\{Y\subseteq\mathfrak{s}_\zeta:(\forall z\in Y)(z<_{\mathfrak{s}_\zeta}y)$ and $\{z\in\mathfrak{s}_\zeta:\xi(z)=\xi(y)$ and $z\in Y\}\in I^{\mathfrak{t}_{\xi(z)}}_y\}$.

$(*)^8_\zeta$ $\mathfrak{s}_\zeta$ is a smooth FSI-template.

(Why? This is just easier than the proof above.)

$(*)^9_\zeta$ If $K\subseteq L^{\mathfrak{s}_\zeta}$ is countable and $t\in L^{\mathfrak{s}_\zeta}$, then the ideal $I^{\mathfrak{s}_\zeta}_t\cap\mathcal{P}(K)$ of subsets of K is generated by a countable family of subsets of K.

(Why? By $(*)^7_\zeta$ and by the definition of $\mathfrak{s}_\zeta$ and of the $\mathfrak{t}_\varepsilon$'s.)

Let $\theta=(2^{\aleph_0})^+$.([10]) We shall prove below by induction on ζ that $\mathfrak{s}_\zeta$ and $\mathfrak{t}_\zeta$ are (λ,θ)-good (see the definition below and Subclaim 3.4). Then we can finish the proof as in Conclusion 2.17 using $\mathfrak{s}_\mu$ (and $(*)^7_\zeta$ and $(*)^9_\zeta$).

Definition 3.2. (1) Assume([11]) that θ is regular uncountable and $(\forall\alpha<\theta)[|\alpha|^{\aleph_0}<\theta]$. We say that a smooth FSI-template $\mathfrak{t}$ is (λ,θ,τ)-*good* if the following condition is

([10]) But if you like to avoid using $(*)^7_\zeta$, $(*)^9_\zeta$ and $\mathcal{W}$ below, just use $\theta=\beth_2^+$. In fact, even without $(*)^7_\zeta$ and $(*)^9_\zeta$ above, countable $\mathcal{W}$ suffice, but then we have to weaken the notion of isomorphism, and there is no point in that.

([11]) We here ignore $\overline{K}$ and $\{(t,\bar{\varphi}_t,\underset{\sim}{\eta}_t):t\in L^{\mathfrak{t}}\}$.

satisfied:

$\oplus$ Assume that $t_{\alpha,n}\in L^{\mathfrak{t}}$ for $\alpha<\theta$ and $n<\omega$, that $\{t_{\alpha,n}:n<\omega\}$ is $\overline{K}$-closed and that $\mathcal{W}$ is a family of subsets of ω such that $2^{|\mathcal{W}|}<\theta$. Then we can find a club C of θ and a pressing-down function h on C such that

$\oplus'$ if $S\subseteq C$ is stationary in θ, $(\forall\delta\in S)[\mathrm{cf}(\delta)>\aleph_0]$ and $h\restriction S$ is constant, then we have:

$\boxtimes^1_S$ for every $\alpha<\beta$ in S, the truth value of the following statements does not depend on (α,β) (but may depend on n, m and $w\in\mathcal{W}$):

(i) $t_{\alpha,n}=t_{\beta,m}$;

(ii) $t_{\alpha,n}<_{L^{\mathfrak{t}}}t_{\beta,m}$;

(iii) $\{t_{\alpha,l}:l\in w\}\in I^{\mathfrak{t}}_{t_{\alpha,m}}$;

(iv) $\{t_{\beta,l}:l\in w\}\in I^{\mathfrak{t}}_{t_{\alpha,n}}$;

(v) $\{t_{\alpha,l}:l\in w\}\in I^{\mathfrak{t}}_{t_{\beta,n}}$;

$\boxtimes^2_S$ let $\delta^*\leqslant\theta$ be such that $\mathrm{cf}(\delta^*)=\tau$ and $\sup(S\cap\delta^*)=\delta^*$; if $\theta\leqslant\beta^*<\lambda$ and $s_{\beta,n}\in L^{\mathfrak{t}}$, for $\beta<\beta^*<\lambda$ and $n<\omega$, then we can find $t_n\in L^{\mathfrak{t}}$ for $n<\omega$ such that for every $\beta<\beta^*$, for every large enough $\alpha\in S\cap\delta^*$, for some $\mathfrak{t}$-partial isomorphism f we have $f(t_n)=t_{\alpha,n}$ and $f(s_{\beta,n})=s_{\beta,n}$.

(2) We say that $\mathfrak{t}$ is *strongly* (λ,θ,τ)*-good* if above we allow $\mathcal{W}=\mathcal{P}(\omega)$ (so if $\theta>\beth_2$, this is the same). In both cases we may omit τ in the notation if $\tau=\theta$.

Observation 3.3. Instead of "h regressive" it is enough to demand that for some sequence $\langle X_\alpha:\alpha<\theta\rangle$ of sets, increasing continuous, $|X_\alpha|<\theta$ and for every (or club of) $\delta<\theta$, if $\mathrm{cf}(\delta)>\aleph_0$ then $h(\delta)\in\mathcal{H}_{<\aleph_1}(X_\delta)$.

SUBCLAIM 3.4. *In the proof of Lemma* 3.1,

(i) $\mathfrak{t}_\zeta$ *is strongly* (λ,θ)*-good*;

(ii) $\mathfrak{s}_\zeta$ *is strongly* $(\lambda,\theta,\aleph_1)$*-good*;

(iii) *if* $\mathrm{cf}(\zeta)\neq\theta$ *then* $\mathfrak{s}_\zeta$ *is also strongly* (λ,θ)*-good.*

Proof. Recall that $\theta=(2^{\aleph_0})^+$, and let $\mathcal{W}$ be given ($2^{|\mathcal{W}|}<\theta$ for the first version; $\mathcal{W}=\mathcal{P}(\omega)$ for the second, using $(*)^7_\zeta$ and $(*)^9_\zeta$ from the proof of Lemma 3.1). We prove this by induction on ζ.

For $\mathfrak{s}_\zeta$. If $\zeta=0$, it is empty. Otherwise, given $t_{\alpha,n}\in\mathfrak{s}_\zeta=\sum_{\varepsilon<\zeta}\mathfrak{t}_\varepsilon$ for $\alpha<\theta$ and $n<\omega$, let $h^*_0(\alpha)$ be the sequence consisting of

(i) $\xi_{\alpha,n}:=\min\{\xi:\xi\in\{\xi(t_{\beta,m}):\beta<\delta\text{ and }m<\omega\}\cup\{\infty\}\text{ and }\xi\geqslant\xi(t_{\alpha,n})\}$ for $n<\omega$;

(ii) $u_\alpha=\{(n,m,l):\xi(t_{\alpha,n})=\xi_{\alpha,m}\ \&\ l=1\text{ or }\xi(t_{\alpha,n})=\xi(t_{\alpha,m})\ \&\ l=2\}$;

(iii) $\mathbf{w}_\alpha=\{(n,w):n<\omega,\ w\subseteq\omega\text{ and }\{t_{\alpha,m}:m\in w\}\in I^{\mathfrak{t}}_{t_{\alpha,n}}\}$;

i.e. $h^*_0(\delta)=\langle u_\alpha,\langle\xi_{\alpha,n}:n<\omega\rangle,\mathbf{w}_\alpha\rangle$. If $S_y=\{\delta:\mathrm{cf}(\delta)\geqslant\aleph_1,\ h^*_0(\delta)=y\}$ is stationary, we define $h^*_1\restriction S_y$ such that it codes $h^*_0(\delta)$, and if $n(*)<\omega$, $\alpha\in S_y\Rightarrow\xi(t_{\alpha,n(*)})=\xi_{y,n(*)}$ and we let $u_{y,n(*)}=\{n:\xi_{\alpha,n}=\xi_{y,n(*)}\}$, then $h^*_1\restriction S_y$ codes a function witnessing the (λ,θ)-goodness of

$\mathfrak{t}_{\xi_{y,n(*)}}$ for $\langle t_{\alpha,n} : n \in u_{y,n(*)}, \alpha \in S_y \rangle$.

