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Partial hyperbolicity and robust transitivity

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1. Introduction

Throughout this paper M denotes a three-dimensional boundaryless compact manifold and $\mathrm{Diff}(M)$ the space of C^1-diffeomorphisms defined on M endowed with the usual C^1-topology. A φ-invariant set Λ is *transitive* if $\Lambda=\omega(x)$ for some $x\in\Lambda$. Here $\omega(x)$ is the *forward limit set of x* (the accumulation points of the positive orbit of x). The *maximal invariant set* of φ in an open set U, denoted by $\Lambda_\varphi(U)$, is the set of points whose whole orbit is contained in U, i.e. $\Lambda_\varphi(U)=\bigcap_{i\in\mathbf{Z}}\varphi^i(U)$. The set $\Lambda_\varphi(U)$ is *robustly transitive* if $\Lambda_\phi(U)$ is transitive for every diffeomorphism ϕ C^1-close to φ.

A diffeomorphism $\varphi\in\mathrm{Diff}(M)$ is *transitive* if $M=\omega(x)$ for some $x\in M$, i.e. if $\Lambda_\varphi(M)=M$ is transitive. Analogously, φ is *robustly transitive* if every ϕ C^1-close to φ also is transitive, i.e. if $\Lambda_\varphi(M)=M$ is robustly transitive.

In this paper we focus our attention on forms of hyperbolicity (uniform, partial and strong partial) of a maximal invariant set $\Lambda_\varphi(U)$ derived from its robust transitivity. Observe that U can be equal to M, and then φ is robustly transitive.

On one hand, in dimension one there do not exist robustly transitive diffeomorphisms: the diffeomorphisms with finitely many hyperbolic periodic points (Morse–Smale) are open and dense in $\mathrm{Diff}(S^1)$. On the other hand, for two-dimensional diffeomorphisms, every robustly transitive set $\Lambda_\varphi(U)$ is a basic set (i.e. $\Lambda_\varphi(U)$ is hyperbolic, transitive, and the periodic points of φ are dense in $\Lambda_\varphi(U)$). In particular, every robustly transitive surface diffeomorphism is Anosov and the unique surface which supports such diffeomorphisms is the torus T^2. These assertions follow from [M3] and [M4].

In dimension bigger than or equal to three, besides Anosov (hyperbolic) diffeomorphisms there are robustly transitive diffeomorphisms of nonhyperbolic type. As far as we know, three types of such diffeomorphisms have been constructed: skew products,

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Derived from Anosov, and deformations of the time-τ map X_τ of the flow of a transitive Anosov vector field X.

Before describing these examples let us recall that the standard *Derived from Anosov* (*DA*) *diffeomorphisms,* defined on the two-torus T^2, are obtained via saddle-node bifurcations of Anosov systems: the unfolding of the bifurcation leads to structurally stable maps (the DA-diffeomorphisms) whose nonwandering set is a source and a nontrivial hyperbolic attractor, see [Sm] and [W]. This two-dimensional construction can be carried to higher dimensions to get DA-diffeomorphisms which are robustly nonhyperbolic and transitive, see [M1] and [C].

Chronologically, the first examples of nonhyperbolic robustly transitive diffeomorphisms were skew products. Such diffeomorphisms were constructed in the four-dimensional torus $T^4 = T^2 \times T^2$ by perturbing the product of a DA-diffeomorphism and an Anosov one, see [Sh]. Nowadays we also know that one can perturb the product of any diffeomorphism Φ having a hyperbolic transitive attractor Λ_Φ and the identity Id on any compact manifold to get G C^1-close to $\Phi \times \mathrm{Id}$ with a robustly nonhyperbolic transitive attractor Λ_G. Moreover, $\Lambda_G = \Lambda_G(U)$ for some neighbourhood U, and $\Lambda_G(U)$ is robustly transitive. In particular, if Φ is Anosov (i.e. $\Lambda_\Phi = M$), then the perturbation G is robustly transitive, see [BD1].

All robustly transitive diffeomorphisms mentioned above (skew products and DA-maps) are nonisotopic to the identity, but there also are robustly nonhyperbolic transitive diffeomorphisms isotopic to the identity: Given any transitive Anosov vector field X let X_τ be the flow of X at time τ. Then one can perturb X_τ to obtain a robustly transitive diffeomorphism, see [BD1].

In dimension bigger than or equal to three, besides the constructions above, one can also obtain robustly nonhyperbolic transitive sets (of semilocal nature) via cycles containing periodic points of different *indices* (dimension of the stable manifold), see [Di1], [Di2] and [DR].

The nonhyperbolic transitive sets $\Lambda_\varphi(U)$ quoted above always contain periodic points with different indices and coincide with the closure of their transverse homoclinic points (i.e. the transverse intersections between the invariant manifolds of a periodic point). The previous examples fit into the category which we call *strong partially hyperbolic* (see the definition below): there is a $D\varphi$-invariant partially hyperbolic splitting of $T_{\Lambda_\varphi(U)}M = E^s \oplus E^c \oplus E^u$ with three nontrivial bundles, where E^s and E^u are hyperbolic directions (contracting and expanding, respectively) and E^c is a nonhyperbolic central direction. On the other hand, recently, see [B] and [BV], there have been constructed examples of robustly transitive diffeomorphisms which do not admit three nontrivial invariant bundles (i.e. either E^s or E^u above is trivial).

It is important to mention that in this paper we are only concerned with transitive sets which are locally maximal. Notice that one can also define transitive sets (in a robust way) as follows: given a hyperbolic saddle P of φ, for every ψ close to φ define Σ_ψ as the closure of the transverse homoclinic points of P_ψ (P_ψ is the continuation of P). Such sets are transitive, but in general they fail to be locally maximal: in some cases sinks or sources accumulate to Σ_ψ, see [BD2]. Even more, they do not admit any nontrivial $D\psi$-invariant splitting, see [BD2] and the constructions in [DU].

Our goal here is to characterize the forms of possible hyperbolicity for a maximal invariant set $\Lambda_\varphi(U)$ which is robustly transitive. We prove that, in the case of three-dimensional compact manifolds, the *robustly transitive sets* $\Lambda_\varphi(U)$ *are generically partially hyperbolic.*

Now let us state precisely our results. We begin by giving some basic definitions. Let φ be a diffeomorphism and Λ a φ-invariant set. A splitting $T_\Lambda M = E\oplus F$ is *dominated* if E and F are $D\varphi$-invariant and there are constants $m>0$ and $K<1$ such that

$$\|(D_x\varphi^m)|_{E_x}\|\cdot\|((D_x\varphi^m)^{-1})|_{F_{\varphi^m(x)}}\| < K \quad \text{for all } x\in\Lambda.$$

A $D\varphi$-invariant bundle E defined on Λ is *uniformly contracting* (resp. *expanding*) if there are $C>0$ and $0<\lambda<1$ such that for every $n>0$ one has

$$\|D_x\varphi^n(v)\| \leqslant C\lambda^n\|v\| \quad (\text{resp. } \|D_x\varphi^{-n}(v)\|\leqslant C\lambda^n\|v\|) \quad \text{for all } x\in\Lambda \text{ and } v\in E.$$

The set Λ is *uniformly hyperbolic,* or shortly *hyperbolic,* if there is a $D\varphi$-invariant splitting $T_\Lambda M = E\oplus F$ such that E is uniformly contracting and F is uniformly expanding. The splitting $E\oplus F$ is called (*uniformly*) *hyperbolic.*

The set Λ is *partially hyperbolic* if there is a dominated splitting $E\oplus F$ of $T_\Lambda M$ such that either E is uniformly contracting or F is uniformly expanding. In the first case we write $T_\Lambda M = E^s\oplus E^{cu}$, otherwise we write $E^u\oplus E^{cs}$. Notice that we can have simultaneously both types of splittings, $T_\Lambda M = E^u\oplus E^{cs} = E^s\oplus E^{cu}$. Then, taking $E^c = E^{cu}\cap E^{cs}$, one has a $D\varphi$-invariant splitting $T_\Lambda M = E^s\oplus E^c\oplus E^u$, with three nontrivial directions, where E^s and E^u are uniformly hyperbolic, and we speak of *strong partial hyperbolicity,* see the precise definition below.

Given an open subset U of M let

$$\mathcal{T}(U) = \{\varphi\in \operatorname{Diff}(M) : \Lambda_\varphi(U) \text{ is robustly transitive}\}.$$

By definition $\mathcal{T}(U)$ is open. In the case of transitive diffeomorphisms we let $\mathcal{T}=\mathcal{T}(M)$ (i.e. $\mathcal{T}$ denotes the set of robustly transitive diffeomorphisms). Our main result is

THEOREM A. *Let U be an open subset of a compact boundaryless three-dimensional manifold M. There is an open and dense subset $\mathcal{A}(U)$ of $\mathcal{T}(U)$ such that $\Lambda_\varphi(U)$ is partially hyperbolic for all $\varphi \in \mathcal{A}(U)$.*

Observe that this statement is trivial when $\Lambda_\varphi(U)$ is finite (actually, in this case $\Lambda_\varphi(U)$ is hyperbolic). Hence, from now on we assume that $\Lambda_\varphi(U)$ is infinite.

Theorem A admits a stronger version in the case of transitive diffeomorphisms. We say that a transitive diffeomorphism is *partially hyperbolic* if the whole manifold is partially hyperbolic. We prove the following

THEOREM B. *Every $\varphi \in \mathcal{T}$ is partially hyperbolic.*

At least in their full scope, these results do not extend directly to higher dimensions. For dimension strictly bigger than three, there are sets $\mathcal{U}$ of robustly transitive diffeomorphisms such that every diffeomorphism φ in $\mathcal{U}$ does not admit a partially hyperbolic splitting, see [BV]. Actually, for such φ one cannot identify any hyperbolic direction. However, we expect that an appropriate reformulation of Theorems A and B holds in any dimension: we conjecture that the robustly transitive sets $\Lambda_\varphi(U)$ generically admit a dominated splitting.

Next we state stronger versions of Theorems A and B which relate the types of hyperbolicity of $\Lambda_\varphi(U)$ (uniform, partial and strong partial) to the indices and the eigenvalues of the periodic points of $\Lambda_\varphi(U)$. We also state the connection between approximation by homoclinic tangencies (associated to points in $\Lambda_\varphi(U)$) and the lack of uniform or strong partial hyperbolicity.

For ergodic properties of partially hyperbolic systems we refer the reader to [BV]. See [GPS] for results in the conservative case.

Finally, in the context of vector fields defined on three-manifolds, we first observe that every C^1-robustly transitive flow is Anosov, see [Do]. On the other hand, every robustly transitive set (a priori different from the ambient manifold and containing singularities) is partially hyperbolic, see [MPP].

Before stating new results let us recall a result due to R. Mañé that holds in any dimension. By a *robust property of φ* we understand a property of φ that holds for every ϕ in a C^1-neighbourhood $\mathcal{V}_\varphi$ of φ.

THEOREM ([M3]). *Let $\Lambda_\varphi(U)$ be a robustly transitive set. Then the following three conditions are equivalent:*

1. *all the hyperbolic periodic points of $\Lambda_\varphi(U)$ are robustly hyperbolic,*
2. *the set $\Lambda_\varphi(U)$ is robustly uniformly hyperbolic,*
3. *all hyperbolic points of $\Lambda_\varphi(U)$ have robustly the same index.*

This theorem means that in our context the relevant case for proving Theorems A and B is exactly the nonhyperbolic one, that is, when $\Lambda_\varphi(U)$ contains robustly hyperbolic points with different indices. So, in the remainder of this section we will assume that $\Lambda_\varphi(U)$ is robustly nonhyperbolic.

In this paper we consider two special types of partially hyperbolic sets: strong partially hyperbolic and volume-expanding/contracting in the central bundle. Let us state these definitions precisely.

Let $\varphi\in\mathrm{Diff}(M)$. A φ-invariant set Λ_φ is *strong partially hyperbolic* if there are a $D\varphi$-invariant splitting of $T_{\Lambda_\varphi}M=E^s\oplus E^c\oplus E^u$, where the bundles E^s and E^u are nontrivial and hyperbolic (uniformly contracting and expanding, respectively), and constants $C>0$ and $0<\lambda<1$, such that

$$\|D_x\varphi^n(v^s)\|\cdot\|D_{\varphi^n(x)}\varphi^{-n}(v^c)\|\leqslant C\lambda^n\|v^s\|\cdot\|v^c\|,$$
$$\|D_{\varphi^n(x)}\varphi^{-n}(v^u)\|\cdot\|D_x\varphi^n(v^c)\|\leqslant C\lambda^n\|v^u\|\cdot\|v^c\|,$$

for all $n>0$, $v^i\in E^i_x$, $i=s,c,u$.

A partially hyperbolic set Λ_φ *expands* (*resp. contracts*) *volume in the central bundle* if E^c is volume-expanding (resp. -contracting). By a *volume-expanding bundle* F of Λ_φ we mean a $D\varphi$-invariant bundle F such that there are constants $C>0$ and $\sigma>1$ such that

$$|\mathrm{Jac}_{F(x)}(\varphi^k)|>C\sigma^k \quad \text{for all } x\in\Lambda_\varphi,\ k\geqslant 1,$$

where $\mathrm{Jac}_{F(x)}\,\varphi$ denotes the Jacobian of φ in the bundle F at x. We say that a $D\varphi$-invariant bundle F is volume-contracting if it is volume-expanding for φ^{-1}.

As we have mentioned, a partially hyperbolic set can also be hyperbolic. Here, to avoid misunderstandings, we adopt the following convention: the partially hyperbolic sets we consider are genuinely partially hyperbolic, meaning that their central directions are nontrivial and nonhyperbolic.

Given $\varphi\in\mathcal{T}(U)$ the set $\Lambda_\varphi(U)$ *has robustly real eigenvalues* if there is a C^1-neighbourhood $\mathcal{U}_\varphi$ of φ such that for every $\phi\in\mathcal{U}_\varphi$ and every periodic point $P\in\Lambda_\phi(U)$ all the eigenvalues of $D_P\phi^n$ are real (n is the period of P). Consider the subset $\mathcal{P}(U)$ of $\mathcal{T}(U)$ of diffeomorphisms φ such that $\Lambda_\varphi(U)$ has robustly real eigenvalues and hyperbolic points of different indices (i.e. $\Lambda_\varphi(U)$ is not uniformly hyperbolic). When $U=M$ we let $\mathcal{P}=\mathcal{P}(M)$.

THEOREM C. *The set $\Lambda_\varphi(U)$ is strong partially hyperbolic for all $\varphi\in\mathcal{P}(U)\cap\mathcal{A}(U)$.*

In the case of transitive diffeomorphisms we have a stronger version of the previous result:

COROLLARY D. *Let $\varphi \in \mathcal{P}$. Then φ is strong partially hyperbolic.*

The existence of periodic points with complex (nonreal) eigenvalues prevents the existence of a splitting having three nontrivial directions (recall that we are considering three-manifolds). Theorem C means that the existence of such points with complex eigenvalues is the unique obstruction for the strong partial hyperbolicity. The next theorem says that if the nonhyperbolic set $\Lambda_\varphi(U)$ has complex eigenvalues then it satisfies a stronger form of partial hyperbolicity: the central bundle is either volume-expanding or -contracting.

Let $\varphi \in \mathcal{T}(U)$. The set $\Lambda_\varphi(U)$ *has complex eigenvalues* if there is some periodic point $P \in \Lambda_\varphi(U)$ such that $D_P\varphi^n$ has two eigenvalues with the same modulus (n is the period of P). We denote by $\mathcal{V}(U)$ (resp. $\mathcal{V}$) the subset of $\mathcal{T}(U)$ (resp. $\mathcal{T}$) of diffeomorphisms φ such that $\Lambda_\varphi(U)$ (resp. φ) is not uniformly hyperbolic and has complex eigenvalues.

THEOREM E. *Let φ be a diffeomorphism in $\mathcal{A}(U)$ that can be approximated by diffeomorphisms in $\mathcal{V}(U)$. Then the central bundle of $T_{\Lambda_\varphi(U)}M$ is two-dimensional and volume-expanding/contracting: if $T_{\Lambda_\varphi(U)}M = E^s \oplus E^{cu}$ then E^{cu} is volume-expanding, and if $T_{\Lambda_\varphi(U)}M = E^u \oplus E^{cs}$ then E^{cs} is volume-contracting.*

In the case of a transitive diffeomorphism Theorem E can be read as

COROLLARY F. *Let $\varphi \in \mathcal{T}$ be a diffeomorphism which can be approximated by diffeomorphisms in $\mathcal{V}$. Then φ is partially hyperbolic and volume-expanding/contracting in the central bundle.*

Note that since $\Lambda_\varphi(U)$ is not uniformly hyperbolic it contains points of indices one and two. Our proof shows that all periodic points with complex eigenvalues have the same index.

Finally, the following corollary gives the connection between the absence of strong partial and uniform hyperbolicity and the approximation by homoclinic tangencies. Recall that a hyperbolic periodic point P has a *homoclinic tangency at x* if the invariant manifolds of P have a nontransverse intersection at x.

COROLLARY G. *Let $\varphi \in \mathcal{A}(U)$ be such that $\Lambda_\varphi(U)$ is neither strong partially hyperbolic nor uniformly hyperbolic. Then φ can be approximated by some ϕ with a homoclinic tangency associated to some hyperbolic periodic point in $\Lambda_\phi(U)$.*

Let us observe that in dimension bigger than two the existence of homoclinic tangencies does not lead to creation of sinks or sources, and thus homoclinic tangencies are not an obstruction for transitivity. We remark that Corollary G can be formulated in the case of transitive diffeomorphisms.

In view of the results above, let us summarize the different types of robustly transitive sets $\Lambda_\varphi(U)$ in three-manifolds. For that let $P(\varphi)$ denote the set of periodic points of φ in $\Lambda_\varphi(U)$. We also consider the subsets $P_{\mathbf{R}}(\varphi)$ (resp. $P_{\mathbf{C}}(\varphi)$) of $P(\varphi)$ of points having only real eigenvalues of different moduli (resp. having two eigenvalues of the same modulus, this case including periodic points with eigenvalues of multiplicity bigger than one and, obviously, periodic points with complex (nonreal) eigenvalues).

