Acta Math., 179 (1997), 197–222

Relative algebraic K-theory and topological cyclic homology

by

RANDY MCCARTHY

University of Illinois
Urbana, IL, U.S.A.

Introduction

In recent years, the study of the algebraic K-theory space $\mathrm{K}(R)$ of a ring R has been approached by the introduction of spaces with a more homological flavor. One collection of such spaces is connected to K-theory by various trace maps and is particularly effective in measuring the relative K-theory $\mathrm{K}(f)$ associated to a surjective ring homomorphism $f: R \to S$ whose kernel is nilpotent. Goodwillie's main theorem in [10] shows that the rational homotopy type of $\mathrm{K}(f)$ can be recovered from cyclic homology. The main theorem of this paper shows that, for any prime p, the p-adic homotopy type of $\mathrm{K}(f)$ can be recovered from topological cyclic homology, $\mathrm{TC}(f)$.

To define the topological cyclic homology $\mathrm{TC}(R)$ for a ring R one must first consider ordinary rings as special types of more general rings up to homotopy. Functors with smash product, or FSP's, were introduced by M. Bökstedt in [2] as useful models for such topological rings and it is for these objects that topological cyclic homology is defined by Bökstedt–Hsiang–Madsen in [4]. Every ring naturally gives rise to an FSP which models the associated Eilenberg–MacLane ring spectrum of the ring, and in this way rings and simplicial rings are naturally embedded into the category of FSP's. One can extend the definition of algebraic K-theory from simplicial rings to FSP's so that the algebraic K-theory of a simplicial ring agrees with the algebraic K-theory of its associated FSP. There is a natural transformation $\mathrm{trc}: \mathrm{K} \to \mathrm{TC}$, called the cyclotomic trace, which was used by Bökstedt–Hsiang–Madsen in [4] to solve the algebraic K-theory analogue of Novikov's conjecture for a large class of discrete groups. In [12], Goodwillie conjectured that for maps f of FSP's such that $\pi_0(f)$ (a ring map) is surjective with nilpotent kernel then the relative cyclotomic trace from $\mathrm{K}(f)$ to $\mathrm{TC}(f)$ would be an equivalence after pro-finite

This work was partially supported by National Science Foundation Grant 1-5-30943.

completion. In this paper we prove the following theorem.

MAIN THEOREM. *If $f: R \to S$ is a surjective ring map with nilpotent kernel then the relative trace map* $\mathrm{K}(f) \xrightarrow{\mathrm{trc}} \mathrm{TC}(f)$ *is an equivalence after p-completion for all primes p.*

We actually prove the slightly more general result that if $f: R \to S$ is a map of simplicial rings such that $\pi_0(f)$ is surjective with nilpotent kernel, then the relative trace map $\mathrm{K}(f) \xrightarrow{\mathrm{trc}} \mathrm{TC}(f)$ is an equivalence after p-completion for all primes p. Recently, Bjørn Dundas has shown how to deduce Goodwillie's original conjecture for all FSP's from this result for maps of simplicial rings ([6]). Essentially, he shows how one can use simplicial rings to approximate arbitrary FSP's when the functors in question are sufficiently analytic.

We will be using the work of Hesselholt and Madsen in [16] for our basic results and terminology of TC. This result was announced in 1994 and has since been used to make several explicit calculations in algebraic K-theory (see for example the computations of Hesselholt and Madsen in [16], Tsalidis in [19], and Bökstedt and Madsen in [5]). For a very nice overview of the subject of trace maps, algebraic K-theory and computations recently obtained we recommend the review by Ib Madsen in [18].

The general scheme for our proof goes as follows. By an argument of Goodwillie in §III of [10], we can reduce the main theorem to the case of a map of ordinary rings admitting a section and having a square-zero kernel. For X a based simplicial set, A a simplicial ring and M a simplicial A-bimodule we let $\widetilde{M}[X]$ be the simplicial A-bimodule $M[X]/M[*]$. We let $A \ltimes \widetilde{M}[X]$ be the new simplicial ring with multiplication $(a,m)(a',m')=(aa', am'+ma')$. Let $\widetilde{\mathrm{K}}(A \ltimes \widetilde{M}[-])$ be the functor obtained by taking the homotopy fiber of the natural map $\mathrm{K}(A \ltimes \widetilde{M}[-]) \to \mathrm{K}(A)$. Similarly, let $\widetilde{\mathrm{TC}}(A \ltimes \widetilde{M}[-])$ be the functor obtained by taking the homotopy fiber of the natural map $\mathrm{TC}(A \ltimes \widetilde{M}[-]) \to \mathrm{TC}(A)$. The functors $\widetilde{\mathrm{K}}(A \ltimes \widetilde{M}[-])$ and $\widetilde{\mathrm{TC}}(A \ltimes \widetilde{M}[-])$ are homotopy functors from based spaces to spectra which are both (-1)-analytic in the sense of calculus (for homotopy functors) as defined by Goodwillie in [14].

If for all primes p and $n \geqslant 1$ the trace map from $\widetilde{\mathrm{K}}(A \ltimes \widetilde{M}[S^n])$ to $\widetilde{\mathrm{TC}}(A \ltimes \widetilde{M}[S^n])$ is $2n$-connected after p-completion then the derivatives of these two homotopy functors agree at a point (as defined by Goodwillie in [11]). It is not too difficult to show that for these two functors this also implies that their differentials agree. By analytic continuation (5.10 of [12]) we can deduce that the trace is an equivalence after p-completion for all based spaces within the radius of convergence for each functor. Since the functors are both (-1)-analytic this implies that this is true for all based spaces. In particular, the p-completed trace is an equivalence for the space S^0 which is the special case we needed.

In [7], B. Dundas and the author prove that there is a natural transformation from

$\widetilde{\mathrm{K}}(A \ltimes \widetilde{M}[-])$ to $\mathrm{TH}(A; \widetilde{M}[S^1 \wedge -])$ which is $(2n+2)$-connected on n-connected spaces. In [15], L. Hesselholt proves that for n-connected spaces and all primes p, $\widetilde{\mathrm{TC}}(A \ltimes \widetilde{M}[-])^{\widehat{}}_p$ is equivalent to $S^1 \wedge \mathrm{TH}(A; \widetilde{M}[-])^{\widehat{}}_p$ in a $2n$-range (where $\widehat{}_p$ is p-completion) by a natural diagram of spaces. Thus, the objective is to "glue" these two arguments together compatibly which we do in §4 using the categorical description of the trace map established by B. Dundas and the author in [8].

I would like to thank Bjørn Dundas and Stavros Tsalidis for many encouraging conversations while working on this paper. Also, I would like to thank Ib Madsen for inviting me to talk on this result in Aarhus during which several improvements to the original argument were made. I am especially indebted to Tom Goodwillie for generously sharing his time and insights with me while I was a Tamarkin assistant professor at Brown University, and specifically for his help in establishing the p-limit condition and analyticity for $\widetilde{\mathrm{TC}}(A \ltimes \widetilde{M}[-])^{\widehat{}}_p$ (§1 and §2).

0. Reduction by work of Tom Goodwillie

We now establish some notation, terminology and a result which will reduce the main theorem to showing three conditions. Let p be a fixed prime number.

Notation. Our conventions for spaces, connectivity and spectra are the same as those in [14]. In particular, a map of spaces is called k-connected if each of its homotopy fibers is $(k-1)$-connected. The empty space is (-2)-connected, every based space is (-1)-connected, path-connected spaces are 0-connected, and so on. For us a *spectrum* (which some authors call a *prespectrum*) is a sequence $\{E(n) \mid n \geqslant 0\}$ of based spaces equipped with based maps $E(n) \to \Omega E(n+1)$ (we will assume that all our spectra are (-1)-connected). A morphism of spectra is a sequence of based maps which strictly respects these structure maps. Following [14, 5.10] we define the p-completion of a spectrum E, written $E^{\widehat{}}_p$, to be the homotopy limit of the tower $E \wedge M([p^n])$ of smash products with Moore spaces.

Let F be a functor from the category of simplicial rings to spectra. For f a map of simplicial rings from R to S we write $F(f)$ for the homotopy fiber of $F(R) \xrightarrow{f_*} F(S)$. The functors F which we will consider satisfy the following three conditions:

Condition 1. There is some integer b such that for k-connected simplicial ring maps f, $F(f)$ is $(k-b)$-connected. In particular, if f is such that $|f|$ is a homotopy equivalence then the map f_* is a weak homotopy equivalence.

Condition 2. If f is a surjective map of simplicial rings whose kernel I satisfies $I^2=0$ then $F(f)$ is naturally equivalent to the realization of the simplicial spectra $[n] \to F(f_{[n]})$.

We note that Condition 2 can be modified to say that $F(f)^{\widehat{}}_p$ is naturally equivalent to the realization of the simplicial spectrum $[n] \to F(f_{[n]})^{\widehat{}}_p$ since each $F(f_{[n]})$ is a spectrum bounded below by -1.

We write $\widetilde{F}(A \ltimes \widetilde{M}[X])$ for $F(\pi)$ when π is the projection $A \ltimes \widetilde{M}[X] \to A$ (sending (a, m) to a). By Condition 1, the functor $\widetilde{F}(A \ltimes \widetilde{M}[-])$ from based spaces to based spaces is a reduced homotopy functor in the sense of [11].

Condition 3. For all rings A, A-bimodules M, the functor $\widetilde{F}(A \ltimes \widetilde{M}[-])^{\widehat{}}_p$ is (-1)-analytic and satisfies the p-limit axiom as defined in [14, 4.2 and 5.10].

We have the following variation of one of the main theorems of T. Goodwillie's calculus of functors.

THEOREM 0.1 (Tom Goodwillie). *Let F and G be functors satisfying Conditions 1, 2 and 3. Let η be a natural transformation from F to G such that for all rings A, A-bimodules M, and $n \geqslant 0$, the natural map produced by η from $\widetilde{F}(A \ltimes \widetilde{M}[S^n])^{\widehat{}}_p \to \widetilde{G}(A \ltimes \widetilde{M}[S^n])^{\widehat{}}_p$ is at least $2n$-connected. Then for any map $f: R \to S$ of simplicial rings such that $\pi_0(f)$ is surjective with nilpotent kernel, the diagram*

$$\begin{tikzcd} F(R) \arrow[r, "\eta"] \arrow[d, "f_*"] & G(R) \arrow[d, "f_*"] \\ F(S) \arrow[r, "\eta"] & G(S) \end{tikzcd}$$

is homotopy Cartesian (a homotopy pull-back) after p-completion.

Reduction step. By Conditions 1 and 2, one can use the argument in §III of [10] to reduce to the special case when f is a map of rings having a section (i.e. f has a right inverse) and whose ideal I is square zero. We briefly recall the steps in this reduction.

Let H be the homotopy fiber of $(\eta)^{\widehat{}}_p$ and suppose that $H(f)$ is an equivalence for surjective ring maps f having a section and a square-zero kernel. We want to deduce that $H(f)$ is an equivalence for simplicial ring maps f such that only $\pi_0(f)$ is a surjective ring map having a nilpotent kernel.

Suppose that $f: R \to S$ is a surjective simplicial ring map with kernel square zero. Take a free resolution $\Phi S \xrightarrow{\simeq} S$ of S (as simplicial rings) and form the fiber product

$$\begin{tikzcd} R' \arrow[r, "g"] \arrow[d, "\simeq"] & \Phi S \arrow[d, "\simeq"] \\ R \arrow[r, "f"] & S. \end{tikzcd}$$

Since H preserves equivalences it suffices to show that $H(g)$ is an equivalence. By Condition 2, $H(g)$ can be computed degreewise and these are given by split extensions (having square-zero kernel) since S is free, and thus $H(g)$ is an equivalence by assumption. By induction applied to the diagrams

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & R/I^n \arrow[dl] \\ & R/I^{n-1} & \end{tikzcd}$$

we see that $H(f)$ is an equivalence for all surjective maps of simplicial rings with nilpotent kernel.

