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The approximation problem for Sobolev maps between two manifolds

by

FABRICE BETHUEL

ENPC, CERMA
Noisy-Le-Grand, France

1. Introduction

The problem of density of smooth maps between two compact manifolds M^n and N^k was first considered by Eells and Lemaire ([EL]). If $p>\dim M^n$, then $W^{1,p}\hookrightarrow C^0$ (by the Sobolev embedding theorem) and it is easy to see (using standard approximation methods) that $C^\infty(M^n, N^k)$ is dense in $W^{1,p}(M^n, N^k)$. Schoen and Uhlenbeck [SU2], [SU3] have proved that smooth maps are dense in the limiting case $p=\dim M^n$. They also gave an example of non density of smooth maps: they showed that $C^\infty(B^3, S^2)$ is not dense in $H^1(B^3, S^2)$; for instance the radial projection π from B^3 to S^2 defined by $\pi(x)=x/|x|$ cannot be approximated by smooth maps.

We consider in this paper two compact Riemannian manifolds M^n and N^k of dimension n and k respectively. N^k is isometrically embedded in $\mathbf{R}^l\,(l\in\mathbf{N}^*)$. M^n may have a boundary, but not N^k. For $1\leqslant p<n$, we consider the Sobolev space $W^{1,p}(M^n, N^k)$ defined by:

$$W^{1,p}(M^n, N^k)=\{u\in W^{1,p}(M^n, \mathbf{R}^l);\, u(x)\in \mathbf{N}^k \text{ a.e.}\}.$$

Since $W^{1,p}(M^n, N^k)$ is included in $W^{1,p}(M^n, \mathbf{R}^l)$, it inherits both strong and weak topology from $W^{1,p}(M^n, \mathbf{R}^l)$. It is moreover clear that $W^{1,p}(M^n, N^k)$ is stable under strong and weak convergence. Note that in our definition we embed N^k in an Euclidean space $\mathbf{R}^l$, in order to define these spaces. This is actually the most convenient way for doing so, and the results do not depend on the way we embed N^k.

The following theorem is the main result of this paper, and gives a necessary and sufficient condition for smooth maps to be dense in $W^{1,p}(M^n, N^k)$.

THEOREM. *Let $1\leqslant p<n$. Smooth maps between M^n and N^k are dense in*

$W^{1,p}(M^n, N^k)$ *if and only if* $\pi_{[p]}(N^k)=0$ ($[p]$ *represents the largest integer less or equal to* p).

The fact that this condition is necessary is proved in [BZ] (Theorem 2) (actually in [BZ], we only gave a proof in the case $M^n=B^n$, the unit ball in $\mathbf{R}^n$; for a proof in the case M^n is any manifold see Theorem A0 of the Appendix). A large part of this paper is devoted to the proof of sufficiency. Theorem 1 settles the problem of density of smooth maps in $W^{1,p}(M^n, N^k)$ for every p, since for $p\geq n$ we have density.

Assume that ∂M^n is not empty. It may also be useful to be able to approximate a map u in $W^{1,p}(M^n, N^k)$ such that the restriction of u to ∂M^n is continuous (resp. smooth), by continuous (resp. smooth) maps from M^n to N^k, which agree with u on the boundary. With some slight modifications in the proof of Theorem 1 we also have:

THEOREM 1 bis. *Let* $1\leq p\leq n$ *and assume* $\pi_{[p]}(N^k)=0$, and $\partial M^n\neq\emptyset$. *Let* u *be in* $W^{1,p}(M^n, N^k)$, *such that* u *restricted to* ∂M^n *is in* $W^{1,p}(\partial M^n, N^k)\cap C^0(\partial M^n, N^k)$ *(resp.* $C^\infty(\partial M^n, N^k)$*). If there is a map* v *in* $C^0(M^n, N^k)$ *(resp.* $C^\infty(M^n, N^k)$*) such that* $u=v$ *on* ∂M^n *then* u *can be approximated in* $W^{1,p}(M^n, N^k)$ *by maps in* $W^{1,p}(M^n, N^k)\cap C^0$ *(resp.* $C^\infty(M^n, N^k)$*) which coincide with* u *on* ∂M^n.

When $\pi_{[p]}(N^k)\neq 0$, by Theorem 1, smooth maps are not dense in $W^{1,p}(M^n, N^k)$. In this case, we are nevertheless able to approximate maps in $W^{1,p}(M^n, N^k)$ by maps which are regular except on a simple set of low dimension. More precisely we consider the class R_p^0 (resp. R_p^∞) of maps in $W^{1,p}(M^n, N^k)$ defined in the following way: $u\in W^{1,p}(M^n, N^k)$ is in R_p^0 (resp. R_p^∞) if and only if u is continuous (resp. smooth) except on a singular set $\Sigma(u)$, where $\Sigma(u)=\bigcup_{j=1}^r \Sigma_i$, $r\in\mathbf{N}^*$, where for $i=1,\dots,r$, Σ_i is a subset of a submanifold of M^n of dimension $n-[p]-1$, and the boundary of Σ_i is smooth; if $p\geq n-1$, Σ_i is a point. Actually, in the case M^n is some domain of $\mathbf{R}^n$, we may assume that Σ_i is a subset of a linear subspace of $\mathbf{R}^n$, of dimension $n-[p]-1$, and the boundary $\partial\Sigma_i$ a subset of a linear subspace of dimension $n-[p]-2$. We have the following:

THEOREM 2. *For every* $1\leq p<n$, R_p^0 (resp. R_p^∞) *is dense in* $W^{1,p}(M^n, N^k)$.

We have also the following, which is the analogue of Theorem 1 bis:

THEOREM 2 bis. *Assume* $1\leq p<n$, *and* $\partial M^n\neq\emptyset$. *Let* u *be in* $W^{1,p}(M^n, N^k)$ *such that* u *restricted to* ∂M^n *is in* $W^{1,p}(\partial M^n, N^k)\cap C^0$ *(resp.* $C^\infty(M^n, N^k)$*). If there is a map* v *in* $C^0(M^n, N^k)$ *(resp.* $C^\infty(M^n, N^k)$*) such that* $u=v$ *on* ∂M^n, *then* u *can be approximated in* $W^{1,p}(M^n, N^k)$ *by maps in* R_p^0 *(resp.* R_p^∞*) which coincide with* u *on* ∂M^n.

When $\pi_{[p]}(N^k) \neq 0$, we also consider the problem of density of smooth maps for the weak topology, induced by the weak topology in $W^{1,p}(M^n, \mathbf{R}^l)$. We have the following theorem:

THEOREM 3. *If $\pi_{[p]}(N^k) \neq 0$, and p is not an integer, then smooth maps are not sequentially dense for the weak topology in $W^{1,p}(M^n, N^k)$. Moreover every map in $W^{1,p}(M^n, N^k)$ which is a weak limit of smooth maps is also a strong limit of smooth maps (in $W^{1,p}(M^n, N^k)$).*

This theorem is useful when trying to minimize a functional in a certain class of maps in $W^{1,p}(M^n, N^k)$ (for instance $C^\infty(M^n, N^k)$, see [W2]). For instance let $M^n = B^n$, let ζ be in $C^\infty(\partial B^n, N^k)$ and consider the energy functional

$$E_p(u) = \int_{B^n} |\nabla u|^p \, dx$$

defined on $W^{1,p}(B^n, N^k)$. If u is a C^1 critical point of E_p, u is called a p-harmonic maps and satisfied the Euler–Lagrange equation related to E_p. Weakly p-harmonic maps are weak solutions of that equation. In the case p is *not* an integer and $\pi_{[p]}(N^k) \neq 0$ a refined version of Theorem 3 then shows that there are infinitely many p-harmonic maps in $W^{1,p}_\zeta(B^n, N^k) = \{u \in W^{1,p}(B^n, N^k),\ u = \zeta \text{ on } \partial B^n\}$ (for a precise statement of the results see Section VI).