Fix S as in $\oplus'$. It is easy to check that this shows $\boxtimes^1_S$ even if $\mathrm{cf}(\zeta)=\theta$. But assume $\mathrm{cf}(\zeta)\neq\theta$ & $\delta^*=\theta$ or $\delta^*<\theta$, $\mathrm{cf}(\delta^*)=\aleph_1$ (or just $\aleph_0<\mathrm{cf}(\delta^*)<\theta$), $\delta^*=\sup(S\cap\delta^*)$; we shall also prove the statement from $\boxtimes^2_S$. Let $w_1=\{n:$ the sequence $\langle \xi(t_{\beta,n}):\beta\in S\rangle$ is strictly increasing$\}$ and $w_0=\{n:\langle \xi(t_{\beta,n}):\beta\in S\rangle$ is constant$\}$. Let

$$\xi(S,n)=\xi_{S,n}=\bigcup\{\xi(t_{\beta,n}):\beta\in S\};$$

as $\mathrm{cf}(\zeta)\neq\theta$ it is less than ζ also when $n\in w_1$.

Given $\langle \bar{s}_\beta:\beta<\beta^*\rangle$, $\beta^*<\lambda$ and $\bar{s}_\beta=\bar{s}=\langle s_{\beta,n}:n<\omega\rangle$ we have to find $\langle t_n:n<\omega\rangle$ as required in $\boxtimes^2_S$. If $n\in w_0$, $w'_{0,n}=\{m\in w_0:\xi(t_{\alpha,n})=\xi(t_{\alpha,m})$ for $\alpha\in S\}$, and to choose $\langle t_m:m\in w'_{0,n}\rangle$ we use the induction hypothesis on $\mathfrak{t}_{\xi(S,n)}$. If $n\in w_1$ then we can find $t^*_n\in\mathfrak{t}_{\xi_{S,n}}$ such that $\{t:t\in\mathfrak{t}_{\xi_{S,n}}$ and $t\leqslant_{\mathfrak{t}_{\xi(S,n)}}t^*\}$ is disjoint to $\{t_{\beta,m}:\beta<\delta^*$ and $m<\omega\}\cup\{s_{\beta,m}:\beta<\beta^*$ and $m<\omega\}$. This is possible because the lower cofinality of $L^{\mathfrak{t}_{\xi(S,n)}}$ is the same as that of L_0 and we have $\lambda>\theta+|\beta^*|$. Then we choose $\eta^*\in\mathbf{J}$ such that $(\forall x)(\langle\zeta\rangle{}^\frown\langle\eta^*\rangle{}^\frown x\in\mathfrak{t}_{\xi(S,n)}\Rightarrow\langle\zeta\rangle{}^\frown\langle\eta^*\rangle{}^\frown\langle x\rangle<_{\mathfrak{t}_{\xi(S,n)}}t^*)$, and we choose also $\langle t_{n'}:n'\in w_1$ and $\xi_{S,n'}=\xi_{S,n}\rangle$ such that $t_n\in\{\langle\zeta\rangle{}^\frown\langle\eta\rangle{}^\frown\langle x\rangle\in\mathfrak{s}_\zeta:\eta<_{\mathbf{J}}\eta^*\}$, taking care of $\mathcal{W}$ (inside $w'_{1,n}:=\{m\in w_1:\xi(t_{\alpha,m})=\xi_{S,m}\}$ and automatically for others, i.e. considering t_{n_1} and t_{n_2} such that $\xi_{S,n_1}\neq\xi_{S,n_2}$). This is immediate.

For $\mathfrak{t}_\zeta$. This case is similar (using $\boxdot_1$ and $\boxdot_2$).

We have proved Subclaim 3.4 and Lemma 3.1. □

We may like to have "$2^{\aleph_0}=\lambda$ is singular", $\mathfrak{a}=\lambda$ and $\mathfrak{b}=\mathfrak{d}=\mu$. Toward this we would like to have a linear order $\mathbf{J}$ such that if $\bar{x}=\langle x_\alpha:\alpha<\theta\rangle$ is monotonic, say decreasing, then for any $\sigma<\lambda$, for some limit $\delta<\theta$ of uncountable cofinality the linear order $\{y\in\mathbf{J}:\alpha<\delta\Rightarrow y<_{\mathbf{J}}x_\alpha\}$ has cofinality $>\sigma$. Moreover, δ can be chosen to suit ω such sequences $\bar{x}$ simultaneously. So every set of ω-tuples from $\mathbf{J}$ of cardinality $\geqslant\theta$ but less than λ can be "inflated".

LEMMA 3.5. *Assume that*

(a) $(2^{\aleph_0})^+<\mu=\mathrm{cf}(\mu)<\lambda=\lambda^{\aleph_0}$, λ *singular*;

(b) $(\forall\alpha<\mu)[|\alpha|^{\aleph_0}<\mu]$;

(c) $\mu\geqslant\aleph_{\mathrm{cf}(\lambda)}$, *or at least that*

(c$^-$) *there is* $f:\lambda\to\mathrm{cf}(\lambda)$ *such that if* $\langle\alpha_\varepsilon:\varepsilon<\mu\rangle$ *is strictly increasing continuous,* $\alpha_\varepsilon<\lambda$ *and* $\gamma<\mathrm{cf}(\lambda)$, *then for some* $\varepsilon<\tau$ *we have* $f(\alpha_\varepsilon)\geqslant\gamma$.

Then for some c.c.c. forcing notion of cardinality λ *we have* $\Vdash_{\mathbf{P}}$ "$2^{\aleph_0}=\lambda$, $\mathfrak{b}=\mathfrak{d}=\varkappa$ *and* $\mathfrak{a}=\lambda$".

Proof. Note that (c) $\Rightarrow$ (c$^-$). Just let $\alpha<\lambda$ & $\mathrm{cf}(\alpha)=\aleph_\varepsilon$ & $\varepsilon<\mathrm{cf}(\lambda)\Rightarrow f(\alpha)=\varepsilon$; clearly there is such a function, and it satisfies clause (c$^-$). So we can assume (c$^-$). Let $\sigma=\mathrm{cf}(\lambda)$

and $\langle \lambda_\varepsilon : \varepsilon<\sigma\rangle$ be a strictly increasing sequence of regular cardinals greater than $\mu+\sigma$ with limit λ. Let L_0, L_0^+, L_0^- be as in the proof of Lemma 3.1, $L_{0,\varepsilon}$ be the unique interval of L_0 of order type (the inverse of λ_ε)$+\lambda_\varepsilon$ (so that $\langle L_{0,\varepsilon}:\varepsilon<\sigma\rangle$ is increasing with union L_0), and $L_{0,\varepsilon}$ is an interval of $L_{0,\xi}$ for $\varepsilon<\xi<\sigma$. We define $g\colon L_0\to \mathrm{cf}(\lambda)$ as follows: if $x\in L_0^+$ then $g(x)=f(\mathrm{otp}(\{y\in L_0^+ : y<_L x\},<))$; if $x\in L_0^-$ and the order type (otp) of $(\{y\in L_0^+ : x<_L y\},<_L)$ is the inverse of γ, then $g(x)=f(\gamma)$; and let

$$\mathbf{J}^* = \{\eta\in {}^{\omega>}(L_0) : \eta(0)\in L_{0,0} \text{ and } \eta(n+1)\in L_{0,g(\eta(n))} \text{ for } n<\omega\},$$

ordered as in the proof of Lemma 3.5.