Now let $f\colon R \to S$ be a map of simplicial rings such that $\pi_0 f$ is surjective with nilpotent kernel. By the diagram of simplicial rings

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ \pi_0 R \arrow[r, "\pi_0 f"] & \pi_0 S \end{tikzcd}$$

we see that $H(f)$ is an equivalence if H of the two vertical maps are equivalences. Thus, we are left to consider the case of a simplicial ring R with I a 0-connected ideal and $f\colon R \to R/I$ the quotient map.

In general, if $I \subseteq R$ is a k-connected simplicial ideal, then by Lemma 1.1.7 of [10] there exists a simplicial ideal $J \subseteq S$ with J $(k+1)$-reduced (its k-skeleton is a point) and a map of simplicial ring-ideal pairs $(S, J) \to (R, I)$ such that both $S \to R$ and $J \to I$ are equivalences. Observe that if J is $(k+1)$-reduced then $J \otimes J$ is $(2k+1)$-connected since the realization of the diagonal simplicial complex is equivalent to the bisimplicial realization which is $(2k+2)$-reduced. If $m\colon J \otimes J \to J^2$ is the multiplication map, then $\ker(m)$ is also $(k+1)$-reduced and so by the short exact sequence

$$0 \to \ker(m) \to J \otimes J \xrightarrow{m} J^2 \to 0$$

we see that J^2 is $(k+1)$-connected when J is $(k+1)$-reduced. Considering the diagram of simplicial rings

$$\begin{tikzcd} S \arrow[rr, "g_2"] \arrow[dr, "g"'] & & S/J^2 \arrow[dl, "g_1"] \\ & S/J & \end{tikzcd}$$

we see by Condition 1 that $H(g_2)$ is $(k+1-b)$-connected and since $H(g_1)$ is an equivalence this implies that $H(g)$ is $(k+1-b)$-connected, which implies (again by Condition 1) that for I k-connected, $H(R\to R/I)$ is $(k+1-b)$-connected. Thus, by induction, if I is 0-connected then $H(R\to R/I)$ is $(n-b)$-connected for all n and hence an equivalence.

This ends the reduction step.

Proof of Theorem 0.1. By the reduction step it suffices to show that for all rings A, A-bimodules M and based sets Y, the diagram

$$\begin{tikzcd} F(A\ltimes\widetilde{M}[Y]) \arrow[r, "\eta"] \arrow[d, "\pi_*"] & G(A\ltimes\widetilde{M}[Y]) \arrow[d, "\pi_*"] \\ F(A) \arrow[r, "\eta"] & G(A) \end{tikzcd} \tag{1}$$

is homotopy Cartesian after p-completion.

Let X be a based simplicial set. Since $A\ltimes\widetilde{M}[X\vee S^n]\cong\langle A\ltimes\widetilde{M}[X]\rangle\ltimes\widetilde{M}[S^n]$, the homotopy fiber of $\widetilde{F}(A\ltimes\widetilde{M}[X\vee S^n])\to\widetilde{F}(A\ltimes\widetilde{M}[X])$ is naturally equivalent to $\widetilde{F}(\langle A\ltimes\widetilde{M}[X]\rangle\ltimes\widetilde{M}[S^n])$. By Condition 2, $\widetilde{F}(\langle A\ltimes\widetilde{M}[X]\rangle\ltimes\widetilde{M}[S^n])\widehat{_p}$ is naturally equivalent to the diagonal of the bi-simplicial space sending $[m]$, $[m']$ to $\widetilde{F}(\langle A\ltimes\widetilde{M}[X_m]\rangle\ltimes\widetilde{M}[S^n_{[m']}])\widehat{_p}$. The same is true for G in place of F and for each fixed m the natural map induced by η from $\widetilde{F}(\langle A\ltimes\widetilde{M}[X_m]\rangle\ltimes\widetilde{M}[S^n])\widehat{_p}$ to $\widetilde{G}(\langle A\ltimes\widetilde{M}[X_m]\rangle\ltimes\widetilde{M}[S^n])\widehat{_p}$ is at least $2n$-connected by assumption. Thus, the map of realizations is also $2n$-connected and the diagram

$$\begin{tikzcd} F(A\ltimes\widetilde{M}[X\vee S^n]) \arrow[r, "\eta"] \arrow[d, "\pi_*"] & G(A\ltimes\widetilde{M}[X\vee S^n]) \arrow[d, "\pi_*"] \\ F(A\ltimes\widetilde{M}[X]) \arrow[r, "\eta"] & G(A\ltimes\widetilde{M}[X]) \end{tikzcd}$$

is $2n$-Cartesian (homotopy pull-back in a $2n$-range) after p-completion for all $n\geqslant 0$.

This shows that the map $\partial_x F(A\ltimes\widetilde{M}[X])\widehat{_p}\to\partial_x G(A\ltimes\widetilde{M}[X])\widehat{_p}$ induced by η is an equivalence for all based spaces (X,x). By Condition 3, the functors $\widetilde{F}(A\ltimes\widetilde{M}[-])\widehat{_p}$ and $\widetilde{G}(A\ltimes\widetilde{M}[-])\widehat{_p}$ are (-1)-analytic and satisfy the p-limit axiom. By Theorem 5.10 of [14] we deduce that diagram (1) is Cartesian after p-completion for all (-1)-connected spaces Y. That is, for all based spaces.

In order to apply Theorem 0.1, we recall that the functor $\mathrm{K}(\cdot)$ satisfies Condition 1 by Proposition 1.1 of [20], and Condition 2 by Lemma I.2.2 of [10]. Actually, one uses 2.7 of [9] to deduce that the diagram on p. 359 of [10] is homotopy Cartesian. Proposition 3.1 establishes Condition 3 for $\mathrm{K}(\cdot)$. We show in §§1 and 2 that the functor TC satisfies Conditions 1–3. Theorem 4.1 establishes the fact that the cyclotomic trace from K to TC defined in [4] satisfies the conditions of η in Theorem 0.1. This will be deduced by showing that the equivalence of stable K-theory and topological Hochschild homology proved in [7] and the p-completed equivalence of stable TC and topological Hochschild homology proved in [15] can be compatibly combined.

Aside. The careful reader may object at this point since the cyclotomic trace defined in [4] is not a natural transformation of functors but a natural homotopy class. This problem can be resolved, for example, by using the definition of the cyclotomic trace in [3, §2] which uses another model for the algebraic K-theory of an FSP which is naturally equivalent to one used in [4]. In [8], the cyclotomic trace is defined for ordinary rings by a natural transformation of functors to spectra (using iterations of the S-construction for deloopings) and we can equivalently use this model for the cyclotomic trace when we want to apply [14, 5.10].

1. The analyticity of $\mathrm{TC}(A \ltimes \widetilde{M}[-])\hat{_p}$

In this section we will show that $\mathrm{TC}(A \ltimes \widetilde{M}[-])$ satisfies Conditions 1 and 2 and the analyticity part of Condition 3 for Theorem 0.1. Unless specified otherwise, we will be using the notation and terminology of [16] and [18]. In particular, for A a ring, we write $\tilde{A}[-]$ for the FSP sending a space (pointed simplicial set) to the realization of the simplicial A-module $[n] \mapsto A[X_n]/A[*]$. We let S^1 (instead of G in [16]) be the circle group and write C_q for the subgroup of S^1 with q elements. We will write R_p for the *restriction*—the natural map $\mathrm{TH}(A)^{C_{p^n}} \to \mathrm{TH}(A)^{C_{p^{n-1}}}$ obtained by restricting mapping spaces to fixed points (written ϕ_p in [4] and [13]) and F_p for the *Frobenius*—the natural map $\mathrm{TH}(A)^{C_{p^n}} \to \mathrm{TH}(A)^{C_{p^{n-1}}}$ obtained by inclusion of fixed points (written i_p in [4] and [13]). Following [18], for a fixed prime p and natural number n, we will call the cofibration sequences

$$\mathrm{TH}(R)_{hC_{p^n}} \to \mathrm{TH}(R)^{C_{p^n}} \xrightarrow{R_p} \mathrm{TH}(R)^{C_{p^{n-1}}}$$

the *fundamental cofibration sequences* (see for example [16, §1] or [18, 2.4.6]). An observation which we will be implicitly using in Lemma 1.1 and Proposition 1.3 is the following. If

$$\ldots \to E_{i+1} \xrightarrow{f_{i+1}} E_i \xrightarrow{f_i} E_{i-1} \to \ldots \to E_1 \xrightarrow{f_1} E_0 = *$$

is a sequence of spectra such that the homotopy fibers of each f_i are k-connected then $\mathrm{holim}_f\, E_i$ is again k-connected.

We now establish Condition 1 for TC. Given a simplicial ring R_*, we define $\widetilde{R}_*[-]$ to be the evident FSP determined by $X \to |[n] \mapsto \widetilde{R}_n[X]|$.

LEMMA 1.1. *If* $f: R_* \to S_*$ *is a map of simplicial rings such that* $|f|$ *is k-connected, then* $\mathrm{TC}(f)$ *is* $(k-2)$*-connected.*

Proof. If $f: R_* \to S_*$ is a map of simplicial rings such that $|f|$ is k-connected, then for all spaces X, $\tilde{f}[X]: \widetilde{R}_*[X] \to \widetilde{S}_*[X]$ is k-connected (see for example 5.1 of [20]) and so $\mathrm{TH}_{[n]}(f)$ is $(k-1)$-connected for each simplicial dimension $[n]$ and hence $\mathrm{TH}(f)$ is $(k-1)$-connected. Since homotopy orbits preserve connectivity, for a fixed prime p, it follows from the fundamental cofibration sequences

$$\mathrm{TH}(R)_{hC_{p^n}} \to \mathrm{TH}(R)^{C_{p^n}} \xrightarrow{R_p} \mathrm{TH}(R)^{C_{p^{n-1}}}$$

that $\mathrm{holim}_{R_p} \mathrm{TH}(f)^{C_{p^n}}$ is $(k-1)$-connected. Thus $\mathrm{TC}(f;p)$ (which is the homotopy fiber of $1-F_p$ acting on $\mathrm{holim}_{R_p} \mathrm{TH}(f)^{C_{p^n}}$) is $(k-2)$-connected and hence $\mathrm{TC}(f)$ is $(k-2)$-connected by [16, Theorem 3.1].

Now we establish Condition 2 for TC. What we prove for TC is actually stronger than Condition 2 since we show that TC always commutes with realizations. We note in contrast that algebraic K-theory does not, in general, have this additional property.

LEMMA 1.2. *For* A_* *a simplicial ring,* $\mathrm{TC}(A_*)$ *is naturally equivalent to*

$$|[n] \mapsto \mathrm{TC}(A_n)|.$$

Proof. For each simplicial dimension $[k]$, $|[n] \mapsto \mathrm{TH}_{[k]}(A_n)| \xrightarrow{\simeq} \mathrm{TH}_{[k]}(A_*)$ and hence

$$|[n] \mapsto \mathrm{TH}(A_n)| \xrightarrow{\simeq} \mathrm{TH}(A_*).$$

Since homotopy orbits commute with realization, by inducting the fundamental cofibration sequences we see that

$$|[n] \mapsto \mathrm{TH}(A_n)^{C_{p^k}}| \xrightarrow{\simeq} \mathrm{TH}(A_*)^{C_{p^k}}$$

for all primes p and natural numbers k. In general, inverse limits do not commute with realizations but the directed inverse limit of functors to connective spectra (of weak CW-type) which do commute with realizations again commutes with realizations (essentially because for connective spectra finite homotopy colimits commute with homotopy inverse limits and $\lim^{(n)}=0$ when $n \geqslant 2$ for directed inverse limits) and hence

$$|[n] \mapsto \underset{R_p}{\mathrm{holim}}\, \mathrm{TH}(A_n)^{C_{p^k}}| \xrightarrow{\simeq} \underset{R_p}{\mathrm{holim}}\, \mathrm{TH}(A_*)^{C_{p^k}}.$$

Thus, $\mathrm{TC}(A_*;p)$, which is the homotopy fiber of $1-F_p$ of this inverse limit (see [16, 3.1.1]), can be computed degreewise and hence by Theorem 3.1 of [16] so can $\mathrm{TC}(A_*)$.