When $\pi_{[p]}(N^k) \neq 0$, and p is an integer, we have the following theorem:

THEOREM 5. *If p is an integer and $\pi_{[p]}(N^k) \neq 0$, then smooth maps between M^n and N^k are dense in $W^{1,p}(M^n, N^k)$ for the weak topology.*

This suggests that every map in $W^{1,p}$ is the weak limit of a sequence of smooth maps. But unfortunately, we are not able to prove this except in the special case $N^k = S^p$ (the unit sphere in $\mathbf{R}^{p+1}$). Adapting the method of [Be1] we are able to prove:

THEOREM 6. *If p is an integer, $1 < p < n$ smooth maps between M^n and S^p are sequentially dense in $W^{1,p}(M^n, S^p)$.*

Some of the results in this paper have been obtained earlier by Zheng and the author in special cases. For example, Theorem 1 has been proved in [BZ] for the case $N^k = S^k$ and $p < k$, Theorem 2 for $N^k = S^2$, $n = 3$, and $p = 2$. Some of the arguments in this paper rely on constructions of White ([W1], [W2]). Escobedo ([E]) has studied the density of smooth maps in the Sobolev spaces $W^{r,p}(M^n, S^k)$ with $r > 1$ and $rp < k$ (here r

need not to be an integer). In another direction, in [Be1], we characterize the strong closure of $C^\infty(B^3, S^2)$ in $H^1(B^3, S^2)$.

Approximation theorems are a useful tool in the study of harmonic maps (see [CG], [H], [BBC]).

This paper is divided as follows. We first consider the case $M^n=[0,1]^n=C^n$. In part I we prove the Main Theorem and Theorem 1 bis, for $M^n=C^n$, and $n-1<p<n$. For this purpose we prove that we may approximate each map in $W^{1,p}(C^n, N^k)$, by maps in R_p^∞, that is maps in $W^{1,p}(C^n, N^k)$ smooth except at most at a finite number of point singularities. This, in fact, is a general result (see Theorem 2; for $n-1<p$) and holds even if $\pi_{[p]}(N^k)\neq 0$ (here $[p]=n-1$). Then we conclude using the fact that $\pi_{n-1}(N^k)=0$ and the following lemma, which is proved in the Appendix (see also Theorem 5 of [BZ]):

LEMMA 1. *Assume $\pi_{n-1}(N^k)=0$ and $p<n$. Let u be a map in $W^{1,p}(M^n, N^k)$ such that u is continuous except at a finite number of point singularities. Then u can be approximated in $W^{1,p}(M^n, N^k)$ by smooth maps between M^n and N^k.*

In part II, we prove Theorem 1 and Corollary 1 for $M^n=C^n$ and $n-2<p\leq n$. We adapt the construction of part I, and, when $p<n-1$, we need the following lemma (which holds even if $\pi_{[p]}(N^k)\neq 0$):

LEMMA 2. (i) *Let $p<n-1$. Let v be some map in $W^{1,p}(C^n, N^k)$ such that v is continuous except at most at a finite number of point singularities (here we do not make any topological assumption concerning N^k). Then, v can be approximated for the $W^{1,p}$ norm by maps in $C^\infty(C^n, N^k)$.*

(ii) *If v restricted to ∂C^n is in $W^{1,p}(\partial C^n, N^k)\cap C^0$ (resp. C^∞) and if there is some map v' in $C^0(C^n, N^k)$ (resp. C^∞) such that $v'=v$ on ∂C^n then v can be approximated for the $W^{1,p}$ norm by maps in $W^{1,p}(C^n, N^k)\cap C^0$* (resp. C^∞) *which coincide with v on the boundary.*

In part III we prove Theorem 1 and Theorem 1 bis for $M^n=C^n$ and $p\leq n-2$. In part IV, we prove Theorem 2 and Theorem 2 bis for $M^n=C^n$. In part V we extend the results obtained for $M^n=C^n$ to any compact manifold M^n of dimension n. Part VI deals with the weak density results, and with the problem of finding infinitely many p-harmonic maps satisfying a given boundary condition. In part VII, we extend some results to the Sobolev space $W^{r,p}(M^n, N^k)$, with r in $\mathbf{N}^*$. In the Appendix we prove technical lemmas.

We recall next some usual notations.

For $n\geq 1$, B^n is the unit open ball in $\mathbf{R}^n$ and $B^n(y;\delta)$ is the ball of radius δ centered at $y\in\mathbf{R}^n$. We set $S^n=\partial B^{n+1}$. We consider also the cubes

$$C^n=[0,1]^n,\quad C'^n=\left[-\frac{1}{2},\frac{1}{2}\right]^n,\quad C'^n(a)=\left[-\frac{a}{2},\frac{a}{2}\right]^n,\ \text{for } a>0.$$

For $\delta>0$, small enough, and for $y_0\in N^k$, we set $\tilde{B}_\varrho(y,\delta)=N^k\cap B^l(y_0,\delta)$. For $x=(x_1,\ldots,x_i,\ldots,x_n)\in\mathbf{R}^n$ we set

$$\|x\|=\max_{i\in\{1,\ldots,n\}}\{|x_i|\}.$$

For u in $W^{1,p}(M^n,N^k)$ we set $E(u)=\int_{M^n}|\nabla u|^p\,dx$. If W is an open subset of M^n we set $E(u;W)=\int_W|\nabla u|^p\,dx$. Likewise if C^s is a submanifold of dimension s of M^n we set $\underline{E}(u;C^s)=\int_{C^s}|\nabla u|^p\,d\sigma$ where $d\sigma$ is the volume measure on C^s induced by the measure on M^n, and when the integral is finite.

For $q\in\mathbf{N}^*$, $\pi_q(N^k)$ is the qth homotopy group of N^k. $K_1,K_2,\ldots$ represent absolute constants depending possibly on M^n, N^k and p. $\mathcal{O}$ is some open neighborhood of N^k in $\mathbf{R}^l$ such that the nearest point projection π from $\mathcal{O}$ to N^k is a smooth fibration.

