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# Inverse scattering on asymptotically hyperbolic manifolds

by

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## 1. Introduction

In this paper, we study scattering for Schrödinger operators on asymptotically hyperbolic manifolds. In particular, we show that the scattering matrix depends meromorphically on the energy $\zeta\in\mathbf{C}$, and for the values of $\zeta$ where it is defined, it is a pseudo-differential operator of order $2\operatorname{Re}\zeta-n$ (really complex order $2\zeta-n$), where the dimension of the manifold is $n+1$. We then show that the total symbol of this operator is determined locally by the metric and the potential, and that, except for a countable set of energies, the asymptotics of either the metric or the potential can be recovered from the scattering matrix at one energy. This also allows us to characterize the total symbol of the scattering matrix in the case where the manifold is of product type modulo terms vanishing to infinite order at the boundary.

We remark that the fact that the scattering matrix at energy $\zeta$ is a pseudo-differential operator is a known result, see for example §8.4 of [29]. However, the proof in the general case does not seem to be available in the literature. Proofs of several particular cases have been given, see for example [6], [13], [14], [17] and [37], and references given there.

Recall that a compact manifold with boundary $(X,\partial X)$ is asymptotically hyperbolic when it is equipped with a metric of the form

$$g=\frac{H}{x^2},$$

where $x$ is a defining function of $\partial X$, and $H$ is a smooth Riemannian metric on $X$, which is smooth and non-degenerate up to $\partial X$. Furthermore we assume that $|dx|_H=1$ at $\partial X$. In other words, $g$ can be expressed by

$$g=\frac{dx^2+h(x,y,dx,dy)}{x^2}, \tag{1.1}$$

for some boundary-defining function $x$, and a product decomposition $X \sim \partial X \times [0, \varepsilon)$ near the boundary, where $h|_{x=0}$ is independent of $dx^2$, and $h|_{x=0}$ induces a Riemannian metric on $\partial X$. As observed in [28] this implies that along a smooth curve in $X \setminus \partial X$, approaching a point $p \in \partial X$, the sectional curvatures of $g$ approach $-1$. We note that this form is invariant under multiplying $x$ by a function of $y$, so there is no canonical metric on $\partial X$ induced by $g$, but there is a natural conformal structure. The simplest example of such a manifold is the hyperbolic space, $\mathbf{H}^{n+1}$, and its quotients by certain discrete group actions.

Let $\Delta_g$ be the (positive) Laplacian on $X$ induced by $g$. It will be shown in §4 that given a function $f \in C^\infty(\partial X)$ and $\zeta \in \mathbf{C} \setminus (-\infty, 0]$, with $2\zeta \in \mathbf{C} \setminus \mathbf{Z}$, which is not a pole of the meromorphic continuation of the resolvent, there exists a unique solution of the equation $(\Delta_g + \zeta(\zeta - n))u = 0$ of the form

$$u = x^\zeta f_+ + x^{n-\zeta} f_-, \tag{1.2}$$

with $f_+, f_- \in C^\infty(X)$ and $f = f_-|_{\partial X}$. This is implicit in [26], [28] and is stated without a proof in [29]. A related result is also stated in the introduction of [28]. The first terms of the expansion with $\operatorname{Re} \zeta = \frac{1}{2}n$ have been established in [6].

It is then natural to define, as in [29], for these values of $\zeta$, the scattering matrix to be the map

$$T(\zeta): f \mapsto f_-|_{\partial X}. \tag{1.3}$$

However, the scattering matrix is then (mildly) dependent on the choice of boundary-defining function $x$, and so we instead define it as a map

$$S(\zeta): C^\infty(\partial X, |N^*(\partial X)|^{n-\zeta}) \to C^\infty(\partial X, |N^*(\partial X)|^\zeta) \tag{1.4}$$

via $S(\zeta)(f|dx|^{n-\zeta}) = (T(\zeta)f)|dx|^\zeta$, and it is then invariant. Whilst this definition can not make sense for $\zeta$ such that $2\zeta \in \mathbf{Z}$, as the decomposition (1.2) can not then be unique, and the uniqueness of the expansion (1.2) is not established in §4 for $\zeta \in (-\infty, 0]$, we shall see in Proposition 4.4 that the scattering matrix can be defined as a restriction of the resolvent. This allows a meromorphic continuation of $S(\zeta)$ across points which are not poles of (4.36). In fact, we show that these results remain valid in the presence of a short-range potential $V$, which in this context means

$$V \in C^\infty(\overline{X}, \mathbf{R}), \quad V = x\widetilde{V}, \quad \widetilde{V} \in C^\infty(\overline{X}, \mathbf{R}). \tag{1.5}$$

As we will see in §3, such a potential does not affect the normal operator of $\Delta_g$, and that allows the construction of [28] to go through without significant change.

For simplicity, we work in a product decomposition such that

$$g=\frac{dx^2+h(x,y,dy)}{x^2}. \tag{1.6}$$

The existence of such a model form is established in §2. This yields a trivialization of the normal bundle which we work with. In §5 we prove

THEOREM 1.1. *Let $(X,\partial X)$ be a smooth manifold with boundary. Suppose that $g$ induces an asymptotically hyperbolic structure on $X$, and that $g=(dx^2+h(x,y,dy))/x^2$ with respect to some product decomposition near $\partial X$. Let $V\in C^\infty(X)$ be a short-range potential and let $\zeta\in\mathbf{C}$ be such that the scattering matrix, $S(\zeta)$, associated to $\Delta_g+V+\zeta(\zeta-n)$ is defined. Then $S(\zeta)\in\Psi^{2\operatorname{Re}\zeta-n}$, and its principal symbol equals $C(\zeta)|\xi|^{2\zeta-n}$, where $|\xi|$ is the length of the covector $\xi$ induced by $h_0=h(0,y,dy)$, and*

$$C(\zeta)=2^{n-2\zeta}\frac{\Gamma\left(\frac{1}{2}n-\zeta\right)}{\Gamma\left(\zeta-\frac{1}{2}n\right)}.$$

This result has been established in [6] for $\operatorname{Re}\zeta=\frac{1}{2}n$. As a direct consequence we obtain

COROLLARY 1.1. *Let $(X,\partial X)$ be a smooth manifold with boundary, and let $p\in\partial X$. Suppose that $g_1$ and $g_2$ induce asymptotically hyperbolic structures on $X$, and that $g_i=(dx^2+h_i(x,y,dy))/x^2$, $i=1,2$, with respect to some product decomposition. Let $V_i$, $i=1,2$, satisfy (1.5), and let $S_i(\zeta)$ be the scattering matrix associated to $\Delta_{g_i}+V_i+\zeta(\zeta-n)$. There exists a discrete set $Q\subset\mathbf{C}$ such that $S_2(\zeta)-S_1(\zeta)\in\Psi^{2\operatorname{Re}\zeta-n-1}$ for $\zeta\in\mathbf{C}\setminus Q$ if and only if $h_1(0,y,dy)=h_2(0,y,dy)$.*

We then analyze the difference of the scattering matrices when the metrics $g_1,g_2$, and the potentials $V_1,V_2$, agree to order $k\geqslant 1$ at the boundary. We also prove in §5

THEOREM 1.2. *Let $(X,\partial X)$ be a smooth manifold with boundary, and let $p\in\partial X$. Suppose that $g_1$ and $g_2$ induce asymptotically hyperbolic structures on $X$, and that $g_i=(dx^2+h_i(x,y,dy))/x^2$, $i=1,2$, with respect to some product decomposition. Moreover suppose that $h_2-h_1=x^kL(y,dy)+O(x^{k+1})$, $k\geqslant 1$, near $p$, and that $V_1,V_2$ are smooth short-range potentials such that $V_2-V_1=x^kW(y)+O(x^{k+1})$ near $p$. Let $S_i(\zeta)$ be the scattering matrix associated to $\Delta_{g_i}+V_i+\zeta(\zeta-n)$. We then have that, near $p$,*

$$S_1(\zeta)-S_2(\zeta)\in\Psi^{2\operatorname{Re}\zeta-n-k}, \tag{1.7}$$

*and the principal symbol of $S_1(\zeta)-S_2(\zeta)$ equals*

$$A_1(k,\zeta)\sum_{i,j}H_{ij}\xi_i\xi_j|\xi|^{2\zeta-n-k-2}+A_2(k,\zeta)\left(W+\tfrac{1}{4}k(k+1)T\right)|\xi|^{2\zeta-n-k}, \tag{1.8}$$

*where* $H=h_0^{-1}Lh_0^{-1}$ *as matrices,* $h_0=h_1|_{x=0}=h_2|_{x=0}$, $T=\mathrm{trace}(h_0^{-1}L)$, $|\xi|$ *is the length of the covector* $\xi$ *induced by* $h_0$, *and* $A_1, A_2$ *are meromorphic functions of* $\zeta$, *given by*

$$\begin{aligned} A_1(k,\zeta) &= -\pi^{n/2}2^{k+2-2\zeta+n}\frac{\Gamma\left(\frac{1}{2}(k+2-2\zeta+n)\right)}{\Gamma\left(-\frac{1}{2}(k+2-2\zeta)\right)}\cdot\frac{C(\zeta)}{M(\zeta)}T_1(k,\zeta), \\ A_2(k,\zeta) &= \pi^{n/2}2^{k-2\zeta+n}\frac{\Gamma\left(\frac{1}{2}(k-2\zeta+n)\right)}{\Gamma\left(-\frac{1}{2}(k-2\zeta)\right)}\cdot\frac{C(\zeta)}{M(\zeta)}T_2(k,\zeta), \end{aligned} \tag{1.9}$$

*where* $C(\zeta)$ *is given by* (5.4), $T_j(k,\zeta)$, $j=1,2$, *is the meromorphic continuation of the functions given by* (5.4), *and* $M(\zeta)$ *is given by Proposition* 4.2.

As an application of Theorem 1.2 we analyze the cases where the manifold is actually hyperbolic and is almost of product type.

THEOREM 1.3. *If* $(X,\partial X)$ *is such that in some product decomposition the metric is a product modulo terms vanishing to infinite order at the boundary, then the scattering matrix is equal to*

$$2^{n-2\zeta}\frac{\Gamma\left(\frac{1}{2}n-\zeta\right)}{\Gamma\left(\zeta-\frac{1}{2}n\right)}\Delta_{\partial X}^{\zeta-n/2},$$

*modulo smoothing. If* $(X,\partial X)$ *is a smooth hyperbolic manifold, the same result holds modulo pseudo-differential operators of order* $2\operatorname{Re}\zeta-n-2$. *Here we have chosen a defining function* $x$ *in order to trivialize the normal bundle and to induce a metric on the boundary, with respect to which we take* $\Delta_{\partial X}$.

We prove the result for almost product-type structures in §6. In the hyperbolic case, the result for principal symbols is due to Perry, [37] (Perry's definition of the scattering matrix was slightly different which caused an extra factor to be present). The result for hyperbolic manifolds is an immediate consequence of Theorem 1.2 and observing that a funnel is product type to second order.

As consequences of Theorem 1.2 we have the inverse results:

COROLLARY 1.2. *Let* $(X,\partial X), g_j, V_j, S_j$, $j=1,2$, *be as in Theorem* 1.2, *let* $p\in\partial X$ *and suppose that* $V_1=V_2$ *near* $p$. *There exists a discrete set* $Q\subset\mathbf{C}$ *such that if* $\zeta\in\mathbf{C}\setminus Q$ *and* $S_1(\zeta)-S_2(\zeta)\in\Psi^{2\operatorname{Re}\zeta-n-k}$, $k\geqslant1$, *near* $p$, *then there exists a diffeomorphism* $\psi$ *of a neighbourhood* $U\subset X$ *of* $p$, *fixing* $\partial X$, *such that* $\psi^*g_1-g_2=O(x^{k-2})$.

COROLLARY 1.3. *Let* $(X,\partial X), g_j, V_j, S_j$, $j=1,2$, *be as in Theorem* 1.2, *let* $p\in\partial X$ *and suppose that* $g_1=g_2$ *near* $p$. *There exists a discrete set* $Q\subset\mathbf{C}$ *such that if* $\zeta\in\mathbf{C}\setminus Q$ *and* $S_1(\zeta)-S_2(\zeta)\in\Psi^{2\operatorname{Re}\zeta-n-k}$, $k\geqslant0$, *near* $p$, *then* $V_1-V_2=O(x^k)$ *near* $p$.

Of course, intersecting over all $k$, we see that off a countable set of energies a metric or potential can be recovered modulo terms vanishing to infinite order at the boundary. We will prove these corollaries in §5, after the proof of Theorems 1.1 and 1.2.

In §7 we give an application of these results, or rather of the methods used to prove them, to inverse scattering on the Schwarzschild and the De Sitter–Schwarzschild model of black holes. We show that the Taylor series at the boundary of certain perturbations of these models can be recovered from the scattering matrix at a fixed energy.

Our approach is heavily influenced by the work of Guillopé and Zworski, [13], [14], [15]. In particular, we compute the scattering matrix as a boundary value of the resolvent. To do this we use the calculus developed by Mazzeo and Melrose, [28], of zero-pseudo-differential operators in order to construct the resolvent.

As in our work on asymptotically Euclidean scattering, [21], [23], [24], a key part of our approach is to consider the principal symbol of the difference of the scattering matrices rather than the lower-order terms of the symbol of a single operator, which allows us to proceed more invariantly. We remark that whilst our results are quite similar to those in the Euclidean case, the proofs and underlying ideas are very different. The fundamental reason for this is that in the asymptotically Euclidean category, as observed by Melrose, [30], and by Melrose and Zworski, [33], there is propagation of growth at infinity, whilst this does not occur in the asymptotically hyperbolic category. This is reflected in the fact, proved in [33], that in the asymptotically Euclidean case the scattering matrix is a Fourier integral operator associated to the geodesic flow at time $\pi$, whilst in the asymptotically hyperbolic manifold case it is a pseudo-differential operator, and in the fact that the principal symbol of the difference of the scattering matrices is locally determined by the perturbation. See [32] for a discussion of a general framework including both cases.

There is a long history of scattering theory on hyperbolic manifolds arising from the observation that the Eisenstein series for a Fuchsian group is a generalized eigenfunction for the Laplacian on the associated quotient of hyperbolic space—the fundamental reference for this is [25] where the finite-volume case is studied. The study of the infinite-volume case was initiated by Patterson in [36]. There has been a wealth of results in both cases, and we refer the reader to [14] for a comprehensive bibliography and to [17] and [29] for a review of the subject. There has been less work on asymptotically hyperbolic spaces. Mazzeo and Melrose, [28], and Mazzeo, [26], [27], studied properties of the Laplacian on such manifolds from which the properties of the scattering matrix proved in §4 are implicit. In [6] Borthwick showed the continuous dependence of the scattering matrix on the metric. Agmon has also studied related questions, see [1], [2]. Recently Borthwick and Perry, [8], showed that, except for possibly finitely many exceptions, the poles of the resolvent agree with multiplicity with those of the scattering matrix. Andersson, Chrusciel and Friedrichs have studied solutions of the Einstein equations and related problems on asymptotically hyperbolic spaces, [3], [4]. There appears to be no results in the literature on the inverse scattering problem on asymptotically hyperbolic manifolds. Perry, [38], has shown that for hyperbolic quotients in three dimensions by convex, cocompact, torsion-free Kleinian groups with non-empty regular set, the scattering matrix determines the manifold. Borthwick, McRae and Taylor have proved an associated rigidity result, [7].

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## 2. A model form

In this section, we establish a model form for asymptotically hyperbolic metrics near infinity (the boundary). This is very similar in statement and proof to the model form for scattering metrics proved in [24]. See also [22] for a similar model for exact $b$-metrics. In the case where all sectional curvatures are equal to $-1$ near the boundary a related normal form has been established in [16].

PROPOSITION 2.1. *Let $(X, \partial X)$ be a smooth manifold with boundary $\partial X$, and suppose that $g$ is a metric on $X$ such that*

$$g=\frac{dx^2+h(x,y,dx,dy)}{x^2}$$

*in some product decomposition near $\partial X$, where $x$ is a defining function of $\partial X$, with $h|_{x=0}$ independent of $dx^2$, and inducing a metric on $\partial X$. Then there exists a product decomposition, $(\bar{x},\bar{y})$, near $\partial X$ such that*

$$g=\frac{d\bar{x}^2+h(\bar{x},\bar{y},d\bar{y})}{\bar{x}^2}. \tag{2.1}$$

*Proof.* First we prove this result modulo terms that vanish to infinite order at $x=0$. It is enough to show the existence of a sequence of diffeomorphisms, $\psi_k$, of $\partial X\times[0,\varepsilon)$ such that

$$\psi_k^* g=\frac{d\bar{x}^2+h(\bar{x},\bar{y},d\bar{y})}{\bar{x}^2}+O(\bar{x}^{k-2}),\quad k\geqslant 1,$$

and

$$f_k = \psi_{k-1}^{-1}\psi_k$$

fixes the boundary to order $k+1$. This is enough, as a diffeomorphism $\psi$ can then be picked, using the Borel lemma, of which the $l$th term in the Taylor series will agree with that of $\psi_k$ for $l\leqslant k$, for all $l$.

First we observe that we can assume that $h(0,y,dx,dy)$ is independent of $dx^2$ and $dx\,dy_j$. Indeed, by defining the vector field $\nabla_x$ as $g(\nabla_x,\cdot\,)=dx$ and denoting $|\nabla_x|^2=g(\nabla_x,\nabla_x)$, the vector field $\Sigma=\nabla_x/|\nabla_x|^2$ is smooth in $X$, and moreover $\Sigma x=1$. For $p\in\partial X$, let $y_j$, $1\leqslant j\leqslant n$, be smooth local coordinates on $\partial X$ valid in a neighbourhood of $p$. Extending $y_j$ to be constant along the integral curves of $\Sigma$, which are perpendicular to $\partial X$, gives a coordinate system in $X$ valid in a neighbourhood of $(0,p)$. It is clear by the construction that $g(\Sigma,\partial_{y_j})=0$ at $\partial X$, $1\leqslant j\leqslant n$. Thus, in these coordinates, $h(0,y,dx,dy)$ is independent of $dx^2$ and $dx\,dy_j$. This gives the map $\psi_1$.