Now we examine the analyticity of $\mathrm{TC}(A\ltimes\widetilde{M}[-])$. We recall from [14, §4] that a homotopy functor F is *ϱ-analytic* if there is some number q such that F satisfies $E_n(n\varrho-q,\varrho+1)$ for all $n\geqslant 1$. Recall that F satisfies $E_n(c,k)$ if $\mathcal{X}:\mathcal{P}(S)\to\mathcal{C}$ is any strongly co-Cartesian $(n+1)$-cube (every face is co-Cartesian, see [14, 2.1]) such that for all $s\in S$ the map $\mathcal{X}(\varnothing)\to\mathcal{X}(s)$ is k_s-connected and $k_s\geqslant k$, then the diagram $F(\mathcal{X})$ is $(-c+\Sigma k_s)$-Cartesian. We simply write $E_n(c)$ for $E_n(c,-1)$.

PROPOSITION 1.3. *If* $\mathrm{TH}(A\ltimes\widetilde{M}[-])$ *satisfies* $E_n(-n)$ *for all* $n\geqslant 1$ *then for all primes* p, $\mathrm{TC}(A\ltimes\widetilde{M}[-])^{\widehat{}}_p$ *is* (-1)*-analytic.*

Proof. Since homotopy orbits preserve homotopy fibrations and connectivity, $\mathrm{TH}(A\ltimes\widetilde{M}[-])_{hC}$ also satisfies $E_n(-n)$ for all finite subgroups C of S^1. Induction with respect to k on the fundamental cofibration sequences shows that $\mathrm{TH}(A\ltimes\widetilde{M}[-])^{C_{p^k}}$ satisfies $E_n(-n)$ for all n. Thus, $\mathrm{holim}_{R_p}\mathrm{TH}(A\ltimes\widetilde{M}[-])^{C_{p^k}}$ also satisfies $E_n(-n)$ for all n. Since $\mathrm{TC}(A\ltimes\widetilde{M}[-];p)$ is the homotopy fiber of $1-F_p$ acting on this, it satisfies $E_n(1-n)$ for all n. Since p-completion preserves fibrations and connectivity, $\mathrm{TC}(A\ltimes\widetilde{M}[-];p)^{\widehat{}}_p$ satisfies $E_n(1-n)$ also. By [16, Theorem 3.1], the natural map $\mathrm{TC}(F)\to\mathrm{TC}(F;p)$ is an equivalence after p-completion for all FSP's F and primes p so $\mathrm{TC}(A\ltimes\widetilde{M}[-])^{\widehat{}}_p$ also satisfies $E_n(1-n)$.

In order to study the analyticity of $\mathrm{TH}(A\ltimes\widetilde{M}[-])$ we first establish some notation for rewriting it into its homogeneous pieces. This rewriting will also be used extensively in §2.

Notation 1.4. Following [15], we let $A\vee M$ be the FSP defined by setting $A\vee M(X)=\tilde{A}[X]\vee\widetilde{M}[X]$ with multiplication

$$\begin{aligned}A\vee M(X)\wedge A\vee M(Y)&\xrightarrow{\alpha}\tilde{A}[X]\wedge\tilde{A}[Y]\vee\tilde{A}[X]\wedge\widetilde{M}[Y]\vee\widetilde{M}[X]\wedge\tilde{A}[Y]\vee\widetilde{M}[X]\wedge\widetilde{M}[Y]\\&\xrightarrow{\beta}\tilde{A}[X\wedge Y]\vee\widetilde{M}[X\wedge Y]\vee\widetilde{M}[X\wedge Y]\xrightarrow{\gamma}A\vee M[X\wedge Y].\end{aligned}$$

The first map α is the canonical homeomorphism, the second map β is $\mu_{X,Y}\vee l_{X,Y}\vee r_{X,Y}\vee *$ and the third map γ is $1\vee\mathrm{fold}$. The unit in $A\vee M$ is the composite $X\xrightarrow{1_X}\tilde{A}[X]\to A\vee M(X)$. It is straightforward to check that $A\vee M$ is an FSP.

We define a morphism of FSP's from $A\vee M$ to $A\ltimes M$ by

$$\tilde{A}[X]\vee\widetilde{M}[X]\xrightarrow{\mathrm{inc}}\tilde{A}[X]\times\widetilde{M}[X]\cong\widetilde{A\ltimes M}[X].$$

Using the canonical homeomorphisms which permute smash products and wedge sums, we obtain a natural map of cyclic spaces from $\bigvee_{a=0}^{\infty}\mathrm{T}_a(A;M)$ to $\mathrm{T}(A\vee M)$ where

$\mathrm{T}_a(A;M)$ is the cyclic subspace of $\mathrm{TH}(A\vee M)$ determined by the a-homogeneous part. That is,

$$\mathrm{T}_a(A;M)_k = \operatorname*{holim}_{I^{k+1}} \Omega^{x_0\sqcup\ldots\sqcup x_k} \bigvee_{0\leqslant j_1<\ldots<j_a\leqslant k} (\widetilde{F_0}[S^{x_0}]\wedge\ldots\wedge\widetilde{F_k}[S^{x_k}]),$$

$$F_t = \begin{cases} M & \text{if } t\in\{j_1,\ldots,j_a\}, \\ A & \text{otherwise.}\end{cases}$$

We see that for all $n\geqslant 0$ the natural inclusion from $\bigvee_{a=0}^{\infty}\mathrm{T}_a(A;M)$ to $\mathrm{T}(A\ltimes M)$ is a map of cyclic spectra and by Theorem 2.1 of [15] this natural inclusion is an equivalence of cyclotomic spectra.

PROPOSITION 1.5. *Let F be a simplicial object in the category of reduced homotopy functors from pointed spaces to (-1)-connective spectra such that*

(i) *$F([v])$ satisfies $E_n(c,\varkappa)$ for all $\varkappa$,*

(ii) *$F([v])=*$ for $0\leqslant v\leqslant q-1$.*

Then $|F|$ is a reduced homotopy functor from pointed spaces to spectra which satisfies $E_n(c-q,\varkappa)$ for all $\varkappa$.

Proof. Let $\mathcal{X}$ be a strongly co-Cartesian $(n+1)$-cube such that $\mathcal{X}(\varnothing)\to\mathcal{X}(s)$ is $k_s(\geqslant\varkappa)$-connected for all $s\in S$. By condition (i), $F([v])(\mathcal{X})$ is a $(-c+\Sigma k_s)$-Cartesian cube of spectra which is a $(n-c+\Sigma k_s)$-co-Cartesian cube of spectra. Since homotopy colimits commute and realization (as we are using it here) is a homotopy colimit the iterated homotopy cofiber is the realization of the simplicial iterated homotopy cofibers degreewise. This simplicial spectrum is q-reduced (i.e. does not have cells in dimensions $<q$) by condition (ii) and is $(n-c+\Sigma k_s)$-connected in all other simplicial dimensions—so it is at least $(n-c+q+\Sigma k_s)$-connected. Thus, $|F|(\mathcal{X})$ is $(n-c+q+\Sigma k_s)$-co-Cartesian and hence $(-c+q+\Sigma k_s)$-Cartesian and $|F|$ satisfies $E_n(c-q,\varkappa)$.

COROLLARY 1.6. $\mathrm{TH}(A\ltimes\widetilde{M}[-])$ *satisfies $E_n(-n)$ for all $n\geqslant 1$.*

Proof. We first note that $\mathrm{T}_a(A;M)$ is $(a-1)$-reduced for all a and hence it is at least $(a-2)$-connected. Thus, $\mathrm{TH}(A\ltimes M)$ is naturally homotopy equivalent to $\prod_a \mathrm{T}_a(A;M)$ and it suffices to show that $\mathrm{T}_a(A;\widetilde{M}[-])$ satisfies $E_n(-n)$ for all $n\geqslant 1$. Fix a. In each simplicial dimension $v\geqslant a-1$, $\mathrm{T}_a(A;\widetilde{M}[-])_v$ is naturally equivalent to

$$X\mapsto \bigvee_{0\leqslant j_1<\ldots<j_a\leqslant v} HA^{\wedge(v+1-a)}\wedge HM^{\wedge a}\wedge\Sigma^\infty[X^{\wedge a}].$$

The functor $X\mapsto\Sigma^\infty[X^{\wedge a}]$ satisfies $E_n(c)$ for any c when $n\geqslant a$ and $E_n(0)$ when $n<a$ by the argument of [14, 4.4]. Thus $\mathrm{T}_a(A;\widetilde{M}[-])_v$ satisfies $E_n(c)$ for any c when $n\geqslant a$ and $E_n(0)$ when $n<a$ also since it is the finite coproduct of functors obtained by composing

$X \mapsto \Sigma^\infty X^{\wedge a}$ with linear functors (smashing with another spectrum). By Proposition 1.5, $\mathrm{T}_a(A;\widetilde{M}[-])$ satisfies $E_n(c)$ for any c when $n \geqslant a$ and $E_n(0+1-a)$ when $n<a$, and hence $E_n(-n)$ for all $n \geqslant 1$.

2. The p-limit axiom for $\widetilde{\mathrm{TC}}(A \ltimes \widetilde{M}[-])$

If F is a homotopy functor from spaces to p-complete spectra, then one says that F satisfies the *p-limit axiom* if, for every CW-complex X, the mod p homotopy groups of $F(X)$ are colimits of mod p homotopy groups of $F(X_\alpha)$, indexed by the finite subcomplexes $X_\alpha \subseteq X$. To show the p-limit axiom for $\widetilde{\mathrm{TC}}(A \vee \widetilde{M}[-])$ we will first make a series of reductions and observations. To ease our notation we will write T_a (or T) for the functor $\mathrm{T}_a(A;-)$ (or $\mathrm{T}(A\vee -)$) from A-bimodules to cyclotomic spectra as defined in Notation 1.4. We also recall that after p-completion, the natural map from TC to $\mathrm{TC}(-;p)$ is an equivalence. Thus, it suffices to show that $\mathrm{TC}(-;p)$ satisfies the p-limit axiom and we will write the functor $\mathrm{TC}(A\vee -;p)$ simply as $\mathrm{TC}(p)$.

General Results 2.1. *Let k be relatively prime to p. Then:*

(a) *T_{kp^n} is at least (kp^n-1)-connected.*

(b) $T^{C_{p^r}} \simeq \bigvee_{a \geqslant 0} \mathrm{T}_a^{C_{p^r}}$.

(c) $F_p\colon \mathrm{T}_{kp^n}^{C_{p^r}} \to \mathrm{T}_{kp^n}^{C_{p^{r-1}}}$.