We define $\mathfrak{s}_\zeta$ and $\mathfrak{t}_\zeta$ as there. We then prove that $\mathfrak{s}_\zeta$ and $\mathfrak{t}_\zeta$ are (τ,θ)-good and (λ,τ)-good as there, and this suffices repeating the proof of Lemma 3.1. □

Discussion 3.6. We may like to separate $\mathfrak{b}$ and $\mathfrak{d}$. So below we adapt the proof of Lemma 3.1 to do this (we can do it also for Lemma 3.5).

A way to do this is to look at the forcing in Lemma 3.1 as the limit of the FS-iteration $\langle \mathbf{P}_i^*, \underset{\sim}{\mathbf{Q}}_j^* : i\leqslant\mu \text{ and } j<\mu\rangle$, so that the memory of $\mathbf{Q}_j^*$ is $\{i : i<j\}$, where $\underset{\sim}{\mathbf{Q}}_j^*$ is $\mathrm{Lim}_{\mathfrak{t}}[\langle \mathbf{Q}_t : t\in L^{\mathfrak{t}_j}\rangle]$. Below we will use the limit of the FS-iteration $\langle \mathbf{P}_i^*, \underset{\sim}{\mathbf{Q}}_j^* : j<\mu\times\mu_1\rangle$, so that $\mathbf{Q}_\zeta^*$ has memory $w_\zeta\subseteq\zeta$, where, e.g., for $\zeta=\mu\alpha+i$, with $i<\mu$, $w_\zeta=\{\varkappa\beta+j : \beta\leqslant\alpha,\, j\leqslant i$ and $(\beta,j)\neq(\alpha,i)\}$. Let $\mathbf{P}^*=\mathbf{P}^*_{\mu\times\mu_1}=\bigcup\{\mathbf{P}_i : i<\mu\times\mu_1\}$.

Of course, $\mathbf{Q}_\zeta$ will be defined as $\mathrm{Lim}_{\mathfrak{t}_\zeta}(\overline{\mathbf{Q}})$, $\mathfrak{t}_\zeta$ defined as above, $\mathfrak{b}=\mu$ and $\mathfrak{d}=\mu_1$. This should be easy. If $\langle \underset{\sim}{A}_\varepsilon : \varepsilon<\varepsilon^{\bar{x}}\rangle$ exemplifies $\mathfrak{a}$ in $\mathbf{V}^{\mathbf{P}^*}$, and thus $\varepsilon^*\geqslant\mu$, then for some $(\alpha^*,\beta^*)\in\mu\times\mu_1$, for $\varkappa\,(=\theta)$ of the names they involve $\{\underset{\sim}{\mathbf{Q}}_{\mu\alpha+\beta} : \alpha\leqslant\alpha^*, \beta\leqslant\beta^*\}$ only.

Using indiscernibility on the pairs (α,β) to make them increase we can finish.

LEMMA 3.7. (1) *In Lemma* 3.1, *if* $\mu=\mathrm{cf}(\mu)\leqslant\mathrm{cf}(\mu_1)$, $\mu_1<\lambda$, *then we can change the conclusion* $\mathfrak{b}=\mathfrak{d}=\mu$ *to* $\mathfrak{b}=\mu$ *and* $\mathfrak{d}=\mu_1$.

(2) *Similarly for Lemma* 3.5.

Proof. First assume that μ_1 is regular.

First proof. Let $\mu_0=\mu$. In the proof of Lemma 3.1, for $l\in\{0,1\}$, using $\mu=\mu_l$ gives $\mathfrak{s}^l_{\mu_l}$, and without loss of generality, $\mathfrak{s}^0_{\mu_0}$ and $\mathfrak{s}^1_{\mu_1}$ are disjoint. Let $\mathfrak{s}$ be $\mathfrak{s}_0+'\mathfrak{s}_1$, by which we mean that $L^{\mathfrak{s}}=L^{\mathfrak{s}^0_{\mu_0}}+L^{\mathfrak{s}^1_{\mu_1}}$, and for $t\in L^{\mathfrak{s}^l_{\mu_l}}$ we let $I^{\mathfrak{s}}_t:=I^{\mathfrak{s}^l_{\mu_l}}_t$ (this is not $\mathfrak{s}_0+\mathfrak{s}_1$ of Claim 2.11). Now the appropriate goodness can be proved, so we can prove $\mathfrak{a}=\lambda$. Easily we get $\mathfrak{d}\geqslant\mu_l$ and $\mathfrak{b}\leqslant\mu_0$. This is enough to get inequality, but to get exact values we turn to the second proof.

Second proof. Instead of starting with $\langle \mathbf{Q}_i : i<\mu\rangle$ with full memory, we start with $\langle \underset{\sim}{\mathbf{Q}}_\zeta : \zeta<\mu\times\mu_1\rangle$, and $\underset{\sim}{\mathbf{Q}}_\zeta$ having the following "memory": if $\zeta=\mu\alpha+i$, with $i<\varkappa$, then

$$w_\zeta = \{\mu\beta+j : \beta\leqslant\alpha,\ j\leqslant i \text{ and } (\beta,j)\neq(\alpha,i)\}.$$

To deal with the case when μ_1 is singular, we should use a μ-directed index set (instead of $\mu_0 \times \mu_1$) as the product of ordered sets. □

4. On related cardinal invariants

Explanation of §4. On Theorem 4.1 you may wonder: $\mathfrak{u}$ has nothing to do with order or quite directed family, so how can we preserve small $\mathfrak{u}$? It is true that using the "directed character" of $\mathfrak{b}$ and $\mathfrak{d}$ has been the idea, i.e. in the end we have that $\mathbf{P}=\langle \mathbf{P}_i : i<\mu\rangle$ is $\lessdot$-increasing, $\mathbf{P}=\bigcup\{\mathbf{P}_i : i<\mu\}$ and $\underset{\sim}{\eta}_i$ is a $\mathbf{P}_{i+1}$-name of a real dominating $\mathbf{V}^{\mathbf{P}_i}$. But what we really need, for $(\mathbf{P}, \underset{\sim}{\bar\eta})$ as above, is that taking ultrapower by the $\varkappa$-complete ultrafilter D preserves the property of $\underset{\sim}{\bar\eta}$, in our present case $\underset{\sim}{\bar\eta}$ has to witness $\mathfrak{u}=\mu$. For being a dominating real this is very natural (Łoś' theorem). But here we shall use $\langle \underset{\sim}{D}_i : i<\mu\rangle$, with $\underset{\sim}{D}_i$ being a $\mathbf{P}_i$-name of an ultrafilter on ω, and demand $\underset{\sim}{\eta}_i$ to be mod finite included in every member of $\underset{\sim}{D}_i$, and moreover $\underset{\sim}{\eta}_i$ to be generic over $\mathbf{V}^{\mathbf{P}_i}$ for a forcing related to $\underset{\sim}{D}_i$. When we like to preserve something in an inductive construction on $\alpha<\lambda$ of $\langle \mathbf{P}^\alpha_i : i<\mu\rangle$, it is reasonable to have a stronger induction hypothesis than needed just for the final conclusion. We here need a condition on $(\mathbf{P}^\alpha_{i+1}, \underset{\sim}{\eta}^\alpha_i, \mathbf{P}^\alpha_i, \underset{\sim}{D}^\alpha_i)$ preserved by the ultrapower (as the relevant forcing is c.c.c., nicely enough defined in this work).