(1) Suppose that $\Lambda_\varphi(U)$ is hyperbolic. Then $P_{\mathbf{C}}(\varphi)$ is robustly empty if and only if for every ϕ $\mathcal{C}^1$-close to φ the set $\Lambda_\phi(U)$ has a hyperbolic splitting with three nontrivial directions.

(2) Suppose that $\Lambda_\varphi(U)$ is robustly nonhyperbolic. Then

- $\Lambda_\varphi(U)$ contains (robustly) points of indices one and two,
- $\Lambda_\varphi(U)$ is (robustly) nonstrong partially hyperbolic if and only if φ can be approximated by diffeomorphisms ϕ with $P_{\mathbf{C}}(\phi)\neq\varnothing$,
- $\Lambda_\varphi(U)$ is robustly nonstrong partially hyperbolic if and only if φ can be approximated by a diffeomorphism ϕ with a homoclinic tangency (associated to some point of $P(\phi)$).

As we have mentioned, the unique surface which supports robustly transitive diffeomorphisms is the two-torus. This means that (at least for surfaces) the existence of such transitive diffeomorphisms gives some topological information about the surface. For higher dimensions we would like to know if it is possible to deduce some topological information about the ambient manifold M from the existence of robustly transitive diffeomorphisms. In the case of three-manifolds, we study the connection between the existence of transitive diffeomorphisms in M and the growth of the fundamental group of M. As an application of Theorem B, we obtain an obstruction for the existence of robustly transitive diffeomorphisms on manifolds with finite fundamental group. The formulation of this obstruction depends on the integrability of the central bundle: note that, to the best of our knowledge, it is an open question whether the central bundle is necessarily integrable, even in the simplest case of three-manifolds.

Let $E^i(\varphi)\oplus E^c(\varphi)$, $i=s$ or u, be a partially hyperbolic splitting of M for $\varphi\in\mathrm{Diff}(M)$, where $E^c(\varphi)$ has dimension two. The splitting is *dynamically coherent* if there exists a foliation $\mathcal{F}^c(\varphi)$ tangent to $E^c(\varphi)$. Notice that, by the hyperbolicity, $E^s(\varphi)$ or $E^u(\varphi)$ (according to the case) is integrable, and then one can define the stable/unstable foliation $\mathcal{F}^i(\varphi)$, tangent to $E^i(\varphi)$, $i=s,u$.

THEOREM H. *Let M be a three-dimensional boundaryless compact manifold. Suppose that M supports a robustly transitive diffeomorphism having a dynamically coherent splitting. Then the fundamental group $\pi_1(M)$ is infinite.*

Let us say a few words about the organization of this paper. The main step of our constructions is the following preliminary result:

THEOREM 1.1. *There is a residual subset $\mathcal{R}(U)$ of $\mathcal{T}(U)$ such that for every $\varphi \in \mathcal{R}(U)$ the set $\Lambda_\varphi(U)$ has a partially hyperbolic splitting $T_{\Lambda_\varphi(U)}M = E^i(\varphi) \oplus E^c(\varphi)$, $i=s$ or u, where $E^i(\varphi)$ is one-dimensional and uniformly hyperbolic.*

We give an outline of the proof of this theorem in §2. In §3 we introduce the different types of perturbations that we use in this paper (perturbation of the derivative and creation of cycles). Theorem 1.1 is proved in §4, which is the main and the longest section of this paper. This section is divided in three parts: estimates on the eigenvalues (§4.1), angular estimates of the bundles (§4.2), and construction of uniformly dominated splittings (§4.3). Finally, in §5 we prove the theorems in this introduction by using Theorem 1.1.

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2. Outline of the proof of Theorem 1.1

To explain the main ideas and difficulties of the proof of Theorem 1.1 (actually, the key result in this paper) let us begin by saying a few words about a stronger two-dimensional version of our result. From now on fix the open set U and denote by $P(\varphi)$ the set of periodic points of φ in U, and by $P_{\mathbf{R}}(\varphi)$ the subset of $P(\varphi)$ of periodic points having all eigenvalues real and different in modulus.

THEOREM ([M3]). *Every C^1-robustly transitive set $\Lambda_\varphi(U)$ of a surface diffeomorphism φ is a basic set (hyperbolic, locally maximal, and with dense periodic points).*

Let us assume that $\Lambda_\varphi(U)$ is infinite, otherwise, as we have mentioned in the introduction, the result is immediate. To prove the result it is enough to see that $P(\varphi)$ is robustly hyperbolic, or equivalently (due to the fact that we are in dimension two) that the number of sinks and sources is finite and constant in a neighbourhood of φ. From the transitivity and since we are assuming that $\Lambda_\varphi(U)$ is infinite, in our case this number is zero. Arguing by contradiction, if $P(\varphi)$ is not hyperbolic then one gets an elementary bifurcation of some periodic point (saddle-node, flip or Hopf). In dimension two, such bifurcations lead to the creation of new sinks or sources, contradicting the fact that the number of sinks and sources is locally constant. We also observe that in dimension two homoclinic tangencies generically lead to the creation of sinks or sources, see [PV]. Thus, in the case of surface diffeomorphisms, such bifurcations are also forbidden.

In higher dimensions the examples quoted in the introduction show that a robustly transitive set $\Lambda_\varphi(U)$ can be nonhyperbolic and its periodic points can bifurcate. Moreover, one can also have homoclinic tangencies. Actually, the main difficulty in the proof of Theorem 1.1 arises from the fact that in dimension three the list of forbidden bifurcations of points in $\Lambda_\varphi(U)$ is rather limited: Hopf bifurcations and sectionally *expansive/dissipative homoclinic tangencies* (i.e. homoclinic tangencies associated to periodic points such that the modulus of the product of any pair of eigenvalues is bigger/less than one). Let us observe that, for example, sectionally dissipative homoclinic tangencies imply the creation of sinks, see [PV], and thus they are forbidden in our context.

The proof of Theorem 1.1 is by contradiction: assuming that $\Lambda_\varphi(U)$ is not partially hyperbolic we create either a sink or a source in U. Since $\Lambda_\varphi(U)$ is infinite this contradicts its robust transitivity. Let us now be much more precise and sketch some key ideas and ingredients of our proof.

An important difficulty in the proof is to find a suitable candidate for the role of $D\varphi$-invariant splitting over $\Lambda_\varphi(U)$. For that we first restrict our attention to the diffeomorphisms φ such that $P_{\mathbf{R}}(\varphi)$ is dense in $\Lambda_\varphi(U)$, and prove that such diffeomorphisms are generic in $\mathcal{T}(U)$ (see Lemma 4.2). For points $P\in P_{\mathbf{R}}(\varphi)$ there is a splitting $T_PM=E^s_P\oplus E^c_P\oplus E^u_P$ with three nontrivial directions (E^i_P is the eigenspace associated to the eigenvalue $\lambda_i(P)$, where $|\lambda_s(P)|<|\lambda_c(P)|<|\lambda_u(P)|$). The problem now is to extend this splitting to the closure of $P_{\mathbf{R}}(\varphi)$. Unfortunately, in general, such an extension does not exist, for instance, if $P(\varphi)$ contains a point with some complex (nonreal) eigenvalue. Using E^s_P, E^c_P and E^u_P we define two new splittings, $T_PM=E^s_P\oplus E^{cu}_P$ and $T_PM=E^u_P\oplus E^{sc}_P$, where $E^{ij}_P=E^i_P\oplus E^j_P$. We show that at least one of these two splittings is uniformly dominated. Then, by [M2], one can extend such a splitting to the closure $\Lambda_\varphi(U)$ of $P_{\mathbf{R}}(\varphi)$ (again, if there are periodic points with complex eigenvalues it is not possible to extend simultaneously both splittings).

The key for obtaining the uniform dominance is to have an appropriate control of the angles between these bundles. More precisely, we prove that if both families of angles $\{\alpha(E^s_P, E^{cu}_P)\}_{P_{\mathbf{R}}(\varphi)}$ and $\{\alpha(E^u_P, E^{cs}_P)\}_{P_{\mathbf{R}}(\varphi)}$, $\alpha(E,F)$ denoting the angle between E and F, are not uniformly bounded away from zero, then after a perturbation of φ we get ϕ and points P and $Q\in P_{\mathbf{R}}(\phi)$ (with the same index, say 2), homoclinically related and such that $\alpha(E^s_P(\phi), E^{cu}_P(\phi))$ and $\alpha(E^u_Q(\phi), E^{cs}_Q(\phi))$ are both small. These features lead to the creation of sinks (if $|\lambda_s(P)\lambda_c(P)\lambda_u(P)|<1$) or sources (if $|\lambda_s(P)\lambda_c(P)\lambda_u(P)|>1$) in U, see Proposition 4.8. In a few words, to obtain the sinks/sources, we first use the angular estimates to get a heteroclinic tangency. Associated to such a tangency we get a *saddle-node* R (i.e. a periodic point having a unique eigenvalue in the unit circle, moreover such an eigenvalue is 1) such that $|\lambda_s(R)|<1=\lambda_c(R)<|\lambda_u(R)|$ and $\alpha(E^s_R, E^u_R)$ is arbitrarily small. After a new perturbation, we create either a sink or a source. We point out that such sinks/sources are not explicitly associated to a tangency.

In this paper we explore the correlation between the dominance (also expansion/contraction of the derivative) of a splitting and the estimates on the angles between the bundles of the splitting. We show that at least one of the splittings we are considering (either $E^s\oplus E^{cu}$ or $E^u\oplus E^{cs}$) is uniformly dominated, see Proposition 4.23. For example, if $E^s\oplus E^{cu}$ is not uniformly dominated then, after perturbation, one obtains a splitting such that the angle between the E^s and E^{cu} is arbitrarily small.

Suppose now that, for instance, $\{E^s_P\oplus E^{cu}_P\}_{P_{\mathbf{R}}(\varphi)}$ is uniformly dominated. Then one can extend such a splitting to a uniformly dominated one defined on the whole $\Lambda_\varphi(U)$. In §4.4, we see that the ergodic closing lemma, see [M3], and the uniform dominance of the splitting imply the uniform hyperbolicity of E^s. This means that $E^s\oplus E^{cu}$ is partially hyperbolic.

In our proof we use some ideas introduced by Mañé in [M3] considering *families of periodic linear maps,* see §3.1. Given φ such that $P_{\mathbf{R}}(\varphi)$ is dense in $\Lambda_\varphi(U)$ we take the family of periodic linear maps $\mathcal{D}(\varphi)=\{D_P\varphi\}_{P\in P_{\mathbf{R}}(\varphi)}$. The robust transitivity of $\Lambda_\varphi(U)$ allows us to deduce some properties for families of linear maps $\mathcal{B}=\{B_P\}_{P\in P_{\mathbf{R}}(\varphi)}$ close to $\mathcal{D}(\varphi)$. Given $B_P\in\mathcal{B}$, write $\widehat{B}_P=B_{\varphi^{n-1}(P)}\dots B_P$ (n is the φ-period of P) and let $\lambda_s(B_P)$, $\lambda_c(B_P)$ and $\lambda_u(B_P)$ be the eigenvalues of $\widehat{B}_P$, $|\lambda_s(B_P)|\leqslant|\lambda_c(B_P)|\leqslant|\lambda_u(B_P)|$ (two of them may be nonreal). If $\lambda_i(B_P)$ is real, $E^i(B_P)$ denotes its eigenspace. In this case, let $E^{kj}(B_P)$, $k,j\neq i$, be the $\widehat{B}_P$-invariant space that does not contain $E^i(B_P)$. Notice that if $\lambda_k(B_P)$ and $\lambda_j(B_P)$ are both real then $E^{kj}(B_p)=E^k(B_p)\oplus E^j(B_P)$.

We prove that families $\mathcal{B}$ close to $\mathcal{D}(\varphi)$ satisfy the following conditions (see Proposition 4.7 and its proof):

(1) Either $E^s(\mathcal{B})\oplus E^{cu}(\mathcal{B})$ is defined for all $\mathcal{B}$ close to $\mathcal{D}(\varphi)$ (i.e. $E^s(B_P)\oplus E^{cu}(B_P)$ is defined for all $B_P\in\mathcal{B}$), or $E^u(\mathcal{B})\oplus E^{cs}(\mathcal{B})$ is defined for all $\mathcal{B}$ close to $\mathcal{D}(\varphi)$.

(2) Assume that the splitting $E^s(\mathcal{B})\oplus E^{cu}(\mathcal{B})$ is defined for all $\mathcal{B}$ close to $\mathcal{D}(\varphi)$. Then the angle $\alpha(E^s(B_P), E^{cu}(B_P))$ is uniformly bounded away from zero ($B_P\in\mathcal{B}$ and $\mathcal{B}$ close to $\mathcal{D}(\varphi)$).

Finally, we see that these properties (definition of the splitting and angular estimates), which hold for families of periodic linear maps $\mathcal{B}$ close to $\mathcal{D}(\varphi)$ (thus indexed by $P_{\mathbf{R}}(\varphi)$), are passed on from $\mathcal{B}$ to diffeomorphisms ψ close to φ. Observe that since the periodic points of φ bifurcate, a priori $P_{\mathbf{R}}(\psi)$ has nothing to do with $P_{\mathbf{R}}(\varphi)$. Hence, the previous assertion is not at all trivial.

3. Perturbations of diffeomorphisms

The existence of a partially hyperbolic splitting for a robustly transitive set $\Lambda_\varphi(U)$ arises from the fact that it is not possible to perturb the diffeomorphism φ to create sinks or sources in U. In this section we introduce the two types of C^1-perturbations that we use: perturbation of the derivative and creation of cycles.

3.1. Perturbation of derivatives: linear maps and diffeomorphisms

We begin by recalling a result due to Franks (see Lemma 3.1 below) which will enable us to perturb the derivative of a diffeomorphism φ at any $x\in\Lambda_\varphi(U)$ along a finite segment of its φ-orbit (preserving such a segment of orbit). Typically, we will apply this lemma to periodic points of φ: given $P\in P(\varphi)$ and a neighbourhood V of the φ-orbit of P there is ψ C^1-close to φ, preserving the φ-orbit of P and coinciding with φ outside V, such that the derivative $D\psi$ at any $\psi^i(P)=\varphi^i(P)$ is the product of $D\varphi$ with some matrix close to the identity. As a consequence of this result we get that the moduli of the strong stable and unstable eigenvalues of points in $P(\varphi)$ are both uniformly bounded away from 1, see Lemmas 4.5 and 4.6.

LEMMA 3.1 (Lemma 1.1 of [F] and Lemma II.2 of [M3]). *Given $\varphi\in\mathrm{Diff}(M)$ and a neighbourhood $\mathcal{U}$ of φ in $\mathrm{Diff}(M)$ there is $\varepsilon>0$ such that for any finite set $F=\{x_1,x_2,...,x_n\}\subset M$, neighbourhod U of F, and linear maps $L_i: T_{x_i}M\to T_{\varphi(x_i)}M$ with $\|L_i-D_{x_i}\varphi\|<\varepsilon$, there is $\phi\in\mathcal{U}$ such that*

(1) *$\phi(x)=\varphi(x)$ for all $x\in F\cup(M\setminus U)$, and*

(2) *$D_{x_i}\phi=L_i$ for every $i=1,...,n$.*

This lemma plays a key role in our proof, and it will allow us to move back and forth between the spaces of linear maps and of C^1-diffeomorphisms: Roughly speaking, consider a diffeomorphism φ and a point P of φ-period m. Then to each family of linear maps $\{A_i\}_{i=0}^{m-1}$ such that every A_i is close to $D_{\varphi^i(P)}\varphi$ we associate a diffeomorphism ψ close to φ, preserving the φ-orbit of P, and such that $D_{\varphi^i(P)}\psi=A_i$, and vice versa, to each ψ close to φ and each periodic point of ψ we associate a family of linear maps (the family of derivatives along the orbit). For that we introduce the notion of *family of linear maps.*

A family $\mathcal{A}=(A_i^\gamma)_{\gamma\in I, i\in\mathbf{Z}}$ of linear maps is *periodic* if for each $\gamma\in I$ there is n_γ such that $A_{i+n_\gamma}^\gamma=A_i^\gamma$ for every i. The eigenvalues (resp. eigenspaces) of the periodic sequence $A^\gamma=(A_i^\gamma)_{i\in\mathbf{Z}}$ are the eigenvalues (resp. eigenspaces) of the product $A_{i+n_\gamma-1}^\gamma\dots A_i^\gamma$, denoted by $\lambda_k(A^\gamma)$ (resp. $E_k(A^\gamma)$).

Two periodic families of linear maps $\mathcal{A}=(A_i^\gamma)_{\gamma\in I, i\in\mathbf{Z}}$ and $\mathcal{B}=\{B_i^\gamma\}_{\gamma\in I, i\in\mathbf{Z}}$ are *ε-close* if they have the same period and

$$\sup_{\substack{\gamma\in I\\ i\in\mathbf{Z}}}\|B_i^\gamma-A_i^\gamma\|<\varepsilon.$$

Given a family of periodic linear maps $\mathcal{A}=(A_i^\gamma)_{\gamma\in I, i\in\mathbf{Z}}$ then an $\mathcal{A}$-invariant splitting $E(\mathcal{A})\oplus F(\mathcal{A})$ is a family of splittings $(E_i^\gamma\oplus F_i^\gamma)_{\gamma\in I, i\in\mathbf{Z}}$ such that $A_i^\gamma(G_i^\gamma)=(G_{i+1}^\gamma)$ and $G_{i+n_\gamma}^\gamma=G_i^\gamma$ for each i, $G=E,F$. An $\mathcal{A}$-invariant splitting $E(\mathcal{A})\oplus F(\mathcal{A})$ is *uniformly dominated* if there are m and $\lambda\in(0,1)$ such that

$$\left\|\left(\prod_{j=0}^{m-1}(A_{i+j}^\gamma)\right)\bigg|_{E_i^\gamma}\right\|\cdot\left\|\left(\prod_{j=0}^{m-1}(A_{i+j}^\gamma)\right)^{-1}\bigg|_{F_{i+m}^\gamma}\right\|<\lambda$$

for all $\gamma\in I$ and $i\in\mathbf{Z}$. Finally, the angle of the splitting $E(\mathcal{A})\oplus F(\mathcal{A})$ is

$$\alpha_0(E(\mathcal{A}),F(\mathcal{A}))=\inf\{\alpha(E_i^\gamma,F_i^\gamma),\gamma\in I, i\in\mathbf{Z}\},$$

where $\alpha(E_i^\gamma,F_i^\gamma)$ denotes the angle between E_i^γ and F_i^γ.