(d) $R_p\colon \mathrm{T}_{kp^n}^{C_{p^r}} \to \mathrm{T}_{kp^{n-1}}^{C_{p^{r-1}}}$ *(where p^{-1} implies $*$).*

(e) *For all n there is a natural cofibration sequence of spectra*

$$(\mathrm{T}_{kp^n})_{hC_{p^r}} \to \mathrm{T}_{kp^n}^{C_{p^r}} \xrightarrow{R_p} \mathrm{T}_{kp^{n-1}}^{C_{p^{r-1}}}.$$

(f) *Let tr_p be a representative for the transfer map from $(\mathrm{T}_{kp^n})_{hC_{p^r}}$ to $(\mathrm{T}_{kp^n})_{hC_{p^{r-1}}}$. Then the following diagram of cofibration sequences commutes up to homotopy:*

$$\begin{tikzcd}
(\mathrm{T}_{kp^n})_{hC_{p^r}} \arrow[r] \arrow[d, "\mathrm{tr}_p"] & \mathrm{T}_{kp^n}^{C_{p^r}} \arrow[r, "R_p"] \arrow[d, "F_p"] & \mathrm{T}_{kp^{n-1}}^{C_{p^{r-1}}} \arrow[d, "F_p"] \\
(\mathrm{T}_{kp^n})_{hC_{p^{r-1}}} \arrow[r] & \mathrm{T}_{kp^n}^{C_{p^{r-1}}} \arrow[r, "R_p"] & \mathrm{T}_{kp^{n-1}}^{C_p^{r-2}}.
\end{tikzcd}$$

(g) *There is a natural map from $(\Sigma \mathrm{T}_{kp^n})_{hS^1}$ to the homotopy inverse limit of $(\mathrm{T}_{kp^n})_{hC_{p^r}}$ obtained from the transfer maps tr_p which is an equivalence after taking p-completion.*

Proof. Part (a) follows from the fact that T_{kp^n} is (kp^n-1)-reduced, (b)–(e) follow from §§ 2.1 and 2.2 of [15]. We obtain (f) from Lemma 3.5 of [16] and (g) (in our generality) is due to Goodwillie and can be found as Lemma 4.4.9 (or Remark 4.4.10) in [18].

For k relatively prime to p, we write $\mathrm{T}(k)$ for $\bigvee_{r\geqslant 0}\mathrm{T}_{kp^r}$ and hence $T\simeq\bigvee_{(k,p)=1}\mathrm{T}(k)$, and by (c) and (d) this decomposition respects both structure maps F_p and R_p. We note that by (a), $\mathrm{T}(k)$ is at least $(k-1)$-connected. Since homotopy orbits preserve connectivity, (a) and induction with (e) imply that $\mathrm{T}_{kp^n}^{C_{p^r}}$ is at least $(kp^{n-r}-1)$-connected for all k, n and r, and so $\mathrm{T}(k)^{C_p^r}$ is also at least $(k-1)$-connected.

LEMMA 2.2. $\mathrm{TC}(p)=[\operatorname{holim}_{R_p}T^{C_{p^r}}]^{hF_p}\simeq\bigvee_{(k,p)=1}[\operatorname{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}]^{hF_p}$.

Proof. By (a) and (e), $\operatorname{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}$ is at least $(k-1)$-connected, and hence $[\operatorname{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}]^{hF_p}$ is at least $(k-2)$-connected. Thus, (using (b)) the following natural maps are weak equivalences:

$$\Big[\operatorname*{holim}_{R_p} T^{C_{p^r}}\Big]^{hF_p}\simeq\Big[\operatorname*{holim}_{R_p}\bigvee_{(k,p)=1}\mathrm{T}(k)^{C_{p^r}}\Big]^{hF_p}\xrightarrow{\simeq}\left[\operatorname*{holim}_{R_p}\prod_{(k,p)=1}\mathrm{T}(k)^{C_{p^r}}\right]^{hF_p},$$

$$\bigvee_{(k,p)=1}\Big[\operatorname*{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}\Big]^{hF_p}\xrightarrow{\simeq}\prod_{(k,p)=1}\Big[\operatorname*{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}\Big]^{hF_p}.$$

This proves the lemma since homotopy inverse limits commute.

To prove the p-limit axiom, it suffices to show that $\mathrm{TC}(p)$ commutes with filtered direct limits after p-completion. Since homotopy colimits commute, we see by Lemma 2.2 that

$$\begin{aligned}\operatorname{holim}\mathrm{TC}(p)&\simeq\operatorname{holim}\bigvee_{(k,p)=1}\Big[\operatorname*{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}\Big]^{hF_p}\\&\simeq\bigvee_{(k,p)=1}\operatorname{holim}\Big[\operatorname*{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}\Big]^{hF_p},\end{aligned}$$

and hence it suffices to show that $[\operatorname{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}]^{hF_p}$ satisfies the p-limit axiom for all fixed k relatively prime to p. Thus, we now fix k.

We now recall an observation of Goodwillie (see for example 3.1.1 of [16]) which allows us to interchange to roles of R_p and F_p:

$$\Big[\operatorname*{holim}_{R_p}\mathrm{T}(k)^{C_{p^r}}\Big]^{hF_p}\simeq\Big[\operatorname*{holim}_{F_p}\mathrm{T}(k)^{C_{p^r}}\Big]^{hR_p}.$$

LEMMA 2.3. *For each fixed n, the functor $\operatorname{holim}_{F_p}\mathrm{T}_{kp^n}^{C_{p^r}}$ satisfies the p-limit axiom.*

Proof. We will induct on n. First, the case $n=0$. By observation (e), we have cofibration sequences

$$(\mathrm{T}_k)_{hC_{p^r}}\to\mathrm{T}_k^{C_{p^r}}\xrightarrow{R_p}*,$$

and since homotopy inverse limits preserve cofibration sequences of spectra we get by observations (f) and (g) a map

$$(\Sigma \mathrm{T}_k)_{hS^1} \to \operatorname*{holim}_{\mathrm{tr}_p} (\mathrm{T}_k)_{hC_{p^r}} \xrightarrow{\simeq} \operatorname*{holim}_{F_p} \mathrm{T}_k^{C_{p^r}}$$

which is an equivalence after taking p-completions. The functor $(\Sigma \mathrm{T}_k)_{hS^1}$ commutes with direct limits because T_k does and homotopy colimits commute.

Since the p-completion of a functor which satisfies the limit axiom satisfies the p-limit axiom (bottom of [14, p. 328]), we see that $\operatorname{holim}_{F_p} \mathrm{T}_k^{C_{p^r}}$ satisfies the p-limit axiom. Now assume that the lemma is true for the case $n-1$. We again obtain a cofibration of spectra

$$\operatorname*{holim}_{\mathrm{tr}_p} (\mathrm{T}_{kp^n})_{hC_{p^r}} \to \operatorname*{holim}_{F_p} \mathrm{T}_{kp^n}^{C_{p^r}} \xrightarrow{R_p} \operatorname*{holim}_{F_p} \mathrm{T}_{kp^{n-1}}^{C_{p^{r-1}}}.$$

The left-hand term satisfies the p-limit axiom since it is equivalent to $(\Sigma \mathrm{T}_{kp^n})_{hS^1}$ after p-completion and T_{kp^n} commutes with direct limits. The term on the right is equivalent to the case $n-1$ by cofinality and hence satisfies the p-limit axiom. Since p-completion preserves cofibrations of spectra and homotopy colimits commute we are done.

Since $\mathrm{T}_{kp^n}^{C_{p^r}}$ is at least $(kp^{n-r}-1)$-connected, for a fixed r, $\bigvee_{n\geqslant 0} \mathrm{T}_{kp^n}^{C_{p^r}} \xrightarrow{\simeq} \prod_{n \geqslant 0} \mathrm{T}_{kp^n}^{C_{p^r}}$, and thus,

$$\begin{aligned}\big[\operatorname*{holim}_{F^p} \mathrm{T}(k)^{C_{p^r}}\big]^{hR_p} &\simeq \big[\operatorname*{holim}_{F_p} \bigvee_{n\geqslant 0} \mathrm{T}_{kp^n}^{C_{p^r}}\big]^{hR_p} \\ &\simeq \left[\operatorname*{holim}_{F_p} \prod_{n\geqslant 0} \mathrm{T}_{kp^n}^{C_{p^r}}\right]^{hR_p} \simeq \left[\prod_{n\geqslant 0} \operatorname*{holim}_{F_p} \mathrm{T}_{kp^n}^{C_{p^r}}\right]^{hR_p}.\end{aligned}$$

If we write H_n for the functor $\operatorname{holim}_{F_p} \mathrm{T}_{kp^n}^{C_{p^r}}$ then the action of R_p takes H_n to H_{n-1} and we get

$$\big[\operatorname*{holim}_{F_p} \mathrm{T}(k)^{C_{p^r}}\big]^{hR_p} \simeq \left[\prod_{n\geqslant 0} H_n\right]^{hR_p} \simeq \operatorname*{holim}_{R_p} H_n.$$

We still need to know if we can pass homotopy colimits pass this homotopy inverse limit after p-completion. The fiber of $H_n \xrightarrow{R_p} H_{n-1}$ is naturally equivalent to $(\Sigma \mathrm{T}_{kp^n})_{hS^1}$ after p-completion which is at least kp^n-connected. Thus, the natural map from $\operatorname{holim}_{R_p} H_n$ to H_n is at least kp^n-connected after p-completion. By Lemma 2.3 the functors H_n satisfy the p-limit axiom for all n and hence $\operatorname{holim}_{R_p} H_n$ satisfies the p-limit axiom also.

3. The functor $\mathrm{K}(A;M)$ and the analyticity of $\mathrm{K}(A\ltimes\widetilde{M}[-])$

We need to recall some of the constructions and results of [7]. We let A be a ring, $\mathcal{P}$ its exact category of finitely generated projective right A-modules and $\mathcal{M}$ its exact category of right modules.

Definition 3.1. For M an A-bimodule, we define $\mathrm{K}(A;M)$ to be

$$\mathrm{K}(A;M)=\Omega\left|\coprod_{\overline{C}\in \mathrm{S}_{\bullet}\mathcal{P}}\mathrm{Hom}_{\mathrm{S}_{\bullet}\mathcal{M}}(\overline{C},\overline{C}\otimes_A M)\right|$$

where S is Waldhausen's S-construction for algebraic K-theory (see [21]). We note that $\mathrm{K}(A;0)\cong\mathrm{K}(A)$, $\mathrm{K}(A;A)\cong\mathrm{K}(\mathrm{End}(\mathcal{P}))$ and that $\mathrm{K}(A;-)$ is a functor of A-bimodules. We also note that $\mathrm{K}(A;M)$ is the usual algebraic K-theory for the exact category with objects the pairs (P,α) consisting of $P\in\mathcal{P}$ and α an A-module homomorphism from P to $P\otimes_A M$ with morphisms

$$\mathrm{Hom}((P,\alpha),(Q,\beta))=\{f\in\mathrm{Hom}_A(P,Q)\mid \beta\circ f=(f\otimes\mathrm{id}_M)\circ\alpha\}.$$

A sequence $(P'',\alpha'')\to(P,\alpha)\to(P',\alpha')$ is exact if and only if $P''\to P\to P'$ is an exact sequence in $\mathcal{P}$.

We extend $\mathrm{K}(A;-)$ to simplicial A-bimodules degreewise. That is, for $M_{\bullet}$ a simplicial A-bimodule, $\mathrm{K}(A;M_{\bullet})$ is the realization of the simplicial space $[n]\to\mathrm{K}(A;M_n)$. Since $|\coprod_{\overline{C}\in\mathrm{S}_{\bullet}\mathcal{P}}\mathrm{Hom}_{\mathrm{S}_{\bullet}\mathcal{M}}(\overline{C},\overline{C}\otimes_A M)|$ is connected, $\mathrm{K}(A;M_{\bullet})$ can equivalently be defined as the loop space of the realization of the associated bi-simplicial set. By the realization lemma, $\mathrm{K}(A;\widetilde{M}[-])$ is a *homotopy functor*: taking homotopy-equivalent spaces to homotopy-equivalent spaces.

We define $\widetilde{\mathrm{K}}(A;\widetilde{M}[X])$ to be the (homotopy) fiber of the natural retraction from $\mathrm{K}(A;\widetilde{M}[X])$ to $\mathrm{K}(A;\widetilde{M}[*])=\mathrm{K}(A)$. Since $\mathrm{K}(A;\widetilde{M}[X])$ is an infinite loop space and the map in question is a map of infinite loop spaces, we see that $\mathrm{K}(A;\widetilde{M}[X])$ is weakly homotopic to $\mathrm{K}(A)\times\widetilde{\mathrm{K}}(A;\widetilde{M}[X])$. We note that $\widetilde{\mathrm{K}}(A;\widetilde{M}[X])$ is naturally equivalent to $\lim_{n\to\infty}\Omega^n\bigvee_{\overline{P}\in\mathrm{S}_{\bullet}^{(n)}\mathcal{P}}(\mathrm{Hom}_{\mathrm{S}_{\bullet}^{(n)}\mathcal{P}}(\overline{P},\overline{P}))^{\sim}[X]$ since this is the underlying space of the cofiber (as spectra) of the natural section from $\mathrm{K}(A)$ to $\mathrm{K}(A;\widetilde{M}[X])$.