Secondly, we need for limit α: if $\mathrm{cf}(\alpha)>\aleph_0$ it is straightforward, if not, being generic for the $\mathbf{Q}_i$ has nice enough properties so that we can complete $\bigcup_{\beta<\alpha} \underset{\sim}{D}^\beta_i$ to a suitable ultrafilter.

This explains to some extent the scope of possible applications; of course, in each case the exact inductive assumption on $(\mathbf{P}^\alpha_{i+1}, \underset{\sim}{\eta}^\alpha_i, \mathbf{P}^\alpha_i, \underset{\sim}{Y}^\alpha_i)$ with $\underset{\sim}{Y}^\alpha_i$ a relevant witness, varies.

THEOREM 4.1. *Assume that*

(a) *$\varkappa$ is a measurable cardinal;*

(b) $\varkappa<\mu=\mathrm{cf}(\mu)<\lambda=\mathrm{cf}(\lambda)=\lambda^\varkappa$.

Then for some c.c.c. forcing notion $\mathbf{P}$ *of cardinality* λ, *in* $\mathbf{V}^{\mathbf{P}}$ *we have* $2^{\aleph_0}=\lambda$, $\mathfrak{u}=\mathfrak{d}=\mathfrak{b}=\mu$ *and* $\mathfrak{a}=\lambda$.

Remark. Recall that

$$\mathfrak{u} = \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\omega]^{\aleph_0} \text{ generates a non-principal ultrafilter on } \omega\}.$$

The proof of Theorem 4.1 is broken into definitions and claims.

Definition 4.2. For a filter D on ω (to which all cofinite subsets of ω belong) let $\mathbf{Q}(D)$ be

$$\{T : T \subseteq {}^{\omega>}\omega \text{ is closed under initial segments, and for some } \mathrm{tr}(T), \text{ the trunk of } T, \text{ we have (i) } l \leqslant \lg(\mathrm{tr}(T)) \Rightarrow T \cap {}^{l}\omega = \{\mathrm{tr}(T) \restriction l\}; \text{ (ii) } \mathrm{tr}(T) \trianglelefteq \eta \in {}^{\omega>}\omega \Rightarrow \{n : \eta\hat{\ }\langle n\rangle \in T\} \in \underset{\sim}{D}\},$$

ordered by inverse inclusion.

Definition 4.3. Let $\mathfrak{K}$ be the family of $\mathfrak{t}$ consisting of

$$\bar{\mathbf{Q}} = \bar{\mathbf{Q}}^{\mathfrak{t}} = \langle \mathbf{P}_i, \underset{\sim}{\mathbf{Q}}_i : i<\mu\rangle = \langle \mathbf{P}_i^{\mathfrak{t}}, \underset{\sim}{\mathbf{Q}}_i^{\mathfrak{t}} : i<\mu\rangle;$$
$$\bar{D} = \bar{D}^{\mathfrak{t}} = \langle \underset{\sim}{D}_i : i<\mu \text{ and } \mathrm{cf}(i)\neq\varkappa\rangle = \langle D_i^{\mathfrak{t}} : i<\mu \text{ and } \mathrm{cf}(i)\neq\varkappa\rangle;$$
$$\bar{\tau}^{\mathfrak{t}} = \langle \underset{\sim}{\tau}_i^{\mathfrak{t}} : i<\mu\rangle$$

such that

(a) $\bar{\mathbf{Q}}$ is an FS-iteration of c.c.c. forcing notions (and $\mathbf{P}_\mu = \mathbf{P}_\mu^{\mathfrak{t}} = \mathrm{Lim}(\bar{\mathbf{Q}}^{\mathfrak{t}}) = \bigcup_{i<\mu} \mathbf{P}_i^{\mathfrak{t}}$);
(b) if $i<\mu$ and $\mathrm{cf}(i)\neq\varkappa$ then $\mathbf{Q}_i = \mathbf{Q}(\underset{\sim}{D}_i)$, see Definition 4.2 above;
(c) $\underset{\sim}{D}_i$ is a P_i-name of a non-principal ultrafilter on ω when $i<\mu$ and $\mathrm{cf}(i)\neq\varkappa$;
(d) $|\mathbf{P}_i| \leqslant \lambda$;
(e) for $i<\mu$ and $\mathrm{cf}(i)\neq\varkappa$ let $\underset{\sim}{\eta}_i$ be the $\mathbf{P}_{i+1}$-name of the $\underset{\sim}{\mathbf{Q}}_i$-generic real,

$$\underset{\sim}{\eta}_i = \bigcup \{\mathrm{tr}(p(i)) : p \in \underset{\sim}{G}_{\mathbf{P}_{i+1}}\},$$

and we demand that for $i<j<\mu$ of cofinality $\neq\varkappa$ we have

$$\Vdash_{\mathbf{P}_j} \text{“}\mathrm{Rang}(\underset{\sim}{\eta}_i) \in \underset{\sim}{D}_j\text{”};$$

(f) $\underset{\sim}{\tau}_i$ is a $\mathbf{P}_i$-name of a function from $\underset{\sim}{\mathbf{Q}}_i$ to $\{h : h$ is a function from a finite set of ordinals to $\mathcal{H}(\omega)\}$ such that $\Vdash_{\mathbf{P}_i}$ “$p, q \in \underset{\sim}{\mathbf{Q}}_i$ are compatible (in $\underset{\sim}{\mathbf{Q}}_i$) if and only if the functions $\underset{\sim}{\tau}_i(p)$ and $\underset{\sim}{\tau}_i(q)$ are compatible (i.e.

$$\tau_i(p) \restriction (\mathrm{Dom}(\underset{\sim}{\tau}_i(p)) \cap \mathrm{Dom}(\underset{\sim}{\tau}_i(q))) = \tau_i(q) \restriction (\mathrm{Dom}(\underset{\sim}{\tau}_i(p)) \cap \mathrm{Dom}(\underset{\sim}{\tau}_i(q))),$$

and then they have a common upper bound r such that $\underset{\sim}{\tau}_i(r) = \underset{\sim}{\tau}_i(p) \cup \underset{\sim}{\tau}_i(q)$”;

(g) if $\mathrm{cf}(i)\neq\varkappa$, $i\in\mathrm{Dom}(p)$, $p\in\mathbf{P}_j$ and $i<j\leqslant\mu$, then $\underset{\sim}{\tau}_i(p(i))$ is $\{\langle 0,\mathrm{tr}(p)\rangle\}$; i.e. this is forced to hold;

(h) we stipulate $\mathbf{P}_i=\{p:p$ is a function with domain a finite subset of i such that for each $j\in\mathrm{Dom}(p)$, $\varnothing_{\mathbf{P}_j}$ forces that $p(j)\in\underset{\sim}{\mathbf{Q}}_j$ and it forces a value to $\underset{\sim}{\tau}_j(p(j))\}$;

(i) $\Vdash_{\mathbf{P}_i}$ "$\underset{\sim}{\mathbf{Q}}_i\subseteq\mathcal{H}_{<\aleph_1}(\gamma)$ for some ordinal γ".