I. Proof of Theorem 1 when $M^n=C^n$ and $n-1<p<n$

We assume throughout this section that $M^n=C^n$, $n-1<p<n$, and $\pi_{n-1}(N^k)=0$. Let u be in $W^{1,p}(C^n,N^k)$. We are going to approximate u by maps u_m which are continuous except at most at a finite number of point singularities (the conclusion then follows from Lemma 1 applied to u_m and the assumption $\pi_{n-1}(N^k)=0$). In order to construct our approximation sequence u_m, we divide, in a convenient way, the cube C^n in $(m+1)^n$ little cubes C_r having an edge of length $1/m$, and we classify these cubes in two categories. The "good cubes" are the cubes such that the energy of u restricted to these cubes, and the energy of u restricted to the boundary of these cubes are small. For these cubes, "most of" the image of u lies in some small geodesic ball of N^k, and we can approximate u using a standard mollifying technique. The bad cubes are the other ones, on which we approximate u by maps having point singularities. Next we present a basic method for dividing C^n in small cubes C_r, in a convenient way.

I.1. *A method for dividing C^n in small cubes C_r.* Without loss of generality we may assume that u restricted to ∂C^n is in $W^{1,p}(\partial C^n,N^k)$ and thus continuous by the Sobolev embedding theorem. Indeed, in Lemma A0 of the Appendix we prove that u can always be approximated in $W^{1,p}(C^n, N^k)$ by maps in $W^{1,p}(C^n, N^k)$ such that their restriction to the boundary is also in $W^{1,p}$. For $k=1,\dots,n$, we set $e_k=(0,\dots,1,0,\dots,0)\in\mathbf{R}^n$. For $1\leq k\leq n$, and a in $[0,1]$ we note $P(a,k)$ the restriction to C^n of the hyperplane passing through the point $A_k(a)=ae_k$ and orthogonal to e_k. For $m\in\mathbf{N}^*$, and for $\alpha\in[1/4m, 3/4m]$ we consider the hyperplanes $P(\alpha+j/m, k)$ for $0\leq j\leq m-1$ and the union of these hyperplanes $W(m,\alpha,k)=\bigcup_{j=0}^{m-1}P(\alpha+j/m,k)$. For almost every α in $[1/4m, 3/4m]$, u restricted to $W(m,\alpha,k)$ is in $W^{1,p}$ and thus continuous by the Sobolev embedding theorem. Moreover, we have clearly

$$\int_{1/4m}^{3/4m} \underline{E}(u; W(m,\alpha,k))\,d\alpha \leq E(u).$$

Thus, there is some α_k in $[1/4m, 3/4m]$ such that u restricted to $W(m,\alpha,k)$ is in $W^{1,p}\hookrightarrow C^0$ and such that

$$\underline{E}(u; W(m,\alpha_k,k)) = \sum_{j=0}^{m-1} \underline{E}\left(u; P\left(\alpha_k+\frac{j}{m},k\right)\right) \leq 2mE(u). \tag{1}$$

Considering now the "slicings" of C^n by the set $W(m,\alpha_k,k)$ obtained by the method described above when we change the slicing direction k, we see that we have divided C^n in $(m+1)^n$ small cubes that we note $C_1, C_2, \dots, C_r, \dots, C_{(m+1)^n}$. The cubes which are not in contact with the boundary have edges of length $1/m$ (and are translates of $[0,1/m]^n$). The cubes which are in contact with the boundary are diffeomorphic to $[0,1/m]^n$ by linear maps f_r such that $|\nabla f_r|\leq 4$, $|\nabla f_r^{-1}|\leq 4$ (these inequalities are due to the technical condition α_k in $[1/4m, 3/4m]$). Inequality (1) then gives us:

$$\sum_{r=1}^{(m+1)^n} \underline{E}(u;\partial C_r) \leq K_1' mE(u)+\underline{E}(u;\partial C^n) \leq K_1 mE(u), \quad \text{for } m \text{ large enough.} \tag{2}$$

For every little cube C_r we define the scaled energy $\tilde{E}_m(u;C_r)$ by: $\tilde{E}_m(u;C_r)= E(\tilde{u}_{m,r}; C'^n)$, where $\tilde{u}_{m,r}$ is the map from C'^n to N^k defined by $\tilde{u}_{m,r}(x)=u(x/m+x_r)$ (where x_r is the barycenter of C_r), for the cubes which are not in contact with the boundary, and in a similar way for the cubes which are in contact with the boundary ($\tilde{u}_{m,r}$ is a "blow-up" map of u restricted to C_r). We also set $\underline{\tilde{E}}_m(u;\partial C_r)=\underline{E}(\tilde{u}_{m,r};\partial C'^n)$. We have the following scaling equalities:

$$\begin{cases} \tilde{E}_m(u;C_r) = m^{n-p}E(u;C_r); \\ \underline{\tilde{E}}_m(u;\partial C_r) = m^{n-p-1}\underline{E}(u;\partial C_r). \end{cases} \tag{3}$$

Next we are going to classify the cubes C_r in two categories, the "good" and the "bad" cubes.

I.2. *Definition of the "good" and of the "bad" cubes.* Let $\varepsilon>0$ be small, to be determined later. We consider first the cubes C_r (for $r=1$ to $(m+1)^n$), such that $\tilde{E}_m(u;\partial C_r)\geqslant\varepsilon$. We note $P_{1,m}$ the union of the cubes C_r which verify this condition, and $I_{1,m}$ the set of indexes r of these cubes (that means $C_r\subset P_{1,m}$ if and only if $r\in I_{1,m}$; $P_{1,m}=\bigcup_{r\in I_{1,m}} C_r$). We consider also the cubes C_r such that $\tilde{E}_m(u;C_r)\geqslant\varepsilon m^{-\nu}$ where ν is some positive constant, which is fixed and small. We note $P_{2,m}$ the union of these cubes and $I_{2,m}$ the set of indexes for these cubes; we have $P_{2,m}=\bigcup_{r\in I_{2,m}} C_r$. We set $P_m=P_{1,m}\cup P_{2,m}$, $I_m=I_{1,m}\cup I_{2,m}$. P_m is the union of the "bad cubes". We also consider $Q_m=\overline{C^n\setminus P_m}$, and $J_m=\{(1,\dots,(m+1)^n\}\setminus I_m$ (the set of indexes for the cubes in Q_m). Q_m is the union of the "good" cubes. We are going to show that the volume of P_m is "small". Indeed, using relation (2) and the scaling equalities (3), it is easy to verify that

$$\# I_{1,m}\leqslant K_2\, m^{n-p}E(u)\,\varepsilon^{-1}.$$

Likewise the equality $\Sigma_{r=1}^{(m+1)^n} E(u;C_r)=E(u)$, and the scaling equality (3) give

$$\# I_{2,m}\leqslant K_2\, m^{n-p+\nu}E(u)\,\varepsilon^{-1}.$$

Thus we see that $(\# I_m)\, m^{-n}=\mathrm{vol}(P_m)\to 0$ when $m\to+\infty$. Hence $E(u;P_m)\to 0$ when $m\to+\infty$, by Lebesgue's theorem.

We are going to approximate u in different ways on P_m and Q_m. Since $E(u;P_m)$ tends to zero, we do not need to approximate u very closely on P_m; we only have to construct u_m on P_m in such a way that $u_m=u$ on ∂P_m, $E(u_m;P_m)\leqslant CE(u;P_m)$, and u_m is continuous, except at most at a finite number of point singularities. This is the purpose of the next construction.