Given $\varphi\in\mathcal{T}(U)$ we define $P_{\mathbf{R}}(\varphi)$ as the subset of periodic points of $\Lambda_\varphi(U)$ having only real eigenvalues of different moduli. We have the following result whose proof we postpone to §4 (see Lemma 4.2).

LEMMA. *Consider the subset $\mathcal{R}$ of diffeomorphisms φ in $\mathcal{T}(U)$ such that $P_{\mathbf{R}}(\varphi)$ is dense in $\Lambda_\varphi(U)$ and every periodic point P of $P_{\mathbf{R}}(\varphi)$ is hyperbolic. Then $\mathcal{R}$ is residual in $\mathcal{T}(U)$.*

This lemma means that to prove Theorem 1.1 it is enough to consider diffeomorphisms φ such that $P_{\mathbf{R}}(\varphi)$ is dense in $\Lambda_\varphi(U)$. For such a φ consider the periodic family of linear maps

$$\mathcal{D}(\varphi)=\{D_P^i\}_{i\in\mathbf{Z},P\in P_{\mathbf{R}}(\varphi)},\quad D_P^i=D_{\varphi^i(P)}\varphi.$$

For $\mathcal{D}(\varphi)$ we have the $\mathcal{D}(\varphi)$-invariant splittings

$$E^{sc}(\mathcal{D}(\varphi))\oplus E^u(\mathcal{D}(\varphi)),\quad E^s(\mathcal{D}(\varphi))\oplus E^{cu}(\mathcal{D}(\varphi)),$$

where $E^j(\mathcal{D}(\varphi))$ is the family of one-dimensional eigenspaces $E_P^j(\varphi)$ associated to the eigenvalue $\lambda_j(P)$ of $D_P\varphi^n$, n is the φ-period of P ($j=s,c,u$), and $E^{kr}(\mathcal{D}(\varphi))$ is the family of spaces $\{E_P^{kr}(\varphi)=E_P^k(\varphi)\oplus E_P^r(\varphi)\}_{P\in P_{\mathbf{R}}(\varphi)}$.

Our goal is to prove that either $E^{sc}(\mathcal{D}(\varphi))\oplus E^u(\mathcal{D}(\varphi))$ is uniformly dominated, or $E^s(\mathcal{D}(\varphi))\oplus E^{cu}(\mathcal{D}(\varphi))$ is uniformly dominated.

3.2. Creation of cycles

The next lemma which we borrow from [H] allows us to create homoclinic/heteroclinic cycles associated to points in $P(\varphi)$. Observe that Lemmas 3.2, 3.3 and 4.5 below hold in any dimension.

Before stating the lemma let us recall that if P is a hyperbolic periodic point of a diffeomorphism φ then for every ϕ close to φ there is a hyperbolic periodic point P_ϕ close to P (given by the implicit function theorem). This point is called the *continuation of* P for ϕ.

LEMMA 3.2 ([H]). *Let* $\varphi \in \mathrm{Diff}(M)$, *and let* P *and* Q *be hyperbolic periodic points of* φ. *Suppose that there are sequences of points* (x_n) *and natural numbers* (k_n) *such that*

$$(x_n) \to p^u \in W^u(P) \quad \textit{and} \quad \varphi^{k_n}(x_n) \to q^s \in W^s(Q).$$

Then there is ϕ *arbitrarily* C^1*-close to* φ *such that*

$$W^u(P_\phi) \cap W^s(Q_\phi) \neq \varnothing,$$

where P_ϕ *and* Q_ϕ *are the continuations of* P *and* Q *for* ϕ.

From now on we denote by $A \pitchfork B$ the transverse intersection between A and B. Let us now state the following result that follows from Lemma 3.2:

LEMMA 3.3. *Let* P *and* Q *be hyperbolic points in* $P(\varphi)$, $\varphi \in \mathcal{T}(U)$, *such that*

$$\operatorname{index}(P) \geqslant \operatorname{index}(Q).$$

Then there is ϕ *close to* φ, $\phi \in \mathcal{T}(U)$, *such that*

(a) *if* $\operatorname{index}(P) = \operatorname{index}(Q)$ *then* $W^s(P_\phi) \pitchfork W^u(Q_\phi) \neq \varnothing$ *and* $W^u(P_\phi) \pitchfork W^s(Q_\phi) \neq \varnothing$,

(b) *if* $\operatorname{index}(P) > \operatorname{index}(Q)$ *then* $W^s(P_\phi) \pitchfork W^u(Q_\phi) \neq \varnothing$, *and there is*

$$x \in W^u(P_\phi) \cap W^s(Q_\phi)$$

such that

$$T_x W^u(P_\phi) + T_x W^s(Q_\phi) = T_x W^u(P_\phi) \oplus T_x W^s(Q_\phi),$$

i.e. x *is a point of quasitransverse intersection.*

Here P_ϕ *and* Q_ϕ *are the continuations of* P *and* Q *for* ϕ. *This lemma also holds for the homoclinic case* $P = Q$.

Proof. Let us suppose that P and Q have different indices. The case $\operatorname{index}(P) = \operatorname{index}(Q)$ follows analogously. We first claim that there is ϕ close to φ such that $W^s(P_\phi)$ and $W^u(Q_\phi)$ have a nonempty transverse intersection.

To prove the claim notice that $\dim(W^s(P))+\dim(W^u(Q))=4>3$, and thus if $W^s(P)\cap W^u(Q)\neq\varnothing$ after perturbing φ one can get a transverse intersection between these invariant manifolds. So let us assume that $W^s(P)\cap W^u(Q)=\varnothing$. By the transitivity of $\Lambda_\varphi(U)$, there is $x\in\Lambda_\varphi(U)$ with a dense orbit in $\Lambda_\varphi(U)$. Thus there are sequences (n_i) and (m_i), $n_i, m_i\to\infty$, with $m_i>n_i$, such that

$$\varphi^{n_i}(x)\to P \quad \text{and} \quad \varphi^{m_i}(x)\to Q.$$

Hence, for fixed fundamental domains D^s of $W^s_{\text{loc}}(P)$ and D^u of $W^u_{\text{loc}}(Q)$ there are new sequences, say $(\tilde{n}_i)$ and $(\widetilde{m}_i)$, $\tilde{n}_i\to\infty$, $\widetilde{m}_i=\tilde{n}_i+k_i$, $k_i>0$, such that

$$\varphi^{\tilde{n}_i}(x)=p_i\to d^s\in D^s \quad \text{and} \quad \varphi^{\widetilde{m}_i}(x)=\varphi^{k_i}(p_i)\to d^u\in D^u.$$

Applying Lemma 3.2 to (p_i) we get $\phi_0\in\mathcal{T}(U)$ close to φ such that

$$W^s(P_{\phi_0})\cap W^u(Q_{\phi_0})\neq\varnothing.$$

After a new perturbation, if necessary, we get a transverse intersection. Moreover, such a transverse intersection persists for every $\phi\in\mathcal{T}(U)$ close to ϕ_0. This ends the proof of the claim.

Since $\phi_0\in\mathcal{T}(U)$, applying the above argument to $W^u(P_{\phi_0})$ and $W^s(Q_{\phi_0})$, we obtain $\phi_1\in\mathcal{T}(U)$ close to ϕ_0 with $W^u(P_{\phi_1})\cap W^s(Q_{\phi_1})\neq\varnothing$. Since

$$\dim(W^u(P_{\phi_1}))+\dim(W^s(Q_{\phi_1}))=2<3,$$

in this case we obtain (after a new perturbation if necessary) a quasitransverse intersection instead of a transverse one. Clearly, $\phi=\phi_1$ satisfies the conclusions of the lemma. The proof of the lemma now is complete. □

4. Proof of Theorem 1.1

As mentioned in the introduction the first difficulty to prove the theorem is to find a suitable candidate for the role of partially hyperbolic splitting of $T_{\Lambda_\phi(U)}M$. To obtain an appropriate splitting we first focus our attention on the periodic points of $\Lambda_\phi(U)$ having real eigenvalues with different moduli, i.e. on the subset $P_{\mathbf{R}}(\phi)$ of $P(\phi)$. For points in $P_{\mathbf{R}}(\phi)$ one has the splitting $T_PM=E^s_P(\phi)\oplus E^c_P(\phi)\oplus E^u_P(\phi)$ with three $D\phi$-invariant one-dimensional directions, corresponding to $\lambda_s(P)$, $\lambda_c(P)$ and $\lambda_u(P)$.

We now prove the genericity (in $\mathcal{T}(U)$) of the diffeomorphisms ϕ such that $P_{\mathbf{R}}(\phi)$ is dense in $\Lambda_\phi(U)$, see Lemma 4.2. In §§ 4.2 and 4.3 we deal with the problem of how to extend this auxiliary splitting, defined only on $P_{\mathbf{R}}(\phi)$, to the whole $\Lambda_\phi(U)$. To prove the density of $P_{\mathbf{R}}(\phi)$, Lemma 4.2, we need the following result:

LEMMA 4.1. *There is a residual subset* $\mathcal{R}$ *of* Diff(M) *such that for every* $\phi\in \mathcal{R}\cap\mathcal{T}(U)=\mathcal{R}_0(U)$ *one has*:

(a) *the periodic points of* ϕ *are dense in* $\Lambda_\phi(U)$,

(b) *every periodic point of* $\Lambda_\phi(U)$ *is hyperbolic.*

Proof. Using that $\Lambda_\phi(U)=\omega(x)$ for some x and the robust transitivity of $\Lambda_\phi(U)$, we get that $\Lambda_\phi(U)$ is generically the closure of the periodic points of ϕ contained in it, see [P]. Moreover, by the Kupka–Smale theorem, all periodic points of ϕ are generically hyperbolic. □

Given a point P denote by H_P the set of transverse homoclinic points of P (the transverse intersection between the invariant manifolds of P). Consider the set

$$\mathcal{R}_{\mathbf{R}}(U)=\{\phi\in\mathcal{T}(U):P_{\mathbf{R}}(\phi) \text{ is dense in } \Lambda_\phi(U), \\ \text{and } H_P \text{ is dense in } \Lambda_\phi(U) \text{ for all } P\in P(\phi)\}.$$

Let $\operatorname{Hyp} P(\phi)$ be the subset of hyperbolic points of $P(\phi)$, and let

$$\operatorname{Hyp} P_{\mathbf{R}}(\phi)=\operatorname{Hyp} P(\phi)\cap P_{\mathbf{R}}(\phi).$$

LEMMA 4.2. *The set* $\mathcal{R}_{\mathbf{R}}(U)$ *is residual in* $\mathcal{T}(U)$.

Proof. Let us first prove the generic density of $P_{\mathbf{R}}(\phi)$ in $\Lambda_\phi(U)$. For each $n\in\mathbf{N}$ consider a finite covering $\mathcal{B}_n$ of M by open balls $B_n(w)$ of radius $1/n$. Let

$$\mathcal{R}_n=\{\phi\in\mathcal{T}(U):B_n(w)\cap\Lambda_\phi(U)\neq\varnothing\Rightarrow B_n(w)\cap\operatorname{Hyp} P_{\mathbf{R}}(\phi)\neq\varnothing\}.$$

By definition, $\mathcal{R}_n(U)$ is open. We claim that it is also dense in $\mathcal{T}(U)$. This claim implies that $P_{\mathbf{R}}(\phi)$ generically is dense in $\Lambda_\phi(U)$: just consider the residual subset $\bigcap_n\mathcal{R}_n(U)$ of $\mathcal{T}(U)$.

Fix n, $B_n(w)\in\mathcal{B}_n$ and $\phi\in\mathcal{R}_0(U)$. Suppose that $B_n(w)\cap\Lambda_\phi(U)\neq\varnothing$. By Lemma 4.1 we have $\operatorname{Hyp} P(\phi)\cap B_n(w)\neq\varnothing$. We prove that ϕ can be approximated by some φ with $\operatorname{Hyp} P_{\mathbf{R}}(\varphi)\cap B_n(w)\neq\varnothing$. This implies the density of $\mathcal{R}_n(U)$ in $\mathcal{T}(U)$.

Suppose that our claim does not hold. Note that if P has two real eigenvalues of the same modulus then one can perturb ϕ to obtain a hyperbolic periodic point with three real hyperbolic eigenvalues of different modulus. Thus we can suppose that every periodic point P of ϕ in $B_n(w)$ has a complex (nonreal) eigenvalue λ.

Take a periodic point P of ϕ (which for simplicity we will assume to be fixed) and suppose, for example, that $|\lambda|<1$. After a perturbation we can suppose that ϕ is linearizable in a neighbourhood $V\subset U$ of P and that $W^s(P,\phi)$ and $W^u(P,\phi)$ have a nonempty transverse intersection. The last assertion follows from Lemma 3.3.

Take a point $x \in V$ of transverse intersection between $W^s_{\mathrm{loc}}(P)$ and $W^u(P)$. Let $y = \phi^{-l}(x) \in W^u_{\mathrm{loc}}(P) \cap V$. By replacing x (resp. y) by some forward (resp. backward) iterate $\bar{x} = \phi^t(x)$ (resp. $\bar{y} = \phi^{-t}(y)$), with big t, we can assume that

$$T_{\bar{z}} W^u(P) \simeq E^u_P \quad \text{and} \quad T_{\bar{z}} W^s(P) \simeq E^{cs}_P, \quad \bar{z} = \bar{x}, \bar{y}, \tag{4.1}$$

where $\simeq$ means that the two spaces are close. After a new perturbation we can suppose that $T_{\bar{z}} W^u(P)$ and $T_{\bar{z}} W^s(P)$ are parallel to E^u_P and E^{cs}_P, respectively.

Consider a normal basis $\{v, w, w^\perp\}$ of $T_{\bar{y}} M$ (resp. $\{v_1, w_1, w_1^\perp\}$ of $T_{\bar{x}} M$) where $w, w^\perp$ are orthogonal vectors in $T_{\bar{y}} W^s(P)$ and $v \in T_{\bar{y}} W^u(P)$ (resp. $w_1, w_1^\perp$ are orthogonal vectors in $T_{\bar{x}} W^s(P)$ and $v_1 \in T_{\bar{x}} W^u(P)$). In these coordinates the derivative $D_{\bar{y}} \phi^m : T_{\bar{y}} M \to T_{\bar{x}} M$, where $m = 2t + l$, is of the form

$$D_{\bar{y}} \phi^m = \begin{pmatrix} c_m & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Observe that we can take (and we do) w and $w^\perp$ such that $A(w)$ and $A(w^\perp)$ are also orthogonal.

There is a sequence of diffeomorphisms $\phi_k \to \phi$ such that every ϕ_k has a periodic point Q_k of period $r_k = n_k + m$ such that $r_k \to \infty$, $Q_k \to \bar{x}$, $\phi^i_k(Q_k) \in V$ for every $0 \leqslant i \leqslant n_k$, and $\phi^{n_k}_k(Q_k) \to \bar{y}$. We claim that after a new perturbation, if necessary, we can assume that

$$D_{Q_k} \phi^{n_k}_k (A(w)) = |\lambda|^{n_k} w \quad \text{and} \quad D_{Q_k} \phi^{n_k}_k (A(w^\perp)) = |\lambda|^{n_k} w^\perp.$$

To prove the claim recall that the eigenvalue λ of $D_P \phi$ is complex and that $w \perp w^\perp$ and $A(w) \perp A(w^\perp)$. After a new perturbation, we can assume that $D_P \phi_k$ has a complex eigenvalue λ_k with argument θ_k such that $n_k \theta_k$ is the angle between $A(w)$ and w. On the other hand, since $\phi^{n_k}_k(Q_k) \to \bar{y}$ there are matrices $J_k \to \mathrm{Id}$ such that

$$D_{Q_k} \phi^m_k = J_k \cdot \begin{pmatrix} c_m & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}.$$

Thus, by the choice of θ_k, in the basis $\{v, w, w^\perp\}$, one has

$$D_{Q_k} \phi^{r_k}_k = J_k \cdot \begin{pmatrix} c_m \lambda_u^{n_k} & 0 & 0 \\ 0 & a_m |\lambda|^{n_k} & 0 \\ 0 & 0 & b_m |\lambda|^{n_k} \end{pmatrix}, \quad J_k \to \mathrm{Id} \text{ as } k \to \infty,$$

where a_m and b_m are real numbers independent of k and n, and λ_u is the expanding real eigenvalue of $D_P \phi$. Using Lemma 3.1 we perturb each ϕ_k at $\phi^{-1}_k(Q_k) = \phi^{r_k - 1}_k(Q_k)$

to obtain a sequence $\psi_k \to \phi$ such that Q_k is a periodic point of period $r_k = n_k + m$ of ψ_k with derivative

$$D_{Q_k}\psi_k^{r_k} = (J_k)^{-1} \cdot D_{Q_k}\phi_k^{r_k} = \begin{pmatrix} c_m \lambda_u^{n_k} & 0 & 0 \\ 0 & a_m |\lambda|^{n_k} & 0 \\ 0 & 0 & b_m |\lambda|^{n_k} \end{pmatrix}. \tag{4.2}$$

Hence $D_{Q_k}\psi_k^{r_k}$ has three real eigenvalues of moduli different from one. After a new perturbation the moduli of the three eigenvalues are all different. By construction $Q_k \in \Lambda_{\phi_k}(U)$ and, if k is big enough, its orbit intersects $B_n(w)$. This implies that $\psi_k \in \mathcal{R}_n(U)$. Finally, by construction, $\psi_k \to \phi$, which gives the density of $\mathcal{R}_n(U)$ in $\mathcal{T}(U)$. This ends the proof of the first part of the lemma.

The second part of the lemma follows similarly: given any periodic point $P \in P(\phi)$ and a ball $B_n(w)$ of the covering $\mathcal{B}_n$ of M intersecting $\Lambda_\phi(U)$ we use Lemma 3.3 to perturb ϕ, and obtain a homoclinic point of P in $B_n(w)$. □

Let us make the following remark to the proof of the lemma before that will be used in the proof of Lemma 4.14.