Next we proceed by induction. Suppose that $\psi_{k-1}$, $k\geqslant 1$, has been constructed. We show how to pick $f_k$ so that

$$f_k^*\psi_{k-1}^*g=\frac{d\bar{x}^2+h(\bar{x},\bar{y},d\bar{y})}{\bar{x}^2}+O(\bar{x}^{k-2}),\quad k\geqslant 2.$$

Putting $\psi_k=\psi_{k-1}f_k$, our result follows.

We work in local coordinates on the boundary. We shall see that the choice of the next term in the Taylor series is actually unique so there is no problem patching these local computations together. So suppose that we have, for $k\geqslant 2$,

$$\psi_{k-1}^*g=\frac{dx^2+h(x,y,dy)}{x^2}+x^{k-3}\alpha(y)\,dx^2+x^{k-3}\sum\beta_j(y)\,dx\,dy_j+O(x^{k-2}).$$

Putting

$$x=\bar{x}+\gamma(\bar{y})\,\bar{x}^l$$

and

$$y=\bar{y}+\delta(\bar{y})\,\bar{x}^l,$$

we have

$$dx=d\bar{x}+l\bar{x}^{l-1}\gamma(\bar{y})\,d\bar{x}+\bar{x}^l\frac{\partial\gamma}{\partial\bar{y}}\,d\bar{y}$$

and

$$dy=d\bar{y}+l\bar{x}^{l-1}\delta(\bar{y})\,d\bar{x}+\bar{x}^l\frac{\partial\delta}{\partial\bar{y}}(\bar{y})\,d\bar{y}.$$

Now if $h(0,y,dy)=\sum h_{ij}(y)\,dy_i\,dy_j$ and $l=k$, we see that the metric becomes, modulo $O(\bar{x}^{k-2})$-terms,

$$\frac{d\bar{x}^2+h(\bar{x},\bar{y},d\bar{y})}{\bar{x}^2}+\bar{x}^{k-3}\alpha(\bar{y})\,d\bar{x}^2+\bar{x}^{k-3}\sum\beta_j(\bar{y})\,d\bar{x}\,d\bar{y}_j$$
$$+2k\gamma(\bar{y})\bar{x}^{k-3}\,d\bar{x}^2+2k\bar{x}^{k-3}\sum h_{ij}\delta_i\,d\bar{y}_j\,d\bar{x}.$$

Pick $\gamma(\bar{y})=-\alpha(\bar{y})/2k$; as the form $h_{ij}(\bar{y})$ is non-degenerate, there is a unique choice of $\delta$ such that $2k\sum h_{ij}\delta_i=-\beta_j$. This cancels the terms of order $k-3$ in $d\bar{x}^2$ and $d\bar{x}\,d\bar{y}$. Equation (2.1), modulo $O(\bar{x}^\infty)$-terms, follows.

Having achieved the modulo form modulo $x^\infty$, which is all that is necessary for the rest of this paper, we now show that this form can be improved to remove this error. If

$$g=\frac{dx^2+h(x,y,dy)+O(x^\infty)}{x^2}$$

then the geodesic flow is generated by the Hamiltonian function

$$\sigma=x^2\tau^2+x^2\sum_{i,j}h^{ij}(x,y)\xi_i\xi_j+O(x^\infty).$$

Now if we work in rescaled zero-coordinates, that is, let $\bar{\tau}=x\tau$, $\bar{\xi}=x\xi$, and leave $(x,y)$ fixed, the canonical one-form $\alpha=\tau\,dx+\xi\cdot dy$ is rescaled to

$${}^0\alpha=\bar{\tau}\frac{dx}{x}+\bar{\xi}\cdot\frac{dy}{x},$$

and the Hamiltonian to

$$\bar{\sigma}=\bar{\tau}^2+\sum_{i,j}h^{ij}(x,y)\bar{\xi}_i\bar{\xi}_j+O(x^\infty).$$

The zero-Hamilton vector field of $\bar{\sigma}$, $H_{\bar{\sigma}}$, is defined by

$$d\,{}^0\alpha(\,\cdot\,,H_{\bar{\sigma}})=d\bar{\sigma}.$$

We then find that, modulo $O(x^\infty)$-terms,

$$H_{\bar{\sigma}}\equiv 2\bar{\tau}\left(x\frac{\partial}{\partial x}+\bar{\xi}\cdot\frac{\partial}{\partial\bar{\xi}}\right)-\left(2h^{-1}+x\frac{\partial}{\partial x}h^{-1}\right)\frac{\partial}{\partial\bar{\tau}}+xH_{h^{-1}},$$

where

$$h^{-1}(x,y,\bar{\xi})=\sum_{i,j}h^{ij}(x,y)\bar{\xi}_i\bar{\xi}_j\quad\text{and}\quad H_{h^{-1}}=\sum_i\left(\partial_{\bar{\xi}_i}h^{-1}\frac{\partial}{\partial y_i}-\frac{\partial}{\partial y_i}h^{-1}\partial_{\bar{\xi}_i}\right).$$

This is of the form

$$H_{\bar\sigma} = 2\bar\tau\left(x\frac{\partial}{\partial x}+\bar\xi\cdot\frac{\partial}{\partial\bar\xi}\right)+O(x^2+|\bar\xi|^2),$$

where $|\bar\xi|^2=h^{-1}(x,y,\bar\xi)$.

Now if we restrict to the cosphere bundle,

$$\bar\tau^2+h^{-1}(x,y,\bar\xi)=1,$$

which is invariant under the flow, we can re-express $\bar\tau$ in terms of $(x,y,\bar\xi)$, and near $\bar\tau=-1$ the vector field becomes

$$H_{\bar\sigma} = -2\left(x\frac{\partial}{\partial x}+\bar\xi\cdot\frac{\partial}{\partial\bar\xi}\right)+O(|\bar\xi|^2+x^2).$$

This forms a sink at $(x,\bar\xi)=0$ and thus, by Theorem 7 of [40], there exist local coordinates $(x',\xi')$, equal to $(x,\bar\xi)$ to second order at $(x,\bar\xi)=(0,0)$, which reduce the vector field to the form

$$H_{\bar\sigma} = -2\left(x'\frac{\partial}{\partial x'}+\xi'\cdot\frac{\partial}{\partial\xi'}\right)+O(|\xi'|^2+(x')^2)\frac{\partial}{\partial y}.$$

We therefore see that any integral curve starting close enough to $(x',\xi')=(0,0)$ will converge to $(0,0)$.

So in particular if we take a hypersurface $S_\varepsilon=\{x=\varepsilon\}$, then the geodesics starting on the unit normals pointing to the boundary will converge to $(x',\xi')=(0,0)$. We also have that the $x'$-derivatives of these geodesics will be non-zero so they can be reparametrized in terms of $x'$. In $x'>0$, we can put $\theta=\xi'/x'$, and use $(x',\theta,y)$ as coordinates. The form of the vector field means that the angular coordinate $\theta$ will be constant on geodesics, and we have

$$\frac{dy}{dx'}=O((x')^2+(x')^2|\theta|^2)=O((x')^2).$$

Now as the finite-time solution of an ODE, the point $y$ on the boundary, which is the limit of the geodesic, will vary smoothly with the start point on $S_\varepsilon$. We also see, as the change in $y$ is the integral of the derivative along the curve, that the Jacobian of the map from $S_\varepsilon$ to the boundary will be invertible for $\varepsilon$ sufficiently small. This will also be true for the map to hypersurfaces $S_{\varepsilon'}$, $\varepsilon'<\varepsilon$.

So if we now take geodesic normal coordinates to the hypersurface $S_\varepsilon$, then these give us a map $\chi:\partial X\times\mathbf{R}_+\to X$, which is smooth up to the boundary, and is a diffeomorphism in a neighbourhood of the boundary. Now the metric in these coordinates is of the form

$$(dx')^2+h(x',y,dy),$$

so if we put $X = e^{-x'}$ we get coordinates in a neighbourhood of infinity such that the metric is of the form

$$\frac{dX^2}{X^2} + h(X, y, dy).$$

Note that the change of coordinates gives the correct compactification at the boundary to ensure that there has been no change of smooth structure there. Thus we know that the transformed metric must be of the form

$$\frac{dX^2 + l(X, y, dy, dX)}{X^2},$$

with $l$ smooth up to $X=0$; so we conclude that $h(X,y,dy) = k(X,y,dy)/X^2$, with $k(X,y,dy)$ smooth up to $X=0$, and we are done. □

We remark that the construction of the Taylor series in the first part of the proof gave a boundary-defining function of the form $\bar{x} = x + O(x^2)$, and that the rest of the Taylor series was then determined; one could however start with a different defining function $\alpha(y)x$. This contrasts with the case of a scattering metric where the $x^1$-term is fixed by the metric but the $x^2$-term can be chosen.

## 3. Constructing the resolvents

In this section, we review the construction of the resolvent on an asymptotically hyperbolic manifold due to Mazzeo and Melrose, and show how to modify it to obtain information about the difference of two resolvents associated to data which agree to some order at the boundary. Our account is necessarily brief and we concentrate on explaining where our construction differs from theirs and refer the reader to their paper [28] for further details. We shall work with half-densities throughout as they give better invariance properties.

We recall that a Riemannian metric $g$ on a manifold $Y$ induces a canonical trivialization of the one-density bundle by taking $\omega = \sqrt{\delta}\,|dy|$ where $\delta$ is the determinant of $g_{ij}$ in the local coordinates $y$ on $Y$. The square root of this is then a natural trivialization of the half-density bundle. We then have a natural (self-adjoint) Laplacian, $\tilde{\Delta}$, acting on half-densities by

$$\tilde{\Delta}(f\omega^{1/2}) = \Delta(f)\omega^{1/2},$$

where $\Delta$ is the Laplacian acting on functions.

Mazzeo and Melrose showed that in an asymptotically hyperbolic manifold, the resolvent

$$R(\zeta) = (\tilde{\Delta} + \zeta(\zeta - n))^{-1}, \quad \operatorname{Re}\zeta > n,$$

which is a well-defined operator for $\operatorname{Re}\zeta>n$, depending holomorphically on $\zeta$, could be meromorphically continued to the entire complex plane, and that its extension could be constructed in a certain class of "zero"-pseudo-differential operators. A "zero"-vector field is a vector vanishing at the boundary, and a "zero"-differential operator is a composition of such vector fields, the most important example being the Laplacian associated to an asymptotically hyperbolic metric.

"Zero"-pseudo-differential operators have kernels living on the blown-up space $X\times_0 X$. This is the space obtained by blowing up $X\times X$ along the diagonal, $\Delta_{\partial X}$, of $\partial X\times\partial X$. We recall that blow-up is really just an invariant way of introducing polar coordinates, and that a function is smooth on the space $X\times_0 X$ if it is smooth in polar coordinates about $\Delta_{\partial X}$. As a set, $X\times_0 X$ is $X\times X$ with $\Delta_{\partial X}$ replaced by the interior pointing portion of its normal bundle. Let

$$\beta\colon X\times_0 X\to X\times X$$

denote the blow-down map. If $(x,y)$ are coordinates in a product decomposition of $X$ near $\partial X$, and we let $(x',y')$ be the corresponding coordinates on a second copy of $X$, then $R=(x^2+(x')^2+(y-y')^2)^{1/2}$ is a defining function for the new face, which we call the front face. The functions $\varrho=x/R$ and $\varrho'=x'/R$ are then defining functions for the other two boundary faces, which we call the top and bottom faces respectively. One advantage of working on this blown-up space is that the lift of the diagonal of $X\times X$ only meets the front face of the blown-up space and is disjoint from the other two boundary faces. See §3 of [28] for a picture.

To define the space of "zero"-pseudo-differential operators, Mazzeo and Melrose defined a bundle $\Gamma_0(X)$, whose sections are smooth multiples of the Riemannian density. Note that for the Riemannian structure (1.1), the natural density is singular at $\partial X$. In local coordinates $(x,y)$, where $x$ is a defining function of the boundary, it is given by

$$h(x,y)\,\frac{dx}{x}\,\frac{dy}{x^n},\quad h\in C^\infty(X),\, h\neq 0.$$

We denote $\Gamma_0^{1/2}(X)$ the analogous bundle of half-densities. Similarly we define the bundle $\Gamma_0^{1/2}(X\times X)$. The bundle $\Gamma_0^{1/2}$ over $X\times_0 X$ is then defined to be the lift of $\Gamma_0^{1/2}(X\times X)$ under the blow-down map.

A "small" zero-pseudo-differential operator of order $m$ is then an operator on $X$ of which the Schwartz kernel when lifted to $X\times_0 X$ vanishes to infinite order at the top and bottom faces, and is the restriction of a section of $\Gamma_0^{1/2}$ over the double across the front face, which is conormal to the lifted diagonal of order $m$. In the interior, these are of course just the usual class of pseudo-differential operators acting on half-densities.

The space of these kernels will be denoted $K_0^m(X)$, and the corresponding operators by $\Psi_0^m(X,\Gamma_0(X))$.

The "large class" $\Psi_0^{m,s,t}(X)$, $s,t\in\mathbf{C}$, is then defined to be operators which have Schwartz kernels that are equal to an element of $K_0^m(X)$ plus a function of the form $\varrho^s(\varrho')^t f$ with $f\in C^\infty(X\times_0 X,\Gamma_0^{1/2})$ and smooth up to the boundary. This space then has three natural filtrations, but it will also be important to consider a fourth which is the order of vanishing at the front face, so we commonly work with operators with kernels in the class $R^k\Psi_0^{m,s,t}(X)$. In [28], Mazzeo and Melrose show that the meromorphic extension of the resolvent to $\mathbf{C}$ lies in $\Psi_0^{-2,\zeta,\zeta}(X)$.

The ordinary symbol map, expressing the lead singularity at the diagonal in the interior, extends to this class and is a homogeneous section of the zero-cotangent bundle—that is, the dual bundle to the space of vector fields vanishing at the boundary. There is also a second natural symbol map, which is called the normal operator. This is obtained by restricting the Schwartz kernel to the front face, and hence expresses the lead term there, which is therefore a section of the bundle $\Gamma_0^{1/2}(X\times_0 X)$ restricted to that face.

Let $p\in\partial X$ and let $X_p$ be the inward-pointing vectors in $T_p(X)$. This is a manifold with boundary and has a metric

$$g_p=(dx)^{-2}h_p,$$

where $g=x^{-2}h$, making it isometric to the hyperbolic upper half-plane. (We regard $h_p$ and $dx$ as linear functions on the tangent space $X_p$.) Mazzeo and Melrose observed that the leaf of the front face above a point $p$ is naturally isomorphic to $X_p$, using a natural group action on the front face. This group action is obtained by lifting the action of the subgroup of the general linear group of the boundary of $X_p$ to the normal bundle of $X_p$, as a leaf of the front face is just a quarter of the normal bundle over $p$.

It is also observed in [28] that the restriction of $\Gamma_0^{1/2}(X\times_0 X)$ to the front face is canonically trivial, and then can act as a convolution operator using the natural group structure on the front face. As mentioned above, the fibre of the front face above a point $p$ can be identified with $X_p$. Let $(x,y)$ be local coordinates near $p\in\partial X$, with $x$ a boundary-defining function, and also denote the natural corresponding linear coordinates on $X_p$ by $(x,y)$. Let $(x',y')$ be the same coordinates on the right factor in $X\times X$, and let $s=x/x'$, $z=(y-y')/x$. Then if the Schwartz kernel of a map $B$ is $k(x',y',s,z)\gamma$ with $\gamma=|ds\,dz\,dx\,dy/sx^{n+1}|^{1/2}$, the normal operator is given at $p=(0,\bar{y})$ by

$$[N_p(B)(f\mu)]=\int k(0,\bar{y},s,z)\,f\Big(\frac{x}{s},y-\frac{x}{s}z\Big)\,\frac{ds}{s}\,dz\cdot\mu, \tag{3.1}$$

where

$$d\mu=\left|\frac{dx}{x}\,\frac{dy}{x^n}\right|^{1/2}.$$

This formula will be very useful in §5.

In fact, Mazzeo and Melrose only used the normal operator for terms in $\Psi_0^{-\infty,s,t}(X)$ but it works equally well for terms in $\Psi_0^{m,s,t}(X)$, see Theorem 4.16 of [28], the main difference being that the normal operator instead of being a smooth half-density on the front face, now has a conormal singularity at the centre, i.e. the intersection of the lift of the diagonal of $X\times X$ with the front face. The normal operator will of course have growth at the boundaries of the front face according to $s,t$. In particular, it will be in the space $\mathcal{A}^{s,t}$ of half-densities growing of order $s$ at the top edge, and of order $t$ at the bottom.

The important fact is that the normal operator of a zero-differential operator is obtained by freezing the coefficients at a point on the boundary, and the normal operator of the Laplacian is just the Laplacian of the induced metric on the space $X_p$. As a short-range potential vanishes at the boundary, if $P(\zeta)=\tilde{\Delta}+V+\zeta(\zeta-n)$, with $V$ short-range, and $Q\in\Psi_0^{m,s,t}(X)$, we thus have that

$$N_p(P(\zeta)Q)=(\tilde{\Delta}_p+\zeta(\zeta-n))N_p(Q), \tag{3.2}$$

see the proof of Proposition 5.19 of [28], with $\tilde{\Delta}_p$ the Laplacian on $X_p$, acting on half-densities, which is the model hyperbolic half-space up to a linear scaling.