I.3. *Construction of the approximation map u_m on P_m.* For the construction of the map u_m on each cube C_r included in P_m we use the following lemma:

LEMMA 3. *Let $n-1<p<n$, $\mu>0$ and v be a map in $W^{1,p}(C^n(\mu),N^k)$ such that v restricted to $\partial C^n(\mu)$ is in $W^{1,p}(\partial C^n(\mu),N^k)\hookrightarrow C^0$. There is an absolute constant K_3 depending only on p and n, and some map w in $W^{1,p}(C^n(\mu),N^k)$ continuous except at most at a finite number of point singularities, such that $w=v$ on ∂C^n and $E(w)\leqslant K_3 E(v)$.*

Before we give the proof of Lemma 3, we complete the construction of u_m on P_m. Defining u_m on each cube C_r in P_m as the map w obtained by Lemma 3 for $\mu=1/m$,

$C^n(\mu)=C_r$, and $v=u$, we see that u_m is continuous on P_m except at most at a finite number of point singularities, u_m and u coincide on ∂C_r and thus on ∂P_m, and we have:

$$E(u_m; P_m) \leqslant K_3 E(u; P_m) \to 0 \quad \text{when} \quad n \to +\infty. \tag{4}$$

Next we give the proof of Lemma 3.

Proof of Lemma 3. Using a simple scaling argument, we may assume without loss of generality that $C^n(\mu)=C^n=[0,1]^n$. We are going to use the method of Section I.1 for dividing C^n in little cubes. Thus, for every $s \in \mathbf{N}^*$, we may divide C^n in $(s+1)^n$ little cubes C_l of length $1/s$ (except those in contact with the boundary, which are linearly diffeomorphic to $[-1/2s, 1/2s]^n$), such that:

$$\sum_{l=1}^{(s+1)^n} \underline{E}(v; \partial C_l) \leqslant K_1 s E(v), \quad \text{for } s \text{ large enough.} \tag{5}$$

On each cube C_l (for $l=1$ to $(s+1)^n$), we define a map w_s by:

$$w_s(x) = v\left[\frac{x-x_l}{2s\|x-x_l\|} + x_l\right],$$

where x_l is the barycenter of C_l, and if C_l is not in contact with the boundary, and in a similar way if C_l is in contact with the boundary (using the fact that, in this case C_l is diffeomorphic to $[-1/2s, 1/2s]^n$ by a linear map f_l such that $|\nabla f_l| \leqslant 4$, $|\nabla f_l|^{-1} \leqslant 4$). It is then easy to show that w_s is in $W^{1,p}(C^n, N^k)$, is continuous, except at the points x_l, $w_s=u$ on ∂C^n, and for each cube C_l we have:

$$E(w_s; C_l) \leqslant K_4 s^{-1} \underline{E}(v; \partial C_l).$$

Adding these inequalities for $l=1$ to $(s+1)^n$, and combining with (5) we obtain:

$$E(w_s) \leqslant \frac{K_4}{s} \sum_{l=1}^{(s+1)} \underline{E}(v; \partial C_l) \leqslant K_4 K_1 E(v) \quad \text{for } s \text{ large enough.}$$

Thus for s large enough, we set $w=w_s$, and w satisfies the conditions of Lemma 3. This completes the proof of Lemma 3. In the next section, we construct the approximation map u_m on Q_m.

Remark. An alternate to the proof of Lemma 3, could be derived by choosing w as a minimizing map for the boundary value v on ∂C^n, and by application of the regularity results of [F], [HL], [L] (it is known that such a map has only point singularities).

I.4. *Construction of the approximation map u_m on Q_m.* On Q_m, we are going to approximate u by maps u_m which are continuous. First we construct a map w_m such that w_m is in $W^{1,p}(C^n, N^k)$, and such that for each cube C_r in Q_m the image of w_m on C_r lies in a small geodesic ball of N_k (then it will be easy to construct on Q_m the map u_m, which is continuous, using a simple mollifying argument). For this purpose we need two technical lemmas:

LEMMA 4. *Let $\delta>0$ be small. Let $q\in\mathbf{N}^*$ and $p>q$. We consider the cube C^q and a map v in $W^{1,p}(C^q, N^k)$. There is some constant $\varepsilon_0(\delta, q, p, N^k)$, depending only on δ, p, q and N^k, such that if $E(v)\leq\varepsilon_0(\delta, q, p, N^k)$, then the image of C^q by v (which is continuous by the Sobolev embedding theorem) lies in some domain $\tilde{B}_\varrho(y, \delta)$ for some $y\in N^k$.*

Note that Lemma 4 is a simple consequence of the Sobolev embedding theorem, since $W^{1,p}(C^q, N^k)\hookrightarrow C^0(C^q, N^k)$. For technical reasons (which will become clear in the sequel), we choose δ_0 such that, for every y in N^k, $B^l(y, 4n\delta_0)$ lies in $\mathcal{O}$. We choose also $\varepsilon=\varepsilon_0(\delta_0/2n, p, n-1, N^k)$ (recall that ε is the constant needed for the definition of P_m and Q_m). We need also the following result:

LEMMA 5. *For $\delta>0$ small enough, there is some constant $\tilde{K}(\delta)$, depending only on N^k and δ, such that there is some smooth map $\varphi(y, \delta)$, for every y in N^k, from N^k to $\tilde{B}_\varrho(y, \delta)$ such that $|\nabla\varphi(y, \delta)|_\infty\leq\tilde{K}(\delta)$ and $\varphi(y, \delta)=\mathrm{Id}$ on $\tilde{B}_\varrho(y, \delta)$.*

The proof of Lemma 5 is given the Appendix. We come back now to the construction of w_m on Q_m. For each cube C_r in Q_m, we have (by the definition of Q_m) $\tilde{E}(u; \partial C_r)\leq\varepsilon$. If we apply Lemma 4 to $C_r, q=n-1, p$, and each face of ∂C_r we see that, (after a "blow-up" of the cube) the image of ∂C_r by u (which is continuous on ∂C_r by the Sobolev embedding theorem) lies in some domain $\tilde{B}_\varrho(y_r, \delta_0)$, for some $y_r\in N^k$.