Remark 4.3. The numbers a_m and b_m in equation (4.2) are independent of n and k.

4.1. Estimates on the eigenvalues

Our next step is to get some estimates on the eigenvalues of the periodic points of a robustly transitive set $\Lambda_\phi(U)$. Let P be a periodic point of period k of ϕ, i.e. $\phi^k(P) = P$ and $\phi^i(P) \neq P$ for all $0 < i < k$. Denote by $\lambda_s(P)$, $\lambda_c(P)$ and $\lambda_u(P)$ the three eigenvalues of $D_P\phi^k$, where

$$|\lambda_s(P)| \leqslant |\lambda_c(P)| \leqslant |\lambda_u(P)|.$$

Take $\phi \in \mathcal{T}(U)$. Since we are assuming that $\Lambda_\phi(U)$ is infinite, the robust transitivity of $\Lambda_\phi(U)$ implies the following result (whose proof is trivial) which we will use repeatedly.

FACT 4.4. *Let $\phi \in \mathcal{T}(U)$. Then ϕ has neither sinks nor sources in U. Moreover,*

$$|\lambda_s(P)| < 1 < |\lambda_u(P)| \quad \textit{for every } P \in P(\phi).$$

We begin by obtaining some estimates on the strong stable and unstable eigenvalues of points $P \in P(\phi)$ which ensure that the moduli of these strong eigenvalues are uniformly bounded away from 1:

LEMMA 4.5. *Let $\phi \in \mathcal{T}(U)$. There is $\delta > 0$ such that*

$$|\lambda_s(P)| \leqslant (1-\delta)^k < (1+\delta)^k \leqslant |\lambda_u(P)|$$

for every $P \in P(\phi)$ of period k.

Proof. The proof is by contradiction. Suppose contrary to our claim that for every $n>0$ there is $P_n \in P(\phi)$ of period m_n such that

$$|\lambda_s(P_n)| > \left(1-\frac{1}{n}\right)^{m_n} \quad \text{or} \quad |\lambda_u(P_n)| < \left(1+\frac{1}{n}\right)^{m_n}.$$

Suppose, for instance, that the first inequality holds for infinitely many n. Taking a subsequence, if necessary, we can assume that

$$|\lambda_s(P_n)| > \left(1-\frac{1}{n}\right)^{m_n} \quad \text{for every } n \text{ and } (P_n) \to Q \in \Lambda_\phi(U).$$

If the m_n are bounded by some m then $Q \in P(\phi)$ and its period k is less than m. Thus, by construction, $|\lambda_s(Q)| \geqslant 1$, contradicting Fact 4.4.

Hence we lose no generality assuming that $(m_n) \to \infty$. For each big n, using Lemma 3.1, we have ϕ_n close to ϕ preserving the ϕ-orbit of P_n such that

$$D_{\phi_n^j(P_n)}\phi_n = \frac{1}{1-1/n} D_{\phi^j(P_n)}\phi.$$

By construction, $\phi_n \to \phi$. Thus $\phi_n \in \mathcal{T}(U)$ for every big n, and $|\lambda_s(P_n, \phi_n)| \geqslant 1$. But Fact 4.4 prevents such a possibility. This completes the proof of the lemma. □

Finally, the estimates on the strong stable and unstable eigenvalues above can be translated into the context of families of linear maps as follows.

Given a family of periodic linear maps $\mathcal{B}$ close to $\mathcal{D}(\varphi)$ and a periodic point $P \in P_{\mathbf{R}}(\varphi)$ of period k, consider the linear map $\widehat{B}_P = B_{\varphi^{k-1}(P)} \dots B_P$, and denote by $\lambda_s(B_P)$, $\lambda_c(B_P)$ and $\lambda_u(B_P)$, $|\lambda_s(B_P)| \leqslant |\lambda_c(B_P)| \leqslant |\lambda_u(B_P)|$, its eigenvalues.

Using Lemmas 4.4 and 3.1 we get uniform estimates on the strong stable and unstable eigenvalues of families $\mathcal{B}$ close to $\mathcal{D}(\varphi)$, $\varphi \in \mathcal{T}(U)$:

LEMMA 4.6. *Let $\varphi \in \mathcal{R}(U)$. There is $\delta > 0$ such that for every family of periodic linear maps $\mathcal{B}$ close to $\mathcal{D}(\varphi)$ one has*

$$|\lambda_s(B_P)| \leqslant (1-\delta)^k < (1+\delta)^k \leqslant |\lambda_u(B_P)|$$

for every $B_P \in \mathcal{B}$ and $P \in P_{\mathbf{R}}(\varphi)$ of period k.

4.2. Angular estimates

In what follows we focus our attention on the residual subset of $\mathcal{T}(U)$ given by

$$\mathcal{R}(U)=\mathcal{R}_{\mathbf{R}}(U)\cap\mathcal{R}_0(U), \quad \mathcal{R}_{\mathbf{R}}(U) \text{ and } \mathcal{R}_0(U) \text{ as in Lemmas 4.2 and 4.1, respectively.}$$

As above, given two linear spaces E and F let $\alpha(E,F)$ denote the angle between E and F. Also, recall that given a family of linear maps $\mathcal{B}$ and a splitting $E(\mathcal{B})\oplus F(\mathcal{B})$, $\alpha_0(E(\mathcal{B}),F(\mathcal{B}))$ denotes the angle of the two bundles of the splitting $E(\mathcal{B})\oplus F(\mathcal{B})$, see §3.1.

PROPOSITION 4.7. *Let $\phi\in\mathcal{R}(U)$. Then there are constants $C,\delta>0$ such that*

- *either $E^s(\mathcal{B})\oplus E^{cu}(\mathcal{B})$ is defined and $\alpha_0(E^s(\mathcal{B}),E^{cu}(\mathcal{B}))>C$ for every family of linear maps $\mathcal{B}$ such that $\|\mathcal{B}-\mathcal{D}(\phi)\|<\delta$,*
- *or $E^u(\mathcal{B})\oplus E^{cs}(\mathcal{B})$ is defined and $\alpha_0(E^u(\mathcal{B}),E^{cs}(\mathcal{B}))>C$ for every family of linear maps $\mathcal{B}$ such that $\|\mathcal{B}-\mathcal{D}(\phi)\|<\delta$.*

The main step of the proof of the proposition is the following result:

PROPOSITION 4.8. *Let $\phi\in\mathcal{R}(U)$. There are $C>0$ and a C^1-neighbourhood $\mathcal{U}$ of ϕ such that for every $\psi\in\mathcal{U}$ one of the two possibilities holds: either $\alpha(E^s_P(\psi),E^{cu}_P(\psi))>C$ for every $P_{\mathbf{R}}(\psi)$, or $\alpha(E^u_P(\psi),E^{cs}_P(\psi))>C$ for every $P\in P_{\mathbf{R}}(\psi)$.*

4.2.1. *Proof of Proposition* 4.8. The proof of this proposition is by contradiction. If the result is false then there are sequences of diffeomorphisms $\psi_n\to\phi$ and of points $P_n,Q_n\in P_{\mathbf{R}}(\psi_n)$ such that

$$\alpha(E^s_{P_n}(\psi_n),E^{cu}_{P_n}(\psi_n))<1/n \quad \text{and} \quad \alpha(E^u_{Q_n}(\psi_n),E^{cs}_{Q_n}(\psi_n))<1/n. \tag{4.3}$$

Remark 4.9. To state Proposition 4.8 we prove that the existence of the sequences in (4.3) leads to the creation (after perturbation) of either sinks or sources in $\Lambda_\phi(U)$, contradicting Fact 4.4.

Take points P_n and Q_n as above and let t_n and r_n be their periods. We begin by observing that $t_n,r_n\to\infty$. Suppose for instance that (t_n) does not go to infinity. Taking a subsequence, if necessary, one gets $P_n\to P\in P_{\mathbf{R}}(\phi)$. One has $\alpha(E^s_P(\phi),E^{cu}_P(\phi))>\tau>0$, which prevents $\alpha(E^s_{P_n}(\psi_n),E^{cu}_{P_n}(\psi_n))\to 0$. This proves our assertion.

Now, since t_n and $r_n\to\infty$, from Lemma 4.5 one gets

$$|\lambda_s(P_n)|,|\lambda_s(Q_n)|\to 0 \quad \text{and} \quad |\lambda_u(P_n)|,|\lambda_u(Q_n)|\to\infty.$$

Our next step is to see that we can take (after a perturbation) the points P_n and Q_n having the same index.

LEMMA 4.10. *Let ψ_n, P_n and Q_n be as in* (4.3). *After a perturbation of ψ_n we can assume that P_n and Q_n have index* 2 *for n large enough.*

Remark 4.11. The arguments in the proof of the lemma show that one can also take both points with index 1, or P_n with index 2 and Q_n with index 1.

Proof. We prove the lemma for the points P_n (for the Q_n one argues analogously). Consider perturbations of the derivative of $D\phi$ at P_n and Q_n. It is important to note that Lemma 3.1 allows us to find such perturbations at P_n and Q_n simultaneously (for the same diffeomorphism).

If there are infinitely many P_n with index 2 we are done. Thus, we lose no generality assuming that every P_n has index 1. For each $\psi \in \mathcal{R}(U)$ close to ϕ let

$$P_{\mathbf{R}}^{1,n}(\psi) = \{P \in P_{\mathbf{R}}(\psi) : P \text{ has index 1 and } \alpha(E_P^s(\psi), E_P^{cu}(\psi)) \leqslant 1/n\}.$$

Write

$$(P_{\mathbf{R}}^{\infty})^{1,n_0}(\psi) = \bigcup_{n \geqslant n_0} P_{\mathbf{R}}^{1,n}(\psi).$$

Since the periods t_n tend to infinity and every P_n is hyperbolic, for each n there is a neighbourhood $\mathcal{V}_n(\phi)$ of ϕ such that

$$P_{\mathbf{R}}^{1,n}(\psi) \neq \varnothing \quad \text{for all } \psi \in \mathcal{V}_n(\phi). \tag{4.4}$$

We claim that there are diffeomorphisms $\xi_n \to \phi$ and points $P_{\xi_n} \in P_{\mathbf{R}}^{1,n}(\xi_n)$ such that

$$|\lambda_c(P_{\xi_n}, \xi_n)| \to 1.$$

Clearly, the lemma follows from the claim: applying Lemma 3.1 to ξ_n we obtain a new sequence $\xi_n' \to \phi$ such that $|\lambda_c(P_{\xi_n}, \xi_n')| < 1$ and $\alpha(E^s_{P_{\xi_n}}(\xi_n'), E^{cu}_{P_{\xi_n}}(\xi_n')) \leqslant 2/n$. So it remains to prove the claim.

We argue by contradiction. Assume that the claim is false. Then for a fixed n_0 (big) we get $\mu > 0$ such that

$$|\lambda_c(P_\psi, \psi)| > (1+\mu)^k, \quad \text{where } k \text{ is the period of } P_\psi,$$

for all ψ close to ϕ and every $P_\psi \in (P_{\mathbf{R}}^{\infty})^{1,n_0}(\psi)$. This assertion follows from Lemma 3.1 by arguing as in the proof of Lemma 4.5. Now, recall that $|\lambda_s(P_n, \psi_n)| < (1-\delta)^{t_n}$ and $|\lambda_u(P_n, \psi_n)| > (1+\delta)^{t_n}$ (see Lemma 4.6). By Lemma II.9 in [M3] (which, in a few words, asserts that *robust hyperbolicity of periodic points implies that the angles between the stable and unstable bundles are bounded away from zero*) there is $\gamma > 0$ such that

$$\alpha(E^s_{P_\psi}(\psi), E^{cu}_{P_\psi}(\psi)) > \gamma \quad \text{for all } \psi \text{ close to } \phi \text{ and every } P_\psi \in (P_{\mathbf{R}}^{\infty})^{1,n_0}(\psi).$$

Taking $n>1/\gamma$ one has $P_{\mathbf{R}}^{1,n}(\psi)=\varnothing$ for all ψ close to ϕ, contradicting (4.4). This ends the proofs of the claim and of the lemma. □

The next step in the proof of the proposition is to get a saddle-node periodic point R such that $\alpha(E^s_R(\varphi), E^u_R(\varphi))$ is small. We use the following claim whose proof we postpone until the end of this section.

CLAIM 4.12. *Suppose that $\phi\in\mathcal{R}(U)$ and that P_n and Q_n are the periodic points of ψ_n in (4.3), P_n and Q_n with index 2. Then there are sequences of diffeomorphisms (ϕ_n), $\phi_n\in\mathcal{T}(U)$ and $\phi_n\to\phi$, and of points $R_n\in P_{\mathbf{R}}(\phi_n)$ of period k_n, such that*

(1) $\max\{\alpha(E^{cs}_{R_n}(\phi_n), E^u_{R_n}(\phi_n)), \alpha(E^s_{R_n}(\phi_n), E^c_{R_n}(\phi_n))\}\leqslant 1/n$,

(2) $\lambda_c(R_n,\phi_n)=1$, *and*

(3) $k_n\to\infty$.

We begin by stating the following algebraic fact that will be frequently applied to the characteristic polynomial of the derivatives of periodic points.

FACT 4.13. *Consider sequences of real numbers (σ_n), (μ_n) and (ξ_n) such that*

(1) $|\xi_n|\to\infty$,

(2) $1<K<|\sigma_n|/|\mu_n|$ *for some constant K.*

For each $\varepsilon>0$ consider the sequence of polynomials

$$\mathcal{P}_{n,\varepsilon}(x)=(x-\sigma_n)(x-(\mu_n+\varepsilon\xi_n(\mu_n-\sigma_n)))-\varepsilon\xi_n\sigma_n(\mu_n-\sigma_n).$$

Then there is ε_n, $|\varepsilon_n|\simeq 1/|\xi_n|\to 0$, such that the roots $\mu_n(\varepsilon_n)$ and $\sigma_n(\varepsilon_n)$ of $\mathcal{P}_{n,\varepsilon_n}(x)$ are both real and satisfy

$$|\mu_n(\varepsilon_n)|=|\sigma_n(\varepsilon_n)|=\sqrt{|\mu_n\sigma_n|}\,.$$

Proof. If the product $\sigma_n\mu_n$ is positive, just take

$$\varepsilon_n=\frac{2\sqrt{|\mu_n\sigma_n|}-\mu_n-\sigma_n}{\xi_n(\mu_n-\sigma_n)},\quad |\varepsilon_n|\simeq\frac{1}{|\xi_n|}\to 0.$$

Then the roots of the polynomial $\mathcal{P}_{n,\varepsilon_n}(x)$ are

$$\mu_n(\varepsilon_n),\,\sigma_n(\varepsilon_n)=\pm\sqrt{\mu_n\sigma_n}\,.$$

Otherwise, if the product is negative, let

$$\varepsilon_n=-\frac{\mu_n+\sigma_n}{\xi_n(\mu_n-\sigma_n)},\quad |\varepsilon_n|\simeq\frac{1}{|\xi_n|}\to 0.$$

Now the roots of the polynomial are

$$\mu_n(\varepsilon_n),\,\sigma_n(\varepsilon_n)=\pm\sqrt{|\mu_n\sigma_n|}\,.$$

This completes the proof of the fact. □

Now we complete the proof of Proposition 4.8. For simplicity let us write $\lambda_i(n)=\lambda_i(R_n,\phi_n)$, $i=s,c,u$. Since $\lambda_c(n)=1$ one has (generically) two possibilities: either $|\lambda_s(n)\lambda_c(n)\lambda_u(n)|=|\lambda_s(n)\lambda_u(n)|>1$ or $|\lambda_s(n)\lambda_u(n)|<1$ for infinitely many n. Suppose, for instance, that the first possibility holds.

In $T_{R_n}M$ consider an orthonormal basis $B_0(n)$ such that there are eigenvectors $v^c(n)$, $v^s(n)$ and $v^u(n)$ associated to $\lambda_c(n)$, $\lambda_s(n)$ and $\lambda_u(n)$, respectively, with coordinates (in the basis $B_0(n)$)

$$v^c(n)=(1,0,0), \quad v^s(n)=(\alpha_n,1,0), \quad v^u(n)=(\beta_n,\gamma_n,1).$$

By the $D_{R_n}(\phi^{k_n})$-invariance of the $E^i_{R_n}(\phi_n)$, one has (in the basis $B_0(n)$)

$$D_{R_n}(\phi_n^{k_n})=\begin{pmatrix}1 & \tau_1(n) & \tau_2(n)\\ 0 & \lambda_s(n) & \tau_3(n)\\ 0 & 0 & \lambda_u(n)\end{pmatrix}, \quad \text{where } \tau_3(n)=(\lambda_u(n)-\lambda_s(n))\gamma_n.$$

By Claim 4.12, $\alpha(E^{cs}_{R_n}(\phi_n),E^u_{R_n}(\phi_n))<1/n\to 0$, and thus

$$\gamma_n^2+\beta_n^2\to\infty.$$

Suppose first that $\gamma_n^2\to\infty$. By Lemma 3.1, given $\varepsilon>0$ we can perturb ϕ_n at $\phi_n^{-1}(R_n)=\phi_n^{k_n-1}(R_n)$ to get $\phi_{\varepsilon,n}$ such that $\phi^j_{\varepsilon,n}(R_n)=\phi^j(R_n)$ for every j and

$$D_{\phi^{-1}_{\varepsilon,n}(R_n)}(\phi_{\varepsilon,n})=I_\varepsilon\cdot D_{\phi_n^{-1}(R_n)}\phi_n, \quad \text{where } I_\varepsilon=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & \varepsilon & 1\end{pmatrix}.$$

Then

$$D_{R_n}(\phi_{\varepsilon,n})^{k_n}=I_\varepsilon\cdot D_{\phi_n^{-1}(R_n)}(\phi_{\varepsilon,n})\cdot D_{R_n}\phi_n^{k_n-1}=\begin{pmatrix}1 & \tau_1(n) & \tau_2(n)\\ 0 & \lambda_s(n) & \tau_3(n)\\ 0 & \varepsilon\lambda_s(n) & \varepsilon\tau_3(n)+\lambda_u(n)\end{pmatrix}.$$

The characteristic polynomial of $D_{R_n}(\phi_{\varepsilon,n})^{k_n}$ is of the form $(1-x)\mathcal{P}_\varepsilon(x)$. Applying Fact 4.13 to $\mathcal{P}_\varepsilon(x)$, with

(1) $\sigma_n=\lambda_u(n)$,

(2) $\mu_n=\lambda_s(n)$,

(3) $|\sigma_n|/|\mu_n|=|\lambda_u(n)|/|\lambda_s(n)|\to\infty$ (recall that $k_n\to\infty$ and Lemma 4.5), and

(4) $\xi_n=\gamma_n$, $|\gamma_n|\to\infty$,

one gets $\varepsilon_n \to 0$ such that the eigenvalues of $D_{R_n}(\phi^{k_n}_{\varepsilon_n,n})$ are $\lambda_1(\varepsilon_n)=1$, $\lambda_2(\varepsilon_n)$ and $\lambda_3(\varepsilon_n)$, where

$$|\lambda_2(\varepsilon_n)| = |\lambda_3(\varepsilon_n)| = \sqrt{|\lambda_u(n)\lambda_s(n)|} > 1.$$

Thus we can perturb $\phi_{\varepsilon_n,n}$ (big n) to get $\psi \in \mathcal{T}(U)$ with a repeller at $R_n \in \Lambda_\psi(U)$, contradicting Fact 4.4. This completes the proof of the proposition when $|\gamma_n| \to \infty$.