Now what we are interested in in this section, and in this paper in general, is the structure of the difference of the resolvents associated to two pieces of data. We begin by proving

PROPOSITION 3.1. *Suppose that $g_1, g_2$ are asymptotically hyperbolic metrics which agree to order $k$ at $\partial X$, i.e. in some product decomposition $X\sim\partial X\times[0,\varepsilon)$ near $\partial X$, $x$ is a defining function of $\partial X$, in which*

$$g_l=\frac{dx^2+h_l(x,y,dy)}{x^2},\quad l=1,2, \tag{3.3}$$

*where*

$$h_2(x,y,dy)=h_1(x,y,dy)+x^kL(x,y,dy)+O(x^{k+1}),\quad k\geqslant 1.$$

*Suppose that $V_1,V_2$ are short-range potentials that satisfy $V_2-V_1=x^kW$, $W\in C^\infty(X)$. Let $\tilde{\Delta}_{g_l}$, $l=1,2$, be the (positive) Laplacian associated to $g_l$ acting on half-densities via the natural trivialization of the half-density bundle given by $g_l$, and let*

$$P_l(\zeta)=\tilde{\Delta}_{g_l}+V_l+\zeta(\zeta-n). \tag{3.4}$$

*Let $h_l(x,y)$ and $L(x,y)$ denote the matrices of coefficients of the tensors $h_l(x,y,dy)$ and $L(x,y,dy)$ respectively. We then have that for $H=h_1(0,y)^{-1}L(0,y)h_1(0,y)^{-1}$ and*

$T = \mathrm{Tr}(h_1(0,y)^{-1}L(0,y))$,

$$P_2 - P_1 = x^k\left(\sum_{i,j=1}^{n} H_{ij}x\partial_{y_i}x\partial_{y_j} + \tfrac{1}{4}k(k+1)T + W\right) + x^{k+1}R, \tag{3.5}$$

*with $R$ a second-order symmetric zero-differential operator.*

*Proof.* In local coordinates $(x,y)$ near $q \in \partial X$ the operator $P_l$ acts on a half-density $f(x,y)\,|dx\,dy|^{1/2}$ as

$$P_l(f(x,y)\,|dx\,dy|^{1/2}) = [\delta_l^{1/4}(\Delta_{g_l} + V + \zeta(\zeta-n))\delta_l^{-1/4}f(x,y)]\,|dx\,dy|^{1/2},$$

where $\Delta_{g_l}$ denotes the Laplacian acting on functions, and $\delta_l$ denotes the determinant of $g_l$.

So we need to consider the operator $\delta^{1/4}\Delta_g\delta^{-1/4} + V + \zeta(\zeta-n)$. Let $g_{ij}$ denote the components of the metric $g$, and $g^{ij}$ its inverse. Denote $z=(x,y)$ with $z_0 = x$, $z_j = y_j$, $1 \leqslant j \leqslant n$. So using the expression of $\Delta_g$ in local coordinates,

$$\Delta_g f = -\delta^{-1/2}\sum_{i,j=0}^{n} \partial_{z_i}(g^{ij}\delta^{1/2}\partial_{z_j}f),$$

we have

$$\delta^{1/4}\Delta_g(\delta^{-1/4}f) = \sum_{i,j=0}^{n} \delta^{-1/4}\partial_{z_i}(g^{ij}(f(\partial_{z_j}\delta^{1/4}) - \delta^{1/4}(\partial_{z_j}f))). \tag{3.6}$$

Recall that

$$\begin{aligned} g_{00} &= \frac{1}{x^2}, \quad g_{i0} = g_{0i} = 0, \quad i \neq 0, \\ g_{ij} &= \frac{1}{x^2}h_{ij}, \quad i,j \neq 0. \end{aligned} \tag{3.7}$$

Therefore

$$\begin{aligned} g^{00} &= x^2, \quad g^{i0} = g^{0i} = 0, \quad i \neq 0, \\ g^{ij} &= x^2 h^{ij}, \quad 1 \leqslant i,j \leqslant n. \end{aligned} \tag{3.8}$$

Using (3.3) we can write

$$h_2 = h_1(I + x^k h_1^{-1}L + O(x^{k+1})) \tag{3.9}$$

and therefore conclude that

$$\begin{aligned} h_2^{-1} &= h_1^{-1} - x^k h_1^{-1}Lh_1^{-1} + O(x^{k+1}), \\ \det h_2 &= \det h_1(1 + x^k\,\mathrm{Tr}(h_1^{-1}L) + O(x^{k+1})). \end{aligned} \tag{3.10}$$

We also deduce from (3.7) that

$$\delta_j = x^{-2(n+1)} \det h_j, \quad j=1,2. \tag{3.11}$$

Hence

$$\begin{aligned} \delta_2 &= \delta_1(1+x^k \operatorname{Tr}(h_1^{-1}L)+O(x^{k+1})), \\ \delta_2^{\pm 1/4} &= \delta_1^{\pm 1/4}(1 \pm x^k \cdot \tfrac{1}{4} \operatorname{Tr}(h_1^{-1}L)+O(x^{k+1})). \end{aligned} \tag{3.12}$$

Examining each term of (3.6) and using (3.8), (3.10), (3.11) and (3.12), we deduce that (3.5) holds. We carry out the computation for the term $i=j=0$. This is the term that will contribute with a multiple of $T$ to (3.5). The computations for the terms involving $y$-derivatives are easier and will be left to the reader.

$$\begin{aligned} \text{first term} &= \delta_2^{-1/4}\partial_x(x^2(f\partial_x\delta_2^{1/4}-\delta_2^{1/4}\partial_x f))-\delta_1^{-1/4}\partial_x(x^2(f\partial_x\delta_1^{1/4}-\delta_1^{1/4}\partial_x f)) \\ &= 2xf(\delta_2^{-1/4}\partial_x\delta_2^{1/4}-\delta_1^{-1/4}\partial_x\delta_1^{1/4})+x^2 f(\delta_2^{-1/4}\partial_x^2\delta_2^{1/4}-\delta_1^{-1/4}\partial_x^2\delta_1^{1/4}). \end{aligned}$$

Using that $v^{-1}\partial_x v=\partial_x \log v$ and that $v^{-1}\partial_x^2 v=\partial_x^2 \log v+(\partial_x \log v)^2$, we obtain

$$\text{first term} = \tfrac{1}{2}xf\partial_x \log\left(\frac{\delta_2}{\delta_1}\right)+\tfrac{1}{4}x^2 f\partial_x^2 \log\left(\frac{\delta_2}{\delta_1}\right)+\tfrac{1}{16}x^2 f\partial_x \log\left(\frac{\delta_2}{\delta_1}\right)\partial_x \log(\delta_2\delta_1).$$

Since $\log(1+u)=u+u^2 O(1)$, (3.12) gives that

$$\text{first term} = \tfrac{1}{2}xf(kx^{k-1}T)+\tfrac{1}{4}x^2 f(k(k-1)x^{k-2}T)+O(x^{k+1}) = \tfrac{1}{4}k(k+1)x^k Tf+O(x^{k+1}).$$

This ends the proof of the proposition. □

Let us denote $P_2-P_1=x^kE$, where $E$ is the operator given by the right-hand side of (3.5). Now let $R_1(\zeta)$ be the resolvent of $P_1$, which by Theorem 7.1 of [28] lies in $\Psi_0^{-2,\zeta,\zeta}(X)$. We then have

$$P_2(R_1-R_2) = P_2R_1-\mathrm{Id} = (P_2-P_1)R_1 = x^kER_1.$$

So to get $R_2$ as a perturbation of $R_1$ we need to solve

$$P_2F = x^kER_1. \tag{3.13}$$

We can rewrite this as

$$P_2((x')^kF_1) = (x')^k s^k ER_1,$$

with $s=x/x'$. As $x'$ commutes with $P_2$, this becomes

$$P_2F_1 = s^kER_1.$$

Now $s^k E R_1$ is in $\Psi_0^{0,\zeta+k,\zeta-k}(X)$, and we look for $F_1 \in \Psi_0^{-2,\zeta,\zeta-k}(X)$. To get improvement on the front face, we use normal operators, (3.2) and the fact that $V$ is short-range, to deduce that

$$(\tilde{\Delta}_p+\zeta(\zeta-n))N_p(F_1)=N_p(s^k E R_1). \tag{3.14}$$

This can be solved near the singularity by using the elliptic calculus, and away from this the right-hand side is in $\mathcal{A}^{\zeta+k,\zeta-k}$. Now Proposition 6.19 of [28] states that this equation has a meromorphic solution in $\mathcal{A}^{\zeta,\zeta-k}$. So we can choose $F_1$ meromorphically to satisfy (3.14).

Putting $F=(x')^k F_1 \in R^k \Psi_0^{-2,\zeta,\zeta}(X)$, we then have that

$$P_2(R_1-F)-\mathrm{Id} \in R^{k+1}\Psi_0^{0,\zeta,\zeta}(X). \tag{3.15}$$

We can then remove the term at the front face iteratively and asymptotically summing obtain

$$P_2(R_1-F')-\mathrm{Id} \in R^{\infty}\Psi_0^{0,\zeta,\zeta}(X),$$

with $F' \in R^k \Psi_0^{-2,\zeta,\zeta}(X)$. The error term now vanishes to infinite order at the front face. The diagonal singularity can be removed by an element of $R^\infty \Psi_0^{-2,\zeta,\zeta}(X)$ by standard symbolic arguments for constructing the parametrix of a pseudo-differential operator. This leaves an error in the class $x^\zeta (x')^\zeta f$, $f \in C^\infty(X\times X, \Gamma_0^{1/2}(X\times X))$. This can be removed using the indicial equation by an element of the same space as in [28].

We observe that the error term in $R^\infty \Psi_0^{-2,\zeta,\zeta}(X)$ is, by definition, the sum of a term in $R^\infty \Psi_0^{-2}(X)$ plus a term whose kernel is of the form $R^\infty \varrho^\zeta (\varrho')^\zeta F$, with $F$ smooth in the blown-up space. The latter term is in fact of the form $x^\zeta (x')^\zeta f$, $f\in C^\infty(X\times X, \Gamma_0^{1/2}(X\times X))$. Since the kernel of an operator in $R^\infty \Psi_0^{-2}(X)$ vanishes to infinite order at the three faces of $X\times_0 X$, this construction gives

THEOREM 3.1. *Let $(X,\partial X)$ be a smooth manifold with boundary and defining function $x$. Suppose that*

$$g_j=\frac{dx^2+h_j(x,y,dy)}{x^2}, \quad j=1,2,$$

*and $V_j$ are smooth real-valued functions vanishing at $\partial X$. Let $R_j(\zeta)$ denote the resolvent of $P_j=\tilde{\Delta}_j+V_j+\zeta(\zeta-n)$ where $\tilde{\Delta}_j$ is the Laplacian associated to $g_j$ acting on half-densities. Suppose that $\zeta$ is not a pole of $R_j(\zeta)$, and that $h_1-h_2$ and $V_1-V_2$ vanish to order $k\geqslant 1$ at $x=0$. Then*

$$R_1(\zeta)-R_2(\zeta)=G_1(\zeta)+G_2(\zeta)+G_3(\zeta), \quad G_i \in \Psi_0^{-2,\zeta,\zeta}(X,\Gamma_0^{1/2}(X)),\ i=1,2,3, \tag{3.16}$$

*where $G_3$ has kernel of the form $x^\zeta(x')^\zeta\gamma$, $\gamma \in C^\infty(X\times X, \Gamma_0^{1/2}(X\times X))$, the lift of the kernel of $G_2$ under $\beta$ is singular at the lift of the diagonal but vanishes to infinite order* *at the top, bottom and front faces of $X\times_0 X$, and the kernel of $G_1$ satisfies*

$$\beta^* G_1(\zeta) = R^k \varrho^\zeta (\varrho')^\zeta \alpha(\zeta), \quad \alpha(\zeta)\in C^\infty(X\times_0 X\setminus\Delta_0, \Gamma_0^{1/2}(X\times_0 X)), \tag{3.17}$$

*is a conormal distribution to the lifted diagonal $\Delta_0$.*

*If $E$ is such that $P_2-P_1=x^k E$, then the restriction of $\alpha(\zeta)$ to the front face, $F=\{R=0\}$, satisfies*

$$(\tilde{\Delta}_{h_0}+\zeta(\zeta-n))N_p(\varrho^\zeta(\varrho')^{\zeta-k}\alpha(\zeta)) = N_p((x/x')^k E)G, \tag{3.18}$$

*where $\tilde{\Delta}_{h_0}$ is the Laplacian on the hyperbolic space with metric $h_0(p)$, acting on half-densities, i.e. in coordinates $(z_0, z')$ where the boundary is given by $\{z_0=0\}$, $\tilde{\Delta}_{h_0}(f\omega^{1/2})=(\Delta_{h_0} f)\omega^{1/2}$, with*

$$\omega = \left|\delta_0^{1/2}\frac{dz_0}{z_0}\frac{dy}{z_0^n}\right|$$

*and*

$$\Delta_{h_0} = z_0^2\sum_{i,j=0}^{n} h^{ij}(p)\partial_{z_i}\partial_{z_j} - (n-1)z_0\partial_{z_0}, \tag{3.19}$$

*and $G$ is the Green function of $\tilde{\Delta}_{h_0}+\zeta(\zeta-n)$.*

Note that the last statement follows from Propositions 2.17 and 5.19 of [28] and the fact that the normal operator of the resolvent is its Green function.

*Remark* 1. In what follows, it is important to realize that there is a unique solution (3.18) which is meromorphic in $\zeta$, is conormal to the centre of the front face, and is in $\mathcal{A}^{\zeta,\zeta-k}$ near the boundaries.

To see this, note that, if we have two choices, $w_1$ and $w_2$, then

$$(\Delta_{h_0}+\zeta(\zeta-n))(w_1-w_2)=0.$$

Since $w_1-w_2$ is conormal to the centre of the front face it must be actually smooth. By Theorem 7.3 of [26], we know that $w_1-w_2=(\varrho\varrho')^\zeta f+(\varrho\varrho')^{n-\zeta} g$ where $f,g$ are distributional coefficients.

On the other hand we also know that $w_1-w_2\in\mathcal{A}^{\zeta,\zeta-k}$, and so is of the form $w_1-w_2=\varrho^\zeta(\varrho')^{\zeta-k} w$ with $w$ smooth up to the boundary. Therefore we conclude that $w_1-w_2=(\varrho\varrho')^\zeta\widetilde{w}$ with $\widetilde{w}$ smooth up to the boundary. Since $\Delta_{h_0}$ has no discrete spectrum, it follows from Proposition 4.3 that for $\zeta\notin(-\infty,\frac{1}{2}n]$, $w_1=w_2$, and thus by meromorphicity everywhere.

## 4. The Poisson operator and the scattering matrix

In this section we extend some of the results of [13] and [14], obtained in the case of Riemann surfaces, to asymptotically hyperbolic manifolds. We show that the kernel of the Poisson operator is a multiple of the Eisenstein function and, as in [13], [14], we obtain a formula for the scattering matrix in terms of the resolvent. Similar results have been established by Borthwick in [6] for $\operatorname{Re}\zeta=\frac{1}{2}n$. The Poisson operator has also been studied by Agmon in [1]. As a consequence of this formula, we prove that the scattering matrix at energy $\zeta$, $\zeta\in\mathbf{C}\setminus Q$, where $Q$ is a discrete subset which is described in Proposition 4.4, is a pseudo-differential operator of order $2\zeta-n$. We also prove the result stated in equation (1.2) of the introduction.

Before proceeding to this, we sketch our argument. The resolvent of the Laplacian acting on half-densities has, by [28], a meromorphic extension to the entire complex plane. Its weighted restriction to $X\times\partial X$, we call the Eisenstein function, $E(\zeta)$, in analogy to previous work on hyperbolic manifolds. This function is automatically in the kernel of $\Delta+\zeta(\zeta-n)$ and we examine its distributional asymptotics. In particular, we see that it has two components, one lead term is a multiple of the delta-function on the diagonal times $x^\zeta$, and the other is a pseudo-differential operator times $x^{n-\zeta}$. This means that upon integration of a suitable multiple of the Eisenstein function against a half-density on the boundary one obtains roughly an eigenfunction of the form $x^\zeta f+x^{n-\zeta}g$ plus lower-order terms, where $g=S(\zeta)f$, with $f$ prescribed, and $S(\zeta)$ a fixed pseudo-differential operator, which is of course the scattering matrix acting on half-densities. So the Eisenstein function is really the Poisson operator for the problem, and our first task is to prove that it has the appropriate distributional asymptotics. The Eisenstein function $E(\zeta)$ plays an analogous rôle to that of the Poisson operator $P(\lambda)$ in [33]. However, it lives on the manifold $X\times\partial X$ blown up along the boundary diagonal, rather than on a microlocally blown-up space.

Recall that $X\times_0X$ is the space obtained from $X\times X$ by blowing up the diagonal $\Delta\subset\partial X\times\partial X$, and that $\beta: X\times_0X\to X\times X$ is the corresponding blow-down map. Theorem 7.1 of [28] states that the resolvent $R(\zeta)$, which is well defined and holomorphic in $\zeta$, for $\operatorname{Re}\zeta>n$, extends to a meromorphic family $R(\zeta)\in\Psi_0^{-2,\zeta,\zeta}(X,\Gamma_0^{1/2}(X))$, $\zeta\in\mathbf{C}$, that satisfies, in terms of the spaces introduced in §3,

$$R(\zeta)=R'(\zeta)+R''(\zeta),\quad R'(\zeta)\in\Psi_0^{-2}(X,\Gamma_0^{1/2}(X))\quad\text{and}\quad R''(\zeta)\in\Psi_0^{-\infty,\zeta,\zeta}(X,\Gamma_0^{1/2}(X)),$$

with the boundary term, $R''(\zeta)$, having Schwartz kernel of a special form:

$$\beta^*K''(\zeta)=\varrho^\zeta(\varrho')^\zeta F(\zeta),\quad F(\zeta)\in C^\infty(X\times_0X;\Gamma_0^{1/2}(X\times_0X)),\tag{4.1}$$

where $\varrho$ and $\varrho'$ are defining functions of the top and bottom faces respectively, and $F(\zeta)$ is meromorphic in $\zeta$.