Then we define w_m in the following way:

$$w_m=\varphi(y_r, 2\delta_0)\circ u \quad \text{on } C_r.$$

Since φ is Lipschitz, by the composition chain rule of maps in $W^{1,p}$, w_m is clearly in $W^{1,p}(C_r, \tilde{B}_\varrho(y_r, 2\delta_0))$. Moreover, since $\varphi(y_r, 2\delta_0)=\mathrm{Id}$ on $\tilde{B}_\varrho(y_r, 2\delta_0)$, and since the image of u restricted to ∂C_r is in $\tilde{B}_\varrho(y_r, \delta_0)$, we have $w_m=u$ on ∂C_r. Thus defining w_m in such a way on each cube C_r in Q_m, we see that $w_m\in W^{1,p}(Q_m, N^k)$ and $w_m=u$ on ∂Q_m. It remains to show that w_m approximates u on Q_m. For $C_r\subset Q_m$, consider the set:

$$\mathcal{U}_{m,r}=\{u\in C_r | u(x)\neq w_m(x)\}.$$

We have:

$$
(6)\qquad \begin{aligned}\int_{C_r} |\nabla(u-w_m)|^p\,dx &= \int_{\mathcal{U}_{m,r}} |\nabla(u-w_m)|^p\,dx \leqslant K_5 \int_{\mathcal{U}_{m,r}} (|\nabla u|^p + |\nabla w_m|^p)\,dx \\ &\leqslant K_5(1+K(2\delta_0)^p) \int_{\mathcal{U}_{m,r}} |\nabla u|^p\,dx.\end{aligned}
$$

Thus, in order to prove that w_m approximates u on Q_m, we only have to show that (meas $\mathcal{U}_{m,r}$) m^n is small. Since it is easier to argue on "blow-up" maps, we consider the maps $\tilde{u}_m$ and $\tilde{w}_m$ defined on C'^n by:

$$
\tilde{u}_m(x) = u(x/m + x_r); \quad \tilde{w}_m(x) = w(x/m + x_r) \quad \text{where } x_r \text{ is the barycenter of } C_r
$$

($\tilde{u}_m$ and $\tilde{w}_m$ are the "blow-up" maps of u_m and w respectively). We also consider the set:

$$
\mathcal{A}_{m,r} = \{x \in C'^n \,|\, \tilde{u}_m(x) \notin B(y_r, 2\delta_0)\}.
$$

It is easy to verify that:

$$
(7)\qquad m^n \operatorname{meas} \mathcal{U}_{m,r} \leqslant \operatorname{meas} \mathcal{A}_{m,r},
$$

since if x is in $\mathcal{U}_{m,r}$, $u(x) \neq w_m(x)$ and thus $u(x) \notin B(y_r, 2\delta_0)$ (the factor m^n in (7) being the scaling factor). We are going to estimate meas $\mathcal{A}_{m,r}$. We *claim that:*

$$
(8)\qquad \operatorname{meas} \mathcal{A}_{m,r} \leqslant m^{-\nu} \varepsilon / \varepsilon_0(\delta_0, N^k).
$$

Proof of the claim. For a in $[-1/2, 1/2]$ we consider the hyperplanes $P(a, 1)$, defined in Section I.1. We have, by Fubini's theorem, and since $C_r \subset Q_m$ implies that $E(\tilde{u}_m; C'^n) \leqslant \varepsilon m^{-\nu}$:

$$
\int_{-1/2}^{1/2} E(\tilde{u}_m; P(a, 1))\,da \leqslant E(\tilde{u}_m; C'^n) \leqslant \varepsilon m^{-\nu}.
$$

It follows that

$$
(9)\qquad \operatorname{meas}\left\{ a \in \left[-\frac{1}{2}, \frac{1}{2}\right] \middle| E(\tilde{u}_m; P(a, 1)) \leqslant \varepsilon_0(\delta_0, N^k) \right\} \geqslant 1 - m^{-\nu} \varepsilon / \varepsilon_0(\delta_0, N^k),
$$

where $\varepsilon_0(\delta_0, N^k)$ is the constant arising in Lemma 4. For every a such that $E(\tilde{u}_m; P(a, 1)) \leqslant \varepsilon_0(\delta_0, N^k)$, we may apply Lemma 4, to $P(a, 1)$ which is a translate of C'^{n-1}, p and $q = n-1$; since $u(\partial C_r) \subset \tilde{B}_\varrho(y_r, \delta_0)$, Lemma 4 then shows that

$\tilde{u}_m(P(a,1))\subset\tilde{B}_\varrho(y_r,2\delta_0)$, and hence $\mathscr{A}_{m,r}\cap P(a,1)=\emptyset$. Inequality (8) follows from this relation and (9), and this completes the proof of the claim. Next we complete the estimate of $\int_{Q_m}|\nabla(u-w_m)|^p\,dx$.

Adding relation (7), for all cubes C_r in Q_m we obtain

$$\begin{aligned}\int_{Q_m}|\nabla(u-w_m)|^p\,dx&\leqslant K_5(1+\tilde{K}(2\delta_0)^p)\int_{\cup_{r\in J_m}\mathscr{U}_{m,r}}|\nabla|^p\,dx\\&\leqslant K_5(1+\tilde{K}(2\delta_0)^p)\,E\big(u;\cup_{r\in J_m}\mathscr{U}_{m,r}\big).\end{aligned}\tag{10}$$

Relations (7) and (8) shows that:

$$\operatorname{meas}\left(\cup_{r\in J_m}\mathscr{U}_{m,r}\leqslant\sum_{r\in J_m}m^{-n}\operatorname{meas}\mathscr{A}_{m,r}\right)\leqslant(\#J_m)\,m^{-n}m^{-\nu}\varepsilon/\varepsilon_0(\delta_0,N^k)$$

since $\#J_m\leqslant(m+1)^n$, we see that $\operatorname{meas}(\cup_{r\in J_m}\mathscr{U}_{m,r})\to0$ when $m\to+\infty$ and thus by (10), and by the dominated convergence theorem, we have:

$$\int_{Q_m}|\nabla(u-w_m)|^p\,dx\to0\quad\text{when}\quad m\to+\infty.\tag{11}$$

Hence w_m approximates u in $W^{1,p}$ on Q_m. In order to approximate u by continuous maps, we have to "smoothen" w_m. For this purpose we are going to use the following lemma, the proof of which is given in the Appendix:

LEMMA 6. *Let $\mu>0$, let $p>1$, and let $\tilde{B}_\varrho(y,\delta)$ be some ball in N^k such that $\tilde{B}_\varrho(y,\delta)\subset\mathcal{O}$. Let v be in $W^{1,p}(C^n(\mu),\tilde{B}_\varrho(y;\delta))$, such that v restricted to ∂C^n is in $W^{1,p}(\partial C^n,\tilde{B}_\varrho(y,\delta))\cap C^0$. Then v can be approximated in $W^{1,p}(C^n(\mu),N^k)$ by maps v_m in $W^{1,p}(C^n(\mu),N^k)\cap C^0$, which coincide with v on $\partial C^n(\mu)$.*

We apply Lemma 6 to each cube C_r in Q_m, w_m and $\tilde{B}_\varrho(y_r,2\delta)$. Lemma 6 provides us a map u_m such that $u_m=w_m=u$ on ∂Q_m, and such that:

$$\int_{Q_m}|\nabla(w_m-u_m)|^p\,dx\leqslant\frac{1}{m}.\tag{12}$$

Combining (12) and (11) we see that

$$\int_{Q_m}|\nabla(u-u_m)|^p\,dx\to0\quad\text{when}\quad m\to\infty.\tag{12}$$

This completes the construction of u_m on Q_m.