We are interested in the functor $\mathrm{K}(A;M)$ because of its relationship with $\mathrm{K}(A\ltimes M)$ which we now recall from [7]. We let $\widetilde{\mathrm{K}}(A\ltimes M)$ be the fiber of the natural map from $\mathrm{K}(A\ltimes M)$ to $\mathrm{K}(A)$ produced by the ring homomorphism sending (a,m) to a. This is a map of infinite loop spaces with a section so $\mathrm{K}(A\ltimes M)$ is weakly homotopic to $\mathrm{K}(A)\times\widetilde{\mathrm{K}}(A\ltimes M)$. We write $\mathrm{B}_{\bullet}M$ for the bar construction naturally considered as a simplicial A-bimodule; in particular $\mathrm{K}(A;\mathrm{B}_{\bullet}M)\cong\mathrm{K}(A;\widetilde{M}[S^1])$.

THEOREM ([7, 4.1]). *For any simplicial A-bimodule M, there exists a natural weak homotopy equivalence*

$$\Psi(A,M)\colon \mathrm{K}(A;\mathrm{B}_{\bullet}M) \xrightarrow{\simeq} \mathrm{K}(A\ltimes M)$$

which factors to give a homotopy equivalence $\widetilde{\Psi}(A,M)$ *from* $\widetilde{\mathrm{K}}(A,\mathrm{B}_{\bullet}M)$ *to* $\widetilde{\mathrm{K}}(A\ltimes M)$.

PROPOSITION 3.2. *The homotopy functor* $X\mapsto \mathrm{K}(A;\widetilde{M}[X])$ *is* 0*-analytic. Thus, the homotopy functor* $\mathrm{K}(A\ltimes\widetilde{M}[-])$ *is* (-1)*-analytic since it is equivalent to the composition of the suspension functor followed by the* 0*-analytic functor* $\mathrm{K}(A;\widetilde{M}[-])$. *Since p-completion preserves analyticity,* $\mathrm{K}(A\ltimes\widetilde{M}[-])^{\widehat{}}_p$ *is also* (-1)*-analytic for all primes p.*

Proof. We show that for each q the functor $\bigvee_{\bar{P}\in S^q_{\bullet}\mathcal{P}}(\mathrm{Hom}_{S^q_{\bullet}\mathcal{P}}(\bar{P},\bar{P}))^{\widetilde{}}[-]$ takes strongly co-Cartesian $(n+1)$-cubes to $(q+n+\Sigma_i x_i)$-co-Cartesian cubes. By taking Ω^q of these and the limit with respect to q we will obtain an $(n+\Sigma_i x_i)$-co-Cartesian diagram of spectra which is equivalent to a $(\Sigma_i x_i)$-Cartesian diagram of spectra (see [14, Remark 1.19]) and hence the result.

Let $\mathcal{X}$ be a strongly co-Cartesian $(n+1)$-cube of spaces. We may assume that the natural maps are inclusions of sub-simplicial sets. Suppose also that the maps $\mathcal{X}(\varnothing)\to\mathcal{X}(\{i\})$ are x_i-connected. Thus, the maps from $\mathcal{X}(\{0,\dots,n\}-\{i\})$ to $\mathcal{X}(\{0,\dots,n\})$ are also x_i-connected. For G any abelian group, the functor $\widetilde{G}[-]$ is linear and preserves connectivity, thus $\widetilde{G}[\mathcal{X}]$ is a strongly Cartesian cube with $\widetilde{G}[\mathcal{X}(\{0,\dots,n\}-\{i\})]$ to $\widetilde{G}[\mathcal{X}(\{0,\dots,n\})]$ being x_i-connected. By Theorem 1.4 of [14], we see that $\widetilde{G}[\mathcal{X}]$ is an $(n+\Sigma_i x_i)$-co-Cartesian cube.

If X is a simplicial subset of Y, then the cofiber of the inclusion map is just the degreewise quotient Y_n/X_n. Thus,

$$\mathrm{cofiber}\langle \bigvee_{\bar{P}\in S^q_{\bullet}\mathcal{P}}(\mathrm{Hom}_{S^q_{\bullet}\mathcal{P}}(\bar{P},\bar{P}))^{\widetilde{}}[\mathcal{X}]\rangle \simeq \bigvee_{\bar{P}\in S^q_{\bullet}\mathcal{P}}\mathrm{cofiber}\langle(\mathrm{Hom}_{S^q_{\bullet}\mathcal{P}}(\bar{P},\bar{P}))^{\widetilde{}}[\mathcal{X}]\rangle$$

which is $(q+n+\Sigma_i x_i)$-connected (since a q-reduced simplicial space of t-connected spaces is $(q+t)$-connected).

4. Connectivity of the p-completed relative trace

For X a pointed set and E a spectrum we let $X\wedge E$ be the new spectrum determined by $(X\wedge E)_n = X\wedge E_n$. For a pointed simplicial set $X_{\bullet}$ we let $X_{\bullet}\otimes E$ be the resulting simplicial spectrum obtained by $[n]\mapsto X_n\wedge E$.

Let $\Delta[1]$ be the simplicial set $\mathrm{Hom}_{\Delta}(-,[1])$ and let S^1 be the simplicial set $\Delta[1]/\partial\Delta[1]$. Thus, the realization of S^1 is the standard CW-decomposition of the circle with one vertex. One can realize the rotation action of S^1 by a simplicial model by observing that S^1 is a cyclic set (see for example [17]). We note that S^1 is a based simplicial set with basepoint 0 and that S^1_+ (obtained by adding a disjoint basepoint $+$) is a based cyclic set. We note that $\widetilde{M}[S^1]$ is the bar construction $\mathrm{B}_\bullet M$ of the abelian group M and that $\widetilde{M}[S^1_+]$ is the cyclic bar construction $\mathrm{N}^{\mathrm{cy}}M$ of M. We recall that if $Y_\bullet$ is a cyclic set then (using the diagonal) there is a simplicial map $S^1_+\wedge Y_\bullet \xrightarrow{\mathrm{ev}} Y_\bullet$ which realizes the usual circle action on $|Y|$. For any spectrum E we have a split cofibration sequence

$$S^0\otimes E \to S^1_+\otimes E \xrightarrow{\pi} S^1\otimes E$$

obtained by identifying $+$ with 0.

We observe that $S^1_+\otimes\mathrm{TH}(A;M)\simeq\mathrm{T}_1(A;M)$ by extending the simplicial inclusion map $\mathrm{TH}(A;M)\to\mathrm{T}_1(A;M)$ to a free cyclic one. We let α be the natural composite map of spectra (in the homotopy category)

$$\widetilde{\widetilde{\mathrm{TH}}}(A\ltimes M)\simeq\widetilde{\widetilde{\mathrm{TH}}}(A\vee M)\xrightarrow{\varrho}\mathrm{T}_1(A;M)\simeq S^1_+\otimes\mathrm{TH}(A;M)\xrightarrow{\pi} S^1\otimes\mathrm{TH}(A;M),$$

where ϱ is the projection map.

THEOREM 4.1. *For M an m-connected simplicial A-bimodule the cyclotomic trace*

$$\widetilde{\mathrm{K}}(A\ltimes M)\xrightarrow{\mathrm{trc}}\widetilde{\widetilde{\mathrm{TC}}}(A\ltimes M)$$

is $2m$-connected after p-completion for all primes p.

Proof. We consider the diagram

$$\begin{tikzcd}
\widetilde{\mathrm{K}}(A\ltimes M) \arrow[rr, "\mathrm{trc}"] \arrow[dr, "\mathrm{tr}"'] & & \widetilde{\widetilde{\mathrm{TC}}}(A\ltimes M) \arrow[dl, "\mathrm{res}"] \\
& \widetilde{\widetilde{\mathrm{TH}}}(A\ltimes M) \arrow[d, "\alpha"] & \\
& S^1\otimes\mathrm{TH}(A;M), &
\end{tikzcd}$$

where tr is the usual Dennis trace map and res is the restriction map. The triangle is known to commute up to homotopy (see for example [18, §2.6]) and the right-hand composite $\alpha\circ\mathrm{res}$ is $2m$-connected after p-completion by the main result of [15] (see the proof of the result). The natural map η from $S^1\otimes\mathrm{TH}(A;M)$ into $\mathrm{TH}(A;\widetilde{M}[S^1])=\mathrm{TH}(A;\mathrm{B}_\bullet M)$ (obtained by including wedges into products) is an equivalence and by Theorems 3.4

and 4.1 of [7], the spectra $\widetilde{\mathrm{K}}(A\ltimes M)$ and $\mathrm{TH}(A;\mathrm{B}_{\bullet}M)$ are $(2m+2)$-equivalent by a natural map. Thus, once we establish that the composite $\eta\circ\alpha\circ\mathrm{tr}$ actually is a $2m$-connected map we will be finished. The rest of the paper is devoted to establishing this fact.

Let τ be the composite map

$$S^1_+\otimes\mathrm{TH}(A;M)\xrightarrow{\simeq}\mathrm{T}_1(A;M)\to\widetilde{\widetilde{\mathrm{TH}}}(A\ltimes M).$$

Thus, τ is $2m$-connected and $\alpha\circ\tau\simeq\pi$. By the Blakers–Massey theorem, the natural map (obtained by sending wedges to products) from $S^1_+\otimes\widetilde{\mathrm{K}}(A;M)\to\widetilde{\mathrm{K}}(A;\mathrm{N}^{\mathrm{cy}}M)$ is at least $2m$-connected. The natural fibration $\mathrm{N}^{\mathrm{cy}}M\xrightarrow{\pi}\mathrm{B}M$ has a section σ defined by $\sigma(m_1,...,m_n)=((-1)\sum_{i=1}^{n}m_i,m_1,...,m_n)$. We also write σ for the natural map from $\widetilde{\mathrm{K}}(A;\mathrm{B}M)$ to $\widetilde{\mathrm{K}}(A;\mathrm{N}^{\mathrm{cy}}M)$ defined by $\widetilde{\mathrm{K}}(A;\sigma)$.

Proposition 4.2. *There exists a natural map (in the homotopy category)*

$$\Phi:\widetilde{\mathrm{K}}(A;\mathrm{N}^{\mathrm{cy}}M)\to\widetilde{\widetilde{\mathrm{TH}}}(A\ltimes M)$$

such the following two squares commute up to homotopy:

$$\begin{tikzcd}
\widetilde{\mathrm{K}}(A\ltimes M) \arrow[r, "\mathrm{tr}"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "(1)"] & \widetilde{\widetilde{\mathrm{TH}}}(A\ltimes M) \arrow[dr, phantom, "(2)"] & S^1_+\otimes\mathrm{TH}(A;M) \arrow[l, "\tau"'] \\
\widetilde{\mathrm{K}}(A;\mathrm{B}M) \arrow[r, "\sigma"] & \widetilde{\mathrm{K}}(A;\mathrm{N}^{\mathrm{cy}}M) \arrow[u, "\Phi"] & S^1_+\otimes\widetilde{\mathrm{K}}(A;M), \arrow[l] \arrow[u, "S^1_+\otimes\beta"']
\end{tikzcd}$$

where β is the natural transformation $\mathrm{K}(R;\cdot)\to\mathrm{TH}(R;\cdot)$ of [7, 3.4]. Though we will not need it, Φ will also be $2m$-connected and defined by a sequence of natural S^1-equivariant maps.