Let $R(\zeta)\in C^{-\infty}(X\times X, \Gamma_0^{1/2}(X\times X))$ also denote the Schwartz kernel of the resolvent, and let $x$ and $x'$ be a boundary-defining function of each copy of $X$ in $X\times X$. We will show that the Eisenstein function, which is defined by

$$E(\zeta)=(x')^{-\zeta+n/2}R(\zeta)|_{x'=0}, \tag{4.2}$$

is a section of $\Gamma_0^{1/2}(X\times\partial X)$ which is smooth in $X\times\partial X$ and has a conormal singularity at $\Delta\subset\partial X\times\partial X$. Notice that it depends on the choice of the defining function $x'$. To make it independent of this choice one can view it as a section of $\Gamma_0^{1/2}(X\times\partial X)\otimes|N^*\partial X|^{\zeta-n/2}$ by defining it as

$$E(\zeta)=(x')^{-\zeta+n/2}R(\zeta)|_{x'=0}\,|dx'|^{\zeta-n/2}. \tag{4.3}$$

This is the analogue of Definition 2.2 of [13]. For simplicity we will work with the definition given by (4.2), and so we fix a product decomposition $X\sim\partial X\times[0,\varepsilon)$ of $X$ near $\partial X$.

Since $R'(\zeta)\in\Psi_0^{-2}(X,\Gamma_0^{1/2}(X))$, its kernel vanishes to infinite order at the top and bottom faces. So we deduce that its kernel satisfies

$$(x')^{-\zeta+n/2}R'(\zeta)|_{x'=0}=0.$$

Therefore, for $K''(\zeta)$ given by (4.1),

$$E(\zeta)=(x')^{-\zeta+n/2}K''(\zeta)|_{x'=0}. \tag{4.4}$$

Since the singularity of $K''$ is better described in $X\times_0 X$, to understand the singularity of $E(\zeta)$ at $\Delta$ we blow up the manifold $X\times\partial X$ along $\Delta$ and analyze the lift of $E(\zeta)$ under the blow-down map. Let $X\times_0\partial X$ be the manifold with corners obtained by blowing up $X\times\partial X$ along the diagonal $\Delta\subset\partial X\times\partial X$, and let

$$\tilde\beta: X\times_0\partial X\to X\times\partial X$$

denote the corresponding blow-down map. It is then clear that $\tilde\beta=\beta|_{(X\times_0\partial X)}$.

Let $\mathcal{F}$ be the new boundary face introduced by the blow-up, the front face, and let $M$ denote the lift of $\partial X\times\partial X\setminus\Delta$ under $\tilde\beta$, i.e.

$$\mathcal{F}=\tilde\beta^{-1}(\Delta),\quad M=\operatorname{clos}\tilde\beta^{-1}(\partial X\times\partial X\setminus\Delta).$$

We refer the reader to Figure 1 for a picture of the two-dimensional local model case $X=[0,1)\times(0,1)$. We observe that in this case $\Delta$ is a submanifold of dimension one of a

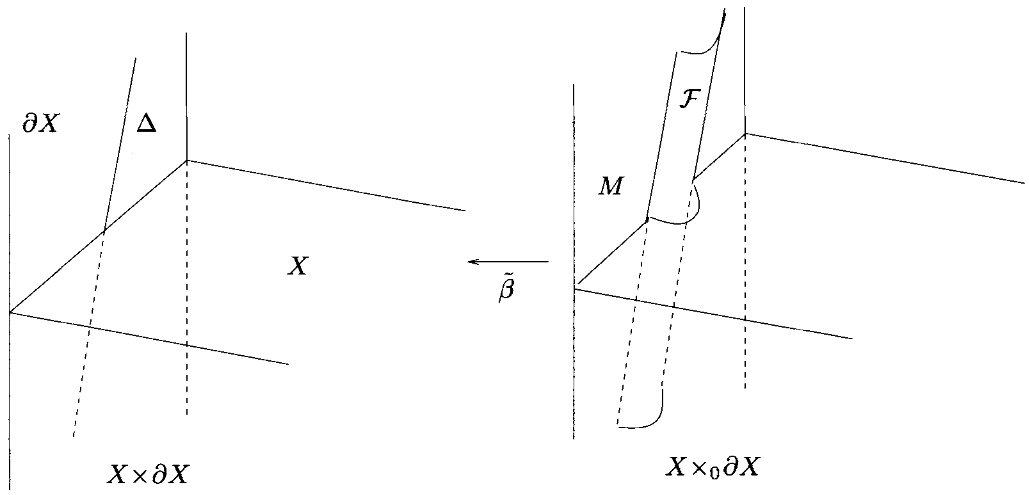


Fig. 1. $X \times_0 \partial X$ for the local model $X = [0,1) \times (0,1)$.

manifold of dimension two, and thus when $\Delta$ is blown up the manifold $M$ is disconnected, as shown in Figure 1. In general, $\Delta$ is a submanifold of dimension $n$ of a manifold of dimension $2n$, so $M$ is not disconnected.

If $R \in C^\infty(X \times_0 X)$ is a defining function of the front face in $X \times_0 X$, $\widetilde{R} = R|_{X \times_0 \partial X}$ is a defining function of $\mathcal{F}$. In local coordinates we have $R = (x^2 + (x')^2 + |y-y'|^2)^{1/2}$ and $\widetilde{R} = (x^2 + |y-y'|^2)^{1/2}$.

Next we consider the lift of $E$ under the map $\tilde{\beta}$. It is actually more convenient to analyze the lift of $x^{-\zeta+n/2}E$. We deduce from (4.1) and (4.4) that

$$\begin{aligned} \tilde{\beta}^*(x^{-\zeta+n/2}E(\zeta)) &= (R\varrho)^{-\zeta+n/2}(R\varrho')^{-\zeta+n/2}\varrho^\zeta(\varrho')^\zeta F(\zeta)|_{\varrho'=0} \\ &= R^{-2\zeta+n}\varrho^{n/2}(\varrho')^{n/2}F(\zeta)|_{\varrho'=0}. \end{aligned} \tag{4.5}$$

As in §3 of [28], it is simpler to do the computations in projective coordinates valid in regions of $X \times_0 X$ which together cover $X \times_0 X$. Following the notation of [28], we use three coordinate patches:

$$\begin{gathered} x', \quad s = \frac{x}{x'}, \quad z = \frac{y-y'}{x'}, \\ x, \quad t = \frac{x'}{x}, \quad z' = -\frac{y-y'}{x}, \\ \varrho = \frac{x}{|y-y'|}, \quad \varrho' = \frac{x'}{|y-y'|}, \quad r = |y-y'|, \quad \omega = \frac{y-y'}{|y-y'|}. \end{gathered} \tag{4.6}$$

We observe that so far we have used $\varrho$ and $\varrho'$ to denote $x/R$ and $x'/R$. However, in the region where the third set of coordinates hold we see that

$$\varrho=\frac{x}{R}=\frac{x}{r(1+(x/r)^2+(x'/r)^2)^{1/2}},\quad \varrho'=\frac{x'}{R}=\frac{x'}{r(1+(x/r)^2+(x'/r)^2)^{1/2}}.$$

So for $x/r$ and $x'/r$ small, $\varrho$ and $\varrho'$ are essentially given by (4.6).

In the region away from the top face, we can use projective coordinates $(x,t,z',y)$, and near the intersection of the top and bottom faces we can use $(r,\varrho,\varrho',y,\omega)$. So we can represent the half-density $F(\zeta)$ in these local coordinates by

$$\begin{aligned} F(\zeta)&=F(\zeta,t,x,y,z')\left|\frac{dx}{x}\frac{dy}{x^n}\frac{dt}{t}\frac{dz'}{t^n}\right|^{1/2},\quad F\in C^\infty,\\ F(\zeta)&=F(\zeta,r,\varrho,\varrho',y,\omega)\left|\frac{dr}{r}\frac{d\varrho}{\varrho}\frac{d\varrho'}{\varrho'}\frac{d\omega\,dy}{\varrho^n(\varrho')^n r^n}\right|^{1/2},\quad F\in C^\infty.\end{aligned}\tag{4.7}$$

We observe that in the respective regions,

$$\begin{gathered} R=(x^2+(x')^2+|y-y'|^2)^{1/2}=x(1+t^2+|z'|^2)^{1/2}=xR',\quad R'\in C^\infty,\\ \varrho=\frac{1}{(1+t^2+|z'|^2)^{1/2}},\quad \varrho'=\frac{t}{(1+t^2+|z'|^2)^{1/2}},\\ R=(x^2+(x')^2+|y-y'|^2)^{1/2}=r(1+(\varrho')^2+\varrho^2)^{1/2}=rR',\quad R'\in C^\infty.\end{gathered}$$

Hence restricting to the bottom face, which is given respectively by $\{t=0\}$ and $\{\varrho'=0\}$, gives

$$\begin{aligned} \tilde\beta^*(x^{-\zeta+n/2}E(\zeta))&=x^{-2\zeta+(n-1)/2}F'(\zeta,x,y,z')\,|dx\,dy\,dz'|^{1/2},\quad F'\in C^\infty,\\ \tilde\beta^*(x^{-\zeta+n/2}E(\zeta))&=r^{-2\zeta+(n-1)/2}\varrho^{-1/2}F'(\zeta,r,\varrho,\varrho',y,\omega)\,|dr\,d\varrho\,d\omega\,dy|^{1/2},\quad F'\in C^\infty.\end{aligned}\tag{4.8}$$

Therefore we have that

$$\tilde\beta^*(x^{-\zeta+n/2}E(\zeta))\in\widetilde{R}^{-2\zeta+(n-1)/2}\varrho^{-1/2}C^\infty(X\times_0\partial X,\Gamma^{1/2}(X\times_0\partial X)),\tag{4.9}$$

where we also denote $\varrho=\varrho|_{X\times_0\partial X}$. Note that

$$\tilde\beta^*\colon x^{-1/2}\Gamma^{1/2}(X\times\partial X)\leftrightarrow\widetilde{R}^{n/2}\varrho^{-1/2}\Gamma^{1/2}(X\times_0\partial X)\tag{4.10}$$

is an isomorphism. Indeed, if we use local coordinates

$$\begin{gathered} x,\quad \omega=\frac{y-y'}{x},\quad y,\\ |y-y'|=r,\quad \varrho=\frac{x}{r},\quad \omega=\frac{y-y'}{r},\end{gathered}\tag{4.11}$$

where the first set is valid away from $M\cap\mathcal{F}$, and the second is valid near $M\cap\mathcal{F}$, respectively. Then the lift of $x^{-1/2}|dx\,dy\,dy'|^{1/2}$ is given by

$$x^{(n-1)/2}|dx\,dy\,d\omega|^{1/2}, \quad r^{(n-1)/2}\varrho^{-1/2}|dr\,d\varrho\,d\omega\,dy|^{1/2}$$

respectively. Therefore the map (4.10) is in fact an isomorphism.

Now that we have found a space which contains $\tilde{\beta}^*(x^{-\zeta+n/2}E(\zeta))$, which is given in (4.9), we consider the push-forward of a smooth section of this space. That, in particular, will establish the properties of $x^{-\zeta+n/2}E(\zeta)$. First we need to introduce some notation. Note that $\mathcal{F}$ and $M$ are manifolds with boundary, and that the restriction of $\tilde{\beta}$ to $M$ induces a map

$$\beta_\partial=\beta|_M\colon M\sim\partial X\times_0\partial X\to\partial X\times\partial X,$$

which corresponds to the blow-up of the manifold $\partial X\times\partial X$ along the diagonal $\Delta\subset\partial X\times\partial X$, see Figure 1.

Given $\widetilde{R}\in C^\infty(X\times_0\partial X)$ and $x\in C^\infty(X)$, defining functions of $\mathcal{F}$ and $\partial X$ respectively, the function $\varrho=x/\widetilde{R}\in C^\infty(X\times_0\partial X)$ is a defining function of $M$. Since $\mathcal{F}$ and $M$ intersect transversally, with $M\cap\mathcal{F}=\partial M=\partial\mathcal{F}$, the functions

$$R_M=\widetilde{R}|_M\in C^\infty(\partial X\times\partial X), \quad \varrho_{\mathcal{F}}=\varrho|_{\mathcal{F}}\in C^\infty(\mathcal{F})$$

are defining functions of $\partial M$ and $\partial\mathcal{F}$ respectively.

Recall that, see for example §3.2 of [18], if $Y$ is a manifold with corners, and $y\in C^\infty(Y)$ is a defining function of a boundary hypersurface of $Y$, then sections of $y^\zeta\Gamma^{1/2}(Y)$, viewed as distributions acting on $\Gamma^{1/2}(Y)$ via

$$\langle y^\zeta F,f\rangle=\int_Y y^\zeta Ff, \quad \text{for } \operatorname{Re}\zeta>-1,\, F\in C^\infty(Y,\Gamma^{1/2}(Y)),\, f\in C_0^\infty(Y,\Gamma^{1/2}(Y)), \tag{4.12}$$

have holomorphic extensions to $\mathbf{C}\setminus-\mathbf{N}$.

We will consider three such half-densities associated to $\widetilde{R}$, $R_M$ and $\varrho_{\mathcal{F}}$ defined on $X\times_0\partial X$, $M$ and $\mathcal{F}$ respectively.

We have fixed a product decomposition $X\sim\partial X\times[0,\varepsilon)$ near $\partial X$, and will prove that the sections of the push-forward of $\widetilde{R}^{-2\zeta+(n-1)/2}\varrho^{-1/2}\Gamma^{1/2}(X\times_0\partial X)$ have distributional asymptotic expansions as $x\downarrow 0$. To do that we define the partial pairing for $u\in\widetilde{R}^{-2\zeta+(n-1)/2}\varrho^{-1/2}C^\infty(X\times_0\partial X,\Gamma^{1/2}(X\times_0\partial X))$, $f\in C^\infty(\partial X\times\partial X,\Gamma^{1/2}(\partial X\times\partial X))$:

$$\langle\beta_{\partial*}u,f\rangle=\int_{\partial X\times\partial X}(\beta_{\partial*}u)(x,y,y')f(y,y'). \tag{4.13}$$

We remark that if $u$ is a smooth section of

$$\widetilde{R}^{-2\zeta+(n-1)/2}\varrho^{-1/2}\Gamma^{1/2}(X\times_0\partial X)$$

then the restriction of $u$ to $M$, denoted by $u|_M$, is well defined as a section of

$$R_M^{-2\zeta+(n-1)/2}\varrho_{\mathcal{F}}^{-1/2}\Gamma^{1/2}(\partial X\times_0\partial X).$$

It is also easy to see that $x^{2\zeta-(n-1)/2}u$ is a smooth section of $\varrho^{2\zeta-n/2}\Gamma^{1/2}(\partial X\times_0\partial X)$. Therefore it can be restricted to $\mathcal{F}=\{\widetilde{R}=0\}$, and $(x^{2\zeta-(n-1)/2}u)|_{\mathcal{F}}$ is a smooth section of $\varrho_{\mathcal{F}}^{2\zeta-n/2}\Gamma^{1/2}(\mathcal{F})$.

We now prove a push-forward theorem which relates the distributional asymptotics of the class of half-densities given by (4.9), which includes the Eisenstein function, to their behaviour at the boundary, cf. Proposition 16 of [33].

PROPOSITION 4.1. *Let $x\in C^\infty(X)$ be a defining function of $\partial X$, and fix a product decomposition $X\sim\partial X\times[0,\varepsilon)$ near $\partial X$. Let $\widetilde{R}\in C^\infty(X\times_0\partial X)$ be a defining function of $\mathcal{F}$, $\varrho=x/\widetilde{R}$, and let $\varrho_{\mathcal{F}}=\varrho|_{\mathcal{F}}$, as above. Let*

$$v=\widetilde{R}^{-2\zeta+(n-1)/2}\varrho^{-1/2}F,\quad F\in C^\infty(X\times_0\partial X,\Gamma^{1/2}(X\times_0\partial X)),\ 2\zeta\in\mathbf{C}\setminus\mathbf{Z}.$$

*Then the push-forward of $v$ under $\tilde\beta$, denoted by $\tilde\beta_*v$, is a section of $x^{-1/2}\Gamma^{1/2}(X\times\partial X)$, which has a conormal singularity at $\Delta$, and moreover it has an asymptotic expansion in $x$ as $x\downarrow 0$, in the sense that if $f\in C^\infty(\partial X\times\partial X,\Gamma^{1/2}(\partial X\times\partial X))$ and $\langle\cdot,\cdot\rangle$ is the partial pairing defined above, then*

$$\langle\tilde\beta_*v,f\rangle=(H_\zeta(x)+x^{n-2\zeta}G_\zeta(x))\,|dx/x|^{1/2}\quad \text{as } x\downarrow 0,\tag{4.14}$$

*where $G_\zeta,H_\zeta\in C^\infty([0,\varepsilon))$ depend holomorphically on $\zeta$. Also, if $v|_M$ and $x^{2\zeta-(n-1)/2}v|_{\mathcal{F}}$ denote the restrictions of these half-densities to $M$ and $\mathcal{F}$ respectively, then*

$$\begin{aligned}H_\zeta(0)&=\langle\beta_{\partial *}(v|_M),f\rangle,\\ G_\zeta(0)&=\langle\langle x^{2\zeta-(n-1)/2}v|_{\mathcal{F}},\varrho_{\mathcal{F}}^{-(n+2)/2}\rangle\delta_\Delta,f\rangle,\end{aligned}\tag{4.15}$$

*where $\delta_\Delta$ is the delta-function of the diagonal, and $\langle x^{2\zeta-(n-1)/2}v|_{\mathcal{F}},\varrho_{\mathcal{F}}^{-1-n/2}\rangle$ is the pairing induced by the trivialization of the half-density bundle $\Gamma^{1/2}(\mathcal{F})$ given by the product structure.*

*Proof.* Since this is a local result and $\tilde\beta$ is a diffeomorphism away from $\Delta$, we only need to work in a neighbourhood of a point $q\in\Delta$. Let $y,y'$ be local coordinates near $q$, and let $\widetilde{R}=(x^2+|y-y'|^2)^{1/2}$, $\varrho=x/\widetilde{R}$ and $\omega=(y-y')/R$. The map $\tilde\beta$ can be described as (see Figure 1)

$$\begin{aligned}\tilde\beta\colon \mathbf{S}^n_+\times[0,\infty)\times[0,\infty)\times\mathbf{R}^n&\to\mathbf{R}_+\times\mathbf{R}^{n-1}\times\mathbf{R}^{n-1},\\ (\omega,\varrho,R,y')&\mapsto(R\varrho,y'+R\omega,y'),\end{aligned}$$

and we will denote

$$v=\widetilde{R}^{-2\zeta+(n-1)/2}\varrho^{-1/2}F(\varrho,\omega,R)\,|d\varrho\,d\omega\,dR|^{1/2},\quad F\in C^\infty(X\times_0\partial X). \tag{4.16}$$

We also set $z=y-y'$. Then the variables $y'$ become parametric, and for simplicity we will ignore them. The diagonal is given by $\Delta=\{x=0,z=0\}$.