Suppose now that γ_n^2 is bounded; thus $\beta_n^2 \to \infty$ and then $\alpha(E^u_{R_n}, E^c_{R_n}) \to 0$ as $n \to \infty$. By Claim 4.12, $\alpha(E^s_{R_n}, E^c_{R_n}) \to 0$, and hence $\alpha(E^u_{R_n}, E^s_{R_n}) \to 0$.

In $E^{su}_{R_n} = E^s_{R_n} \oplus E^u_{R_n}$ let us consider an orthonormal basis $\{v^u(n), v^\perp(n)\}$. Since $\alpha(E^s_{R_n}, E^u_{R_n}) \to 0$ there is an eigenvector $v^s(n)$ of $\lambda_s(n)$ of the form

$$v^s(n) = v^u(n) + \varkappa_n v^\perp(n), \quad \varkappa_n \to 0 \text{ as } n \to \infty.$$

Now, consider the basis $B'_0(n) = \{v^c(n), v^u(n), v^\perp(n)\}$. In this basis,

$$D_{R_n}\phi_n^{k_n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_u(n) & \mu_3(n) \\ 0 & 0 & \lambda_s(n) \end{pmatrix}, \quad \mu_3(n) = \frac{\lambda_s(n) - \lambda_u(n)}{\varkappa_n}, \quad |\mu_3(n)| \to \infty.$$

As before, using Lemma 3.1, for a fixed $\varepsilon > 0$ we perturb ϕ_n at $\phi_n^{-1}(R_n)$ to obtain $\phi_{\varepsilon,n}$ with $\phi^i_{\varepsilon,n}(R_n) = \phi^i_n(R_n)$ for all i satisfying

$$D_{\phi^{-1}_{\varepsilon,n}(R_n)}(\phi_{\varepsilon,n}) = I_\varepsilon \cdot D_{\phi_n^{-1}(R_n)}\phi_n,$$

I_ε as above. Then

$$D_{R_n}(\phi_{\varepsilon,n})^{k_n} = I_\varepsilon \cdot D_{\phi_n^{-1}(R_n)}(\phi_{\varepsilon,n}) \cdot D_{R_n}\phi_n^{k_n-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_u(n) & \mu_3(n) \\ 0 & \varepsilon\lambda_s(n) & \varepsilon\mu_3(n) + \lambda_u(n) \end{pmatrix}.$$

As before, applying Fact 4.13 to the characteristic polynomial of $D_{R_n}(\phi_{\varepsilon,n})^{k_n}$, one has $\varepsilon_n \to 0$ such that the eigenvalues of $D_{R_n}(\phi^{k_n}_{\varepsilon_n,n})$ are $\lambda_1(\varepsilon_n)=1$, $\lambda_2(\varepsilon_n)$ and $\lambda_3(n)$, where

$$|\lambda_2(\varepsilon_n)| = |\lambda_3(\varepsilon_n)| = \sqrt{|\lambda_u(n)\lambda_s(n)|} > 1.$$

Thus, perturbing $\phi_{\varepsilon_n,n}$ (big n) one gets $\psi \in \mathcal{T}(U)$ with a repeller at $R_n \in \Lambda_\psi(U)$, contradicting Fact 4.4. This completes the proof of Proposition 4.8 assuming Claim 4.12. So it remains to prove the claim.

Proof of Claim 4.12. Take periodic points P_n and Q_n of ψ_n with index 2 satisfying

$$\alpha(E^s_{P_n}(\psi_n), E^{cu}_{P_n}(\psi_n)) < \frac{1}{n} \quad \text{and} \quad \alpha(E^u_{Q_n}(\psi_n), E^{cs}_{Q_n}(\psi_n)) < \frac{1}{n}.$$

By Lemma 3.3 we can assume that P_n and Q_n are homoclinically related (their invariant manifolds meet transversely). Thus there are segments of arcs $\gamma_m^u \subset W^u(P_n, \psi_n)$ and disks $\delta_m^s \subset W^s(P_n, \psi_n)$ such that

$$\gamma_m^u \to W_{\mathrm{loc}}^u(Q_n, \psi_n), \quad \delta_m^s \to W_{\mathrm{loc}}^s(Q_n, \psi_n), \quad m \to \infty.$$

Since $\alpha(E_{Q_n}^u(\psi_n), E_{Q_n}^{cs}(\psi_n)) < 1/n$, there is a homoclinic point x_n associated to P_n such that the angle between $T_{x_n} W^s(P_n, \psi_n)$ and $T_{x_n} W^u(P_n, \psi_n)$ is small (less than $2/n$). Applying Lemma 3.1 to x_n and considering suitable compact parts of the invariant manifolds of P_n, we can perturb ψ_n to get a diffeomorphism ξ_n with a homoclinic tangency at x_n (associated to P_n). In this way we get a point of quadratic contact x_n between $W^s(P_n, \xi_n)$ and $W^u(P_n, \xi_n)$. Now, from [R] and a standard argument on unfolding of homoclinic tangencies, we get φ_n (close to ξ_n) with a periodic saddle-node R_n (associated to the tangency of $W^s(P_n, \xi_n)$ and $W^u(P_n, \xi_n)$) with eigenvalues $\lambda_c(n) = 1$, $|\lambda_s(n)| < 1 < |\lambda_u(n)|$. Moreover, this saddle-node can be chosen arising from some periodic point of a horseshoe such that the angles between their invariant manifolds are small. Observe that such horseshoes appear in the unfolding of a tangency. Thus we can assume that

$$E_{R_n}^u(\varphi_n) \sim E_{Q_n}^u(\varphi_n), \quad E_{R_n}^{cs}(\varphi_n) \sim E_{Q_n}^{cs}(\varphi_n) \quad \text{and} \quad \alpha(E_{R_n}^c(\varphi_n), E_{R_n}^s(\varphi_n)) \text{ is small.}$$

This completes the proof of the claim (and thus the proof of Proposition 4.8). □

4.2.2. *Proof of Proposition* 4.7. We divide the proof of the proposition into two steps: existence of the splittings and estimates on the angles.

First step: Existence of the splittings $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ or $E^u(\mathcal{B}) \oplus E^{cs}(\mathcal{B})$.

LEMMA 4.14. *Let $P \in P(\psi)$, $\psi \in \mathcal{T}(U)$, such that*

(1) *P has index two and $P \in P_{\mathbb{C}}(\psi)$ (i.e. P has two contracting eigenvalues of the same modulus),*

(2) *the set of transverse homoclinic points of P is not empty.*

Then given $\delta > 0$ there are $\varphi \in \mathcal{T}(U)$ close to ψ and $R \in P(\varphi)$ with

$$\alpha(E_R^s(\varphi), E_R^{cu}(\varphi)) < \delta.$$

Remark 4.15. The lemma holds for periodic points having an expanding complex eigenvalue. In this case one gets $\alpha(E_R^u(\varphi), E_R^{cs}(\varphi)) < \delta$.

Proof. Suppose that $P \in U$ (which for simplicity we assume to be a fixed point) has a pair of contracting nonreal eigenvalues. The case in which the eigenvalues are real (one eigenvalue with multiplicity two or different eigenvalues with the same modulus) follows analogously. Let λ_s, λ_c and λ_u, $|\lambda_s|=|\lambda_c|<1<|\lambda_u|$, be the eigenvalues of $D_P\psi$. As in the proof of Lemma 4.2 we construct diffeomorphisms ψ_k, $\psi_k\to\psi$, having periodic points Q_k (in the continuation of some horseshoe of P) of period $r_k=n_k+m$ ($n_k\to\infty$) such that the derivative $D_{Q_k}\psi^{r_k}$ is (see equation (4.2))

$$D_{Q_k}(\psi_k^{r_k})=\begin{pmatrix} c_m\lambda_u^{n_k} & 0 & 0\\ 0 & a_m|\lambda_s|^{n_k} & 0\\ 0 & 0 & b_m|\lambda_s|^{n_k}\end{pmatrix}$$

for some big k. Consider the basis $\{v,w,w^\perp\}$ in the proof of Lemma 4.2 and unit vectors v_i, w_i and $w_i^\perp$ in the directions of $D_{Q_k}\psi_k^i(v)$, $D_{Q_k}\psi_k^i(w)$ and $D_{Q_k}\psi_k^i(w^\perp)$. Using Lemma 3.1 we get ξ_k preserving the ψ_k-orbit of Q_k such that

$$\begin{aligned} D_{\xi_k^i(Q_k)}\xi_k(w_i) &= \frac{1}{|a_m|^{1/r_k}}D_{\psi^i(Q_k)}\psi_k(w_i),\\ D_{\xi_k^i(Q_k)}\xi_k(w_i^\perp) &= \frac{1}{|b_m|^{1/r_k}}D_{\psi^i(Q_k)}\psi_k(w_i^\perp),\\ D_{\psi_k^i(Q_k)}\xi_k(v_i) &= D_{\psi_k^i(Q_k)}\psi_k(v_i). \end{aligned}$$

Since a_m and b_m are independent of r_k (see Remark 4.3), and r_k can be taken arbitrarily big, every ξ_k is close to ψ_k and

$$D_{Q_k}\xi_k^{r_k}=\begin{pmatrix} c_m\lambda_u^{n_k} & 0 & 0\\ 0 & \pm|\lambda_s|^{n_k} & 0\\ 0 & 0 & \pm|\lambda_s|^{n_k}\end{pmatrix}.$$

For a fixed $\delta>0$, a new application of Lemma 3.1 gives diffeomorphisms $\xi_{k,\delta}$, $\xi_{k,\delta}\to\xi_k$ as $\delta\to 0^+$, preserving the ξ_k-orbit of Q_k, with

$$D_{Q_k}(\xi_{k,\delta}^{r_k})=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & \delta\\ 0 & 0 & 1+\delta^2\end{pmatrix}\cdot D_{Q_k}(\xi_k^{r_k})=\begin{pmatrix} c_m\lambda_u^{n_k} & 0 & 0\\ 0 & \pm|\lambda_s|^{n_k} & \pm\delta|\lambda_s|^{n_k}\\ 0 & 0 & \pm|\lambda_s|^{n_k}(1+\delta^2)\end{pmatrix}.$$

A straightforward calculation now shows that the eigenspaces $E^c_{Q_k}(\xi_{k,\delta})$ (associated to $\pm(1+\delta^2)|\lambda_s|^{n_k}$) and $E^s_{Q_k}(\xi_{k,\delta})$ (associated to $\pm|\lambda_s|^{n_k}$) of $D_{Q_k}(\xi_{k,\delta}^{r_k})$ are spanned by $(0,1,\pm\delta)$ and $(0,1,0)$. Thus the angle between $E^s_{Q_k}(\xi_{k,\delta})$ and $E^c_{Q_k}(\xi_{k,\delta})$ is of order of δ. This ends the proof of the lemma. □

The proof of Lemma 4.14 provides immediately the following remark which we will use in the proof of Proposition 4.23.

Remark 4.16. Under the hypotheses of Lemma 4.14, let Σ_P be any horseshoe (non-trivial hyperbolic set) containing P. Then the periodic point R of φ in Lemma 4.14 such that $\alpha(E^s_R(\varphi),E^{cu}_R(\varphi))$ is small can be taken in the continuation $\Sigma_P(\varphi)$ of Σ_P.

LEMMA 4.17. *Let* $\phi \in \mathcal{R}(U)$. *Then*

- *either* $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ *is defined for all* $\mathcal{B}$ *close to* $\mathcal{D}(\phi)$,
- *or* $E^u(\mathcal{B}) \oplus E^{cs}(\mathcal{B})$ *is defined for all* $\mathcal{B}$ *close to* $\mathcal{D}(\phi)$.

Proof. The first step to prove the lemma is to see that for every $\mathcal{B}$ close to $\mathcal{D}(\phi)$ at least one of the splittings $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ and $E^u(\mathcal{B}) \oplus E^{cs}(\mathcal{B})$ is well defined. The second step is to prove that we can take the same type of splitting for all $\mathcal{B}$ close to $\mathcal{D}(\phi)$.

First part of Lemma 4.17: *Existence of splittings.* We argue by contradiction. Suppose that there are families $\mathcal{B}$ arbitrarily close to $\mathcal{D}(\phi)$ such that the splittings $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ and $E^u(\mathcal{B}) \oplus E^{cs}(\mathcal{B})$ are both not defined. Then there are sequences $\mathcal{B}_n \to \mathcal{D}(\phi)$, and (P_n) and (Q_n), $P_n, Q_n \in P_{\mathbf{R}}(\phi)$, such that for each n there are $(B_{P_n})_{i \in \mathbf{Z}}, (B_{Q_n})_{i \in \mathbf{Z}} \in \mathcal{B}_n$ such that

(1) $E^{cu}(B_{P_n})$ is not defined; thus B_{P_n} has a complex (nonreal) contracting eigenvalue,

(2) $E^{cs}(B_{Q_n})$ is not defined; thus B_{Q_n} has a complex (nonreal) expanding eigenvalue.

Using the correspondence between diffeomorphisms and linear maps in Lemma 3.1, we get diffeomorphisms $\psi_n \to \phi$ such that P_n and Q_n are periodic points of ψ_n with

$$D_{P_n} \psi_n^{t_n} = B_{\psi^{t_n - 1}(P_n)} \dots B_{P_n} \quad \text{and} \quad D_{Q_n} \psi_n^{r_n} = B_{\psi^{r_n - 1}(Q_n)} \dots B_{r_n},$$

where t_n and r_n are the periods of P_n and Q_n. In particular, $D_{P_n} \psi_n^{t_n}$ (resp. $D_{Q_n} \psi_n^{r_n}$) has a contracting (resp. expanding) complex eigenvalue. Using Lemma 3.3 we can assume that the sets of transverse homoclinic points of P_n and Q_n are both nonempty.

Lemma 4.14 and Remark 4.15 imply that there are sequences of diffeomorphisms $\varphi_n \to \phi$ and periodic points $\widehat{P}_n$ and $\widehat{Q}_n \in P(\varphi_n)$ such that $\alpha(E^u_{\widehat{P}_n}(\varphi_n), E^{cs}_{\widehat{P}_n}(\varphi_n)) < 1/n$ and $\alpha(E^s_{\widehat{Q}_n}(\varphi_n), E^{cu}_{\widehat{Q}_n}(\varphi_n)) < 1/n$, contradicting Proposition 4.8.

Second part of Lemma 4.17: *Splitting of the same type.* We prove that either $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ is well defined for all $\mathcal{B}$ close to $\mathcal{D}(\phi)$, or $E^u(\mathcal{B}) \oplus E^{cs}(\mathcal{B})$ is well defined for all $\mathcal{B}$ close to $\mathcal{D}(\phi)$, that is, we can take the same type of splitting for every family of linear maps close to $\mathcal{D}(\phi)$. If not, there are sequences $\mathcal{B}_n, \mathcal{C}_n \to \mathcal{D}(\phi)$ such that

(1) it is not possible to define $E^s(\mathcal{B}_n) \oplus E^{cu}(\mathcal{B}_n)$; then there are $P_n \in P_{\mathbf{R}}(\phi)$ and $B_{P_n} \in \mathcal{B}_n$ with a contracting complex eigenvalue,

(2) it is not possible to define $E^u(\mathcal{C}_n) \oplus E^{cs}(\mathcal{C}_n)$; then there are $Q_n \in P_{\mathbf{R}}(\phi)$ and $C_{Q_n} \in \mathcal{C}_n$ with an expanding complex eigenvalue.

Now, using Lemma 3.1 and recalling that we can perform the perturbations at P_n and Q_n simultaneously, arguing exactly as in the first part of the proof of the lemma (existence of the splittings) we get diffeomorphisms $\psi_n \to \phi$ and points $R_n, T_n \in P_{\mathbf{R}}(\phi)$ such that

$$\alpha(E^s_{R_n}(\psi_n), E^{cu}_{R_n}(\psi_n)) < \frac{1}{n} \quad \text{and} \quad \alpha(E^u_{T_n}(\psi_n), E^{cs}_{T_n}(\psi_n)) < \frac{1}{n},$$

contradicting Proposition 4.8. This concludes the proof of Lemma 4.17. □

Second step: *Angular estimates.* To obtain the uniform angular estimate suppose, for instance, that $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ is defined for every $\mathcal{B}$ close to $\mathcal{D}(\phi)$.