We note that once we have established Proposition 4.2 the result will follow from the following diagram (all unlabeled figures commute by naturality):

$$\begin{tikzcd}
\widetilde{\mathrm{K}}(A\ltimes M) \arrow[r, "\mathrm{tr}"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "(1)"] & \widetilde{\widetilde{\mathrm{TH}}}(A\ltimes M) \arrow[dr, phantom, "(2)"] & S^1_+\otimes\mathrm{TH}(A;M) \arrow[l, "\tau\,(\simeq 2m)"'] \arrow[dr, "="] & \\
\widetilde{\mathrm{K}}(A;\mathrm{B}M) \arrow[r, "\sigma"] \arrow[dr, "="'] & \widetilde{\mathrm{K}}(A;\mathrm{N}^{\mathrm{cy}}M) \arrow[u, "\Phi\,(\simeq 2m)"] \arrow[d, "\pi"] & S^1_+\otimes\widetilde{\mathrm{K}}(A;M) \arrow[l, "\simeq 2m"'] \arrow[u, "S^1_+\otimes\beta\,(\simeq 2m)"'] \arrow[r, "S^1_+\otimes\beta\,(\simeq 2m)"] \arrow[d, "\pi"] & S^1_+\otimes\mathrm{TH}(A;M) \arrow[d, "\pi"] \\
& \widetilde{\mathrm{K}}(A;\mathrm{B}M) & S^1\otimes\widetilde{\mathrm{K}}(A;M) \arrow[l, "\simeq 2m"'] \arrow[r, "S^1\otimes\beta\,(\simeq 2m)"] & S^1\otimes\mathrm{TH}(A;M)
\end{tikzcd}$$

Reduction. It is convenient to restrict our attention to free modules. It follows by cofinality and Nakayama's lemma that the homotopy types of both $\widetilde{K}(A\ltimes M)$ and

$\widetilde{K}(A; B.M)$ are not changed if we use only the subcategory $\mathcal{F}\subseteq\mathcal{P}$ of free modules (see for example p. 697 of [7]). We can also assume that $\mathcal{F}$ is an exact category of finitely generated free modules with one object for each nonnegative integer.

TH *of exact categories.* In order to construct Φ, we will be using the techniques developed in [8] (also outlined in §3.2 of [18]). For $\mathcal{A}$ a small category, the *cyclic nerve*, $\mathrm{N^{cy}}\mathcal{A}$, is the cyclic set

$$[n]\mapsto \coprod_{A_0,\ldots,A_n\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(A_1,A_0)\times\mathrm{Hom}_{\mathcal{A}}(A_2,A_1)\times\ldots\times\mathrm{Hom}_{\mathcal{A}}(A_0,A_n)$$

with operators like those for Hochschild homology. For $\mathcal{C}$ a small linear category, we can define a cyclic spectrum $\mathrm{TH}(\mathcal{C})$ where we use the $\mathrm{Hom}_{\mathcal{C}}$-abelian groups to form FSP's with several objects. In particular,

$$[n]\mapsto \operatorname*{holim}_{x_0,\ldots,x_n\in I^{\times n+1}} \mathrm{Map}\big(S^{x_0}\wedge\ldots\wedge S^{x_n}, \bigvee_{A_0,\ldots,A_n\in\mathcal{A}} (\mathrm{Hom}_{\mathcal{A}}(A_1,A_0))\tilde{}\,[S_0^x]\wedge\ldots\wedge\mathrm{Hom}_{\mathcal{A}}(A_0,A_n)\tilde{}\,[S^{x_n}]\big),$$

where I is the category of finite nonempty sets having one object for each isomorphism class and for $x\in I$, S^x is the sphere indexed on x. The operators are again like those for Hochschild homology (see [8, 1.3.6] for more details). There is a natural map from $\mathrm{N^{cy}}\mathcal{C}$ to $\mathrm{TH}(\mathcal{C})$ given by sending $\alpha_0\times\ldots\times\alpha_n$ to $\alpha_0\wedge\ldots\wedge\alpha_n$.

We can incorporate the S-construction of [21] into our construction TH as follows. For $\mathcal{C}$ an exact category, each $\mathrm{S}_n\mathcal{C}$ can also be considered as a category with the morphisms the natural transformations of functors. We can further consider this as an exact category by declaring a sequence $\overline{C}''\to\overline{C}\to\overline{C}'$ to be exact if the associated sequences for all $i\leqslant j$ are exact as sequences of $\mathcal{C}$. With these conventions, we can consider $\mathrm{S.}\mathcal{C}$ not only as a simplicial set but as a simplicial category or even as a simplicial exact category. We consider the composed functor $\mathrm{TH}(\mathrm{S}\mathcal{P})$. This is a cyclic simplicial space whose realization is naturally an S^1-space by first realizing the simplicial direction and then giving the realization of the resulting cyclic space its usual S^1-action. We list below several propositions whose proofs can be found in [8].

FACTS 4.3. *Let A be a ring, M an A-bimodule and $\mathcal{P}$ the exact category of finitely generated projective right A-modules. We will also write M for the functor $\mathrm{Hom}_A(-,-\otimes_A M)$.*

(1) [8, 2.1.5] *If we consider A as a category with one object $*$, then the natural linear functor from A to $\mathcal{P}$ (given by sending $*$ to $A\in\mathcal{P}$) produces a homotopy equivalence $\mathrm{TH}(A,M)\to\mathrm{TH}(\mathcal{P},M)$.*

(2) [8, 2.1.3] *For all $k \geqslant 0$ there is a natural homotopy equivalence*

$$|\mathrm{TH}(\mathcal{P},M)| \xrightarrow{\simeq} \Omega^k|\mathrm{TH}(\mathrm{S}^{(k)}\mathcal{P},M)|.$$

(3) [8, 2.2.3] *For all $k \geqslant 0$, the natural map by degeneracies*

$$\mathrm{TH}_0(\mathrm{S}^{(k)}\mathcal{P},M) \xrightarrow{\deg} \mathrm{TH}(\mathrm{S}^{(k)}\mathcal{P},M)$$

is $2k$-connected.

(4) [8, 2.1.6] *The trace map from $\mathrm{K}(A)$ to $\mathrm{TH}(A)$ can be recovered by the composite*

$$\mathrm{K}(A) = \Omega|\mathrm{S}\mathcal{P}| \xrightarrow{1} \Omega|\mathrm{TH}_0(\mathrm{S}\mathcal{P})| \to \Omega|\mathrm{TH}(\mathrm{S}\mathcal{P})| \xleftarrow{\simeq} |\mathrm{TH}(\mathcal{P})| \xleftarrow{\simeq} |\mathrm{TH}(A)|.$$

(5) [8, 2.1.1] *The inclusion functor gives an equivalence $\mathrm{TH}(\mathcal{F}) \xrightarrow{\simeq} \mathrm{TH}(\mathcal{P})$ and the trace map factors up to homotopy through the K-theory of $\mathcal{F}$ via the commuting diagram*

$$\begin{tikzcd}
\mathrm{K}(A) = \Omega|\mathrm{S}\mathcal{P}| \arrow[r, "1"] & \Omega|\mathrm{TH}_0(\mathrm{S}\mathcal{P})| \arrow[r] & \Omega|\mathrm{TH}(\mathrm{S}\mathcal{P})| & |\mathrm{TH}(\mathcal{P})| \arrow[l, "\simeq"'] & |\mathrm{TH}(A)| \arrow[l, "\simeq"'] \\
\Omega|\mathrm{S}\mathcal{F}| \arrow[u] \arrow[r, "1"] & \Omega|\mathrm{TH}_0(\mathrm{S}\mathcal{F})| \arrow[u, "\simeq"'] \arrow[r] & \Omega|\mathrm{TH}(\mathrm{S}\mathcal{F})| \arrow[u, "\simeq"'] & |\mathrm{TH}(\mathcal{F})| \arrow[l, "\simeq"'] \arrow[u, "\simeq"'] & |\mathrm{TH}(A)|. \arrow[l, "\simeq"'] \arrow[u, equal]
\end{tikzcd}$$

Suppose that we have a subcategory $t\mathcal{C}$ of $\mathcal{C}$ with the same set of objects and whose morphisms are always isomorphisms. Note that $\mathrm{N}_0 t\mathcal{C}$ is naturally isomorphic to $\mathcal{C}$ and that $\mathrm{N}_{\bullet} t\mathcal{C}$ can also be considered as a simplicial linear category with morphisms the appropriate commutative diagrams.

LEMMA 4.4 (after [21, 1.4.1]). *The natural maps by degeneracies $\mathrm{N}^{\mathrm{cy}}(\mathcal{C}) \to \mathrm{N}^{\mathrm{cy}}(\mathrm{N}_{\bullet} t\mathcal{C})$ and $\mathrm{TH}(\mathcal{C}) \to \mathrm{TH}(\mathrm{N}_{\bullet} t\mathcal{C})$ are homotopy equivalences.*

Proof. We do only the statement for TH as the other is similar. By the realization lemma, it suffices to show that $\mathrm{TH}(\mathcal{C}) \to \mathrm{TH}(\mathrm{N}_n t\mathcal{C})$ (given by $s = \mathrm{TH}(s_0 \ldots s_0)$) is an equivalence for all n. Fix n. The map s is a section to $d = \mathrm{TH}(d_0 \ldots d_0)$ and thus it suffices to show that $s \circ d$ is homotopic to the identity. This composite is equal to the map induced by the linear functor which takes $A_0 \xleftarrow{\alpha_1} \ldots \xleftarrow{\alpha_n} A_n$ in $\mathrm{N}_n t\mathcal{C}$ to $A_n = \ldots = A_n$. Since this functor is naturally isomorphic to the identity we are done by 1.6.2 of [8].

Now we define an inverse to the first equivalence in Lemma 4.4. We define a simplicial map ϕ from the diagonal of $\mathrm{N}^{\mathrm{cy}}(\mathrm{N}_{\bullet}t\mathcal{C})$ to $\mathrm{N}^{\mathrm{cy}}(\mathcal{C})$ as follows:

$$
\begin{array}{ccccccccc}
\bar{A}_0 & \xleftarrow{\bar{\gamma}_0} & \bar{A}_1 & \xleftarrow{\bar{\gamma}_1} \cdots \xleftarrow{\bar{\gamma}_{i-1}} & \bar{A}_i & \xleftarrow{\bar{\gamma}_i} \cdots \xleftarrow{\bar{\gamma}_n} & \bar{A}_0 \\
\| & & \| & & \| & & \| \\
\overbrace{A_{0,0}} & \xleftarrow{\gamma_0} & \overbrace{A_{0,1}} & \xleftarrow{\gamma_1} \cdots \xleftarrow{\gamma_{i-1}} & \overbrace{A_{0,i}} & \xleftarrow{\gamma_i} \cdots \xleftarrow{\gamma_n} & \overbrace{A_{0,0}} \\
\downarrow{\scriptstyle \alpha_1(0)} & & \downarrow{\scriptstyle \alpha_1(1)} & & \downarrow{\scriptstyle \alpha_1(i)} & & \downarrow{\scriptstyle \alpha_1(0)} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle \alpha_n(0)} & & \downarrow{\scriptstyle \alpha_n(1)} & & \downarrow{\scriptstyle \alpha_n(i)} & & \downarrow{\scriptstyle \alpha_n(0)} \\
\underbrace{A_{n,0}} & \longleftarrow & \underbrace{A_{n,1}} & \longleftarrow \cdots \longleftarrow & \underbrace{A_{n,i}} & \longleftarrow \cdots \longleftarrow & \underbrace{A_{n,0}}
\end{array}
$$

$$\Big\downarrow{\scriptstyle \phi}$$

$$A_{0,0} \xleftarrow{\beta_0} A_{n,1} \xleftarrow{\beta_1} A_{n-1,2} \xleftarrow{\beta_2} \cdots \xleftarrow{\beta_{n-1}} A_{1,n} \xleftarrow{\beta_n} A_{0,0},$$

$$\beta_i = \begin{cases} \gamma_0[\alpha_n(1) \dots \alpha_1(1)]^{-1} & \text{if } i=0, \\ [\alpha_{n-i+1}(i) \dots \alpha_1(i)]\, \gamma_i [\alpha_{n-i}(i+1) \dots \alpha_1(i+1)]^{-1} & \text{if } 1 \leqslant i \leqslant n-1, \\ \alpha_1(n)\gamma_n & \text{if } i=n. \end{cases}$$

It is straightforward to check that ϕ is a simplicial map and that the composite $\mathrm{N}^{\mathrm{cy}}(\mathcal{C}) \xrightarrow{\mathrm{deg}} \mathrm{N}^{\mathrm{cy}}(\mathrm{N}_{\bullet}t\mathcal{C}) \xrightarrow{\phi} \mathrm{N}^{\mathrm{cy}}(\mathcal{C})$ is the identity. Thus, ϕ is a homotopy inverse to the inclusion by degeneracies.