It is easy to prove that $\tilde{\beta}_*(\widetilde{R}^{-2\zeta}F)$ is conormal to $\Delta$. Just observe that the vector fields tangent to $\Delta$ are spanned over $C^\infty(X\times\partial X)$ by

$$x\partial_x,\quad z_j\partial_{z_k},\quad x\partial_{z_k},\quad z_k\partial_x,$$

and it can be proven, by using projective coordinates as in (4.11) above, that these vector fields lift under $\tilde{\beta}$ to smooth vector fields that are tangent to $\mathcal{F}$. Thus repeated applications of these vector fields to $\tilde{\beta}_*(\widetilde{R}^{-2\zeta}F)$ will not change its Sobolev regularity. This shows that $\tilde{\beta}_*(\widetilde{R}^{-2\zeta}F)$ is conormal to $\Delta$. Next we establish the asymptotic expansion.

We observe that the radial vector field is given by

$$\widetilde{R}\frac{\partial}{\partial\widetilde{R}}=\tilde{\beta}^*(x\partial_x+z\cdot\partial_z). \tag{4.17}$$

Thus, since $x(x^2+|z|^2)^{-1/2}$ is homogeneous of degree zero with respect to the action $(x,z)\mapsto(\lambda x,\lambda z)$, $\lambda\in\mathbf{R}_+$, we have

$$(x\partial_x+z\cdot\partial_z)(x(x^2+|z|^2)^{-1/2})=0,\quad (x,z)\neq 0.$$

Therefore,

$$\widetilde{R}\frac{\partial}{\partial\widetilde{R}}\tilde{\beta}^*(x^k(x^2+|z|^2)^{-k/2})=0,\quad k\in\mathbf{N}. \tag{4.18}$$

We will also use that

$$\partial_x(x^2+|z|^2)^{1/2}=x(x^2+|z|^2)^{-1/2}=x/\widetilde{R} \tag{4.19}$$

and that $\widetilde{R}\tilde{\beta}^*(\partial_x)$ is a smooth vector field in $X\times_0\partial X$ which is tangent to $\mathcal{F}$.

We deduce from (4.17) and (4.19) that

$$x\partial_x(x\partial_x+z\cdot\partial_z+2\zeta)\tilde{\beta}_*(\widetilde{R}^{-2\zeta}F)=x\tilde{\beta}_*(\widetilde{R}^{-2\zeta}F_1),\quad F_1\in C^\infty(X\times_0\partial X),$$

and using (4.17), (4.18) and (4.19) we obtain

$$(x\partial_x-1)(x\partial_x+z\cdot\partial_z+2\zeta-1)x\tilde{\beta}_*(\widetilde{R}^{-2\zeta}F_1)=x^2\tilde{\beta}_*(\widetilde{R}^{-2\zeta}F_2),\quad F_2\in C^\infty(X\times_0\partial X).$$

Similarly it follows that

$$(x\partial_x - k)(x\partial_x + z\cdot\partial_z + 2\zeta - k)x^k \tilde{\beta}_*(\widetilde{R}^{-2\zeta} F_k) = x^{k+1}\tilde{\beta}_*(\widetilde{R}^{-2\zeta} F_{k+1}),\quad F_{k+1} \in C^\infty(X \times_0 \partial X). \tag{4.20}$$

Induction and (4.20) give

$$\prod_{j=0}^{M}(x\partial_x - j)(x\partial_x + z\cdot\partial_z + 2\zeta - j)\tilde{\beta}_*(\widetilde{R}^{-2\zeta} F) = x^{M+1}\tilde{\beta}_*(\widetilde{R}^{-2\zeta} F_M),\quad F_M \in C^\infty(X \times_0 \partial X). \tag{4.21}$$

Since the map defined in (4.10) is an isomorphism, it follows that the push-forward of (4.16) can be written in local coordinates $x, z$ as

$$\tilde{\beta}_*(\widetilde{R}^{-2\zeta} F)(x,z)\left|\frac{dx}{x}\,dz\right|^{1/2}.$$

Let $f \in C_0^\infty(\mathbf{R}^n)$. Then

$$\langle \tilde{\beta}_*(v), f(z)\,|dz|^{1/2}\rangle = \left(\int_{\mathbf{R}^n} \tilde{\beta}_*(\widetilde{R}^{-2\zeta} F)(x,z) f(z)\,dz\right)\left|\frac{dx}{x}\right|^{1/2}.$$

Using (4.21) and the identity $\operatorname{div}(zu(z)) = nu(z) + z\cdot\partial_z u(z)$ we deduce that the function $u(x)$ given by

$$u(x) = \int_{\mathbf{R}^n} \tilde{\beta}_*(\widetilde{R}^{-2\zeta} F)(x,z) f(z)\,dz, \quad x > 0,$$

satisfies

$$\prod_{j=0}^{M}(x\partial_x - j)(x\partial_x + 2\zeta - (n+j))u(x) = x^{M+1}\int_{\mathbf{R}^n} \tilde{\beta}_*(\widetilde{R}^{-2\zeta} F_M)(x,z) f(z)\,dz, \quad x > 0. \tag{4.22}$$

Let

$$u_M(x) = \prod_{j=0}^{M}(x\partial_x + 2\zeta - (n+j))u(x).$$

Then we deduce from (4.22) that there exists $s \in \mathbf{R}$, independent of $M$, such that

$$\left|\partial_x \prod_{j=1}^{M}(x\partial_x - j)u_M\right| \leqslant Cx^{M+s}, \quad x > 0. \tag{4.23}$$

Thus, for $M+s>0$, there exists $C_0\in\mathbf{C}$ such that

$$\lim_{x\downarrow 0}\prod_{j=1}^{M}(x\partial_x-j)u_M=C_0. \tag{4.24}$$

From (4.23) we obtain

$$\left|\partial_x\left(\prod_{j=1}^{M}(x\partial_x-j)u_M-C_0\right)\right|\leqslant Cx^{M+s},\quad x>0.$$

It follows from (4.24) that

$$\left|\prod_{j=1}^{M}(x\partial_x-j)u_M-C_0\right|\leqslant Cx^{M+s+1},\quad x>0.$$

Since $(x\partial_x-j)C_0=-jC_0$ we have, for $a_0=C_0/(-1)^M M!$,

$$\left|\prod_{j=1}^{M}(x\partial_x-j)(u_M-a_0)\right|\leqslant Cx^{M+s+1},\quad x>0.$$

Proceeding by induction, and using that $(x\partial_x-j)\cdot=x^{j+1}\partial_x(x^{-j}\cdot)$, we find that for $M+s-p+1>0$, there exist $a_m\in\mathbf{C}$, $0\leqslant m\leqslant p-1$, depending on $\zeta$, such that

$$\left|\prod_{j=p}^{M}(x\partial_x-j)\left(u_M-\sum_{m=0}^{p-1}a_m x^m\right)\right|\leqslant Cx^{M+s+1},\quad x>0. \tag{4.25}$$

Let us denote $V_p=\prod_{j=p+1}^{M}(x\partial_x-j)(u_M-\sum_{m=0}^{p-1}a_m x^m)$. Then

$$|\partial_x x^{-p}V_p(x)|\leqslant Cx^{M+s-p}.$$

Integrating this equation from $x$ to 1 and using that $M+s+1-p>0$ we find that $|V_p(x)|\leqslant Cx^p$.

Now we observe that if

$$|(x\partial_x-\alpha)u|\leqslant Cx^{\beta},\quad 0<x<1,\ 0<\beta<\operatorname{Re}\alpha,$$

then

$$|u(x)|\leqslant Cx^{\beta}. \tag{4.26}$$

To see that, just notice that $u$ satisfies

$$\begin{aligned}|u(1)-x^{-\alpha}u(x)|&=\left|\int_x^1\partial_s(s^{-\alpha}u(s))\,ds\right|=\left|\int_x^1 s^{-\alpha-1}(s\partial_s-\alpha)u(s)\,ds\right|\\&\leqslant C\int_x^1 s^{\beta-\operatorname{Re}\alpha-1}\,ds\leqslant\frac{C}{|\beta-\operatorname{Re}\alpha|}\,|1-x^{\beta-\operatorname{Re}\alpha}|=\frac{Cx^{\beta-\operatorname{Re}\alpha}}{\operatorname{Re}\alpha-\beta}(1-x^{\operatorname{Re}\alpha-\beta}).\end{aligned}$$

Hence

$$|u(x)-x^{\alpha}u(1)|\leqslant \frac{C}{\operatorname{Re}\alpha-\beta}x^{\beta}(1-x^{\operatorname{Re}\alpha-\beta})\leqslant Cx^{\beta}.$$

Thus (4.26) follows.

Applying (4.26) repeatedly with $\beta=p$ and $\alpha=j$, with $p+1\leqslant j\leqslant M$, we deduce that, for $M+s+1-p>0$,

$$\left|u_M(x)-\sum_{m=0}^{p-1}a_m x^m\right|\leqslant Cx^p,\quad x>0. \tag{4.27}$$

Notice that $(x\partial_x+2\zeta-n-j)x^m=(m+2\zeta-n-j)x^m$. Since $2\zeta\notin\mathbf{Z}$, $m+2\zeta-n-j\neq0$, we deduce from (4.27) that for $d_m\prod_{j=0}^{M}(m+2\zeta-n-j)=a_m$, $v_p=\sum_{m=0}^{p-1}d_m x^m$ satisfies

$$\left|\prod_{j=0}^{M}(x\partial_x+2\zeta-n-j)(u-v_p)\right|\leqslant Cx^p,\quad x>0. \tag{4.28}$$

This gives that

$$\left|x^{-2\zeta+n+1}\partial_x x^{2\zeta-n}\prod_{j=1}^{M}(x\partial_x+2\zeta-n-j)(u-v_p)\right|\leqslant Cx^p,\quad x>0.$$

Thus, for $p+2\operatorname{Re}\zeta-n-1>0$, there exists $b_0\in\mathbf{C}$ such that

$$\lim_{x\downarrow0}x^{2\zeta-n}\prod_{j=1}^{M}(x\partial_x+2\zeta-n-j)(u-v_p)=b_0.$$

Since $2\zeta\notin\mathbf{Z}$, we can proceed as above to deduce that there exists $\gamma_0$ such that

$$\left|\prod_{j=1}^{M}(x\partial_x+2\zeta-n-j)(u-v_p-\gamma_0x^{n-2\zeta})\right|\leqslant Cx^p,\quad x>0.$$

Using induction we find that for $p+2\operatorname{Re}\zeta-n-q>0$, there exist $\gamma_m$, $0\leqslant m\leqslant q-1$, depending on $\zeta$, such that

$$\left|\prod_{j=q}^{M}(x\partial_x+2\zeta-n-j)\left(u-v_p-\sum_{m=0}^{q-1}\gamma_m x^{m+n-2\zeta}\right)\right|\leqslant Cx^p,\quad x>0.$$

Proceeding as above we find that

$$\left|\prod_{j=q+1}^{M}(x\partial_x+2\zeta-n-j)\left(u-v_p-\sum_{m=0}^{q-1}\gamma_m x^{m+n-2\zeta}\right)\right|\leqslant Cx^{n+q-2\operatorname{Re}\zeta},\quad x>0.$$

Applying (4.26) with $\alpha=n+j-2\zeta$, $q+1\leqslant j\leqslant M$ and $\beta=n+q-2\operatorname{Re}\zeta$, we obtain, for arbitrary $M\in\mathbf{N}$, and $p,q$ satisfying respectively $M+s-p+1>0$, $p+2\operatorname{Re}\zeta-n-q>0$,

$$\left|u(x)-\sum_{m=0}^{p-1} d_m x^m - x^{n-2\zeta}\sum_{m=0}^{q-1}\gamma_m x^m\right|\leqslant Cx^{n+q-2\operatorname{Re}\zeta},\quad x>0.$$

Now Borel's lemma gives the desired result. It is clear from the construction that $H_\zeta$ and $G_\zeta$ depend holomorphically on $\zeta$, provided $2\zeta\notin\mathbf{Z}$. This method of proving the existence of an expansion goes back to Euler and has been used in similar contexts in [19], [20] and also [31].

Next we need to compute $G_\zeta(0)=\gamma_0$ and $H_\zeta(0)=d_0$. Since these are holomorphic functions of $\zeta$, we only need to compute $H_\zeta(0)$ for $2\operatorname{Re}\zeta-n<0$, and $G_\zeta(0)$ for $2\operatorname{Re}\zeta-n>0$.

In the coordinates above we have, for $f\in C_0^\infty(\mathbf{R}^n)$ and $\widetilde{R}=(x^2+|z|^2)^{1/2}$,

$$\langle\tilde\beta_* v, f\,|dz|^{1/2}\rangle=\int_{\mathbf{R}^n}\widetilde{R}^{-2\zeta}F\left(\widetilde{R},\frac{x}{\widetilde{R}},\frac{z}{\widetilde{R}}\right)f(z)\,dz\,|dx|^{1/2}. \tag{4.29}$$

It follows from the dominated convergence theorem that for $2\operatorname{Re}\zeta-n<0$,

$$\lim_{x\downarrow 0}\int_{\mathbf{R}^n}\widetilde{R}^{-2\zeta}F\left(\widetilde{R},\frac{x}{\widetilde{R}},\frac{z}{\widetilde{R}}\right)f(z)\,dz=\int_{\mathbf{R}^n}|z|^{-2\zeta}F\left(|z|,0,\frac{z}{|z|}\right)f(z)\,dz=\langle\beta_{\partial *}(v|_M),f\rangle.$$

To compute $G_\zeta(0)$ for $2\operatorname{Re}\zeta-n>0$ we set $z=xw$. Observing that in these coordinates $\varrho_{\mathcal{F}}=(1+|w|^2)^{-1/2}$, we deduce from (4.29) that

$$\langle\tilde\beta_*(R^{-2\zeta}F),f\rangle=x^{n-2\zeta}\int_{\mathbf{R}^n}\varrho_{\mathcal{F}}^{2\zeta}F\left(x\varrho_{\mathcal{F}}^{-1},\varrho_{\mathcal{F}},\frac{w}{(1+|w|^2)^{1/2}}\right)f(xw)\,dw.$$

Again by the dominated convergence theorem,

$$\begin{aligned}\lim_{x\downarrow 0}\int_{\mathbf{R}^n}\varrho_{\mathcal{F}}^{2\zeta}F\left(x\varrho_{\mathcal{F}}^{-1},\varrho_{\mathcal{F}},\frac{w}{(1+|w|^2)^{1/2}}\right)f(xw)\,dw&\\ =f(0)\int_{\mathbf{R}^n}\varrho_{\mathcal{F}}^{2\zeta}F\left(0,\varrho_{\mathcal{F}},\frac{w}{(1+|w|^2)^{1/2}}\right)dw.&\end{aligned}$$

Using the map $\mathbf{R}^n\ni w\mapsto SP(w)=((1+|w|^2)^{-1/2},w(1+|w|^2)^{-1/2})\in\mathbf{S}^n$ we have, for $g\in C^\infty(\mathbf{S}^n_+)$,

$$\int_{\mathbf{S}^n_+}g\,d\sigma=\int_{\mathbf{R}^n}g(SP(w))(1+|w|^2)^{-(n+1)/2}\,dw.$$

Therefore

$$\begin{aligned}\int_{\mathbf{R}^n}\varrho_{\mathcal{F}}^{2\zeta}F(0,\varrho_{\mathcal{F}},\varrho_{\mathcal{F}}w)\,dw&=\int_{\mathbf{S}^n_+}\varrho_{\mathcal{F}}^{-1-n/2}\varrho_{\mathcal{F}}^{2\zeta-n/2}F\,d\sigma\\&=\langle x^{2\zeta-(n-1)/2}v|_{\mathcal{F}},\varrho_{\mathcal{F}}^{-1-n/2}\rangle.\end{aligned} \tag{4.30}$$

Since $2\operatorname{Re}\zeta-n>0$ the integral converges. This concludes the proof of the proposition. $\square$

Next we compute the coefficient of $\delta_\Delta$ in the second equation of (4.15) when $v$ is the lift of the Eisenstein function. This will be important in the definition of the scattering matrix.