Let $\gamma(\mathfrak{t})$ be the minimal ordinal γ such that

$$i<\mu \implies \Vdash_{\mathbf{P}_i}\text{"if } x\in\underset{\sim}{\mathbf{Q}}_i \text{ then } \mathrm{dom}(\underset{\sim}{\tau}_i(x))\subseteq\gamma\text{"}.$$

We let $\tau_i^{\mathfrak{t}}$ be the function with domain $\mathbf{P}_i$ such that $\tau_i^{\mathfrak{t}}(p)$ is a function with domain $\{\gamma(\mathfrak{t})j+\beta : j\in\mathrm{Dom}(p)$ and $p\restriction j\Vdash_{\mathbf{P}_j}$ "$\beta\in\mathrm{Dom}(\underset{\sim}{\tau}_j(p(j)))$"$\}$, and we let $\tau_i^{\mathfrak{t}}(\gamma(\mathfrak{t})j+\beta)$ be the value which $p\restriction j$ forces on $\underset{\sim}{\tau}_j^{\mathfrak{t}}(\beta)$.

Obviously we have the following inequality.

Subclaim 4.4. $\mathfrak{K}\neq\varnothing$.

Recall the following result.

Subclaim 4.5. *If in a universe* $\mathbf{V}$, D *is a non-principal ultrafilter on* ω, *then*

(a) $\Vdash_{\mathbf{Q}(D)}$ "$\{\mathrm{tr}(p)(l) : l<\lg(\mathrm{tr}(p))$ *and* $p\in\underset{\sim}{G}_{\mathbf{Q}(D)}\}$ *is an infinite subset of* ω, *almost included in every member of* D";

(b) $\mathbf{Q}(D)$ *is a c.c.c. forcing notion, even* σ*-centered*;

(c) $\underset{\sim}{\eta}_i=\bigcup\{\mathrm{tr}(p):p\in\underset{\sim}{G}_{\mathbf{Q}(D)}\}\in{}^{\omega}\omega$ *is forced to dominate* $({}^{\omega}\omega)^{\mathbf{V}}$;

(d) $\{p\in\mathbf{Q}[D]:\mathrm{tr}(p)=\eta\}$ *is a directed subset of* $\mathbf{Q}[D]$.

(*Note that this, in particular clause* (c), *does not depend on additional properties of* D; *but as we naturally add many Cohen reals* (*by the nature of the support*), *we may add more and can then demand, e.g., that* $\underset{\sim}{D}_i$ ($\mathrm{cf}(i)\neq\varkappa$) *is a Ramsey ultrafilter.*)

Definition 4.6. (1) We define $\leqslant_{\mathfrak{K}}$ by saying that $\mathfrak{t}\leqslant_{\mathfrak{K}}\mathfrak{s}$ if ($\mathfrak{t},\mathfrak{s}\in\mathfrak{K}$ and)

(i) $i\leqslant\mu\Rightarrow\mathbf{P}_i^{\mathfrak{t}}\lessdot\mathbf{P}_i^{\mathfrak{s}}$;

(ii) $i<\mu\ \&\ \mathrm{cf}(i)\neq\varkappa\Rightarrow\Vdash_{\mathbf{P}_i^{\mathfrak{s}}}$ "$\underset{\sim}{D}_i^{\mathfrak{t}}\subseteq\underset{\sim}{D}_i^{\mathfrak{s}}$";

(iii) $i<\mu\Rightarrow\Vdash_{\mathbf{P}_i^{\mathfrak{s}}}$ "$\underset{\sim}{\tau}_i^{\mathfrak{t}}\subseteq\underset{\sim}{\tau}_i^{\mathfrak{s}}$".

(2) We say that $\mathfrak{t}$ is a *canonical* $\leqslant_{\mathfrak{K}}$*-upper bound* of $\langle\mathfrak{t}_\alpha:\alpha<\delta\rangle$ if

(i) $\mathfrak{t},\mathfrak{t}_\alpha\in\mathfrak{K}$;

(ii) $\alpha\leqslant\beta<\delta\Rightarrow\mathfrak{t}_\alpha\leqslant_{\mathfrak{K}}\mathfrak{t}_\beta\leqslant_{\mathfrak{K}}\mathfrak{t}$;

(iii) $i<\mu$ and $\mathrm{cf}(i)=\varkappa$ implies $\Vdash_{\mathbf{P}_i^{\mathfrak{t}}}$ "$\underset{\sim}{\mathbf{Q}}_i^{\mathfrak{t}}=\bigcup_{\alpha<\delta}\underset{\sim}{\mathbf{Q}}_i^{\mathfrak{t}_\alpha}$".

Note that if $\mathrm{cf}(\delta)>\aleph_0$ then $\Vdash_{\mathbf{P}_i^{\mathfrak{t}}}$ "$\underset{\sim}{\mathbf{Q}}_i^{\mathfrak{t}}=\bigcup_{\alpha<\delta}\underset{\sim}{\mathbf{Q}}_i^{\mathfrak{t}_\alpha}$" for every $i<\mu$, so $\mathfrak{t}$ is totally determined.

(3) We say that $\langle\mathfrak{t}_\alpha:\alpha<\alpha^*\rangle$ is $\leqslant_{\mathfrak{K}}$*-increasing continuous* if $\alpha<\beta<\alpha^*\Rightarrow\mathfrak{t}_\alpha\leqslant_{\mathfrak{K}}\mathfrak{t}_\beta$ and for limit $\delta<\alpha^*$, $\mathfrak{t}_\delta$ is a canonical $\leqslant_{\mathfrak{K}}$-upper bound of $\langle\mathfrak{t}_\alpha:\alpha<\delta\rangle$. Note that we have not

said "the canonical $\leqslant_{\mathfrak{K}}$-upper bound", as for $\delta<\alpha^*$ and $\mathrm{cf}(\delta)=\aleph_0$ we have some freedom in completing $\bigcup\{\underset{\sim}{D}_i^{t_\alpha}:\alpha<\delta\}$ to an ultrafilter (on ω in $\mathbf{V}^{\mathbf{P}_i^{t}}$, when $i<\mu$, $\mathrm{cf}(i)\neq\varkappa$).

SUBCLAIM 4.7. *If $\mathbf{P}_1\lessdot\mathbf{P}_2$ and $\underset{\sim}{D}_l$ is a $\mathbf{P}_l$-name of a non-principal ultrafilter on ω for $l=1,2$ and $\Vdash_{\mathbf{P}_2}$ "$\underset{\sim}{D}_1\subseteq\underset{\sim}{D}_2$", then $\mathbf{P}_1*\mathbf{Q}(\underset{\sim}{D}_1)\lessdot\mathbf{P}_2*\mathbf{Q}(\underset{\sim}{D}_2)$.*

Proof. First, we can force with $\mathbf{P}_1$, so without loss of generality, $\mathbf{P}_1$ is trivial and $D_1\in\mathbf{V}$ is a non-principal ultrafilter on ω. Now clearly $p\in\mathbf{Q}(D_1)\Rightarrow p\in\mathbf{Q}(\underset{\sim}{D}_2)$ and $\mathbf{Q}(D_1)\vDash p\leqslant q\Rightarrow\mathbf{Q}(\underset{\sim}{D}_2)\vDash p\leqslant q$, and if $p,q\in\mathbf{Q}(D_1)$ are incompatible in $\mathbf{Q}(D_1)$, then they are incompatible in $\mathbf{Q}(\underset{\sim}{D}_2)$. Lastly, in $\mathbf{V}$, let $\mathcal{I}=\{p_n:n<\omega\}\subseteq\mathbf{Q}(D_1)$ be predense in $\mathbf{Q}(D_1)$. We shall prove that $\mathcal{I}$ is predense in $\mathbf{Q}(\underset{\sim}{D}_2)$. For this it suffices to note that

⊠ if D_1 is a non-principal ultrafilter on ω, $\mathcal{I}\subseteq\mathbf{Q}(D_1)$ and $\eta\in{}^{\omega>}\omega$, then the following conditions are equivalent:

(a_η) there is no $p\in\mathbf{Q}(D_1)$ incompatible with every $q\in\mathcal{I}$ which satisfies $\mathrm{tr}(p)=\eta$;

(b_η) there is a set T such that

(i) $\nu\in T\Rightarrow\eta\trianglelefteq\nu\in p$;

(ii) $\eta\trianglelefteq\nu\trianglelefteq\varrho\in T\Rightarrow\nu\in T$;

(iii) if $\nu\in T$ then either $\{n:\nu^\frown\langle n\rangle\in T\}\in D_1$ or $(\forall n)(\nu^\frown\langle n\rangle\notin T)$ & $(\exists q\in\mathcal{I})(\nu=\mathrm{tr}(q))$;

(iv) there is a strictly decreasing function $h:T\to\omega_1$;

(v) $\eta\in p$.