LEMMA 4.18. *Suppose that $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ is defined for every $\mathcal{B}$ close to $\mathcal{D}(\phi)$. Then there is $C>0$ such that $\alpha_0(E^s(\mathcal{B}), E^{cu}(\mathcal{B}))>C$ for every $\mathcal{B}$ close to $\mathcal{D}(\phi)$.*

Proof. We argue by contradiction. Suppose that for every $C>0$ there is $\mathcal{B}$ close to $\mathcal{D}(\phi)$ such that $\alpha_0(E^s(\mathcal{B}), E^{cu}(\mathcal{B}))<C$. Then one gets $\mathcal{B}_n \to \mathcal{D}(\phi)$ such that $\alpha_0(E^s(\mathcal{B}_n), E^{cu}(\mathcal{B}_n))<1/n$. Thus there are periodic points $P_n \in P_{\mathbf{R}}(\phi)$ and linear maps $B_{P_n} \in \mathcal{B}_n$ such that

$$\alpha(E^s(B_{P_n}), E^{cu}(B_{P_n})) < \frac{1}{n}. \tag{4.5}$$

We now use the following fact:

FACT 4.19. *Let $\phi \in \mathcal{R}(U)$ such that for every family of linear maps $\mathcal{B}$ close to $\mathcal{D}(\phi)$ the splitting $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ is defined. Then*

$$|\lambda_s(B_P)| < |\lambda_c(B_P)| \leqslant |\lambda_u(B_P)|$$

for all $P \in P_{\mathbf{R}}(\phi)$.

Proof. The idea of the proof is that if $|\lambda_s(B_P)|=|\lambda_c(B_P)|$, then we can perturb ϕ in such a way that the continuation of some $R \in P_{\mathbf{R}}(\phi)$ has a contracting complex (nonreal) eigenvalue. This contradicts the fact that $E^s(\mathcal{B}) \oplus E^{cu}(\mathcal{B})$ is well defined for all $\mathcal{B}$ close to $\mathcal{D}(\phi)$.

To prove the fact we argue by contradiction. Suppose that $|\lambda_s(B_P)|=|\lambda_c(B_P)|$ for some $B_P \in \mathcal{B}$, $\mathcal{B}$ close to $\mathcal{D}(\phi)$. Since, by hypothesis, $E^s(B_P) \oplus E^{cu}(B_P)$ is defined, $\lambda_s(B_P)$ and $\lambda_c(B_P)$ are both real. If $\lambda_s(B_p)=\lambda_c(B_P)$, we perturb B_P to obtain a family $\mathcal{C}$ close to $\mathcal{D}(\phi)$ such that C_P has a complex (nonreal) contracting eigenvalue, contradicting the hypothesis of the lemma. So it remains to consider the case $\lambda_s(B_P)=-\lambda_c(B_P)$. First, by Lemma 3.1, there is φ close to ϕ such that $D_P\varphi=B_P$. Using the arguments in the proof of Lemma 4.2, we get ψ close to φ (hence close to ϕ), and a periodic point R_φ which is the continuation of some $R \in P_{\mathbf{R}}(\phi)$ (R in the homoclinic class of P), with a contracting (real) eigenvalue of multiplicity two. After a new perturbation, we get ξ such that R_ξ has a complex (nonreal) contracting eigenvalue. This provides $\mathcal{C}$ close to $\mathcal{D}(\phi)$, such that $C_R = D_{R_\xi}\xi$, and thus with complex contracting eigenvalues. Thus the splitting $E^s(\mathcal{C}) \oplus E^{cu}(\mathcal{C})$ is not defined. □

We claim that we can also suppose that the eigenvalues $\lambda_c(B_{P_n})$ and $\lambda_u(B_{P_n})$ of the points P_n in (4.5) are both real (and thus with different modulus). If $\lambda_c(B_{P_n})$ and $\lambda_u(B_{P_n})$ are complex the arguments in the proof of Lemma 4.2 give ψ close to ϕ, and a periodic point R_ψ which is the continuation of some $R_n \in P_{\mathbf{R}}(\phi)$ with three real eigenvalues. Moreover, $E^s_{R_\psi}(\psi)$ is close to $E^s(B_{P_n})$ (resp. $E^{cu}_{R_\psi}(\psi)$ is close to $E^{cu}(B_{P_n})$). This gives families $\mathcal{C}_n$ of periodic linear maps close to $\mathcal{D}(\phi)$ such that $\alpha(E^s(C_{R_n}), E^{cu}(C_{R_n})) < 2/n$ and C_{R_n} has real eigenvalues. This completes the proof of the claim.

We are now ready to finish the proof of Lemma 4.18. Let v_i^n be a unit vector which spans $E^i(B_{P_n})$, $i = s, c, u$. Take a normal basis $\{v_u^n, v_c^n, v_3^n\}$, where v_3^n is orthogonal to $E^{cu}(B_{P_n})$. In this basis $v_s^n = (a_u^n, a_c^n, a_3^n)$, $a_3^n \neq 0$. We have two possibilities: either $a_c^n \neq 0$ or not. Note that since $\alpha(E^s(B_{P_n}), E^{cu}(B_{P_n})) \to 0$ then

$$\left(\frac{a_u^n}{a_3^n}\right)^2 + \left(\frac{a_c^n}{a_3^n}\right)^2 \to \infty.$$

In the first case, arguing as in the proof of Proposition 4.8, we get $\mathcal{C}$ close to $\mathcal{D}(\phi)$ having two contracting eigenvalues of the same modulus, contradicting Fact 4.19.

In the second case, $\alpha(E^s(B_{P_n}), E^u(B_{P_n}))$ is small. Again as in the proof of Proposition 4.8, there is $\mathcal{C}$ close to $\mathcal{B}$ such that C_{P_n} has eigenvalues λ_1, λ_2 and $\lambda_c(B_{P_n})$, with $|\lambda_1| = |\lambda_2|$. If $|\lambda_c(B_{P_n})| > 1$ then, from Lemma 4.6, $|\lambda_1| = |\lambda_2| < 1$, contradicting Fact 4.19. Suppose now that $|\lambda_c(B_{P_n})| < 1$. Consider an isotopy $(L^t)_{t \in [0,1]}$ from B_{P_n} to C_{P_n} preserving the central direction $E^c(B_{P_n})$, i.e. $L^0 = B_{P_n}$, $L^1 = C_{P_n}$ and $L^t(v_c) = B_{P_n}(v_c)$. Using L^t we define in the natural way a parametrized family of periodic linear maps $\mathcal{L}^t$ close to $\mathcal{B}$. Let

$$\mu = \inf\{t \in [0,1] : L^t \text{ has two eigenvalues with the same modulus}\}.$$

By hypothesis, $\mu \leqslant 1$. Moreover, by construction, $|\lambda_s(L^\mu_{P_n})| = |\lambda_c(B_{P_n})| = |\lambda_c(L^\mu_{P_n})| < 1$, contradicting Fact 4.19. This ends the proof of the lemma. □

Now the proof of Proposition 4.7 is complete.

4.3. Uniformly dominated splittings

In view of Proposition 4.7, we have that for every ϕ in the residual subset $\mathcal{R}(U)$ of $\mathcal{T}(U)$ there are $C > 0$ and $\delta > 0$ such that either $\alpha_0(E^s(\mathcal{B}), E^{cu}(\mathcal{B})) > C$ for every family of periodic linear maps $\mathcal{B}$ δ-close to $\mathcal{D}(\phi)$ or $\alpha_0(E^u(\mathcal{B}), E^{cs}(\mathcal{B})) > C$ for every family of periodic linear maps $\mathcal{B}$ δ-close to $\mathcal{D}(\phi)$. In the sequel let us assume that we have fixed ϕ and that the first possibility holds:

$$\alpha_0(E^s(\mathcal{B}), E^{cu}(\mathcal{B})) > C \quad \text{for all } \mathcal{B} \ \delta\text{-close to } \mathcal{D}(\phi). \tag{4.6}$$

Our goal is to prove that if (4.6) holds and $E^s(\phi)\oplus E^{cu}(\phi)$ is (hyperbolically) dominated on the period (see Lemma 4.24 below) then the splitting is uniformly dominated.

Let us begin with the following two-dimensional lemma about perturbations of linear maps.

LEMMA 4.20. *Consider sequences of* (2×2)*-diagonal matrices* $\{(A_{i,m})_{i=1}^m\}_{m\geqslant 0}$,

$$A_{i,m} = \begin{pmatrix} a_{i,m} & 0 \\ 0 & b_{i,m} \end{pmatrix},$$

such that

(a) $\prod_{i=1}^m A_{i,m} = A_m = \mathrm{Id}^\pm$ *for every* m,

(b) *there is a constant* $c>0$ *such that* $c^{-1}\leqslant |a_{i,m}|, |b_{i,m}|\leqslant c$, *for every* i *and* m.

Then given $\varepsilon>0$ *and* $\varkappa>0$ *there is* m_0 *such that for every* $m\geqslant m_0$ *there are families of triangular matrices* $(\bar{A}_{i,m})_{i=1}^m$ *satisfying*

(i) $\|A_{i,m}-\bar{A}_{i,m}\|<\varepsilon$ *for every* $0\leqslant i\leqslant m$, *and*

(ii) *either*

$$\prod_{i=1}^m \bar{A}_{i,m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} K \\ \pm 1 \end{pmatrix} \quad \text{and} \quad \prod_{i=1}^m \bar{A}_{i,m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

or

$$\prod_{i=1}^m \bar{A}_{i,m}\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ K \end{pmatrix} \quad \text{and} \quad \prod_{i=1}^m \bar{A}_{i,m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

for some K *with* $|K|\geqslant\varkappa$.

Proof. Consider perturbations $\tilde{A}_{i,m}$ and $\hat{A}_{i,m}$ of $A_{i,m}$ of the form

$$\tilde{A}_{i,m} = \begin{pmatrix} a_{i,m} & \tilde{\delta}_{i,m}\delta_m \\ 0 & b_{i,m} \end{pmatrix}, \quad \hat{A}_{i,m} = \begin{pmatrix} a_{i,m} & 0 \\ \hat{\delta}_{i,m}\delta_m & b_{i,m} \end{pmatrix},$$

for some $\tilde{\delta}_{i,m}, \hat{\delta}_{i,m}=\pm 1$ and $\delta_m\geqslant 0$ to be determined later. Arguing inductively and bearing in mind that $\prod_{i=1}^m a_{i,m}=1$ and $\prod_{i=1}^m b_{i,m}=\pm 1$, a straightforward calculation gives

$$\tilde{A}_m = \prod_{i=1}^m \tilde{A}_{i,m} = \begin{pmatrix} 1 & \tilde{*}_m \\ 0 & \pm 1 \end{pmatrix}, \quad \hat{A}_m = \prod_{i=1}^m \hat{A}_{i,m} = \begin{pmatrix} 1 & 0 \\ \hat{*}_m & \pm 1 \end{pmatrix},$$

where

$$\tilde{*}_m = \sum_{j=1}^{m} \left(\prod_{i=1}^{j-1} a_{i,m} \right) \delta_m \tilde{\delta}_{j,m} \left(\prod_{i=j+1}^{m} b_{i,m} \right) = \sum_{j=1}^{m} \frac{\delta_m \tilde{\delta}_{j,m}}{a_{j,m}} \left(\prod_{i=j+1}^{m} \frac{b_{i,m}}{a_{i,m}} \right) = \delta_m \sum_{j=1}^{m} \frac{\tilde{\delta}_{j,m} \tilde{c}_{j,m}}{a_{j,m}}$$

and

$$\tilde{c}_{j,m} = \prod_{i=j+1}^{m} \frac{b_{i,m}}{a_{i,m}}.$$

We choose (inductively) the $\tilde{\delta}_{j,m}$ such that

$$\tilde{\delta}_{j,m} \frac{\tilde{c}_{j,m}}{a_{j,m}} = \frac{|\tilde{c}_{j,m}|}{|a_{j,m}|}.$$

Similarly, we have

$$\hat{*}_m = \sum_{j=1}^{m} \left(\prod_{i=1}^{j-1} b_{i,m} \right) \delta_m \hat{\delta}_{j,m} \left(\prod_{i=j+1}^{m} a_{i,m} \right) = \sum_{j=1}^{m} \frac{\delta_m \hat{\delta}_{j,m}}{b_{i,m}} \left(\prod_{i=j+1}^{m} \frac{a_{i,m}}{b_{i,m}} \right) = \delta_m \sum_{j=1}^{m} \frac{\hat{\delta}_{j,m} \hat{c}_{j,m}}{b_{j,m}}$$

and

$$\hat{\delta}_{j,m} \frac{\hat{c}_{j,m}}{b_{j,m}} = \frac{|\hat{c}_{j,m}|}{|b_{j,m}|}, \quad \hat{c}_{j,m} = \prod_{i=j+1}^{m} \frac{a_{i,m}}{b_{i,m}}.$$

That is,

$$|\tilde{c}_{j,m}| = |\hat{c}_{j,m}|^{-1}.$$

Notice that, by construction,

$$\frac{\tilde{\delta}_{j,m} \tilde{c}_{j,m}}{a_{j,m}} > 0 \quad \text{and} \quad \frac{\hat{\delta}_{j,m} \hat{c}_{j,m}}{b_{j,m}} > 0.$$

Consider now the sums

$$\widetilde{S}_m = \sum_{j=1}^{m} \frac{\tilde{\delta}_{j,m} \tilde{c}_{j,m}}{a_{j,m}}, \quad \widehat{S}_m = \sum_{j=1}^{m} \frac{\hat{\delta}_{j,m} \hat{c}_{j,m}}{b_{j,m}}.$$

Since the $|a_{j,m}|$ and $|b_{j,m}|$ are bounded, we have that these sums cannot be bounded simultaneously. Suppose, for instance, that the first one is not bounded. Then there is m such that $\widetilde{S}_m > 2\varkappa/\varepsilon$.

Observe that $\tilde{*}_m = \delta_m \widetilde{S}_m$. Thus taking $\delta_m \in (\varkappa/\widetilde{S}_m, \frac{1}{2}\varepsilon)$ we have

$$\tilde{*}_m = \delta_m \widetilde{S}_m > \varkappa.$$

Clearly, by the definition of $\tilde{A}_{j,m}$,

$$\prod_{j=1}^{m} \tilde{A}_{j,m}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}, \quad \prod_{j=1}^{m} \tilde{A}_{j,m}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\tilde{*}_m\\\pm 1\end{pmatrix}, \quad \tilde{*}_m>\varkappa.$$

By the definition of δ_m one gets $\|\tilde{A}_{j,m}-A_{j,m}\|<\varepsilon$. Now it is enough to take $\bar{A}_{j,m}=\tilde{A}_{j,m}$. This completes the proof when $\widetilde{S}_m$ is not bounded.

Finally, if $\widetilde{S}_m$ is bounded then $\widehat{S}_m$ is not bounded. Then the proof of the lemma follows as before by considering lower triangular matrices instead of the upper triangular ones. □

We have the following reformulations of the previous lemma and of Proposition 4.7 in terms of cone fields whose proofs are immediate.

LEMMA 4.21. *Let $((A_{i,m})_{i=1}^{m})_{m\geqslant 0}$ be families of linear maps satisfying the hypotheses of Lemma 4.20. Consider any pair of cones $\mathcal{C}^h$ and $\mathcal{C}^v$ around $\mathbf{i}=(1,0)$ and $\mathbf{j}=(0,1)$, respectively, such that $\mathbf{i}\notin\mathcal{C}^v$ and $\mathbf{j}\notin\mathcal{C}^h$. Then the perturbations $(\bar{A}_{i,m})$ in Lemma 4.20 can be taken satisfying*

(a) *either $\prod_{i=1}^{m}\bar{A}_{i,m}(\mathcal{C}^h)\subset\mathcal{C}^v$,*

(b) *or $\prod_{i=1}^{m}\bar{A}_{i,m}(\mathcal{C}^v)\subset\mathcal{C}^h$.*

LEMMA 4.22. *There are $\tau,\delta>0$ such that for every family of linear maps $\mathcal{B}$ δ-close to $\mathcal{D}(\phi)$ it holds*

$$E^s(B_P)\notin\mathcal{C}_\tau(E^{cu}(B_P))$$

for every $P\in P_{\mathbf{R}}(\phi)$. Here $\mathcal{C}_\tau(F)$ denotes the cone field of size τ around F.