PROPOSITION 4.2. *Let $g$ and $V$ satisfy the hypotheses of Theorem* 1.1. *The coefficient of $\delta_\Delta$ in the second equation of* (4.15),

$$M(\zeta)=\langle\tilde{\beta}^*(x^{2\zeta-(n-1)/2}E(\zeta))|_{\mathcal{F}},\varrho_{\mathcal{F}}^{-1-n/2}\rangle,$$

*is equal to $|h_0|^{1/2}$ times a function which is independent of the base point of the fibre $\mathcal{F}$, and is also independent of $g$ and $V$.*

*Proof.* According to (4.30), (4.5) and (4.10), $M(\zeta)$ depends only on the value of $F|_{\mathcal{F}}$ where $F$ is given by (4.1) and $\mathcal{F}$ is as above.

We recall from the construction of $R(\zeta)$ in §3 and the proof of Proposition 7.4 in [28] that the normal operator of $R(\zeta)$ is just $R_0(\zeta)$, the Green function of the operator $\tilde{\Delta}_{h_0}+\zeta(\zeta-n)$ given by (3.19), where as observed in [28], the fibre of the front face over a point $p\in\partial M$ can be naturally identified with the hyperbolic space $\mathbf{H}^n$ with linear metric induced by $h_0$. Thus in order to compute $M(\zeta)$ we need only compute for $R_0(\zeta)$. It is well known, see for example Lemma 2.1 of [16], that

$$\begin{aligned}&R_0(\zeta,x,x',y,y')\\&=\left(\frac{1}{2}\pi^{-n/2}\frac{\Gamma(\zeta)}{\Gamma\big(\zeta-\frac{1}{2}(n-2)\big)}\right)\frac{x^\zeta(x')^\zeta}{(x^2+(x')^2+|y-y'|_0^2)^\zeta}\left||h_0|\,\frac{dx}{x}\,\frac{dy}{x^n}\,\frac{dx'}{x'}\,\frac{dy'}{(x')^n}\right|^{1/2}+S_1,\\&S_1\in\mathcal{A}^{\zeta+1,\zeta+1},\end{aligned}\tag{4.31}$$

where $|y-y'|_0$ is the distance in the $h_0$-metric and $|h_0|$ denotes its volume element. (Here we have multiplied by the appropriate half-density.) Since $\varrho=x(x^2+(x')^2+|y-y'|^2)^{-1/2}$ and $\varrho'=x'(x^2+(x')^2+|y-y'|^2)^{-1/2}$ we deduce from (4.31) that $R_0(\zeta)=\varrho^\zeta(\varrho')^\zeta\mu$ where $\mu$ is the half-density induced on the front face. By an abuse of notation we denote the restrictions of $\varrho$ and $\varrho'$ to the front face also by $\varrho$ and $\varrho'$. Thus $F|_{\mathcal{F}}$ is just the half-density induced on $\mathcal{F}$. This concludes the proof of the proposition. $\square$

Now it follows from (4.9) and Proposition 4.1 that $x^{-\zeta+n/2}E(\zeta)$, with $2\zeta\notin\mathbf{Z}$ and $\zeta$ not a pole of $R(\zeta)$, is a section of $x^{-1/2}\Gamma^{1/2}(X\times\partial X)$ which is smooth in the interior of $X\times\partial X$ and has a conormal singularity at $\Delta$. Since $x^{-(n+1)/2}\Gamma^{1/2}(X\times\partial X)=\Gamma_0^{1/2}(X\times\partial X)$, we have that $x^{-\zeta}E$ is a smooth section of $\Gamma_0^{1/2}(X\times\partial X)$. Therefore we have from (4.14) that

COROLLARY 4.1. *For $2\zeta\notin\mathbf{Z}$ and $\zeta$ not a pole of $R(\zeta)$, the Eisenstein function, $E(\zeta)$, defined by (4.2), is a smooth section of $\Gamma_0^{1/2}(X\times\partial X)$ which is holomorphic in $\zeta$. Moreover, for any product decomposition $X\sim\partial X\times[0,\varepsilon)$, and for any $f,g\in C^\infty(\partial X,\Gamma^{1/2}(\partial X))$, we have that, as $x\downarrow 0$,*

$$\langle E(\zeta),f\otimes g\rangle(x)=x^{-n/2}(x^\zeta h_{1,\zeta}(x)+x^{n-\zeta}h_{2,\zeta}(x))\left|\frac{dx}{x}\right|^{1/2},\quad x>0, \tag{4.32}$$

*where $h_{i,\zeta}\in C^\infty([0,\varepsilon))$, $i=1,2$, depend holomorphically on $\zeta$.*

We observe that, as an element of $C^\infty(X\times\partial X,\Gamma_0^{1/2}(X\times\partial X))$, $E(\zeta)$ defines, by duality, a map

$$\begin{gathered} E(\zeta)\colon C^\infty(\partial X,\Gamma^{1/2}(\partial X))\to C^\infty(X,\Gamma_0^{1/2}(X)),\\ \langle E(\zeta)(f),v\rangle=\langle E(\zeta),f\otimes v\rangle,\quad f\in C^\infty(\partial X,\Gamma^{1/2}(\partial X)),\ v\in C^\infty(X,\Gamma_0^{1/2}(X)). \end{gathered}$$

By definition of the resolvent, $R(\zeta)$, the kernel of $(\tilde\Delta_{g(x)}+V(x)+\zeta(\zeta-n))R(\zeta)$ is supported on the diagonal in $X\times X$. In particular, we find from the definition of $E(\zeta)$ that if $f\in C^\infty(\partial X,\Gamma^{1/2}(\partial X))$,

$$(\tilde\Delta_{g(x)}+V(x)+\zeta(\zeta-n))(E(\zeta)f)=0\quad\text{in } X.$$

Moreover, it follows from (4.15) and (4.32) that for any $f\in C^\infty(\partial X,\Gamma^{1/2}(\partial X))$,

$$(E(\zeta)f)(x,\cdot\,)=x^\zeta f_+ + x^{n-\zeta}f_-,\quad f_\pm\in C^\infty(X,\Gamma_0^{1/2}(X)),\quad x^{n/2}f_-|_{\partial X}=M(\zeta)f,$$

where $M(\zeta)$ is given by Proposition 4.2. This shows that $(1/M(\zeta))E(\zeta)$ is the Schwartz kernel of the Poisson operator.

For completeness, as the general result does not seem to be in the literature, we prove the uniqueness of the generalized eigenfunction $E(\zeta)f$. The case $\operatorname{Re}\zeta=\frac12 n$ has been proved by Borthwick in [6]. Our proof, which is based on an argument of [30], is not very different from his. For simplicity we consider the Laplacian acting on functions. The half-density case is identical.

PROPOSITION 4.3. *Let $\zeta\in\mathbf{C}$ be such that $2\zeta\notin\mathbf{Z}$, $\zeta\notin(-\infty,\frac12 n]$ and $\zeta(\zeta-n)$ is not in the point spectrum of $\Delta_g$. Suppose that $u=x^\zeta f+x^{n-\zeta}f'$, with $f,f'\in C^\infty(X)$, satisfies $(\Delta_g+\zeta(\zeta-n))u=0$. If $f'|_{\partial X}=0$, then $u=0$.*

*Proof.* Substituting $u=x^\zeta f+x^{n-\zeta}f'$ in the equation $(\Delta_g-\zeta(n-\zeta))u=0$, equating the powers of $x$, and using that $2\zeta\notin\mathbf{Z}$, we deduce that if $f'|_{\partial X}=0$ then, in fact, $f'$ vanishes to infinite order at $\partial X$, and so can be absorbed into $f$. So we may assume that $u=x^\zeta f$.

If $\operatorname{Re}\zeta>\frac{1}{2}n$ then $u$ is an $L^2$-eigenfunction and, by our assumption on $\zeta$, must be zero.

To analyze the case $\operatorname{Re}\zeta\leqslant\frac{1}{2}n$, we proceed as in [30]. Let $\phi\in C^\infty(\mathbf{R})$, $\phi(t)\geqslant 0$, $\phi'(t)\geqslant 0$, with $\phi(t)=0$ for $t<1$ and $\phi(t)=1$ for $t>2$, and let $(x,y)$ define a product decomposition near the boundary as in Proposition 2.1. Since in this product decomposition

$$\Delta_g=-(x\partial_x)^2+nx\partial_x-x^2F(x,y)\partial_x+x^2Q(x,y,\partial_y),$$

with $F$ smooth, $\phi(\varepsilon^{-1}x)u\in C^\infty(X)$ vanishes near $\partial X$, and $\Delta_g$ is self-adjoint, we obtain

$$\begin{aligned}\int_X([\Delta_g,\phi(\varepsilon^{-1}x)]u)\bar u\,dg&=2i\operatorname{Im}[\zeta(\zeta-n)]\int_X\phi(\varepsilon^{-1}x)|u|^2\,dg\\&=\int_X([-(x\partial_x)^2+nx\partial_x-x^2F(x,y)\partial_x,\phi(\varepsilon^{-1}x)]u)\bar u\,\frac{dx}{x^{n+1}}\,dh,\end{aligned}$$

where $dg$ is the Riemannian measure induced by the density, and $h$ is the natural density induced by $g$ in $y$.

Now if we have $u=x^\zeta f$, then, after setting $x=\varepsilon\tau$, integrating by parts, and using that $\phi(2)=1$ and $\phi(1)=0$, we obtain

$$\begin{aligned}&\left(-2i(2\varepsilon)^{2\operatorname{Re}\zeta-n}\operatorname{Im}\zeta+2i\varepsilon^{2\operatorname{Re}\zeta-n}\operatorname{Im}\zeta(2\operatorname{Re}\zeta-n)\int_1^2\tau^{2\operatorname{Re}\zeta-n-1}\phi(\tau)\,d\tau\right)\\&\qquad\times\int_{\partial X}|f|^2(0,y)\,dh+O(\varepsilon^{2\operatorname{Re}\zeta-n+1})\\&\qquad\qquad=2i\operatorname{Im}[\zeta(\zeta-n)]\int_X\phi(\varepsilon^{-1}x)|u|^2\,dg.\end{aligned}\tag{4.33}$$

Observe that

$$\begin{aligned}\int_X\phi(\varepsilon^{-1}x)|u|^2\,dg&=\int_\varepsilon^{2\varepsilon}\int_{\partial X}\phi(\varepsilon^{-1}x)x^{2\operatorname{Re}\zeta-n-1}|f|^2(x,y)\,dx\,dh+O(1)\\&=\varepsilon^{2\operatorname{Re}\zeta-n}\int_1^2\phi(\tau)\tau^{2\operatorname{Re}\zeta-n-1}\,d\tau\int_{\partial X}|f|^2(0,y)\,dh\\&\qquad+O(\varepsilon^{2\operatorname{Re}\zeta-n+1})+O(1).\end{aligned}\tag{4.34}$$

Since $\operatorname{Im}\zeta(2\operatorname{Re}\zeta-n)=\operatorname{Im}[\zeta(\zeta-n)]$, we deduce from (4.33) and (4.34) that

$$-2i(2\varepsilon)^{2\operatorname{Re}\zeta-n}\operatorname{Im}\zeta\int_{\partial X}|f|^2(0,y)\,dh+O(\varepsilon^{2\operatorname{Re}\zeta-n+1})=2i\operatorname{Im}[\zeta(\zeta-n)]O(1).\tag{4.35}$$

When $2\operatorname{Re}\zeta<n$, since this holds as $\varepsilon\to 0$, we deduce that $f|_{\partial X}=0$. Observe that when $2\operatorname{Re}\zeta=n$, $\operatorname{Im}[\zeta(\zeta-n)]=0$. Thus the right-hand side of (4.35) vanishes. Letting $\varepsilon\to 0$, we also deduce that $f|_{\partial X}=0$.

Once $f|_{\partial X}$ vanishes it follows, using the indicial equation and the fact that $2\zeta\notin\mathbf{Z}$, that $u$ must vanish to infinite order at the boundary, thus that $u\in L^2(X)$ and therefore $u=0$. □

The scattering matrix, acting on half-densities, can then be defined, for the values of $\zeta$ as in Proposition 4.3, and such that $M(\zeta)\neq 0$, as the map

$$S(\zeta)\colon \Gamma^{1/2}(\partial X)\to\Gamma^{1/2}(\partial X),$$
$$S(\zeta)f=\frac{1}{M(\zeta)}x^{n/2}f_+|_{\partial X},$$

with $M(\zeta)$ defined as above. Thus it follows from the first equation in (4.15):

PROPOSITION 4.4. *For the values of $\zeta$ as in Proposition 4.3, and such that $M(\zeta)\neq 0$, the scattering matrix $S(\zeta)$ is a pseudo-differential operator in $\partial X$, acting on half-densities, which is meromorphic in $\zeta$. Moreover its kernel, which we also denote by $S(\zeta)$, satisfies*

$$\beta_\partial^* S(\zeta)=\frac{1}{M(\zeta)}\beta^*(x^{-\zeta+n/2}(x')^{-\zeta+n/2}R(\zeta))|_{T\cap B}, \tag{4.36}$$

*where $T\cap B$ is the intersection of the top and bottom faces, and $M(\zeta)$ is defined in Proposition 4.2.*

Observe that the right-hand side of (4.36) gives a meromorphic extension of $S(\zeta)$ for values of $\zeta$ that are not poles of $(1/M(\zeta))R(\zeta)$.

As pointed out in the introduction, this definition of the scattering matrix is dependent on the choice of the defining function $x$. There is a standard way to remove it, see for example [14], [37], and view it as an operator

$$S(\zeta)\colon C^\infty(\partial X,\Gamma^{1/2}(\partial X)\otimes|N^*(\partial X)|^{n-\zeta})\to C^\infty(\partial X,\Gamma^{1/2}(\partial X)\otimes|N^*(\partial X)|^{\zeta}).$$

Now this scattering matrix is not quite the same as the one defined in the introduction, as this is the scattering matrix associated to the operator acting on half-densities rather than on functions. Let $\omega_0$ denote the canonical density over the boundary induced by $h$. To get the appropriate Eisenstein function for functions we take $\omega^{-1/2}(x,y)E(\zeta)\omega_0^{1/2}(y')$. We thus see that the scattering matrix on functions is obtained by trivializing the half-density bundle over the boundary by $\omega_0^{1/2}$. Note that conjugating the scattering matrix by the trivializing half-density will not affect the principal symbol, nor will it affect the principal symbol of the difference of two scattering matrices associated to differing metrics which agree at the boundary. It follows therefore that in the next section, where we establish our inverse result, it is irrelevant which definition we use.

## 5. The principal symbol

We compute the principal symbols of $S(\zeta)$ and $S_1(\zeta)-S_2(\zeta)$. Throughout this section we assume that $\zeta$ is not a pole of the right-hand side of (4.36). We also fix a product structure in which

$$\begin{gathered} g_j=\frac{dx^2+h_j(x,y,dy)}{x^2}, \quad V_j\in C^\infty(X), \quad V_j(0,y)=0, \quad j=1,2, \\ h_2(x,y,dy)-h_1(x,y,dy)=x^kL(y,dy)+O(x^{k+1}), \quad k\geqslant 1, \\ V_2-V_1=x^kW(y)+O(x^{k+1}), \quad k\geqslant 1. \end{gathered} \tag{5.1}$$

First, we prove Theorem 1.1.

*Proof.* It follows from (4.36) and (4.29) that the leading singularity of $\beta_\partial^*S(\zeta)$ is given by $(1/M(\zeta))F|_{T\cap B}R^{-2\zeta}$. As observed in the proof of Proposition 4.2, $F|_{T\cap B}$ is the induced half-density on $T\cap B$. Thus, pushing forward to $\partial X\times\partial X$ gives that the leading singularity of $S(\zeta)$ is given by $(1/M(\zeta))|y-y'|^{-2\zeta}$ times the half-density induced by $h$. The density term in $M(\zeta)$ cancels with that of $h$. Taking the Fourier transform we find that the principal symbol of $S(\zeta)$ is given by $C(\zeta)|\xi|^{2\zeta-n}$, where $|\xi|$ is the length of the covector $\xi$ with respect to the metric induced by $h$. Note that the principal symbol could also be computed by observing that it must agree with that in the almost product case, and that doing so gives the explicit value of the constant—we have proceeded in the other way in order to prepare the ground for our next result. □

As a consequence of Theorem 3.1 and Proposition 4.4 we obtain

Proposition 5.1. *Let $g_j,V_j$, $j=1,2$, satisfy* (5.1). *Let $S_j(\zeta)$, $j=1,2$, be the scattering matrix corresponding to $g_j,V_j$, and let $M(\zeta)$ be defined as above. Then*

$$S_1(\zeta)-S_2(\zeta)=\frac{1}{M(\zeta)}(\Lambda_1(\zeta)+\Lambda_2(\zeta)),$$

*where $\Lambda_2\in\Psi^{-\infty}(\partial X,\Gamma^{1/2}(\partial X))$ and the Schwartz kernel of $\Lambda_1$ satisfies*

$$\beta_\partial^*\Lambda_1(\zeta)=(R^{k-2\zeta+n}\varrho^{n/2}(\varrho')^{n/2}\alpha(\zeta))|_{\varrho=\varrho'=0},$$

*with $\alpha(\zeta)$ defined by* (3.17).