A key observation for the commuting of diagram (1) is that when $\mathcal{C}$ is a groupoid (every morphism is an isomorphism) then the diagram

$$
\begin{array}{ccc}
\mathrm{N}_{\bullet}\mathcal{C} & \xrightarrow{1} & \mathrm{N}_0^{\mathrm{cy}}\mathrm{N}_{\bullet}\mathcal{C} \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \mathrm{deg}} \\
\mathrm{N}_{\bullet}^{\mathrm{cy}}\mathcal{C} & \xleftarrow{\phi} & \mathrm{N}_{\bullet}^{\mathrm{cy}}\mathrm{N}_{\bullet}\mathcal{C}
\end{array}
$$

commutes where "1" is the map which takes every object of a category to its identity endomorphism as an element in $\mathrm{N}_0^{\mathrm{cy}}$ and σ is the map defined in Proposition 4.2. Observe that by Facts 4.3 (4) one can factor the trace map through the cyclic bar construction using the natural commuting diagram

$$\begin{tikzcd} \Omega|\mathrm{S}\mathcal{P}| \arrow[r, "1"] & \Omega|\mathrm{N}_0^{\mathrm{cy}}\mathrm{S}\mathcal{P}| \arrow[r] \arrow[d, "\deg"] & \Omega|\mathrm{TH}_0(\mathrm{S}\mathcal{P})| \arrow[d, "\deg"] \\ & \Omega|\mathrm{N}_{\bullet}^{\mathrm{cy}}\mathrm{S}\mathcal{P}| \arrow[r] & \Omega|\mathrm{TH}_{\bullet}(\mathrm{S}\mathcal{P})|. \end{tikzcd}$$

On the relationship of $\mathrm{K}(A\ltimes M)$ *and* $\mathrm{K}(A;M)$. Now we recall some of the details used in [7] to establish the relationship between $\mathrm{K}(A\ltimes M)$ and $\mathrm{K}(A;\mathrm{B}_{\bullet}M)$ in order to obtain a similar result for TH. We can consider $\mathcal{F}_A$ as a subcategory of $\mathcal{F}_{A\ltimes M}$ with all of the objects (one for each natural number), but having only the morphisms $(\beta,0)$. Note, however, that for $q>1$ the subcategory $S_q\mathcal{F}_A$ of $S_q\mathcal{F}_{A\ltimes M}$ does *not* have all the objects.

For short, write $B=A\ltimes M$ and note that as an A-module $B=A\oplus M$. For $q,q'\geqslant 0$ we get

$$\begin{aligned}\mathrm{Hom}_B(A^q\otimes_A B, A^{q'}\otimes_A B) &\cong \mathrm{Hom}_A(A^q, A^{q'}\otimes_A B)\\ &\cong \mathrm{Hom}_A(A^q, A^{q'}\otimes_A (A\oplus M))\\ &\cong \mathrm{Hom}_A(A^q, A^{q'})\oplus \mathrm{Hom}_A(A^q, A^{q'}\otimes_A M).\end{aligned}$$

We will write a morphism of $\mathcal{F}_{A\ltimes M}$ as a pair (α,β) via this natural identification.

Let U be the exact functor from $\mathcal{F}_{A\ltimes M}$ to itself defined by the identity on objects and $U(\alpha,\beta)=(\alpha,0)$. It is a retraction ($UU=U$). Recall that $\mathrm{S}_q\mathcal{F}_{A\ltimes M}$ is a (full) subcategory of the functor category $(\mathcal{F}_{A\ltimes M})^{\mathrm{Ar}[q]}$ and we let $\widetilde{\mathrm{S}}_q\mathcal{F}_{A\ltimes M}$ be the image of $\mathrm{S}_q\mathcal{F}_{A\ltimes M}$ under the endofunctor

$$U^{\mathrm{Ar}[q]}\colon (\mathcal{F}_{A\ltimes M})^{\mathrm{Ar}[q]}\to(\mathcal{F}_{A\ltimes M})^{\mathrm{Ar}[q]}.$$

Let $\mathcal{T}$ be the class of isomorphisms of the form $(1,\beta)$. These are precisely the morphisms which U takes to the identity maps. The functor $\mathcal{F}_A\to\mathcal{F}_{A\ltimes M}$, given by extension of scalars, induces a bijection of the sets of isomorphism classes of objects ([1, III.2.12]). This shows that we may choose a common skeleton category for the categories $\mathcal{F}_A$ and $U\mathcal{F}_{A\ltimes M}$, and more generally, that (when this is done) $\mathrm{S}\mathcal{F}_A$ and $\widetilde{\mathrm{S}}\mathcal{A}_{A\ltimes M}$ have the same set of objects. It follows that there is an isomorphism of bisimplicial sets

$$\mathrm{N}_*t\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M}\cong\coprod_{\overline{F}\in\mathcal{F}_A}\mathrm{B}_*\mathrm{Hom}_{\mathrm{S}\mathcal{M}_A}(\overline{F},\overline{F}\otimes_A M).$$

Every object $\overline{C}$ of $\mathrm{S}_q\mathcal{F}_{A\ltimes M}$ is t-isomorphic to an object of $\widetilde{\mathrm{S}}_q\mathcal{F}_{A\ltimes M}$, namely $\mathrm{S}_q(U)(\overline{C})$. This follows from the fact that in $\mathcal{F}_{A\ltimes M}$ every short exact sequence splits. (Every filtered object $\overline{C}$ is a split object, and so its isomorphism class is determined by the isomorphism classes of its subquotients $\overline{C}(i+1/i)$, $1\leqslant i\leqslant q$.) If η is an isomorphism from $\overline{C}$ to $\mathrm{S}_q(U)(\overline{C})$ then, putting $\widehat{\eta}=\mathrm{S}_q(U)(\eta)$, $\widehat{\eta}^{-1}\circ\eta$ is a t-isomorphism because $\mathrm{S}_q(U)(\widehat{\eta}^{-1}\circ\eta)=\mathrm{S}_q(U)(\widehat{\eta})^{-1}\circ\mathrm{S}_q(U)(\eta)=\widehat{\eta}^{-1}\circ\widehat{\eta}=1$. Thus, $\mathrm{N}.t\widetilde{\mathrm{S}}_q\mathcal{F}_{A\ltimes M}$ is equivalent to $\mathrm{N}.t\mathrm{S}_q\mathcal{F}_{A\ltimes M}$ and we obtain

$$\begin{aligned}\mathrm{K}(\mathcal{F}_{A\ltimes M})&=\Omega|\mathrm{S}\mathcal{F}_{A\ltimes M}|\xrightarrow{\simeq}\Omega|\mathrm{N}_*t\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M}|\cong\Omega\Bigg|\coprod_{\overline{F}\in\mathrm{S}\mathcal{F}_A}\mathrm{B}_*\operatorname{Hom}_{\mathrm{S}\mathcal{F}_A}(\overline{F},\overline{F}\otimes_A M)\Bigg|\\&=\mathrm{K}(\mathcal{F}_A;\mathrm{B}.M).\end{aligned}$$

LEMMA 4.5. *There is a commuting diagram of equivalences:*

$$\begin{array}{ccc}\mathrm{TH}.\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M} & \xrightarrow{\simeq} & \mathrm{TH}.\mathrm{S}\mathcal{F}_{A\ltimes M}\\ \downarrow{\simeq} & & \downarrow{\simeq}\\ \mathrm{TH}.\mathrm{N}.t\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M} & \xrightarrow{\simeq} & \mathrm{TH}.\mathrm{N}.t\mathrm{S}\mathcal{F}_{A\ltimes M},\end{array}$$

Proof. The vertical maps are equivalences by Lemma 4.4. Since $\mathrm{N}.t\widetilde{\mathrm{S}}_q\mathcal{F}_{A\ltimes M}$ is simplicial homotopy equivalent to $\mathrm{N}.t\mathrm{S}_q\mathcal{F}_{A\ltimes M}$ by linear functors and $\mathrm{TH}_{[p]}$ is functorial for all $[p]$, the realization lemma tells us that the bottom map is a homotopy equivalence and hence the top one is also.

Definition. We define Φ to be the natural map of cyclic spaces from $\widetilde{\mathrm{K}}(A;\mathrm{N}^{\mathrm{cy}}M)$ to $\widetilde{\widetilde{\mathrm{TH}}}(A\ltimes M)$ as follows. Let $\widehat{\Phi}\colon\mathrm{K}(A,N^{\mathrm{cy}}M)\to\Omega\mathrm{TH}(\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M})$ be the map determined by

$$\begin{gathered}\widehat{\Phi}_n\colon\coprod_{\overline{F}\in\mathrm{S}\mathcal{F}}\operatorname{Hom}_{\mathrm{S}\mathcal{F}}(\overline{F},\overline{F}\otimes_A M)^{n+1}\to\mathrm{TH}_n\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M},\\(\overline{F};\alpha_0,\dots,\alpha_n)\to((1,\alpha_0)\wedge\dots\wedge(1,\alpha_n)),\\(1,\alpha_i)\in\operatorname{Hom}_{\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M}}(\overline{F}\otimes_A(A\ltimes M),\overline{F}\otimes_A(A\ltimes M)).\end{gathered}$$

We observe that $\hat{\Phi}$ can be written as a composite

$$\coprod_{\overline{F}\in\mathrm{S}\mathcal{F}_A}\mathrm{N}^{\mathrm{cy}}\operatorname{Hom}_{\mathrm{S}\mathcal{M}_A}(\overline{F},\overline{F}\otimes_A M)\to\mathrm{N}^{\mathrm{cy}}(\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M})\to\mathrm{TH}(\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M}).$$

We define Φ (in the homotopy category) by the diagram

$$\widetilde{\mathrm{K}}(A;N^{\mathrm{cy}}M)\xrightarrow{\hat{\Phi}}\Omega\widetilde{\widetilde{\mathrm{TH}}}(\widetilde{\mathrm{S}}\mathcal{F}_{A\ltimes M})\xrightarrow{\simeq}\Omega\widetilde{\widetilde{\mathrm{TH}}}(\mathrm{S}\mathcal{F}_{A\ltimes M})\xleftarrow{\simeq}\widetilde{\widetilde{\mathrm{TH}}}(A\ltimes M)$$

(where the first equivalence is by Lemma 4.5 and the second is by Facts 4.3).