I.5. *Proof of Theorem* 1 *completed in the case* $M^n=C^n$ *and* $n-1<p<n$. In Section I.3 we have constructed u_m on P_m, such that u_m is continuous except a finite number of point singularities, and such that $u_m=u$ on ∂P_m. In Section I.4 we have constructed u_m on Q_m, such that u_m is in $W^{1,p}(Q_m; N^k)\cap C^0$ and $u_m=u$ on ∂Q_m. Thus, since $u_m=u$ on $\partial P_m\cap\partial Q_m$, and since $P_m\cup Q_m=C^n$, the map u_m is in $W^{1,p}(C^n, N^k)$, and moreover is continuous except at finite number of points. We have:

$$\int_{C^n}|\nabla(u-u_m)|^p\,dx\leq\int_{Q_m}|\nabla(u-u_m)|^p\,dx+K_5[E(u;P_m)+E(u_m;P_m)].$$

Thus combining (5) and (13) we see that $\int_{C^n}|\nabla(u-u_m)|^p\,dx\to 0$. This proves that we may approximate u by maps in $W^{1,p}(C^n, N^k)$ continuous except at a finite number of point singularities. Theorem 1 then follows, in the case considered here, from the hypothesis $\pi_{n-1}(N^k)=0$ and Lemma 1.

I.6. *Proof of Theorem* 1 *bis for* $M^n=C^n$, *and for* $n-1<p\leq n$. For $n-1<p<n$ the proof of Theorem 1 shows that the approximation sequence u_m agrees with u on ∂C^n. This completes the proof of Theorem 1 bis.

For $n=p$, we apply the result of Schoen and Uhlenbeck [SU1] and adapt the proof of Lemma 6. This result is stated as Lemma 6 bis of the Appendix.

In the next section, we prove Theorem 1 in the case $M^n=C^n$ and $n-2<p\leq n-1$.

II. Proof of Theorem 1 in the case $M^n=C^n$ and $n-2<p\leq n-1$

Throughout Section II we assume that the hypothesis of Theorem 1 concerning N^k is satisfied, that is, we assume $\pi_{n-1}(N^k)=0$ if $p=n-1$ and $\pi_{n-1}(N^k)=0$ if $n-2<p<n-1$. A new difficulty is that, in this section, we cannot apply Lemma 4 with $q=n-1$ (the Sobolev embedding theorem does not hold in this situation). Note that in Section I.1, we have considered a $(n-1)$-skeleton of C^n, namely $\bigcup_{r=1}^{(m+1)^n}\partial C_r$. In order to treat the case $n-2<p\leq n-1$, and to apply Lemma 4, we need to consider a $(n-2)$-skeleton of C^n (this idea will be generalized in Section III). Another difficulty, when $n-2<p<n-1$ is that we have to "eliminate" point singularities, even if we do not assume $\pi_{n-1}(N^k)=0$. This difficulty can be overcome using Lemma 2. In this section, the proof of Theorem 1 follows the same steps as in Section I. First, we show how to modify the method for dividing C^n in little cubes C_r, in order to find a convenient $(n-2)$-skeleton of C^n.

II.1. *Division of* C^n *in little cubes and the definition of a* $(n-2)$*-skeleton of* C^n. Let u be in $W^{1,p}(C^n, N^k)$. As in Section I.1 we may assume without loss of generality that u restricted to the boundary is in $W^{1,p}(\partial C^n, N^k)$. As in Section I.1 we slice C^n by hyperplanes $P(a,k)$ such that u restricted to $P(a,k)$ is in $W^{1,p}$. This leads to a division of C^n in $(m+1)^n$ cubes C_r. Each of these cubes has $2n$ faces that we denote by $S_{r,1}, \ldots, S_{r,i}, \ldots, S_{r,2n}$. We set for $i=1,\ldots,2n$, $A_{r,i}=\partial S_{r,i}$, the boundary of $S_{r,i}$, which is a union of $2(n-1)$ cubes of dimension $n-2$. We note $Z_r=\bigcup_{i=1}^{2n} A_{r,i}$, Z_r is a $(n-2)$-skeleton of C_r, and we consider the $(n-2)$-skeleton of C^n $\bigcup_{r=1}^{(m+1)^n} Z_r$. Adapting the slicing method of Section I.1, we may divide C^n in $(m+1)^n$ cubes C_r in such a way that:

(14) The cubes C_r are translates of $[-1/2m, 1/2m]^n$ except those in contact with the boundary which are diffeomorphic to $[-1/2m, 1/2m]^n$ by a linear map f_r such that $|\nabla f_r|\leq 4$, $|\nabla f_r^{-1}|\leq 4$.

(15) u restricted to ∂C_r is in $W^{1,p}(\partial C_r, N^k)$ for every r in $[1,(m+1)^n]$.

(16) u restricted to Z_r is in $W^{1,p}$ and thus continuous on Z_r, by the Sobolev embedding theorem.

The following inequalities hold:

(17) $\sum_{r=1}^{(m+1)^n} \underline{E}(u;\partial C_r)\leq K_6\, mE(u)+\underline{E}(u;\partial C^n)\leq K_7\, mE(u)$ for m large enough.

(18) $\sum_{r=1}^{(m+1)^n} E(u;Z_r)\leq K_6(m^2E(u)+m\underline{E}(u;\partial C^n)+\underline{E}(u;Z))\leq K_7\, m^2E(u)$ for m large enough.

We have set $Z=\bigcup_{i=1}^{2n}\partial S_i$, where for $i=1,\ldots,2n$, S_i are the faces of C^n. We have assumed, and that is not a restriction, that u restricted to Z is in $W^{1,p}$ (see Lemma A0 of the Appendix). In the next section, we are going to adapt the definition of the sets P_m and Q_m, introduced in Section I.2.

II.2. *Definition of P_m and Q_m*. Let $\varepsilon>0$ be small, to be fixed later. As in part I.2 we consider the following cubes:

- The cubes C_r such that $\underline{\tilde{E}}_m(u;Z_r)\geq\varepsilon$. We note $P_{1,m}$ the union of these cubes, and $I_{1,m}$ the set of indexes for these cubes, that is $P_{1,m}=\bigcup_{r\in I_{1,m}} C_r$.
- The cubes C_r such that $\underline{\tilde{E}}_m(u;\partial C_r)\geq\varepsilon m^{-\nu}$. We note $P_{2,m}$ the union of these cubes, and $I_{2,m}$ the set of indexes for these cubes, that is $P_{2,m}=\bigcup_{r\in I_{2,m}} C_r$ ($\nu>0$ is some fixed constant sufficiently small).
- The cubes C_r such that $\tilde{E}_m(u, C_r)\geq\varepsilon m^{-\nu}$. We note $P_{3,m}$ the union of these cubes and $I_{3,m}$ the set of indexes for these cubes, that is $P_{3,m}=\bigcup_{r\in I_{3,m}} C_r$.

Finally we set

$$P_m = P_{1,m} \cup P_{2,m} \cup P_{3,m}, \quad Q_m = \overline{C^n \setminus P_m},$$

$$I_m = I_{1,m} \cup I_{2,m} \cup I_{3,m} \quad \text{and} \quad J_m = \{1, \ldots, (m+1)^n\} \setminus I_m.$$

P_m is the set of "bad" cubes, Q_m the set of good cubes. As in Section II.2 the relations (17), (18), the scaling equalities (3) and $\tilde{E}_m(u; Z_r) = m^{n-p-2} \underline{E}(u; Z_r)$ imply that meas $P_m \to 0$. We are going to approximate u in different ways on P_m and Q_m. Since the method in this section is a little more involved than the one of Section I, our approximation sequence, which we denote by $w(m, \mu, \eta)(x)$, will depend on three parameters: $m \in \mathbf{N}^*$ (which goes to $+\infty$), $\mu > 0$ and $\eta > 0$ (which will go to zero). Roughly speaking, we want $w(m, \mu, \eta)$ to be located, locally on Q_m, in small balls of N^k, whereas we allow $w(m, \mu, \eta)$ to have "singularities" on P_m: but these singularities can be eliminated using Lemma 2. First we are going to construct the approximation map $w(m, \mu, \eta)$ on Q_m.