*Proof.* We will apply (3.16) and (3.17) to (4.36). Since the lift of the Schwartz kernel of $G_2$, defined in (3.16) and (3.17), under $\beta$ vanishes to infinite order at the top and bottom faces, it does not contribute to the difference of the scattering matrices. Also notice that if $\gamma\in C^\infty(X\times X,\Gamma_0^{1/2}(X\times X))$, then

$$(x^{-\zeta+n/2}(x')^{-\zeta+n/2}x^\zeta(x')^\zeta\gamma)|_{x=x'=0}\in C^\infty(\partial X\times\partial X,\Gamma^{1/2}(\partial X\times\partial X)).$$

So $G_3(\zeta)$ contributes to the difference of the scattering matrices with a smoothing operator. Finally observe that

$$\beta^*(x^{-\zeta+n/2}(x')^{-\zeta+n/2}G_1)|_{T\cap B} = (R^{k-2\zeta+n}\varrho^{n/2}(\varrho')^{n/2}\alpha(\zeta))|_{\varrho=\varrho'=0}. \tag{5.2}$$

This concludes the proof of the proposition. □

Next we compute the leading singularity of $S_1(\zeta)-S_2(\zeta)$. The main part of the calculation is

LEMMA 5.1. *Let $g_j, V_j$, $j=1,2$, satisfy (5.1), and let $S_j$, $j=1,2$, be the scattering matrix corresponding to $g_j, V_j$. Let $p\in\partial X$ and assume that, after a linear transformation, $h_0(p)=\mathrm{Id}$. Let $S_j(\zeta)$, $j=1,2$, be the scattering matrices acting on half-densities. Then, for $M(\zeta)$ as above,*

$$S_1(\zeta)-S_2(\zeta) = \frac{1}{M(\zeta)}(B_1(\zeta)+B_2(\zeta)),$$

*where in local coordinates $x, y'$, valid near $p=y$, with $Y=y-y'$, $r=|y-y'|$, $\varrho=x/r$, $\varrho'=x'/r$, $w=(y-y')/r$, valid near $T\cap B$, the lift of the kernels of $B_1$ and $B_2$ under $\beta_\partial$ are given by*

$$\begin{aligned}&\beta_\partial^* B_1 = r^{k-2\zeta+n}\alpha(\zeta,0,w,y,0,0)\left|\frac{dr}{r}\,\frac{dw}{r^n}\,dy'\right|^{1/2},\\ &\beta_\partial^* B_2 = O(r^{k-2\zeta+n+1}).\end{aligned} \tag{5.3}$$

*Moreover, for $2\operatorname{Re}\zeta\geqslant\max(k+2, n-k+1)$,*

$$\begin{aligned}\alpha(\zeta,0,w,y,0,0) &= \alpha(\zeta,0,Y/|Y|,y,0,0)\\ &= C(\zeta)\Big[T_1(k,\zeta)\sum_{i,j=1}^{n} H_{ij}(y)|Y|^{2\zeta-k}\partial_{Y_i}\partial_{Y_j}|Y|^{k+2-2\zeta}\\ &\qquad + T_2(k,\zeta)\big(W(y)+\tfrac14 k(k+1)T(y)\big)\Big]\end{aligned} \tag{5.4}$$

$$C(\zeta) = \left(\frac12\pi^{-n/2}\frac{\Gamma(\zeta)}{\Gamma\big(\zeta-\frac12(n-2)\big)}\right)^2,$$

$$T_l(k,\zeta) = \int_0^\infty\int_{\mathbf{R}^n}\frac{u^{2\zeta+k+3-2l-n}}{(u^2+|V|^2)^\zeta(u^2+|e_1-V|^2)^\zeta}\,dV\,du,\quad e_1=(1,0,\dots,0),\ l=1,2.$$

*Proof.* In these coordinates, (5.2) is given by

$$\beta^*(x^{\zeta+n/2}(x')^{\zeta+n/2}G_1)|_{T\cap B} = r^{k-2\zeta+n}\alpha(\zeta,r,w,y,0,0)\left|\frac{dr}{r}\,\frac{dw}{r^n}\,dy'\right|^{1/2}. \tag{5.5}$$

Now we use Proposition 5.1 and observe that $\beta_\partial=\beta|_{T\cap B}$. Equation (5.3) is just the first-order Taylor expansion in $r$ of the function $\alpha(\zeta,r,w,y,0,0)$.

We observe that $\alpha(\zeta,0,w,y,0,0)\,|dw\,dy'|^{1/2}$ is the restriction of $R^{n/2}\varrho^{n/2}(\varrho')^{n/2}\alpha(\zeta)$ to the the intersection of the top, bottom and front faces, $T\cap B\cap F=\{R=\varrho=\varrho'=0\}$, so the second part of the lemma is to compute this value. We know from Theorem 3.1 that the half-density $R^{n/2}\alpha(\zeta)$, restricted to the front face, satisfies (3.18). By Remark 1 (after Theorem 3.1) this equation has a unique solution in $\mathcal{A}^{\zeta,\zeta-k}$. After solving it directly, we find the value of its solution at $T\cap B\cap F$.

Instead of using coordinates $(r,\varrho,\varrho',\omega,y')$, it is easier to solve (3.18) in coordinates $s=x/x'$, $z=(y-y')/x'$. The front face is then given by $x'=0$, and we have

$$\begin{aligned}\varrho&=\frac{x}{(x^2+(x')^2+|y-y'|^2)^{1/2}}=\frac{s}{(1+s^2+|z|^2)^{1/2}},\\ \varrho'&=\frac{x'}{(x^2+(x')^2+|y-y'|^2)^{1/2}}=\frac{1}{(1+s^2+|z|^2)^{1/2}}.\end{aligned}\tag{5.6}$$

The intersection of the top, bottom and front faces, $T\cap B\cap F$, is then given by $\{x'=s=0, |z|=\infty\}$, and since $h_0=\mathrm{Id}$, equation (3.18) is reduced to

$$(\tilde\Delta+\zeta(\zeta-n))(s^\zeta(1+s^2+|z|^2)^{(k-2\zeta)/2}\alpha(s,z))=N_p(s^kE)G,\tag{5.7}$$

where $\tilde\Delta$ is the Laplacian in the hyperbolic space acting on half-densities. Hence we have

$$s^\zeta(1+s^2+|z|^2)^{(k-2\zeta)/2}\alpha(s,z)=G(N_p(s^kE)G)(s,z).\tag{5.8}$$

It follows from (3.5) that

$$N_p(E)=\sum_{i,j=1}^{n}H_{ij}(y)s\partial_{z_i}s\partial_{z_j}+\big(W(y)+\tfrac14k(k+1)T(y)\big).\tag{5.9}$$

We recall from Lemma 2.1 of [16] that

$$G(s,z)=\left(\frac12\pi^{-n/2}\frac{\Gamma(\zeta)}{\Gamma\big(\zeta-\frac12(n-2)\big)}\cdot\frac{s^\zeta}{(1+s^2+|z|^2)^\zeta}\right)\left|\frac{ds}{s}\frac{dz}{s^n}\,dy'\right|^{1/2}+E_1,\tag{5.10}$$

where $E_1$ has a conormal singularity at $\{s=1,z=0\}$ and, near the boundary, $G_1\in\mathcal{A}^{\zeta+1,\zeta+1}$, where $\mathcal{A}^{a,b}$ denotes the space of half-densities of the form

$$s^a|z|^{-b}\left|\frac{ds}{s}\frac{dz}{s^n}\,dy'\right|^{1/2}.$$

It follows from Proposition 6.19 of [28] that $G(\mathcal{A}^{\zeta+k+1,\zeta+1})\subset\mathcal{A}^{\zeta,\zeta-k+1}$. Since, as in equation (4.12) of [28], see also equation (3.1), $G$ acts as a convolution operator with respect to the group action defined in §3 of that paper, we find that

$$\begin{aligned} s^{\zeta}(1+s^2+|z|^2)^{(k-2\zeta)/2}\alpha(s,z)\left|\frac{ds}{s}\frac{dz}{s^n}dy'\right|^{1/2} \\ = C(\zeta)\bigg(\sum_{i,j=1}^{n}H_{ij}(y)\partial_{z_i}\partial_{z_j}I_1(k,\zeta,s,z) \qquad (5.11) \\ +\big(W(y)+\tfrac{1}{4}k(k+1)T(y)\big)I_2(k,\zeta,s,z)\bigg)\left|\frac{ds}{s}\frac{dz}{s^n}dy'\right|^{1/2}+\beta, \end{aligned}$$

where

$$C(\zeta)=\left(\frac{1}{2}\pi^{-n/2}\frac{\Gamma(\zeta)}{\Gamma\big(\zeta-\frac{1}{2}(n-2)\big)}\right)^2,\quad \beta\in\mathcal{A}^{\zeta,\zeta-k+1},$$

and

$$I_l(k,\zeta,s,z)=\int_0^\infty\int_{\mathbf{R}^n}\frac{t^{\zeta}}{(1+t^2+|U|^2)^{\zeta}(1+s^2/t^2+|z-(s/t)U|^2)^{\zeta}}\left(\frac{s}{t}\right)^{\zeta+k+4-2l}\frac{dt}{t}\,dU.$$

Recall that our goal is to compute the restriction of $R^{n/2}\varrho^{n/2}(\varrho')^{n/2}\alpha(\zeta)$ to $T\cap B\cap F$. In these coordinates

$$R^{n/2}\varrho^{n/2}(\varrho')^{n/2}=(x')^{n/2}\frac{s^{n/2}}{(1+s^2+|z|^2)^{n/4}}.$$

So, after restricting to the front face, which is given by $\{x'=0\}$, we have to restrict

$$\frac{s^{n/2}}{(1+s^2+|z|^2)^{n/4}}\alpha(s,z)\left|\frac{ds}{s}\frac{dz}{s^n}dy'\right|^{1/2}$$

to the corner $T\cap B\cap F=\{s=0,|z|=\infty\}$. This is the same as the restriction of

$$\frac{s^{n/2}}{|z|^{n/2}}\alpha(s,z)\left|\frac{ds}{s}\frac{dz}{s^n}dy'\right|^{1/2}.$$

Notice that for $w=z/|z|$, we have $dw=dz/|z|^n$. Thus it follows by (5.11) that the value of $s^{n/2}|z|^{-n/2}\alpha$ at $T\cap B\cap F$ is then given by $A(w)\,|dw\,dy'|^{1/2}$, where

$$A(w)=\lim_{\substack{s\to0\\|z|\to\infty}}\frac{1}{s^{\zeta}(1+s^2+|z|^2)^{(k-2\zeta)/2}}\left[\sum_{i,j=1}^{n}H_{ij}(y)\partial_{z_i}\partial_{z_j}I_1+\big(W(y)+\tfrac{1}{4}k(k+1)T(y)\big)I_2\right]. \tag{5.12}$$

Set $|z|u=s/t$ and $U=(t/s)|z|V$, and observe that $I_l(k,\zeta,s,z)=I_l(k,\zeta,s,|z|)$, so we can also set $z=|z|e_1$, $e_1=(1,0,...,0)$. Then

$$\begin{aligned} &I_l(k,\zeta,s,z)\\ &=s^{\zeta}|z|^{-2\zeta+k+4-2l}\int_0^\infty\int_{\mathbf{R}^n}\frac{u^{2\zeta+k+3-2l-n}}{(u^2+s^2/|z|^2+|V|^2)^{\zeta}(1/|z|^2+u^2+(e_1-V)^2)^{\zeta}}\,dV\,du. \end{aligned}$$

To analyze the limit of $I_l(k,\zeta,s,z)$ as $s\to0$ and $|z|\to\infty$, we begin by proving

LEMMA 5.2. *For $k\geqslant 1$, and for $2\operatorname{Re}\zeta\geqslant\max(n-k+1,k+2)$, we have*

$$J(l,k,\zeta)=\int_0^\infty\int_{\mathbf{R}^n}\frac{u^{2\operatorname{Re}\zeta+k+3-2l-n}}{(u^2+|V|^2)^{\operatorname{Re}\zeta}(u^2+(e_1-V)^2)^{\operatorname{Re}\zeta}}\,dV\,du<\infty.$$

*Proof.* We carry out the proof for $n>1$. The case $n=1$ is actually simpler. Observe that for $V=(v,V')$, $V'\in\mathbf{R}^{n-1}$ and $|V'|=\varrho$,

$$J(l,k,\zeta)=|\mathbf{S}^{n-2}|\int_0^\infty\int_0^\infty\int_{\mathbf{R}}\frac{u^{2\operatorname{Re}\zeta+k+3-2l-n}\varrho^{n-2}}{(u^2+\varrho^2+v^2)^{\operatorname{Re}\zeta}(u^2+\varrho^2+(v-1)^2)^{\operatorname{Re}\zeta}}\,dv\,du\,d\varrho.$$

Setting $v=R\cos\phi$, $u=R\sin\phi\cos\theta$, $\varrho=R\sin\phi\sin\theta$, $0<\phi<\pi$, $0<\theta<\frac{1}{2}\pi$, we obtain

$$J(l,k,\zeta)=K_l(\zeta)\int_0^\infty\int_0^\pi\frac{R^{k+3-2l}(\sin\phi)^{2\operatorname{Re}\zeta+k+2-2l}}{[(R-\cos\phi)^2+(\sin\phi)^2]^{\operatorname{Re}\zeta}}\,d\phi\,dR,$$

$$K_l(\zeta)=|\mathbf{S}^{n-2}|\int_0^{\pi/2}(\cos\theta)^{2\operatorname{Re}\zeta+k+3-2l}(\sin\theta)^{n-2}\,d\theta.$$

Thus, for $k\geqslant 2$ and $\zeta$ as above, we have that

$$\begin{aligned}J(l,k,\zeta)\leqslant K_1(\zeta)\biggl(&\int_0^4\int_0^\pi R^{k+3-2l}(\sin\phi)^{k+2-2l}\,d\phi\,dR\\&+K_2(\zeta)\int_4^\infty\int_0^\pi R^{-2\operatorname{Re}\zeta+k+3-2l}(\sin\phi)^{2\operatorname{Re}\zeta+k+2-2l}\,d\phi\,dR\biggr)<\infty.\end{aligned}$$

The same argument can be used to show that $J(1,1,\zeta)<\infty$. When $k=1$ and $l=2$, another argument has to be used. Setting $R=\cos\phi+t\sin\phi$ we find that

$$J(2,1,\zeta)\leqslant K(\zeta)\int_{-\infty}^\infty\int_0^\pi(1+t^2)^{-\operatorname{Re}\zeta}\,d\phi\,dt<\infty.$$

This concludes the proof of the lemma. □

Thus the dominated convergence theorem gives that for

$$T_l(k,\zeta,s,z)=s^{-\zeta}|z|^{2\zeta-k-4+2l}I_l(k,\zeta,s,z),$$

we have

$$\lim_{\substack{s\to 0\\|z|\to\infty}}T_l(s,z)=T_l(k,\zeta)=\int_0^\infty\int_{\mathbf{R}^n}\frac{u^{2\zeta+k+3-2l-n}}{(u^2+|V|^2)^\zeta(u^2+(e_1-V)^2)^\zeta}\,dV\,du. \tag{5.13}$$

By identical considerations we deduce that

$$\partial_{z_j}T_l(s,z)=O(s^2/|z|^3),\quad \partial_{z_m}\partial_{z_j}T_l(s,z)=O(s^2/|z|^4).$$

Hence

$$\partial_{z_i}\partial_{z_j} I_1(s,z) = C_1(\zeta) s^{\zeta}(\partial_{z_i}\partial_{z_j}|z|^{k+2-2\zeta}) T_1(s,z) + O(s^{\operatorname{Re}\zeta}|z|^{k-2-2\operatorname{Re}\zeta}). \tag{5.14}$$

Using that $z=(y-y')/x'$ we find that $z/|z|=w=(y-y')/|y-y'|$. Therefore (5.4) follows directly form (5.11), (5.12), (5.13) and (5.14). This concludes the proof of the proposition. □

Now we can prove Theorem 1.2.

*Proof.* It follows from (5.3) and (5.4) that the leading singularity of the kernel of $S_1(\zeta)-S_2(\zeta)$ is given by

$$\frac{C(\zeta)}{M(\zeta)}\left(T_1(k,\zeta)\sum_{i,j=1}^{n} H_{ij}(y)\partial_{Y_i}\partial_{Y_j}|Y|^{k+2-2\zeta} + T_2(k,\zeta)\big(W(y)+\tfrac{1}{4}k(k+1)T(y)\big)|Y|^{k-2\zeta}\right) \tag{5.15}$$

times a non-vanishing smooth half-density, where $C(\zeta)$ is given by (5.4) and $M(\zeta)$ by Proposition 4.2. We obtain (1.8) by taking the Fourier transform in $Y$ of (5.15), and observing that (5.4) was obtained under the assumption that $h_0=\mathrm{Id}$, and using the fact that $h_0$ is symmetric. The coefficients of $T_j(k,\zeta)$, $j=1,2$, in (1.9) arise when we take the Fourier transform of the corresponding power of $|Y|$. See for example p. 363 of [12]. This ends the proof of the theorem. □

We now prove Corollaries 1.2 and 1.3. The proof of Corollary 1.3 is a direct consequence of the fact that, for every $k$, $A_2(k,\zeta)\neq 0$ for at least one value of $\zeta$. The proof of Corollary 1.2 requires a more delicate analysis due to the presence of the term involving $T(y)$.

*Proof.* As we are working modulo diffeomorphism invariance we can take a product decomposition such that each $g_j$ is of the form (1.6). Suppose that $g_1$ equals $g_2$ to order $k$ near $p$, and suppose that the principal symbol of $S_1(\zeta)-S_2(\zeta)$ of order $2\operatorname{Re}\zeta-n-k$ is equal to zero at $p$. Since $V_1=V_2$ near $p$, we find that $W=0$. By a linear change of variables on the tangent space to $\partial X$ at $p$ we may assume that $h_0=\mathrm{Id}$. It is clear from (1.8) that if the trace is zero and $A_1(k,\zeta)\neq 0$ then $L_{ij}(p)=0$ is zero, so we need only show that off a discrete set these hold. By taking $\xi=e_j=(0,\dots,0,1,0,\dots,0)$, 1 in the $j$th entry, we deduce from (1.8) that

$$A_1(k,\zeta)L_{ij}(p)+\tfrac{1}{4}k(k+1)A_2(k,\zeta)T(p)=0, \quad 1\leqslant i,j\leqslant n.$$

By taking $i=j$ and adding in $j$ we obtain, for all $\zeta$ which is not a pole of $A_j(k,\zeta)$, $j=1,2$,

$$\big(A_1(k,\zeta)+\tfrac{1}{4}nk(k+1)A_2(k,\zeta)\big)T(p)=0.$$

Using the formulas for $A_1$ and $A_2$ given by (1.9) and the fact that $\Gamma(z+1)=z\Gamma(z)$ we have, again for all $\zeta$ which is not a pole of $A_j(k,\zeta)$, $j=1,2$,

$$\big(T_1(k,\zeta)(k+2-2\zeta)(k-2\zeta+n)+\tfrac{1}{4}nk(k+1)T_2(k,\zeta)\big)T(p)=0. \tag{5.16}$$

(This can also be deduced directly from (5.15).) It follows from the meromorphicity of the scattering matrix that the coefficient, $Z(\zeta)$, of $T(p)$ in (5.16) has a meromorphic extension to $\mathbf{C}$, and we will show that it is not identically zero. Hence, if the symbol of order $2\operatorname{Re}\zeta-n-k$ of the difference of the scattering matrices vanishes at an energy $\zeta$, which is not one of the zeros of $Z(\zeta)$, it follows that $T(p)=0$.