Proof of Proposition 4.2. We first show that the following natural diagram commutes (up to homotopy):

$$\begin{tikzcd}
\widetilde{\mathrm{K}}(A \ltimes M) \arrow[r, "\mathrm{tr}"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "(1)"] & \widetilde{\mathrm{TH}}(A \ltimes M) \\
\widetilde{\mathrm{K}}(A; \mathrm{B}_{\bullet} M) \arrow[r, "\sigma"] & \widetilde{\mathrm{K}}(A; \mathrm{N}^{\mathrm{cy}} M). \arrow[u, "\Phi"']
\end{tikzcd}$$

By Lemma 4.4 and Facts 4.3 (5), it suffices to note that the following diagram commutes (by inspection):

$$\begin{tikzcd}
\mathrm{N}_* t\widetilde{\mathrm{S}}\mathcal{F}_{A \ltimes M} \arrow[r, "\deg \circ 1"] \arrow[d, "\simeq"'] & \mathrm{N}^{\mathrm{cy}}_* \mathrm{N}_* t\widetilde{\mathrm{S}}\mathcal{F}_{A \ltimes M} \arrow[d, "\phi"] \\
\coprod_{\overline{F} \in \mathrm{S}\mathcal{F}_A} \mathrm{B}_* \operatorname{Hom}_{\mathrm{S}\mathcal{F}_A}(\overline{F}, \overline{F} \otimes_A M) \arrow[r, "\sigma"] & \coprod_{\overline{F} \in \mathrm{S}\mathcal{F}_A} \mathrm{N}^{\mathrm{cy}}_* \operatorname{Hom}_{\mathrm{S}\mathcal{F}_A}(\overline{F}, \overline{F} \otimes_A M).
\end{tikzcd}$$

We now establish that the following natural diagram commutes up to homotopy:

$$\begin{tikzcd}
\widetilde{\mathrm{TH}}(A \ltimes M) \arrow[dr, phantom, "(2)"] & S^1_+ \otimes \mathrm{TH}(A; M) \arrow[l, "r"'] \\
\widetilde{\mathrm{K}}(A; \mathrm{N}^{\mathrm{cy}} M) \arrow[u, "\Phi"] & S^1_+ \otimes \widetilde{\mathrm{K}}(A; M). \arrow[l] \arrow[u, "S^1_+ \otimes \beta"']
\end{tikzcd}$$

For each $q \geqslant 0$ and $\overline{F}, \overline{Q} \in \mathrm{S}_q\mathcal{F}$, we obtain group homomorphisms

$$\operatorname{Hom}_{\mathrm{S}_q\mathcal{M}_A}(\overline{F}, \overline{Q} \otimes_A M) \to \operatorname{Hom}_{\widetilde{\mathrm{S}}_q\mathcal{F}_{A \ltimes M}}(\overline{F} \otimes_A (A \ltimes M), \overline{Q} \otimes_A (A \ltimes M)) \quad (\alpha \mapsto (0, \alpha)),$$
$$\operatorname{Hom}_{\mathrm{S}_q\mathcal{F}_A}(\overline{F}, \overline{Q}) \to \operatorname{Hom}_{\widetilde{\mathrm{S}}_q\mathcal{F}_{A \ltimes M}}(\overline{F} \otimes_A (A \ltimes M), \overline{Q} \otimes_A (A \ltimes M)) \quad (\beta \mapsto (\beta, 0)).$$

These homomorphisms produce maps of simplicial spaces from $\mathrm{TH}(\mathrm{S}_q\mathcal{F}, M)$ to $\mathrm{TH}(\widetilde{\mathrm{S}}_q\mathcal{F}_{A \ltimes M})$ which are natural with respect to the S-operators and hence assemble to give a map of bisimplicial spaces $\psi(k)$ for each $k \geqslant 0$:

$$\mathrm{TH}(\mathrm{S}^{(k)}\mathcal{F}, M) \xrightarrow{\psi(k)} \mathrm{TH}(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A \ltimes M}).$$

We recall that

$$\widetilde{\mathrm{K}}(A;M)=\lim_k \Omega^k \bigvee_{\bar{F}\in \mathrm{S}^{(k)}\mathcal{F}_A} \mathrm{Hom}_{\mathrm{S}^{(k)}\mathcal{M}_A}(\bar{F},\bar{F}\otimes_A M),$$

$$\begin{aligned}\mathrm{TH}_0(\mathrm{S}^{(k)}\mathcal{F}_A;M) &= \operatorname*{holim}_{x\in I} \mathrm{Map}\big(S^x, \bigvee_{\bar{F}\in \mathrm{S}^{(k)}\mathcal{F}_A} (\mathrm{Hom}_{\mathrm{S}^{(k)}\mathcal{M}_A}(\bar{F},\bar{F}\otimes_A M))^{\sim}[S^x]\big)\\ &\xrightarrow{\simeq} \bigoplus_{\bar{F}\in \mathrm{S}^{(k)}\mathcal{F}_A} \mathrm{Hom}_{\mathrm{S}^{(k)}\mathcal{M}_A}(\bar{F},\bar{F}\otimes_A M),\end{aligned}$$

$$\begin{aligned}&\mathrm{TH}_0(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M})\\ &\quad= \operatorname*{holim}_{x\in I} \mathrm{Map}\big(S^x, \bigvee_{\bar{F}\in \widetilde{\mathrm{S}}^{(k)}\mathcal{F}_A} (\mathrm{Hom}_{\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M}}(\bar{F}\otimes_A(A\ltimes M),\bar{F}\otimes_A(A\ltimes M)))^{\sim}[S^x]\big)\\ &\quad\xrightarrow{\simeq} \bigoplus_{\bar{F}\in \widetilde{\mathrm{S}}^{(k)}\mathcal{F}_A} \mathrm{Hom}_{\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M}}(\bar{F}\otimes_A(A\ltimes M),\bar{F}\otimes_A(A\ltimes M)).\end{aligned}$$

The map β in [7] from $\widetilde{\mathrm{K}}(A;M)$ to $\lim_k \Omega^k \mathrm{TH}_0(\mathrm{S}^{(k)}\mathcal{F}_A;M)$ is given by the natural map from the wedge into the direct sum of abelian groups and then taking limits with respect to iterations of the S-construction. Thus, composition with ψ_0, up to natural homotopy equivalence, is the map from $\widetilde{\mathrm{K}}(A;M)$ to $\lim_k \Omega^k \mathrm{TH}_0(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M})$ determined by the map $(\alpha\mapsto(0,\alpha))$ and the inclusion of the wedge into the direct sum. We let β' be the map from $\widetilde{\mathrm{K}}(A;M)$ to $\lim_k \Omega^k \mathrm{TH}_0(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M})$ determined by the map $(\alpha\mapsto(1,\alpha))$ and the inclusion of the wedge into the direct sum. The maps $\psi_0\circ\beta$ and β' do not produce homotopy-equivalent maps to $\Omega^k \mathrm{TH}_0(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M})$ but they are homotopy equivalent after composing with the projection to $\Omega^k \widetilde{\mathrm{TH}}(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M})$.

For $k\geqslant 0$ we let $\mu(k)$ be the composite map

$$\begin{aligned}S^1_+\otimes \mathrm{TH}_0(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M}) &\xrightarrow{S^1_+\otimes \deg} S^1_+\otimes \mathrm{TH}(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M}) \to S^1_+\otimes \widetilde{\mathrm{TH}}(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M})\\ &\xrightarrow{\mathrm{ev}} \widetilde{\mathrm{TH}}(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M}).\end{aligned}$$

Using the above remarks, we have a natural diagram (commuting up to homotopy)

$$\begin{tikzcd}
\lim_k \Omega^k \widetilde{\mathrm{TH}}(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M}) & S^1_+\otimes \lim_k \Omega^k \mathrm{TH}_0(\mathrm{S}^{(k)}\mathcal{F}_A;M) \arrow[l, "\mu\circ(S^1_+\otimes\psi_0)"'] \\
\widetilde{\mathrm{K}}(A;\mathrm{N}^{\mathrm{cy}}M) \arrow[u, "\Phi"] & S^1_+\otimes \widetilde{\mathrm{K}}(A;M) \arrow[l, "\deg"'] \arrow[u, "S^1_+\otimes\beta"']
\end{tikzcd}$$

since $\Phi\circ\deg=\mu\circ(S^1_+\otimes\beta')$ and $S^1_+\otimes\beta'\simeq S^1_+\otimes(\psi_0\circ\beta)$.

Let A be the category with one object and morphisms the elements of A. We obtain a commuting diagram

$$\begin{tikzcd}
\widetilde{\mathrm{TH}}(A\ltimes M) \arrow[d, "\simeq"] & S^1_+\otimes\mathrm{TH}(A;M) \arrow[l, "\tau"'] \arrow[d, "\simeq"] \\
\Omega^k\widetilde{\mathrm{TH}}(\widetilde{\mathrm{S}}^{(k)}\mathcal{F}_{A\ltimes M}) & S^1_+\otimes\Omega^k\mathrm{TH}(\mathrm{S}^{(k)}\mathcal{F}_A;M) \arrow[l, "\mathrm{ev}\circ(S^1_+\otimes\psi)"'] \\
 & S^1_+\otimes\lim_k\Omega^k\mathrm{TH}_0(\mathrm{S}^{(k)}\mathcal{F}_A;M). \arrow[u, "\simeq"'] \arrow[ul, "\mu\circ(S^1_+\otimes\psi_0)"]
\end{tikzcd}$$

The vertical arrows are equivalences by Proposition 4.2 and Lemma 4.5, and the commuting (up to homotopy) of (2) follows.

References

[1] Bass, H., *Algebraic K-theory*. Benjamin, New York–Amsterdam, 1968.

[2] Bökstedt, M., Topological Hochschild homology. Preprint, Bielefeld.

[3] Bökstedt, M., Carlsson, G., Cohen, R., Goodwillie, T., Hsiang, W.-C. & Madsen, I., On the algebraic K-theory of simply connected spaces. *Duke Math. J.*, 84 (1996), 541–563.

[4] Bökstedt, M., Hsiang, W.-C. & Madsen, I., The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111 (1993), 865–940.

[5] Bökstedt, M. & Madsen, I., Algebraic K-theory of local number fields: the unramified case. Preprint no. 20, Aarhus University, 1994.

[6] Dundas, B., Relative K-theory and topological cyclic homology. *Acta Math.*, 179 (1997), 223–242.

[7] Dundas, B. & McCarthy, R., Stable K-theory and topological Hochschild homology. *Ann. of Math.*, 140 (1994), 685–701.

[8] — Topological Hochschild homology of ring functors and exact categories. *J. Pure Appl. Algebra*, 109 (1996), 231–294.

[9] Fiedorowicz, Z., Ogle, C. & Vogt, R. M., Volodin K-theory of A_∞-ring spaces. *Topology*, 32 (1993), 329–352.

[10] Goodwillie, T., Relative algebraic K-theory and cyclic homology. *Ann. of Math.*, 124 (1986), 347–402.

[11] — Calculus I: the first derivative of pseudoisotopy theory. *K-Theory*, 4 (1990), 1–27.

[12] — The differential calculus of homotopy functors, in *Proceedings of the International Congress of Mathematicians* (*Kyoto*, 1990), pp. 621–630. Math. Soc. Japan, Tokyo, 1991.

[13] — Notes on the cyclotomic trace. Notes from a lecture series given at MSRI during the spring of 1990.

[14] — Calculus II: analytic functors. *K-Theory*, 5 (1992), 295–332.

[15] Hesselholt, L., Stable topological cyclic homology is topological Hochschild homology. *Astérisque*, 226 (1994), 178–192.

[16] Hesselholt, L. & Madsen, I., On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36 (1997), 29–101.

[17] LODAY, J. L., Opérations sur l'homologie cyclique des algèbres commutatives. *Invent. Math.*, 96 (1989), 205–230.

[18] MADSEN, I., Algebraic K-theory and traces, in *Current Developments in Mathematics*, pp. 191–321. International Press, Cambridge, MA, 1995.

[19] TSALIDIS, S., On the topological cyclic homology of the integers. *Amer. J. Math.*, 119 (1997), 103–125.

[20] WALDHAUSEN, F., Algebraic K-theory of generalized free products. *Ann. of Math.*, 108 (1978), 135–256.

[21] — Algebraic K-theory of spaces, in *Algebraic and Geometric Topology* (*Rutgers*, 1993), pp. 318–419. Lecture Notes in Math., 1126. Springer-Verlag, Berlin–New York, 1985.

RANDY MCCARTHY
Department of Mathematics
University of Illinois
Urbana, IL 61801
U.S.A.
randy@math.uiuc.edu

Received July 29, 1996