The proof of ⊠ is straightforward.

So, as in $\mathbf{V}$, $\mathcal{I}\subseteq\mathbf{Q}(D_1)$ is predense, for every $\eta\in{}^{\omega>}\omega$ we have (a_η) for D_1, and hence by ⊠ we also have $\eta\in{}^{\omega>}\omega\Rightarrow(\mathrm{b}_\eta)$. But clearly if T_η witnesses $(\mathrm{b})_\eta$ in $\mathbf{V}$ for D_1, it witnesses (b_η) in $\mathbf{V}^{\mathbf{P}_2}$ for D_2. Hence applying ⊠ again we get $\eta\in{}^{\omega>}\omega\Rightarrow(\mathrm{a}_\eta)$ in $\mathbf{V}^{\mathbf{P}_2}$ for D_2, and so $\mathcal{I}$ is predense in $\mathbf{Q}(D_2)$ in $\mathbf{V}^{\mathbf{P}_2}$. We have proved Subclaim 4.7. □

SUBCLAIM 4.8. *If $\bar{\mathfrak{t}}=\langle\mathfrak{t}_\alpha:\alpha<\delta\rangle$ is $\leqslant_{\mathfrak{K}}$-increasing continuous and $\delta<\lambda^+$ is a limit ordinal, then it has a canonical $\leqslant_{\mathfrak{K}}$-upper bound.*

Proof. By induction on $i<\mu$, we define $\mathbf{P}_i^{\mathfrak{t}}$, and if $i<\mu$ we then have $\underset{\sim}{\mathbf{Q}}_i^{\mathfrak{t}}$, $\underset{\sim}{\tau}_i$ and $\underset{\sim}{D}_i$ (if $\mathrm{cf}(i)\neq\varkappa$) such that the relevant demands (for $\mathfrak{t}\in\mathfrak{K}$ and for being a canonical $\leqslant_{\mathfrak{K}}$-upper bound of $\bar{\mathfrak{t}}$) hold.

Defining $\mathbf{P}_i^{\mathfrak{t}}$ is obvious: for $i=0$ trivially, if $i=j+1$ it is $\mathbf{P}_j^{\mathfrak{t}}*\underset{\sim}{\mathbf{Q}}_j^{\mathfrak{t}}$, and if i is limit it is $\bigcup\{\mathbf{P}_j^{\mathfrak{t}}:j<i\}$.

If $\mathbf{P}_i^{\mathfrak{t}}$ has been defined and $\mathrm{cf}(i)=\varkappa$, we let $\underset{\sim}{\mathbf{Q}}_i^{\mathfrak{t}}=\bigcup_{\alpha<\delta}\mathbf{Q}_i^{\mathfrak{t}_\alpha}$ and $\underset{\sim}{\tau}_i^{\mathfrak{t}}=\bigcup_{\alpha<\delta}\underset{\sim}{\tau}_i^{\mathfrak{t}_\alpha}$. It is easy to check that they are as required. If $\mathbf{P}_i^t$ has been defined and $\mathrm{cf}(i)\neq\varkappa$, then $\bigcup_{\alpha<\delta}D_i^t$ is a filter on ω containing the cobounded subsets, and we complete it to an ultrafilter.

Note that there is such a $\underset{\sim}{D}_i^{t}$ because

(a) for $\alpha<\delta$ we have $\mathbf{P}^{\mathrm{t}_\alpha}\lessdot\mathbf{P}^{\mathrm{t}}_i$, and hence $\Vdash_{\mathbf{P}^{\mathrm{t}}_i}$ "$\underset{\sim}{D}^{\mathrm{t}_\alpha}_i$ is a filter on ω to which all cofinite subsets of ω belong, and it increases with α".

Note that there will be no need for new values of the $\underset{\sim}{\tau}_i$'s, nor any freedom in defining them. As we have proved the relevant demands on $\mathbf{P}^{\mathrm{t}}_j$ and $\underset{\sim}{\mathbf{Q}}^{\mathrm{t}}_j$ for $j<i$, clearly $\mathbf{P}^{\mathrm{t}}_i$ is c.c.c. by using $\langle\underset{\sim}{\tau}_j : j<i\rangle$, and clearly $\langle\mathbf{P}^{\mathrm{t}}_\zeta, \underset{\sim}{\mathbf{Q}}^{\mathrm{t}}_\xi : \zeta\leqslant i \text{ and } \xi<i\rangle$ is an FS-iteration. Now we shall prove that $\alpha<\delta \Rightarrow \mathbf{P}^{\mathrm{t}_\alpha}_i\lessdot\mathbf{P}^{\mathrm{t}}_i$.

So let $\mathcal{I}$ be a predense subset of $\mathbf{P}^{\mathrm{t}_\alpha}_i$ and $p\in\mathbf{P}^{\mathrm{t}}_i$, and we should prove that p is compatible with some $q\in\mathcal{I}$ in $\mathbf{P}^{\mathrm{t}}_i$; we divide the proof into three cases.

Case 1: *i is a limit ordinal.* If $j\notin\mathrm{Dom}(p)$, it is trivial. Otherwise $p\in\mathbf{P}^{\mathrm{t}}_j$ for some $j<i$. Let $\mathcal{I}'=\{q{\restriction}j : q\in\mathcal{I}\}$, so that clearly $\mathcal{I}'$ is a predense subset of $\mathbf{P}^{\mathrm{t}_\alpha}_j$ (as $\mathrm{t}_\alpha\in\mathfrak{K}$). By the induction hypothesis, in P^{t}_j the condition p is compatible with some $q'\in\mathcal{I}'$; so let $r'\in\mathbf{P}^{\mathrm{t}}_j$ be a common upper bound of q' and p, and let $q'=q{\restriction}j$, where $q\in\mathcal{I}$. Then $r\cup(q{\restriction}[j,i))\in\mathbf{P}^{\mathrm{t}}_i$ is a common upper bound of q and p as required.

Case 2: *$i=j+1$ and* $\mathrm{cf}(j)=\varkappa$. Then without loss of generality, for some $\beta<\delta$, $p(j)$ is a $\mathbf{P}^{\mathrm{t}_\beta}_j$-name of a member of $\underset{\sim}{\mathbf{Q}}^{\mathrm{t}_\beta}_j$; and also without loss of generality $\alpha\leqslant\beta<\delta$. By the induction hypothesis, $\mathbf{P}^{\mathrm{t}_\beta}_j\lessdot\mathbf{P}^{\mathrm{t}}_j$. Hence there is $p'\in\mathbf{P}^{\mathrm{t}_\beta}_j$ such that $[p'\leqslant p''\in\mathbf{P}^{\mathrm{t}_\beta}_j\Rightarrow p''$, $p{\restriction}j$ are compatible in $\mathbf{P}^{\mathrm{t}}_j]$.