II.3. *Construction of the approximation map $w(m, \mu, \eta)$ of u on Q_m.* For $p \leq n-1$ and C_r in Q_m the image by u of ∂C_r may not lie in some domain $\tilde{B}_\varrho(y_r, \delta)$ even if $\tilde{E}_m(u; \partial C_r)$ is small; for this reason, $w(m, \mu, \eta)$ will not agree with u on ∂C_r (but will agree with u on Z_r). For $0 < \alpha < 1$, we consider the set $C_r(\alpha)$ defined by:

$$C_r(\alpha) = \left\{ x \in C_r \,\middle|\, \|x - x_r\| \leq \frac{1}{2m}(1-\alpha) \right\},$$

where x_r is the barycenter of C_r. Using linear interpolations, it is easy to construct a bilipschitz map $\Phi(r, \mu, \eta)$ from C_r to $C_r(\eta)$ such that, for $\mu > 0$, $\eta > 0$ and $2\eta < \mu < 1$, we have:

$$\Phi(r, \mu, \eta)(x) = x \quad \text{on } C_r(\mu);$$

$$\Phi(r, \mu, \eta)(x) = \frac{x - x_r}{2\|x - x_r\| m}(1-\eta) + x_r \quad \text{on } \partial C_r; \quad \text{and} \tag{19}$$

$$|\nabla \Phi(r, \mu, \eta)| \leq K_8\left(\frac{\eta}{\mu - \eta} + 1\right); \quad |\nabla \Phi^{-1}(r, \mu, \eta)| \leq K_8\left(\frac{\eta}{\mu - \eta} + 1\right).$$

We shall choose η of the form $\eta = 1/q$, $q \in \mathbf{N}^*$. For $C_r \subset Q_m$, we are going to construct $w(m, \mu, \eta)$ (we simply note $w(\mu, \eta)$ when there is no confusion possible) on C_r. For $C_r \subset Q_m$ recall that we have, by definition.

$$\tilde{E}_m(u; C_r) \leq \varepsilon m^{-\nu}; \tag{20}$$

$$\tilde{\underline{E}}_m(u; \partial C_r) \leq \varepsilon m^{-\nu}; \tag{21}$$

$$\tilde{\underline{E}}_m(u; Z_r) \leq \varepsilon. \tag{22}$$

Let δ_0 be small. Since Z_r is a union of $(n-2)$-dimensional cubes, we may apply Lemma 4 with $q=n-2, p$ (recall that here $p>n-2$) to each of the cubes composing Z_r. Thus if we choose $\varepsilon \leqslant \varepsilon_0(\delta_0/4n^2, n-2, p, N^k)$, Lemma 4 and relation (22) show that the image of u on Z_r lies in some domain $\tilde{B}_\varrho(y_r, \delta)$ of N^k, $y_r \in N^k$. For technical reasons, we shall give two different definitions of $w(m, \mu, \eta)$ on $C_r(\eta)$ first, then on $C_r \setminus C_r(\eta)$.

Definition of $w(\mu, \eta)$ *on* $C_r(\eta)$. The idea of the construction of $w(\mu, \eta)$ on $C_r(\eta)$ is essentially the same as in Section I.4, with some slight technical modifications. We set

$$w(\mu, \eta) = \varphi(y_r, 8n\delta_0) \circ u \circ \Phi^{-1}(r, \mu, \eta) \quad \text{on } C_r(\eta) \tag{23}$$

(δ_0 to be determined later). Hence on $C_r(\mu)$ we have $w(\mu, \eta) = \varphi(y_r, 8n\delta_0) \circ u$, since $\Phi(r, \mu, \eta) = \mathrm{Id}$ on $C_r(\mu)$. (This corresponds to the definition given in Section I.4, for w_m on C_r in Q_m.) The definition of $w(\mu, \eta)$ on $C_r \setminus C_r(\eta)$ is more involved.

Definition of $w(\mu, \eta)$ *on* $C_r \setminus C_r(\eta)$. (In fact, the definition of $w(\mu, \eta)$ we are going to give holds only in the case $\partial C_r \cap \partial P_m = \varnothing$. In the case $\partial C_r \cap \partial P_m \neq \varnothing$ the definition is slightly different, see Section II.4.) For $i=1, \ldots, 2n$, we consider the faces $S_{r,i}$ of ∂C_r and we may consider, for simplicity, that for every i, $S_{r,i}$ is a tranlate of $[0, 1/m]^{m-1}$. The idea of the construction of $w(\mu, \eta)$ on $C_r \setminus C_r(\eta)$ is the following: In a first step, we want to define $w(\mu, \eta)$ on $S_{r,i}$. For this purpose, we show by adapting the method of Section I.1, that we may divide $S_{r,i}$ in $(q+1)^{n-1}$ $(n-1)$-dimensional cubes, such that the image by u of the boundary of these cubes lies in $\tilde{B}_\varrho(y_r, 2\delta_0)$. We then define $w(\mu, \eta)$ on the boundary of these cubes by $w(\mu, \eta) = u$. In a second step, we extend the value of $w(\mu, \eta)$ to ∂C_r first and then to the interior of $C_r \setminus C_r(\eta)$ in such a way that the definition is compatible with the one given on $\partial C_r(\eta)$. First we present the division of $S_{r,i}$ in convenient cubes:

First step of the construction of $w(m, \mu, \eta)$ *on* $C_r \setminus C_r(\eta)$: *division of the faces* $S_{r,i}$ *in* $(n-1)$*-dimensional cubes*. For $k=1, \ldots, n-1$, and $a \in [1/4mq, 3/4mq]$ (recall that $\eta = 1/q$), let $P^{n-2}(a, k)$ be the restricton to $S_{r,i}$ of the $(n-2)$-dimensional hyperplane orthogonal to $A_k(a) = ae_k$ (here we choose coordinates such that $S_{r,i} = [0, 1/m]^{n-1} \times \{0\}$). For each $k=1, \ldots, n-1$, there is some α_k in $[1/4mq, 3/4mq]$ such that u restricted to $\bigcup_{j=1}^{q-1} P^{n-2}(\alpha_k + j/mq, k)$ is in $W^{1,p} \hookrightarrow C^0$, and such that:

$$\sum_{j=0}^{q-1} \underline{E}\left(u; P^{n-2}\left(\alpha_k + \frac{j}{mq}, k\right)\right) \leqslant q\underline{E}(u; S_{r,i}). \tag{24}$$