Denote by

$$P_{\mathbf{R}}^{n}(\psi)=\{P\in P_{\mathbf{R}}(\psi) \text{ of period } m\geqslant n\}.$$

PROPOSITION 4.23. *Assume that hypothesis (4.6) holds. Then there is n_0 such that the $D\psi$-invariant splitting $(E^s_x(\psi)\oplus E^{cu}_x(\psi))_{x\in P_{\mathbf{R}}^{n_0}(\psi)}$ is uniformly dominated for every $\psi\in\mathcal{R}(U)$ close to ϕ.*

Proof. We first prove the *dominance in the period* for periodic points:

LEMMA 4.24. *Under the hypothesis of Proposition 4.23 there are λ, $0<\lambda<1$, and n_0 such that*

$$\|D_x\psi^n|_{E^s_x(\psi)}\|\cdot\|D_{\psi^n(x)}\psi^{-n}|_{E^{cu}_x(\psi)}\|<\lambda^n \quad \textit{for every } x\in P_{\mathbf{R}}^{n_0}(\psi) \textit{ of period } n\geqslant n_0.$$

Proof. We argue by contradiction and suppose that the lemma is false. Then there are sequences of points $P_m\in P_{\mathbf{R}}(\psi)$, P_m of period n_m, and of increasing numbers $(k_m)\to 1^-$ such that

$$\|D_{P_m}\psi^{n_m}|_{E^s_{P_m}(\psi)}\|\cdot\|(D_{P_m}\psi^{n_m})^{-1}|_{E^{cu}_{\psi^{n_m}(P_m)}(\psi)}\|>(k_m)^{n_m} \quad \text{for all } m. \tag{4.7}$$

On the other hand, from Lemma 4.6,

$$\begin{aligned} \|D_{P_m}\psi^{n_m}(v^s)\| &\leqslant (1-\delta)^{n_m}\|v^s\|, \quad v^s \in E^s_{P_m}(\psi), \\ \|(D_{P_m}\psi^{n_m})^{-1}(v^u)\| &\leqslant (1+\delta)^{-n_m}\|v^u\|, \quad v^u \in E^u_{\psi^{n_m}(P_m)}. \end{aligned} \tag{4.8}$$

Now, if k_m is close enough to 1 one has

$$(1-\delta)^{n_m} < k_m^{n_m}.$$

Therefore, there is $w \in E^{cu}_{\psi^{n_m}(P_m)}$, $\|v^s\| = \|w\| = 1$, such that

$$\|D_{P_m}\psi^{n_m}(v^s)\| \cdot \|(D_{P_m}\psi^{n_m})^{-1}(w)\| = (k_m)^{n_m}. \tag{4.9}$$

Let

$$\tau_m = (k_m)^{1/n_m}, \quad \tau_m \to 1 \text{ as } m \to \infty.$$

We now perturb the derivative of $D\psi$ along the orbit $P_m, \psi(P_m), ..., \psi^{n_m-1}(P_m)$ multiplying the action of the derivative $D_{\psi^i(P_m)}\psi$ in the direction $D_{P_m}\psi^{i-1}(v^s)$ (without modifying the derivative in the direction spanned by $D_{P_m}\psi^{i-1}(w)$) by the factor τ_m. In this way we obtain families $\mathcal{B}_m \to \mathcal{D}(\psi)$ such that

$$\|B_{\psi^{n_m-1}(P_m)} \dots B_{P_m}(v^s)\| \cdot \|B^{-1}_{P_m} \dots B^{-1}_{\psi^{n_m-1}(P_m)}(w)\| = 1. \tag{4.10}$$

Take the unit vector $w' \in E^{cu}(B_{P_m})$ in the direction

$$B^{-1}_{P_m} \dots B^{-1}_{\psi^{n_m-1}(P_m)}(w)$$

and write

$$\mu^s_m = \|B_{\psi^{n_m-1}(P_m)} \dots B_{P_m}(v^s)\|. \tag{4.11}$$

For each $0 \leqslant j \leqslant n_m - 1$ consider the linear space E_j spanned by $v^s_j = B_{\psi^{j-1}(P_m)} \dots B_{P_m}(v^s)$, $w'_j = B_{\psi^{j-1}(P_m)} \dots B_{P_m}(w')$, and its normalized basis β_j,

$$\beta_j = \{v^s_j/\|v^s_j\|, w'_j/\|w'_j\|\}. \tag{4.12}$$

In the sequel we will focus our attention on the space E_j. Recall that $\alpha(E^s(\mathcal{B}), E^{cu}(\mathcal{B})) > C > 0$ for every $\mathcal{B}$ close to $\mathcal{D}(\psi)$, see (4.6). Thus, there is a metric g_1, equivalent to the initial metric of M, such that for every $0 \leqslant j \leqslant n_m - 1$ the vectors v^s_j and w'_j are orthogonal, and $\|v^s_j\|_1 = \|v^s_j\|$ and $\|w'_j\|_1 = \|w'_j\|$ ($\|\cdot\|_1$ denotes the norm associated to g_1). In other words, in the metric g_1 the basis β_j is orthonormal. From (4.11), the restriction of $T_m = B_{\psi^{n_m-1}(P_m)} \dots B_{P_m}$ (in the basis β_0 and β_{n_m}) is

$$T_m = \prod_{i=0}^{n_m-1} T_{i,m} \colon E_0 \to E_{n_m}, \quad T_m = \mu^s_m \,\mathrm{Id}^{\pm}, \quad \mathrm{Id}^{\pm} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad T_{i,m} = B_{\psi^i(P_m)}.$$

We now apply Lemmas 4.20 and 4.21 to the sequence $T_{i,m}$. Consider the spaces E_j, $j=0,\dots,n_m$, spanned by v_j^s and w_j' (see (4.12)). For a fixed $\tau>0$ satisfying Lemma 4.22 we consider the cone fields (see Lemma 4.21)

$$\mathcal{C}^v(B_{P_m})=[\mathcal{C}_\tau(E^{cu}(B_{P_m}))]^c \quad \text{and} \quad \mathcal{C}^h(B_{P_m})=[\mathcal{C}_\tau(E^{cu}(B_{P_m}))],$$

(A^c denotes the complement of A) and the matrices $(T_{i,m})$. Suppose, for instance, that we can perturb the $(T_{i,m})$ according to case (a) in Lemma 4.21. By perturbing $\mathcal{B}$ (which is close to $\mathcal{D}(\psi)$) along the segment of ψ-orbit $\{P_m,\psi(P_m),\dots,\psi^{n_m-1}(P_m)\}$ we obtain $\mathcal{C}$ close to $\mathcal{B}$ such that

(1) $C_{\psi^j(P_m)}E_j=E_{j+1}$,

(2) $E^u(C_{\psi^j(P_m)})=E^u(B_{\psi^j(P_m)})$ for every j,

(3) $C_{\psi^i(P_m)}|_{E_i}=\bar{T}_{i,m}$, where $\bar{T}_{i,m}$ is the perturbation of $T_{i,m}$ in Lemma 4.21.

Therefore,

(4) $\prod_{j=0}^{n_m-1}C_{\psi^j(P_m)}|_{E_j}=\prod_{i=1}^{m}\bar{T}_{i,m}=\bar{T}_m$.

On one hand, by Lemma 4.22 and by definition,

$$E^s(C_{\psi^j(P_m)})\subset[\mathcal{C}_\tau(E^{cu}(C_{\psi^j(P_m)}))]^c=\mathcal{C}^v(C_{\psi^j(P_m)}) \quad \text{for all } 0\leqslant j\leqslant n_m.$$

On the other hand, let $C=C_{\psi^{n_m-1}(P_m)}\dots C_{P_m}$ and take a unit vector v_0^u in the strong unstable direction of P_m. Recall that we are assuming that the $(T_{i,m})$ can be perturbed according to (a) in Lemma 4.21. Since these matrices are triangular (expressed in the basis $\beta_j=\{v_j^s, D_{P_m}\psi^j(w')\}$) we have that in the basis $\{v_0^u,w',v_0^s\}$ the linear map C is given by

$$\begin{aligned}
C(v_0^u)&=\lambda_u v_0^u,\\
C(w')&=\varkappa_1 v_0^u+\varkappa_2 w',\\
C(v_0^s)&=\lambda_s v_0^s+KC(w')=\lambda_s v_0^s+K\varkappa_1 v_0^u+K\varkappa_2 w',
\end{aligned}$$

where $|\varkappa_1|$ and $|\varkappa_2|$ are both small (this follows from the fact that the modulus of $C(w')$ is of order of $|\lambda_s|$, that also is small). Observe that the angle between v_0^u and w' is uniformly bounded from below for all m (otherwise, using that $|\lambda_u|\to\infty$ and $|\lambda_s|\to 0$ we get a contradiction). So after a change of metric (of bounded size) we can assume that the basis $\beta=\{v_0^u,w',v_0^s\}$ is orthonormal.

Observe that λ_s is an eigenvalue of C. Consider an eigenvector associated to λ_s of the form $(a,b,1)$. A straightforward calculation shows that

$$b=\frac{K\varkappa_1}{\varkappa_1-\lambda_s}.$$

That is, if K is sufficiently big, we have

$$C_{\psi^{n_m-1}(P_m)} \dots C_{P_m}(E^s(C_{P_m})) = E^s(C_{\psi^{n_m}(P_m)}) \subset \mathcal{C}_\tau(E^{cu}(C_{\psi^{n_m}(P_m)})),$$

contradicting Lemma 4.22. This completes the proof of Lemma 4.24. $\square$

We now finish the proof of Proposition 4.23. We argue by contradiction. Suppose that $\{E^s_P(\psi) \oplus E^{cu}_P(\psi)\}_{P^n_{\mathbf{R}}(\psi)}$ is not uniformly dominated for all n. Then there are sequences of points $P_m \in P_{\mathbf{R}}(\psi)$, P_m of period n_m, $n_m \to \infty$, and of increasing numbers $(k_m) \to 1^-$ such that

$$\|D_{P_m}\psi^m|_{E^s_{P_m}(\psi)}\| \cdot \|(D_{P_m}\psi^m)^{-1}|_{E^{cu}_{\psi^m(P_m)}(\psi)}\| > k_m \quad \text{for all } m. \tag{4.13}$$

We claim that we can take the points P_m in the sequence with periods $n_m > m$ for infinitely many m (then, taking a subsequence we can suppose that $n_m > m$ for all m). We prove this claim by contradiction. Suppose that $n_m \leqslant m$ for every m sufficiently big. Then $m = kn_m + r_m$, $1 \leqslant k$, $0 \leqslant r_m < n_m$. We have

$$\begin{aligned} k_m &< \|D_{P_m}\psi^m|_{E^s_{P_m}}\| \cdot \|(D_{P_m}\psi^m)^{-1}|_{E^{cu}_{\psi^m(P_m)}}\| \\ &\leqslant \|D_{P_m}\psi^{kn_m}|_{E^s_{P_m}}\| \cdot \|D_{\psi^{kn_m}(P_m)}\psi^{r_m}|_{E^s_{\psi^{kn_m}(P_m)}}\| \\ &\quad \times \|(D_{P_m}\psi^{kn_m})^{-1}|_{E^{cu}_{\psi^{kn_m}(P_m)}}\| \cdot \|(D_{\psi^{kn_m}(P_m)}\psi^{r_m})^{-1}|_{E^{cu}_{\psi^{kn_m}(P_m)}}\| \\ &\leqslant (\lambda)^{kn_m} \cdot \|D_{\psi^{kn_m}(P_m)}\psi^{r_m}|_{E^s_{\psi^{kn_m}(P_m)}}\| \cdot \|(D_{\psi^{kn_m}(P_m)}\psi^{r_m})^{-1}|_{E^{cu}_{\psi^{kn_m}(P_m)}}\|, \end{aligned} \tag{4.14}$$

where the last inequality follows from Lemma 4.24. This equation gives

$$\|D_{\psi^{kn_m}(P_m)}\psi^{r_m}|_{E^s_{\psi^{kn_m}(P_m)}}\| \cdot \|(D_{\psi^{kn_m}(P_m)}\psi^{r_m})^{-1}|_{E^{cu}_{\psi^{kn_m}(P_m)}}\| \geqslant \frac{k_m}{(\lambda)^{n_m k}} \, (> k_m).$$

Since $k_m \to 1$, $0 < \lambda < 1$ and $n_m \to \infty$, this implies that $r_m \to \infty$. Now it is enough to take $\psi^{n_m k}(P_m)$ instead of P_m and $m = r_m < n_m$ (i.e. reindex the sequence) in the definition of dominance. This ends the proof of the claim.

For each k_m consider the point P_m of period $n_m > m$ in (4.13). Take $\sigma > k_m$ such that

$$\sigma^{-1}\|v\| \leqslant \|D_x\psi^{\pm 1}(v)\| \leqslant \sigma\|v\|$$

for all $x \in M$ and $v \in T_x M$. Since $n_m > m$, from Lemma 4.24 for each m there is t_m, $m \leqslant t_m < n_m$, such that

$$\sigma^{-3} \leqslant \sigma^{-2}k_m \leqslant \|D_{P_m}\psi^{t_m}(v^s)\| \cdot \|(D_{P_m}\psi^{t_m})^{-1}(w)\| \leqslant \sigma^2 k_m \leqslant \sigma^3$$

for some unit vectors $w \in E^{cu}_{\psi^{t_m}(P_m)}$ and $v^s \in E^s_{P_m}$. Since the angle between $E^s(\psi)$ and $E^{cu}(\psi)$ is uniformly bounded from below, using Lemma 3.1 and arguing as in the proof of Lemma 4.24 we get families of linear maps $\mathcal{B}_m \to \mathcal{D}(\psi)$ such that

$$\|B_{\psi^{t_m}(P_m)} \dots B_{P_m}(v^s)\| \cdot \|(B_{\psi^{t_m}(P_m)} \dots B_{P_m})^{-1}(w)\| = 1$$

for some unit vectors $v^s \in E^s(B_{P_m})$ and $w \in E^{cu}(B_{\psi^{t_m}(P_m)})$. Arguing as in the proof of Lemma 4.24 this leads to contradiction. This completes the proof of Proposition 4.23. $\square$

4.4. End of the proof of Theorem 1.1

We are now ready to construct a partially hyperbolic splitting for $T_{\Lambda_\psi(U)}M$ for every ψ in $\mathcal{R}(U)$ close to ϕ. By Proposition 4.23, $\{E^s_P(\psi)\oplus E^{cu}_P(\psi)\}_{P\in P^{n_0}_{\mathbf{R}}(\psi)}$ is uniformly dominated. Proposition 1.3 in [M2] allows us to extend this splitting to a uniformly dominated one defined on the closure of $P^{n_0}_{\mathbf{R}}(\psi)$, which is $\Lambda_\psi(U)$ (recall that $\psi\in\mathcal{R}(U)$). We also denote these extensions by $E^s(\psi)$ and $E^{cu}(\psi)$.

The hyperbolicity of $E^s(\psi)$ follows from the ergodic closing lemma (see Theorems A and B in [M3]) and Lemma 4.5 by using standard techniques (see the proof of Theorem B in [M3]). Moreover, by the dominance of the splitting $E^s(\psi)\oplus E^{cu}(\psi)$, $E^s(\psi)$ is the strong stable bundle. This completes the proof of Theorem 1.1.

5. Proofs of the theorems

5.1. Proof of Theorem A

It is enough to see that given any $\phi\in\mathcal{R}(U)$ and its partially hyperbolic splitting from Theorem 1.1, say $T_xM=E^s_x(\phi)\oplus E^c_x(\phi)$, $x\in\Lambda_\phi(U)$, one can extend it for every ψ in a neighbourhood $\mathcal{V}_\phi$ of ϕ in $\mathcal{T}(U)$. Then it is enough to consider $\mathcal{A}(U)=\bigcup_{\phi\in\mathcal{R}(U)}\mathcal{V}_\phi$. To get the neighbourhood $\mathcal{V}_\phi$ first define the map

$$\Psi\colon \mathcal{T}(U)\to P(M),\quad \xi\mapsto\Lambda_\xi(U),$$

where $P(M)$ denotes the set of subsets of M endowed with the usual Hausdorff metric. By the definition of $\Lambda_\xi(U)$ and the continuity of ξ, we have that if $\Lambda_\xi(U)\subset V\subset U$ for some neighbourhood V of $\Lambda_\xi(U)$, then the same holds for every φ close to ξ (i.e. $\Lambda_\varphi(U)\subset V$). This means that the map Ψ is upper semicontinuous in $\mathcal{T}(U)$.

Due to the partial hyperbolicity of $(E^s_x(\phi)\oplus E^c_x(\phi))_{x\in\Lambda_\phi(U)}$, there is an extension of it, say $(\widetilde{E^s}_x(\phi)\oplus\widetilde{E^c}_x(\phi))_{x\in W}$, defined on a small neighbourhood W of $\Lambda_\phi(U)$ such that in the coordinates induced by $(\widetilde{E^s}_x(\phi)\oplus\widetilde{E^c}_x(\phi))_{x\in W}$ the matrix $D\phi^m$ (m as in the definition of partial hyperbolicity) is of the form

$$D_x\phi^m=\begin{pmatrix} a_{1,1} & \varrho_{1,2} & \varrho_{1,3}\\ \varrho_{2,1} & a_{2,2} & a_{2,2}\\ \varrho_{3,1} & a_{3,2} & a_{3,3}\end{pmatrix},\quad A=\begin{pmatrix} a_{2,2} & a_{2,2}\\ a_{3,2} & a_{3,3}\end{pmatrix},$$

where

$$a_{1,1}\cdot\|A^{-1}\|<\varkappa<1\quad\text{and}\quad \max\{\varrho_{i,j}\}\ll\min\{a_{1,1},\|(A)^{-1}\|\}\quad\text{for all } x\in W.$$

By the upper semicontinuity of Ψ we have

$$\Lambda_\psi(U) = \Psi(\psi) \subset W \quad \text{for every } \psi \text{ close to } \phi.$$

Using the splitting $(\widetilde{E^s}_x(\phi) \oplus \widetilde{E^c}_x(\phi))_{x\in W}$, which is not necessarily $D\psi$-invariant, one has

$$D_x\psi^m = \begin{pmatrix} a'_{1,1} & \varrho'_{1,2} & \varrho'_{1,3} \\ \varrho'_{2,1} & a'_{2,2} & a'_{2,2} \\ \varrho'_{3,1} & a'_{3,2} & a'_{3,3} \end{pmatrix}, \quad A' = \begin{pmatrix} a'_{2,2} & a'_{2,2} \\ a'_{3,2} & a'_{3,3} \end{pmatrix}.$$

By continuity, if ψ is close to ϕ, one has

$$\max\{\varrho'_{i,j}\} \ll \min\{a'_{1,1}, \|(A')^{-1}\|\}.$$

Since

$$a_{1,1} \cdot \|A^{-1}\| < \varkappa < 1$$

we obtain a $D\psi$-invariant dominated splitting of ψ, $E^s(\psi) \oplus E^c(\psi)$ in the maximal invariant set of ψ in W, which is exactly $\Lambda_\psi(U)$, see [HPS] for details.

Finally, by construction and the C^1-proximity of ψ to ϕ, the bundle $E^s(\psi)$ is uniformly hyperbolic. This completes the proof of Theorem A.

5.2. Proof of Theorem B

We begin by explaining the obstruction for extending the splitting in Theorem A, defined only for diffeomorphisms in the open and dense subset $\mathcal{A}(U)$ of $\mathcal{T}(U)$, to the whole $\mathcal{T}(U)$ when $U \neq M$. The obstruction arises from the fact that the map $\varphi \mapsto \Lambda_\varphi(U)$ is (in general) only upper semicontinuous. Clearly, if $U = M$ this map is constant, $\varphi \mapsto \Lambda_\varphi(M) = M$, and hence continuous. In other words, given $\varphi \in (\mathcal{T}(U) \setminus \mathcal{A}(U))$ and $x \in \Lambda_\varphi(U)$ a priori we do not know if there are diffeomorphisms $\varphi_n \in \mathcal{A}(U)$ and points $x_n \in \Lambda_{\varphi_n}(U)$ such that

$$\varphi = \lim \varphi_n \quad \text{and} \quad x = \lim x_n.$$

When $U = M$, since $\Lambda_\varphi(M) = M$, this property holds automatically.