Let

$$\mathcal{J}=\left\{q'{\restriction}j : q'\in\mathbf{P}^{\mathrm{t}_\beta}_i,\ q' \text{ is above some member of } \mathcal{I} \text{ and } q'{\restriction}j\Vdash_{\mathbf{P}^{\mathrm{t}_\beta}_j}\text{“}p(j)\leqslant^{\underset{\sim}{\mathbf{Q}}^{\mathrm{t}_\beta}_j}q'(j)\text{”}\right\}.$$

Now $\mathcal{J}$ is a dense subset of $\mathbf{P}^{\mathrm{t}_\beta}_j$ (since if $q\in\mathbf{P}^{\mathrm{t}_\beta}_j$ then $q\cup\{\langle j,p(j)\rangle\}$ belongs to $\mathbf{P}^{\mathrm{t}_\beta}_i$ and is hence compatible with some member of $\mathcal{I}$).

Hence p' is compatible with some $q''\in\mathcal{J}$ (in $\mathbf{P}^{\mathrm{t}_\beta}_j$), so there is r such that $p'\leqslant r\in\mathbf{P}^{\mathrm{t}_\beta}_j$ and $q''\leqslant r$. As $q''\in\mathcal{J}$ there is $q'\in\mathbf{P}^{\mathrm{t}_\beta}_i$ such that $q'{\restriction}j=q''$, q' is above some $q^*\in\mathcal{I}$ and

$$q'{\restriction}j\Vdash\text{“}p(j)\leqslant^{\underset{\sim}{\mathbf{Q}}^{\mathrm{t}_\beta}_j}q'(j)\text{”}.$$

As $\mathbf{P}^{\mathrm{t}_\beta}_j\vDash$ "$p'\leqslant r$ & $q'{\restriction}j=q''\leqslant r$" and by the choice of p' there is $p^*\in\mathbf{P}^{\mathrm{t}}_j$ above r (hence above p' and above $q''=q'{\restriction}j$) and above $p{\restriction}j$. Now let $r^*=p^*\cup(q''{\restriction}\{j\})$. Clearly $r^*\in\mathbf{P}^{\mathrm{t}}_i$ is above $p{\restriction}j$, and $r^*{\restriction}j$ forces that $r^*(j)$ is above $p{\restriction}\{j\}$. Clearly $r^*{\restriction}j$ is above r, and r^* is also above $q^*\in\mathcal{I}$, so we are done.

Case 3: *$i=j+1$ and* $\mathrm{cf}(j)\neq\varkappa$. Use Subclaim 4.7 above.

So we have dealt with $\alpha<\delta\Rightarrow\mathbf{P}^{\mathrm{t}_\alpha}_i\lessdot\mathbf{P}^{\mathrm{t}}_i$.

Clearly we are done with the proof of Subclaim 4.8. □

SUBCLAIM 4.9. *If* $\mathfrak{t}\in\mathfrak{K}$ *and* E *is a* $\varkappa$*-complete non-principal ultrafilter on* $\varkappa$*, then we can find* $\mathfrak{s}$ *such that*

(i) $\mathfrak{t}\leqslant_{\mathfrak{K}}\mathfrak{s}\in\mathfrak{K}$;

(ii) *there is* $\langle \mathbf{k}_i,\mathbf{j}_i : i<\mu$ *and* $\mathrm{cf}(i)\neq\varkappa\rangle$ *such that*

(α) $\mathbf{k}_i$ *is an isomorphism from* $(\mathbf{P}^{\mathfrak{t}}_i)^{\varkappa}/E$ *onto* $\mathbf{P}^{\mathfrak{s}}_i$;

(β) $\mathbf{j}_i$ *is the canonical embedding of* $\mathbf{P}^{\mathfrak{t}}_i$ *into* $(\mathbf{P}^{\mathfrak{t}}_i)^{\varkappa}/E$;

(γ) $\mathbf{k}_i\circ\mathbf{j}_i$ *equals the identity on* $\mathbf{P}^{\mathfrak{t}}_i$;

(iii) $\underset{\sim}{D}^{\mathfrak{s}}_i$ *is the image of* $(\underset{\sim}{D}_i)^{\varkappa}/E$ *under* $\mathbf{k}_i$*, and similarly* $\underset{\sim}{\tau}^{\mathfrak{s}}_i$ *if* $i<\mu$ *and* $\mathrm{cf}(i)\neq\varkappa$;

(iv) *if* $i<\mu$ *and* $\mathrm{cf}(i)=\varkappa$*, then* $\underset{\sim}{\tau}^{\mathfrak{s}}_i$ *is defined such that for* $j<\varkappa$ *and* $\mathrm{cf}(j)\neq\varkappa$ *we have that* $\mathbf{k}_j$ *is an isomorphism from* $(\mathbf{P}^{\mathfrak{t}}_i,\gamma',\tau^{\mathfrak{t}}_i)^{\varkappa}/D$ *onto* $(\mathbf{P}^{\mathfrak{s}}_i,\gamma'',\tau^{\mathfrak{s},i}_i)$ *for some ordinals* γ' *and* γ'' *(except that we do not require that the map from* γ' *to* γ'' *preserves order).*

Proof. This is straightforward. Note that if $\mathrm{cf}(i)=\varkappa$ and $i<\mu$, then $\underset{\sim}{\mathbf{Q}}^{\mathfrak{s}}_i$ is isomorphic to $\mathbf{P}^{\mathfrak{s}}_{i+1}/\mathbf{P}^{\mathfrak{s}}_i$, which is c.c.c., as by Łoś' theorem for the logic $\mathbf{L}_{\varkappa,\varkappa}$ we have $\bigcup_{j<i}(\mathbf{P}^{\mathfrak{t}}_j)^{\varkappa}/E\lessdot(\mathbf{P}^{\mathfrak{t}}_{i+1})^{\varkappa}/E$; similarly for $\underset{\sim}{\tau}_i$, which guarantees that the quotient is c.c.c. too (actually $\underset{\sim}{\tau}_i$ is not needed for the c.c.c. here). □

SUBCLAIM 4.10. *If* $\mathfrak{t}\in\mathfrak{K}$ *then* $\Vdash_{\mathbf{P}^{\mathfrak{t}}_\mu}$ "$\mathfrak{u}=\mathfrak{b}=\mathfrak{d}=\mu$".

Proof. In $\mathbf{V}^{\mathbf{P}^{\mathfrak{t}}_\mu}$, the family $\mathcal{D}=\{\mathrm{Rang}(\underset{\sim}{\eta}_i):i<\mu \text{ and } \mathrm{cf}(i)\neq\varkappa\}\cup\{[n,\omega):n<\omega\}$ generates a filter on $\mathcal{P}(\omega)^{\mathbf{V}[\mathbf{P}^{\mathfrak{t}}_\mu]}$, as $\mathrm{Rang}(\underset{\sim}{\eta}_i)\in[\omega]^{\aleph_0}$ and $i<j<\mu\ \&\ \mathrm{cf}(i)\neq\varkappa\ \&\ \mathrm{cf}(j)\neq\varkappa\Rightarrow\mathrm{Rang}(\underset{\sim}{\eta}_j)\subseteq^*\mathrm{Rang}(\underset{\sim}{\eta}_i)$.