We know from Lemma 5.2 that for $k\geqslant 1$, and for $2\operatorname{Re}\zeta\geqslant\max(n-k+1,k+2)$, $T_1(k,\zeta)$ and $T_2(k,\zeta)$ are finite. In particular they are finite for $2\zeta=k+n$, as long as $n\geqslant 2$. It is clear from the definition of $T_j$ that for $2\zeta=k+n$, $T_j(k,\zeta)>0$, $j=1,2$. Hence $T(p)=0$ and $H_{ij}=0$.

For $n=1$, we can apply the same argument except that we take $2\zeta=k+2$ instead of $k+1$. This ends the proof of the corollary. □

## 6. Almost product-type metrics

In this section, we examine the scattering matrix for metrics which take the form

$$g=\frac{dx^2+h(y,dy)}{x^2}+O(x^\infty) \tag{6.1}$$

for some product decomposition. Our approach is analogous to that of Christiansen, [11], and Parnovski, [35], in the asymptotically Euclidean setting. The computation is also closely related to that of Hislop, [17, §2.3], for $\mathbf{H}^n$.

As we have shown in previous sections that if two metrics agree to infinite order then the associated scattering matrices differ by a smoothing operator, it is sufficient to compute for the manifold $\mathbf{R}_+\times\partial X$ with metric $(dx^2+h(y,dy))/x^2$. The Laplacian is then

$$-\left(x\frac{\partial}{\partial x}\right)^2+nx\frac{\partial}{\partial x}+x^2\Delta_{\partial X},$$

where $\Delta_{\partial X}$ is the Laplacian associated to $h$ on $\partial X$. Let $\psi_j$ be a complete orthonormal basis of eigenfunctions for $\Delta_{\partial X}$ with $\psi_j$ of eigenvalue $\lambda_j^2$.

We then look for solutions of $(\Delta+\zeta(\zeta-n))u=0$ of the form $x^{n/2}a(x)\psi_j(y)$. Computing as in [17] we deduce that $a$ satisfies

$$\left(x^2\frac{\partial^2}{\partial x^2}+x\frac{\partial}{\partial x}-\left[x^2\lambda_j^2+\left(\zeta-\tfrac{1}{2}n\right)^2\right]\right)a(x)=0.$$

This is a modified Bessel equation, and taking the solutions which are regular at infinity, we see that $a$ has an asymptotic expansion as $x \to 0$, and its lead term is of the form

$$\frac{1}{\Gamma\left(1-(\zeta-\frac{1}{2}n)\right)}\left(\tfrac{1}{2}\lambda_j x\right)^{n/2-\zeta}-\frac{1}{\Gamma\left(1+(\zeta-\frac{1}{2}n)\right)}\left(\tfrac{1}{2}\lambda_j\right)^{\zeta-n/2}.$$

It now follows immediately that $S(\zeta)$ applied to $\psi_j$ multiplies it by the ratio of these coefficients:

$$-\frac{\left(\frac{1}{2}\lambda_j\right)^{\zeta-n/2}/\Gamma\left(1+(\zeta-\frac{1}{2}n)\right)}{\left(\frac{1}{2}\lambda_j\right)^{n/2-\zeta}/\Gamma\left(1-(\zeta-\frac{1}{2}n)\right)}=2^{n-2\zeta}\frac{\Gamma\left(\frac{1}{2}n-\zeta\right)}{\Gamma\left(\zeta-\frac{1}{2}n\right)}\lambda_j^{2\zeta-n}.$$

As the functions $\psi_j$ form an orthonormal basis, we have now proven the second part of Theorem 1.3.

## 7. Inverse scattering for black holes

We consider two models for the exterior of a static black hole, the Schwarzschild and the De Sitter–Schwarzschild models. These are given by

$$(Y,g),\quad Y=\mathbf{R}_t\times X,\quad \text{where } g=\alpha^2\,dt^2-\alpha^{-2}\,dr^2-r^2\,|d\omega|^2.$$

$|d\omega|^2$ is the standard metric on $\mathbf{S}^2$. In the Schwarzschild model,

$$X=(r_+,\infty)_r\times\mathbf{S}^2_\omega \quad\text{and}\quad \alpha=\left(1-\frac{2m}{r}\right)^{1/2},\quad r_+=2m<r, \tag{7.1}$$

and in the De Sitter–Schwarzschild model,

$$X=(r_+,r_{++})_r\times\mathbf{S}^2_\omega \quad\text{and}\quad \alpha=\left(1-\frac{2m}{r}-\frac{1}{3}\Lambda r^2\right)^{1/2},\quad r_+<r<r_{++}. \tag{7.2}$$

The parameter $m>0$ denotes the mass of the black hole. In (7.2), $\Lambda$, with $0<9m^2\Lambda<1$, is the cosmological constant, and $r_+$, $r_{++}$ are the two solutions to $\alpha=0$.

These are semi-Riemannian metrics on the manifold with boundary $Y$, so their Laplacians are in fact hyperbolic operators, and we denote them $\Box_g$. We have

$$\Box_g=\alpha^{-2}\left(D_t^2-\alpha^2r^{-2}D_r(r^2\alpha^2)D_r-\alpha^2r^{-2}\Delta_\omega\right), \tag{7.3}$$

where $D_\bullet=(1/i)\partial_\bullet$, and $\Delta_\omega$ is the positive Laplacian on $\mathbf{S}^2$. Therefore stationary scattering phenomena are governed by the operator

$$P=\alpha^2r^{-2}D_r(r^2\alpha^2)D_r-\alpha^2r^{-2}\Delta_\omega. \tag{7.4}$$

Scattering theory for the operator $P$ has been extensively studied, see for example [5], [9], [10], [34], [39] and the references cited there. It was observed in [39] that, after a change of $C^\infty$-structure on $X$, in the De Sitter–Schwarzschild model the operator $P$ can be viewed as a zero-differential operator which is elliptic, and whose normal operator is, after a linear change of variables, a multiple of the Laplacian on the hyperbolic space. This change in $C^\infty$-structure is simply the addition of the square root of the boundary-defining function and therefore only affects smoothness up to the boundary and not smoothness in the interior. Thus the methods of [28] directly apply, and it was shown in [39] that $R(\lambda)=\left(P-\lambda^2-\frac{1}{4}n^2\right)^{-1}$ has a meromorphic continuation to $\mathbf{C}$. It also follows from the discussion in [39], and the methods of §4, that the scattering matrix can be defined in this situation.

The case of the Schwarzschild model is more complicated. At one end, $\alpha=0$, which is the black hole, the operator $P$ behaves as in the De Sitter–Schwarzschild model, i.e. after a change in the $C^\infty$-structure of $X$, it is an elliptic zero-differential operator, and its normal operator is essentially the hyperbolic Laplacian. On the other end, as $r\to\infty$, $\alpha\to 1$, and the metric $g$ tends to the Lorentz metric. Thus the operator $P$ tends to the Euclidean Laplacian. This is the case of an asymptotically Euclidean metric. To study the scattering matrix at this end one proceeds as in [33]. Since the construction of the symbol of the scattering matrix at each end only depends on the metric in a neighbourhood of each boundary, see [33] and §4, it follows that, modulo smoothing operators, the scattering matrices at each boundary are independent.

It was shown in [5] that the resolvent $R(\lambda)$, for the Schwarzschild model, as an operator from $\mathcal{C}_0^\infty(\mathring{X})$ to $\mathcal{C}^\infty(\mathring{X})$, has a meromorphic continuation from $\operatorname{Im}\lambda>0$ to $\mathbf{C}\setminus i\overline{\mathbf{R}}_-$. It is not known whether its poles might accumulate at the origin.

In this section we will prove that the Taylor series of certain perturbations of both models, at $\alpha=0$, are determined from the scattering matrix at a fixed energy. The analogous result at $x=0$ also holds for the Schwarzschild model. However, since its proof relies on the methods of [24], we will not carry it out here.

THEOREM 7.1. *Let $(X,\partial X)$ be a smooth manifold with boundary with dimension $n+1$, and let $p\in\partial X$. Suppose that $g$ induces an asymptotically hyperbolic structure on $X$, and that $g=(dx^2+h(x,y,dy))/x^2$, with respect to some product decomposition near $\partial X$. Suppose that $P$ is a positive, smooth, elliptic, zero-differential operator of second order, and that its normal operator satisfies*

$$N_q(P)=KN_q(\Delta_g) \quad \textit{for all } q\in\partial X, \tag{7.5}$$

*where $K>0$ is a constant on each component of $\partial X$. Then for each $\lambda\in\mathbf{R}\setminus Q$, $Q$ a discrete subset, and $f\in C^\infty(\partial X)$ there exists a unique $u$ satisfying $\left(P-\lambda^2-\frac{1}{4}n^2\right)u=0$ of* *the form*

$$u = x^{i\lambda+n/2} f_+ + x^{-i\lambda+n/2} f_-, \quad f_+|_{\partial X} = f.$$

*Moreover the scattering matrix given by*

$$S(\lambda) f = f_-|_{\partial X}$$

*is a pseudo-differential operator of order* $2i\lambda$.

*Furthermore if* $P_2$ *is another smooth elliptic zero-differential operator of second order that satisfies* (7.5), *and is such that*

$$P - P_2 = x^k \left( \sum_{i,j=1}^{n} H_{ij}(x) x\partial_{y_i} x\partial_{y_j} + W(x) \right) + O(x^{k+1}), \tag{7.6}$$

*where* $H=(H_{ij})$ *is a smooth symmetric matrix, then*

$$S(\lambda) - S_2(\lambda) \in \Psi^{2i\lambda-k}$$

*and the principal symbol of* $S(\lambda)-S_2(\lambda)$ *equals*

$$A_1(k,\lambda) \sum_{i,j} H_{ij} \xi_i \xi_j |\xi|^{2i\lambda-k-2} + A_2(k,\lambda) W |\xi|^{2i\lambda-k}, \tag{7.7}$$

*where* $h_0 = h|_{x=0}$, $|\xi|$ *is the length of the covector* $\xi$ *induced by* $h_0$, *and* $A_1, A_2$ *are functions of* $\lambda$ *which are not identically zero.*

*Proof.* A line-by-line inspection of the proof of Theorem 1.2 with $\zeta = \frac{1}{2}n + i\lambda$ gives the result. □

As an application of Theorem 7.1 we will prove

THEOREM 7.2. *Let* $X$ *and* $\alpha$ *be given by either* (7.1) *or* (7.2). *Let* $a_{ij}(r,\omega) \in C^\infty(\overline{X})$, $0 \leqslant i,j \leqslant 2$, *and let*

$$g = \alpha^2\, dt^2 - \alpha^{-2}(1+\alpha a_{00}(r,\omega))\, dr^2 - \sum_{j=1}^{2} a_{0j}\, dr\, d\omega_j - r^2 \sum_{i,j=1}^{2} (\delta_{ij} + \alpha a_{ij})\, d\omega_i\, d\omega_j \tag{7.8}$$

*be a perturbation of the models above. Let* $X_{1/2}$ *be the manifold* $X$ *with the new* $C^\infty$*-structure in which* $\alpha \in C^\infty(X_{1/2})$ *is the new boundary-defining function. Then the operator* $P_a$ *defined by*

$$\Box_{g_a} = \alpha^{-2}(D_t^2 - P_a) \tag{7.9}$$

*satisfies the hypotheses of Theorem 7.1 at the boundary, $\{\alpha=0\}$, and there exists a product decomposition $(\widetilde{\alpha},\widetilde{\omega})$, with $\widetilde{\omega}=\omega$ at $\alpha=0$, near $X$ such that for $\lambda\in\mathbf{R}\setminus Q$, $Q$ a countable subset, its scattering matrix at energy $\lambda$ determines the Taylor series of $a_{ij}$ in coordinates $(\widetilde{\alpha},\widetilde{\omega})$ at $\{\widetilde{\alpha}=0\}$.*

Note that as before we can recover to finite order off a discrete subset but to infinite order off a countable subset.

*Proof.* We will only carry out the proof for the Schwarzschild model. The other case is very similar, although the computations are more tedious, but are essentially done in [39].

First we check the statement about the normal operator of $P_a$. Since $\alpha^2=1-2m/r$ we find that $dr=\alpha(r^2/m)\,d\alpha$. Hence $g$ is given by

$$g=\alpha^2\,dt^2-\frac{r^4}{m^2}(1+\alpha a_{00}(r,\omega))\,d\alpha^2-\alpha\frac{r^2}{m^2}a_{0j}\,d\alpha\,d\omega_j-r^2\sum_{i,j=1}^{2}(\delta_{ij}+\alpha a_{ij})\,d\omega_i\,d\omega_j. \tag{7.10}$$

Let $A_0=(a^0_{ij})$, where $a^0_{00}=\alpha^{-2}$, $a^0_{22}=a^0_{33}=r^2$ and $a^0_{ij}=0$, $i\neq j$. Let $A_1=(a^1_{ij})$, where $a^1_{00}=\alpha^{-2}a_{00}$, $a^1_{0j}=a^1_{j0}=a_{j0}$, $a^1_{ij}=a_{ij}$, $1\leqslant i,j\leqslant 2$. Let $A=A_0+\alpha A_1$. Then we have $A=A_0(\mathrm{Id}+\alpha A_0^{-1}A_1)$ and hence

$$\begin{aligned}
\det(A)&=\det(A_0)\det(I+\alpha A_0^{-1}A_1)=\det(A_0)(1+\alpha T+O(\alpha^2)),\\
T&=a_{00}+a_{11}+a_{22},\\
A^{-1}&=A_0^{-1}+\alpha A_0^{-1}A_1A_0^{-1}.
\end{aligned}\tag{7.11}$$

Using (7.11) and the definition of $P_a$ we find that the normal operator of $P_a$ at a point $p$ at the boundary $\alpha=0$ is

$$N_p(P_a)=\frac{1}{16m^2}(4(\alpha\partial_\alpha)^2+\alpha^2\Delta_p),$$

where $\Delta_p$ is the Laplacian at the tangent plane to $\mathbf{S}^2$ at $p$. Thus $N_p(P_a)$ satisfies (7.5).

Next we consider two perturbations of the Schwarzschild metric $F$ and $H$ satisfying

$$\begin{aligned}
F_{00}&=\frac{r^4}{m^2}(1+\alpha f_{00}), & F_{1j}&=F_{j1}=\frac{r^2}{m^2}f_{1j}, & F_{ij}&=r^2(\delta_{ij}+\alpha r^{-2}f_{ij}),\\
H_{00}&=\frac{r^4}{m^2}(1+\alpha h_{00}), & H_{1j}&=H_{j1}=\frac{r^2}{m^2}h_{1j}, & H_{ij}&=r^2(\delta_{ij}+\alpha r^{-2}h_{ij}).
\end{aligned}$$

Let $g_F$ and $g_H$ be defined by (7.10), where $f_{ij}$ and $h_{ij}$ play the rôle of $a_{ij}$. Let $S_F$ and $S_H$ be the scattering matrices corresponding to $P_F$ and $P_H$. It follows from the computation of the determinant above that, for $\alpha$ small and $f_{ij}, h_{ij}$ smooth,

$$G_F = \frac{r^4}{m^2}(1+\alpha f_{00}(r,\omega))\,d\alpha^2 + \alpha\frac{r^2}{m^2}\sum_{j=1,2} f_{0j}\,d\alpha\,d\omega_j + r^2\sum_{i,j=1}^{2}(\delta_{ij}+\alpha f_{ij})\,d\omega_i\,d\omega_j,$$

$$G_H = \frac{r^4}{m^2}(1+\alpha h_{00}(r,\omega))\,d\alpha^2 + \alpha\frac{r^2}{m^2}\sum_{j=1,2} h_{0j}\,d\alpha\,d\omega_j + r^2\sum_{i,j=1}^{2}(\delta_{ij}+\alpha h_{ij})\,d\omega_i\,d\omega_j$$

are Riemannian metrics near $\partial X$.

Let $(\widetilde{\alpha},\widetilde{\omega})$ be a product decomposition of $X$ near $\partial X$ in which

$$G_F = d\alpha^2 + \widetilde{f_{ij}}\,d\omega_i\,d\omega_j, \quad G_H = d\alpha^2 + \widetilde{h_{ij}}\,d\omega_i\,d\omega_j.$$

Suppose that, in these coordinates, $\widetilde{f_{ij}} - \widetilde{h_{ij}} = \widetilde{\alpha}^k\widetilde{u_{ij}}$. Therefore

$$P_F - P_H = \widetilde{\alpha}^k(\widetilde{u_{ij}}\,\partial_{\omega_i}\partial_{\omega_j}) + O(\widetilde{\alpha}^{k+1}).$$

So it follows from Theorem 7.1 that the $k$th-order symbol of $S_F(\lambda) - S_H(\lambda)$ determines and is determined by $\widetilde{u_{ij}}$.

This ends the proof of the theorem. □

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