Take $\varphi \in \mathcal{T}$ and a sequence $\varphi_n \to \varphi$, $\varphi_n \in \mathcal{R} = \mathcal{R}(M)$, $\mathcal{R}(U)$ as in Theorem 1.1. For each n and $y \in P_{\mathbf{R}}(\varphi_n)$ define $m_n^{s,u}(y)$ by

$$m_n^s(y) = \min\{m : \|D_y\varphi_n^k|_{E_y^s(\varphi_n)}\| \cdot \|(D_y\varphi_n^k)^{-1}|_{E^{cu}_{\varphi_n^k(y)}(\varphi_n)}\| < \tfrac{1}{2} \text{ for all } k \geqslant m\},$$

$$m_n^u(y) = \min\{m : \|D_y\varphi_n^k|_{E_y^u(\varphi_n)}\| \cdot \|(D_y\varphi_n^k)^{-1}|_{E^{cs}_{\varphi_n^k(y)}(\varphi_n)}\| > \tfrac{1}{2} \text{ for all } k \geqslant m\}.$$

Finally, let

$$m_n^{s,u} = \sup\{m_n^{s,u}(y) : y \in P_{\mathbf{R}}(\varphi_n)\}.$$

Since we have, by Theorem 1.1, that at least one of the splittings $E^s(\varphi_n) \oplus E^{cu}(\varphi_n)$ and $E^u(\varphi_n) \oplus E^{cs}(\varphi_n)$ is uniformly dominated, then for each n, either m_n^s or m_n^u is finite.

LEMMA 5.1. *There is a subsequence (n_k) such that either $(m^s_{n_k})$ or $(m^u_{n_k})$ is uniformly bounded from above.*

Proof. The lemma follows from the arguments in Proposition 4.23. If the lemma is false, taking n big enough one gets ϕ close to φ_n, and points P and Q in $P_{\mathbf{R}}(\phi)$ such that $\alpha(E^s_P(\phi), E^{cu}_P(\phi))$ and $\alpha(E^s_Q(\phi), E^{cu}_Q(\phi))$ are both arbitrarily small, contradicting Proposition 4.8. $\square$

Observe that, in view of Lemma 5.1, there are $i=s$ or u, and a subsequence $\{n_k\}$ such that $m^i_{n_k}$ is bounded. So let us assume (taking a subsequence if necessary) that, for instance, $m^s_n<m<\infty$ for all n. So we get m such that

$$\|D_y\varphi_n^m|_{E^s_y(\varphi_n)}\|\cdot\|(D_y\varphi_n^m)^{-1}|_{E^{cu}_{\varphi_n^m(y)}(\varphi_n)}\|<\tfrac{1}{2}\quad\text{for all } y\in P_{\mathbf{R}}(\varphi_n)\text{ and (big) } n. \tag{5.1}$$

Since $P_{\mathbf{R}}(\varphi_n)$ is dense in M, given x, taking a subsequence if necessary, we can suppose that there are $x_n\in P_{\mathbf{R}}(\varphi_n)$ with $x_n\to x$. Now let

$$E^1_x(\varphi)=\lim E^s_{x_n}(\varphi_n)\quad\text{and}\quad E^2_x(\varphi)=\lim E^{cu}_{x_n}(\varphi_n).$$

Next we extend the splitting to the whole φ-orbit of x. Now (5.1) implies that the splitting is uniformly dominated. Thus, it is the unique $D\varphi$-invariant splitting of type $E^s\oplus E^{cu}$ over the φ-orbit of x. In other words, $E^1_x=E^s_x$ and $E^2_x=E^{cu}_x$.

Finally, as in the proof of Theorem A we have that $E^s(\varphi)$ is uniformly hyperbolic (contracting). This completes the proof of Theorem B.

5.3. Proof of Theorem C: strong partial hyperbolicity

Take $\phi\in\mathcal{R}(U)$ and suppose that it satisfies (4.6):

$$\alpha(E^s_x(\phi), E^{cu}_x(\phi))\geqslant C>0\qquad\text{for every } x\in\Lambda_\phi(U). \tag{5.2}$$

Our goal is to prove that, under the assumptions of Theorem C, the same kind of angular estimates hold for the splitting $E^u(\phi)\oplus E^{cs}(\phi)$. Observe that the arguments in the proof of Proposition 4.7 and Theorem 1.1 imply

LEMMA 5.2. *Let $\phi\in\mathcal{R}(U)$. Suppose that there are C and $\delta>0$ such that every family of linear maps $\mathcal{B}$ δ-close to $\mathcal{D}(\phi)$ satisfies*

$$\alpha_0(E^u(\mathcal{B}), E^{cs}(\mathcal{B}))\geqslant C>0.$$

Then $\{E^u_P(\phi)\oplus E^{cs}_P(\phi)\}_{P\in P_{\mathbf{R}}(\phi)}$ can be extended to a partially hyperbolic splitting defined on the whole $\Lambda_\phi(U)$.

By Lemma 5.2, Theorem C follows from the following claim:

CLAIM 5.3. *Let $\varphi \in \mathcal{P}(U)$. Then there are $C, \delta > 0$ such that*

$$\alpha_0(E^u(\mathcal{B}), E^{cs}(\mathcal{B})) \geqslant C$$

for every family of linear maps $\mathcal{B}$ δ-close to $\mathcal{D}(\varphi)$.

Before proving the claim let us end the proof of Theorem C. By Lemma 5.2 and Claim 5.3 we have two partially hyperbolic splittings in $\Lambda_\varphi(U)$,

$$E^s_x(\varphi) \oplus E^{cu}_x(\varphi) \quad \text{and} \quad E^{sc}_x(\varphi) \oplus E^u_x(\varphi),$$

where

$$\dim(E^{cu}_x(\varphi)) = \dim(E^{sc}_x(\varphi)) = 2.$$

Now take

$$E^c_x(\varphi) = E^{sc}_x(\varphi) \cap E^{cu}_x(\varphi),$$

which is $D\varphi$-invariant and one-dimensional. Then

$$T_{\Lambda_\varphi(U)}M = E^s(\varphi) \oplus E^c(\varphi) \oplus E^u(\varphi).$$

Finally, $E^s(\varphi)$ (resp. $E^u(\varphi)$) dominates $E^c(\varphi)$. This follows from the dominance of $E^s(\varphi)$ (resp. $E^u(\varphi)$) over $E^{cu}(\varphi)$ (resp. $E^{sc}(\varphi)$). This means that $\Lambda_\varphi(U)$ is strong partially hyperbolic.

This ends the proof of the theorem for diffeomorphisms in $\mathcal{P}(U) \cap \mathcal{R}(U)$. Now the proof of Theorem 1.1 allows us to extend this bundle with three invariant directions to $\mathcal{A}(U)$ (which is open and dense in $\mathcal{T}(U)$).

To prove the claim first observe that if $\varphi \in \mathcal{P}(U)$ and $P \in P(\varphi)$, then by Fact 4.19,

$$|\lambda_i(P, \varphi)| \neq |\lambda_j(P, \varphi)| \quad \text{for all } i \neq j, \ i, j \in \{s, c, u\}. \tag{5.3}$$

We argue by contradiction. If the claim is false, using Lemma 3.1, we get diffeomorphisms $\varphi_n \to \varphi$ ($\varphi_n \in \mathcal{P}(U) \cap \mathcal{R}(U)$) and points $P_n \in P_{\mathbf{R}}(\varphi_n)$ with $\alpha(E^{cs}_{P_n}(\varphi_n), E^u_{P_n}(\varphi_n)) \to 0$.

Suppose first that there are infinitely many P_n with index two. We have (see Lemma 4.2) that there is a (nontrivial) transverse intersection between $W^s(P_n, \varphi_n)$ and $W^u(P_n, \varphi_n)$. Then, as in the proof of Claim 4.12, we perturb φ_n to obtain a tangency associated to P_n. Now the unfolding of this tangency leads to the creation of periodic points with complex (nonreal) eigenvalues (see [PV] for details), contradicting (5.3).

So we are left to consider the case in which every P_n (big n) has index one. First recall that we are assuming that $\alpha(E^s_x(\varphi_n), E^{cu}_x(\varphi_n)) > C > 0$ for all $x \in \Lambda_{\varphi_n}(U)$, see (5.2).

Therefore, $\alpha(E^u_x(\varphi_n), E^{cs}_x(\varphi_n)) \to 0$ implies that $\alpha(E^u_x(\varphi_n), E^c_x(\varphi_n)) \to 0$. We now have two possibilities:

(1) $\dfrac{|\lambda_u(P_n,\varphi_n)|}{|\lambda_c(P_n,\varphi_n)|} \to \infty$,

(2) $\dfrac{|\lambda_u(P_n,\varphi_n)|}{|\lambda_c(P_n,\varphi_n)|}$ is uniformly bounded from above.

In the first case, using Fact 4.13 and Lemma 3.1, we get ψ_n close to φ_n such that $|\lambda_c(P_n,\psi_n)| = |\lambda_u(P_n,\psi_n)|$, which contradicts (5.3).

Finally, if $|\lambda_u(P_n,\varphi_n)|/|\lambda_c(P_n,\varphi_n)|$ is uniformly bounded, we apply Lemma 3.1 to perturb φ_n through the orbit of P_n to get ψ_n with $|\lambda_c(P_n,\psi_n)| = |\lambda_u(P_n,\psi_n)|$, contradicting (5.3).

5.4. Proof of Corollary D

The corollary follows applying the arguments in the proof of Theorem B to the splittings $E^s \oplus E^{cu}$ and $E^u \oplus E^{cs}$.

5.5. Proofs of Theorem E and Corollary G

We divide the proof of the theorem into two parts: approximation by homoclinic tangencies (which will imply the corollary) and expansion/contraction of the volume.

5.5.1. *Approximation by homoclinic tangencies. Proof of Corollary G.* By hypotheses there are ϕ close to φ, and P and $Q \in \operatorname{Hyp} P(\phi)$ of indices 2 and 1, respectively, such that Q has an expanding complex eigenvalue (recall Remark 4.15). The next lemma immediately implies the corollary.

LEMMA 5.4. *Let ϕ be as above. Then given any $R \in \operatorname{Hyp} P(\phi)$ of index two there is ψ close to ϕ with a homoclinic tangency associated to R_ψ, where R_ψ is the continuation of R for ψ.*

Proof. By Lemma 3.3 there is ξ close to ϕ such that

(1) $W^s(R_\xi,\xi)$ and $W^u(Q_\xi,\xi)$ have a nontrivial transverse intersection, and

(2) $W^u(R_\xi,\xi)$ and $W^s(Q_\xi,\xi)$ have a point of quasitransverse intersection.

Since Q_ξ has an expanding complex eigenvalue, $W^s(R_\xi,\xi)$ spirals around $W^s(Q_\xi,\xi)$. Finally, since $W^u(R_\xi,\xi)$ and $W^s(Q_\xi,\xi)$ are quasitransverse we can perturb ξ to obtain ψ with a homoclinic tangency associated to R_ψ. □

5.5.2. *The bundle E^{cu} is volume-expanding. Proof of Theorem E.* The theorem follows immediately from the proposition below.

PROPOSITION 5.5. *Let $\varphi \in \mathcal{A}(U)$. Then we have*:

(1) *if $T_{\Lambda_\varphi(U)} = E^s(\varphi) \oplus E^{cu}(\varphi)$, then $E^c(\varphi) = E^{cu}(\varphi)$ is volume-expanding,*

(2) *if $T_{\Lambda_\varphi(U)} = E^u(\varphi) \oplus E^{cs}(\varphi)$, then $E^c(\varphi) = E^{cs}(\varphi)$ is volume-contracting.*

Remark 5.6. This proposition does not hold (in general) if $\varphi \in \mathcal{P}(U)$. To see this observe that in such a case there are two possible choices for the central bundle, either $E^c = E^{cs}$ or $E^c = E^{cu}$. Now just take φ having a fixed point P of index two with $|\lambda_c(P)\lambda_u(P)| < 1$ (then $E^c = E^{cu}$ is not volume-expanding), and a fixed point Q of index 1 such that $|\lambda_c(Q)\lambda_s(Q)| > 1$ (then $E^c = E^{cs}$ is not volume-contracting).

Proof. Let us suppose, for instance, that we have $T_{\Lambda_\varphi(U)} = E^s(\varphi) \oplus E^{cu}(\varphi)$. The case $T_{\Lambda_\varphi(U)} = E^u(\varphi) \oplus E^{cs}(\varphi)$ follows similarly.

To prove that the Jacobian of φ is expanding in the central direction let us first observe that $|\mathrm{Jac}_P^{cu}\varphi| > 1$ for every $P \in P(\varphi)$, that is, $|\lambda_u(P)\lambda_c(P)| > 1$. This is trivial for points P with index 1. We claim that this inequality also holds for points of index 2. Suppose contrary to our claim that there is P with index 2 such that $|\lambda_u(P)\lambda_c(P)| < 1$. By Lemma 5.4, after a perturbation, we can assume that $W^s(P,\varphi)$ and $W^u(P,\varphi)$ have a homoclinic tangency. Such a tangency is sectionally dissipative (the modulus of the product of any pair of eigenvalues is less than one), and thus its unfolding leads to sinks in U (see [PV]), contradicting Fact 4.4.

Assume by contradiction that $E^{cu}(\varphi)$ is not volume-expanding. Then there are sequences $x_n \in \Lambda_\varphi(U)$, $k_n \in \mathbf{N}$ and τ_n, with $k_n \to \infty$ and $\tau_n \to 1^+$, such that

$$|\mathrm{Jac}_{x_n}^{cu}(\varphi^{k_n})| < \tau_n^{k_n},$$

where $\mathrm{Jac}_x^{cu}(\varphi^k)$ is the Jacobian of $D_x\varphi^k|_{E_x^{cu}}$. In other words, one has

$$\frac{1}{k_n}\sum_{i=0}^{k_n-1} \log(|\mathrm{Jac}_{\varphi^i(x_n)}^{cu}(\varphi)|) < \log(\tau_n).$$

Note that we can take k_n such that $\varphi^j(x_n) \neq \varphi^i(x_n)$ for all $j \neq i$, $j, i \in \{0, ..., k_n\}$. Thus for each n we can consider the Dirac measure δ_n supported on the segments of orbits $\{x_n, ..., \varphi^{k_n}(x_n)\}$. Taking a subsequence we can suppose that (δ_n) converges to an invariant measure μ such that

$$\int \log|\mathrm{Jac}_x^{cu}(\varphi)|\, d\mu(x) \leqslant 0.$$

Using the ergodic decomposition theorem, one gets ν (ergodic and φ-invariant) such that

$$\int \log|\mathrm{Jac}_x^{cu}(\varphi)|\, d\nu(x) \leqslant 0.$$

By the ergodic closing lemma [M3], given $\varepsilon>0$ there is ϕ close to φ and a ϕ-periodic point y such that

$$\frac{1}{m_\varepsilon}\sum_{i=0}^{m_\varepsilon-1}\log(|\operatorname{Jac}^{cu}_{\phi^i(y)}(\phi)|)<\varepsilon,$$

where m_ε is the period of y. Observe that $m_\varepsilon\to\infty$ as $\varepsilon\to 0$. By Lemma 3.1, taking $\varepsilon>0$ arbitrarily small and m_ε big, we get ψ close to ϕ such that $\psi^{m_\varepsilon}(y)=y$ and

$$\frac{1}{m_\varepsilon}\sum_{i=0}^{m_\varepsilon-1}\log(|\operatorname{Jac}^{cu}_{\psi^i(y)}(\psi)|)<0.$$

In other words,

$$|\operatorname{Jac}^{cu}_y(\psi^{m_\varepsilon})|<\lambda<1.$$

This last inequality means that y is a sectionally dissipative periodic point of ψ. Now the proposition follows from the comments above. □

5.6. Proof of Theorem H

Let $\varphi\in\operatorname{Diff}(M)$ be a robustly transitive diffeomorphism. If φ is Anosov (uniformly hyperbolic) then $M=T^3$ and we are done, see [N]. Otherwise, by Theorem B, φ has a partially hyperbolic splitting, say $E^s(\varphi)\oplus E^{cu}(\varphi)$. By hypothesis, $E^s(\varphi)$ and $E^{cu}(\varphi)$ are both integrable. Denote by $\mathcal{F}^i(\varphi)$ the integral foliation of $E^i(\varphi)$, $i=s$ or cu.

We argue by contradiction. Suppose that $\pi_1(M)$ is finite. By the $\mathcal{C}^0$-Novikov theorem (see [CL]), $\mathcal{F}^{cu}(\varphi)$ has a compact leaf F. Moreover, the strong stable foliation $\mathcal{F}^s(\varphi)$ is transverse to F. Taking a convenient finite covering $\widetilde{M}$ of M (for instance the universal one) we can suppose that the lift $\widetilde{F}$ of F is orientable, and that the lifts $\widetilde{\mathcal{F}}^s(\varphi)$ and $\widetilde{\mathcal{F}}^{cu}(\varphi)$ of $\mathcal{F}^s(\varphi)$ and $\mathcal{F}^{cu}(\varphi)$, respectively, are transversally orientable. Moreover, since the covering is finite, one has that $\widetilde{F}$ is compact.

We claim that there is a curve α which intersects $\widetilde{F}$ transversally infinitely many times (always with the same orientation). Now, using this curve, it is not hard to construct infinitely many closed curves that are not homotopic to each other. But this is impossible when $\pi_1(\widetilde{M})$ is finite.

Let us now prove the claim. We first take ϕ close to φ such that $P_{\mathbf{R}}(\phi)$ is hyperbolic and dense in M, see Lemma 4.2. Then there is a periodic point P of ϕ such that its strong stable manifold $W^{ss}(P,\phi)$, which is part of a leaf of $\mathcal{F}^s(\phi)$, intersects F (transversally).

Let us first suppose that P has index 1. Thus we have $W^s(P,\phi)=W^{ss}(P,\phi)$. From Lemma 4.2, P has a homoclinic transverse point. Then, by the λ-lemma, $W^s(P,\phi)$ accumulates onto itself. This implies that $W^s(P,\phi)$ intersects F infinitely many times.

Take as α the lift of $W^s(P,\phi)$. Since the covering is finite one has that α intersects $\widetilde{F}$ (transversally) infinitely many times.

Finally, suppose that P has index 2. Since ϕ is not hyperbolic there is a periodic point Q of ϕ with index 1. By Lemma 3.3 we can assume that $W^s(Q,\phi)$ and $W^u(P,\phi)$ meet transversally, and that $W^u(Q,\phi)$ and $W^s(P,\phi)$ intersect quasitransversally. This means that, generically, $W^s(Q,\phi)$ accumulates on $W^{ss}(P,\phi)$. Thus, $W^s(Q,\phi)$ intersects F transversally. Now we can construct the curve α as before. This ends the proof of the theorem.

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