Also it is an ultrafilter, as $\mathcal{P}(\omega)^{\mathbf{V}[\mathbf{P}^{\mathfrak{t}}_\mu]}=\bigcup_{i<\mu}\mathcal{P}(\omega)^{\mathbf{V}[\mathbf{P}^{\mathfrak{t}}_i]}$, and if $i<\mu$ then $\mathrm{Rang}(\underset{\sim}{\eta}_{i+1})$ induces an ultrafilter on $\mathcal{P}(\omega)^{\mathbf{V}[\mathbf{P}^{\mathfrak{t}}_{i+1}]}$. So we have $\mathfrak{u}\leqslant\mu$. Also $({}^{\omega}\omega)^{\mathbf{V}[\mathbf{P}^{\mathfrak{t}}_\mu]}=\bigcup_{i<\mu}({}^{\omega}\omega)^{\mathbf{V}[\mathbf{P}^{\mathfrak{t}}_i]}$, $({}^{\omega}\omega)^{\mathbf{V}[\mathbf{P}^{\mathfrak{t}}_i]}$ is increasing with i, and if $\mathrm{cf}(i)\neq\varkappa$ then $\underset{\sim}{\eta}_i\in{}^{\omega}\omega$ dominates $({}^{\omega}\omega)^{\mathbf{V}[\mathbf{P}^{\mathfrak{t}}_i]}$ by Subclaim 4.5, and so $\mathfrak{b}=\mathfrak{d}=\mu$ as in previous cases. Lastly, always $\mathfrak{u}\geqslant\mathfrak{b}$, and hence $\mathfrak{u}=\mu$. □

Now we define $\mathfrak{t}_\alpha\in\mathfrak{K}$ for $\alpha\leqslant\lambda$, by induction on α, satisfying that $\langle\mathfrak{t}_\alpha:\alpha\leqslant\lambda\rangle$ is $\leqslant_{\mathfrak{K}}$-increasing continuous and such that $\mathfrak{t}_{\alpha+1}$ is obtained from $\mathfrak{t}_\alpha$ as in Subclaim 4.9. Let $\mathbf{P}=\mathbf{P}^{\mathfrak{t}_\lambda}_\mu$. Then $|\mathbf{P}|\leqslant\lambda$, hence $(2^{\aleph_0})^{\mathbf{V}^{\mathbf{P}}}\leqslant(\lambda^{\aleph_0})^{\mathbf{V}}$, and equality easily holds.

We finish by the following subclaim.

SUBCLAIM 4.11. $\Vdash_{\mathbf{P}_\lambda}$ "$\mathfrak{a}\geqslant\mathrm{cf}(\lambda)$".

Proof. Assume toward a contradiction that $\theta<\mathrm{cf}(\lambda)$, $p\in\mathbf{P}$ and $p\Vdash_{\mathbf{P}}$ "$\underset{\sim}{\mathcal{A}}=\{\underset{\sim}{A}_i:i<\theta\}$ is a MAD family", where $\underset{\sim}{\mathcal{A}}$ is a *MAD family* if

(i) $A_i\in[\omega]^{\aleph_0}$;

(ii) $i\neq j\Rightarrow|\underset{\sim}{A}_i\cap\underset{\sim}{A}_j|<\aleph_0$;

(iii) under (i) and (ii), $\underset{\sim}{\mathcal{A}}$ is maximal.

Without loss of generality, $\Vdash_{\mathbf{P}}$ "$\underset{\sim}{A}_i\in[\omega]^{\aleph_0}$". As $\mathfrak{a}\geqslant\mathfrak{b}=\mu$ by Subclaim 4.10, we have $\theta\geqslant\mu$. For each $i<\theta$ and $m<\omega$ there is a maximal antichain $\langle p_{i,m,n}:n<\omega\rangle$ of $\mathbf{P}$ and there is a sequence $\langle \mathbf{t}_{i,m,n} : n<\omega \rangle$ of truth values such that $p_{i,m,n} \Vdash$ "$(m \in \underset{\sim}{A}_i) \equiv \mathbf{t}_{i,m,n}$". We can find countable $w_i \subseteq \mu$ such that $\bigcup_{m,n<\omega} \mathrm{Dom}(p_{i,m,n}) \subseteq w_i$. Possibly increasing w_i and retaining countability, we can find $\langle R_{i,\gamma} : \gamma \in w_i \rangle$ such that

(α) w_i has a maximal element and $\gamma \in w_i \setminus \{\max(w_i)\} \Rightarrow \gamma+1 \in w_i$;

(β) $R_{i,\gamma}$ is a countable subset of $\mathbf{P}^{\mathbf{t}_\lambda}_\gamma$ and $q \in R_{i,\gamma} \Rightarrow \mathrm{Dom}(q) \subseteq w_i \cap \gamma$;

(γ) for $\gamma_1 < \gamma_2$ in w_i, $q \in R_{i,\gamma_2} \Rightarrow q \restriction \gamma_1 \in R_{i,\gamma_1}$;

(δ) for $\gamma_1 \in w_i$, $\gamma \in \gamma_1 \cap w_i$ and $q \in R_{i,\gamma_1}$, the $\mathbf{P}^{\mathbf{t}}_\gamma$-name $q(\gamma)$ involves $\aleph_0$ maximal antichains all included in $R_{i,\gamma}$;

(ε) $\{p_{i,m,n} : m,n\} \subseteq R_{i,\max(w_i)}$.

Since $\mathrm{cf}(\lambda) > \aleph_0$ (as $\mu < \lambda = \mathrm{cf}(\lambda)$ by the assumption of Theorem 4.1), we have $\mathbf{P}^{\mathbf{t}}_\mu = \bigcup_{\alpha<\lambda} \mathbf{P}^{\mathbf{t}_\alpha}_\mu$. Clearly for some $\alpha<\lambda$ we have $\bigcup \{R_{i,\gamma} : i<\theta,\ \gamma \in w_i\} \subseteq \mathbf{P}^{\mathbf{t}_\alpha}_\mu$. But $\mathbf{P}^{\mathbf{t}_\alpha}_\mu \lessdot \mathbf{P}^{\mathbf{t}_\lambda}_\mu$, and so $\Vdash_{\mathbf{P}^{\mathbf{t}_\alpha}_\mu}$ "$\underset{\sim}{\mathcal{A}} = \{\underset{\sim}{A}_i : i<\theta\}$ is MAD".

Now, letting $\mathbf{j}$ be the canonical elementary embedding of $\mathbf{V}$ into $\mathbf{V}^\varkappa/D$, we know that

($*$) in $\mathbf{V}^\varkappa/D$, $\mathbf{j}(\underset{\sim}{\mathcal{A}})$ is a $\mathbf{j}(\mathbf{P}^{\mathbf{t}_\alpha}_\mu)$-name of a MAD family.

As $\mathbf{V}^\varkappa/D$ is $\varkappa$-closed, for c.c.c. forcing notions things are absolute enough; but $\{\mathbf{j}(i) : i<\mu\}$ is not $\{i : \mathbf{V}^\varkappa/D \models i<\mathbf{j}(\mu)\}$, so in $\mathbf{V}$, it is forced for $\Vdash_{\mathbf{j}(\mathbf{P}^{\mathbf{t}}_\mu)}$ that $\{\mathbf{j}(\underset{\sim}{A}_i) : i<\mu\}$ is not MAD!

Chasing arrows, clearly $\Vdash_{\mathbf{P}^{\mathbf{t}_{\alpha+1}}_\mu}$ "$\{\underset{\sim}{A}_i : i<\theta\}$ is not MAD", as required.

The proof of Subclaim 4.11, and hence of Theorem 4.1, is complete. □

Discussion 4.12. We can now look at other problems, e.g. what can be the order and equalities among $\mathfrak{d}$, $\mathfrak{b}$, $\mathfrak{a}$ and $\mathfrak{u}$; we have not considered it. We have considered having $\mathfrak{i} = \mu$, but there was a problem.

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