Since for every a $P^{n-2}(a, k)$ may be considered as an $(n-2)$-dimensional cube, and is in fact a translate of $[0, 1/m]^{n-2}$, we may apply Lemma 4 to $q=n-2$, p and $P^{n-2}(a, k)$. Thus if $P^{n-2}(a, k)$ is such that:

$$\tilde{E}_m(u; P^{n-2}(a, k)) \leqslant \varepsilon_0(\delta_0, n-2, p, N^k) \leqslant \varepsilon,$$

the image by u of $P^{n-2}(a, k)$ lies in $\tilde{B}_\varrho(y_i, 2\delta_0)$ (since $u(\partial P^{n-2}(a, k)) \subset \tilde{B}_\varrho(y_i, \delta_0)$ for some $y_i \in N_k$, since $\tilde{E}(u; S_{r,i}) \leqslant \varepsilon$). On the other hand, since $C_r \subset Q_m$ we have:

$$\tilde{E}_m(u; S_{r,i}) \leqslant \varepsilon m^{-\nu},$$

which implies that, by the same argument as in Section I.4 (proof of the claim);

$$\text{meas}\left\{ a \in \left[0, \frac{1}{m}\right] \middle| \tilde{E}_m(u; P^{n-2}(a, k)) \geqslant \varepsilon \right\} \leqslant m^{-(1+\nu)}. \tag{25}$$

We now consider, for any k in $[1, n-1]$, the planes given by (24), such that $\tilde{E}_m(u; P^{n-2}(\alpha_k + j/mq, k)) \geqslant \varepsilon$. If m is large enough, relation (25) implies that there is some $\beta_{k,j}$ in $[\alpha_k + j/mq - 1/10qm, \alpha_k + j/mq + 1/10qm]$ such that: $\tilde{E}_m(u; P^{n-2}(\beta_{k,j}, k)) \leqslant \varepsilon$. Thus we may replace the hyperplane $P^{n-2}(\alpha_k + j/mq, k)$ by the hyperplane $P^{n-2}(\beta_{k,j}, k)$ in our slicing method. Doing this for all the directions, for $k = 1, \dots, n-1$, we see that we, having divided the face $S_{r,i}$ of C_r in $(q+1)^{n-1}$ $(n-1)$-dimensional "cubes", which we note $C_{l,i}^{n-1}$, for $l=1$ to $(q+1)^{n-1}$: in fact these cubes are not "perfect" cubes, but nevertheless they are all diffeomorphic to $[-1/2qm, 1/2qm]^{n-1}$ by linear maps $f_{l,i}$ such that $|\nabla f_{l,i}| \leqslant 5$, $|\nabla f_{l,i}^{-1}| \leqslant 5$ and such that u restricted to $\partial C_{l,i}^{n-1}$ is in $W^{1,p} \hookrightarrow C^0$, and takes value on $\tilde{B}_\varrho(y_i, 2\delta_0)$ (note that y_i depends only on the value of u on $S_{r,i}$). Moreover (24), clearly remains true if we replace $P^{n-2}(\alpha_k + j/m, k)$ by $P^{n-1}(\beta_{k,j}, k)$, and gives:

$$\sum_{l=1}^{(q+1)^{n-1}} \tilde{E}(u; \partial C_{l,i}^{n-1}) \leqslant K_{10}\, q \tilde{E}(u; S_{r,i}) + E(u; \partial S_{r,i}) \quad \text{for} \quad i = 1, \dots, 2n. \tag{26}$$

On $\bigcup_{l=1}^{(q+1)^{n-1}} \partial C_{l,i}^{n-1}$ we define $w(\mu, \eta)$ in the following way

$$w(\mu, \eta) = u \quad \left(\text{on } \bigcup_{l=1}^{(q+1)^{n-1}} \partial C_{l,i}^{n-1}, \quad \text{for } i = 1, \dots, 2n\right). \tag{27}$$

In particular $w(\mu, \eta) = u$ on Z_r.

Next we extend $w(\mu, \eta)$ to $S_{r,i}$, for $i=1$ to $2n$ (that is we are going to define $w(m, \mu, \eta)$ on ∂C_r).

Second step. Definition of $w(\mu, \eta)$ *on* $\bigcup_{i=1}^{2n} S_{r,i} = \partial C_r$. u restricted to each cube $C_{l,i}^{n-1}$ is in $W^{1,p}(C_{l,i}^{n-1}, N^k)$ and, u restricted to $\partial C_{l,i}^{n-1}$ is in $W^{1,p} \hookrightarrow C^0$. Since $\pi_{[p]}(N^k) = 0$ (by assumption), we may apply Theorem 1 bis, of Section I.6, to $C_{l,i}^{n-1}$ and u: thus u restricted to $C_{l,i}^{n-1}$ can be approximated, on $C_{l,i}^{n-1}$ by maps in $W^{1,p}(C_{l,i}^{n-1}, N^k)\cap C^0$ which coincide with u on $\partial C_{l,i}^{n-1}$. Let $w(\mu,\eta)$ be on $C_{l,i}^{n-1}$ such an approximation map of u satisfying

$$\begin{cases} w(\mu,\eta)=u \text{ on } \partial C_{l,i}^{n-1} \quad \text{and} \quad w(\mu,\eta)\in W^{1,p}(C_{l,i}^{n-1}, N^k)\cap C^0; \\ E(w(\mu,\eta); C_{l,i}^{n-1})\leq 2E(u; C_{l,i}^{n-1}). \end{cases} \tag{28}$$

Note that this definition is compatible with (27) and that, if we define $w(\mu,\eta)$ in the previous way on all the cubes $C_{l,i}^{n-1}$, $w(\mu,\eta)$ is in $W^{1,p}(\partial C_r, N^k)\cap C^0$ and for each face $S_{r,i}$ we have

$$E(w(\mu,\eta); S_{r,i})\leq 2E(u; S_{r,i}). \tag{29}$$

Finally, we only have to extend $w(\mu,\eta)$ to the interior of $C_r\setminus C_r(\eta)$ ($w(\mu,\eta)$ has yet been defined on ∂C_r and $C_r(\eta)$, thus on $\partial(C_r\setminus C_r(\eta))$).

Third step. Definition of (μ,η) *in the interior of* $C_r\setminus C_r(\eta)$. We consider the map π_r from $C_r\setminus C_r(\eta)$ to ∂C_r defined by:

$$\pi_r(x)=\frac{x-x_r}{2\|x-x_r\|\, m}+x_r,$$

and we consider the set $\mathcal{M}_{\mu,\eta}$ defined by

$$\mathcal{M}_{\mu,\eta}=\pi_r^{-1}\left(\bigcup_{i=1}^{2n}\bigcup_{q=1}^{(q+1)^{n-1}}\partial C_{l,i}^{n-1}\right).$$

$\mathcal{M}_{\mu,\eta}$ is a union of portions of $(n-1)$-dimensional planes. We consider the set $\mathcal{N}_{\mu,\eta}=\mathcal{M}_{\mu,\eta}\cup\partial C_r\cup\partial C_r(\eta)$. We may consider that $\mathcal{N}_{\mu,\eta}$ is the union of the boundaries of $2n(q+1)^{n-1}$ cubes $\mathscr{C}_{l,i}^n$ (which are n-dimensional), such that:

$$\bigcup_{i=1}^{2n}\bigcup_{l=1}^{(q+1)^{n-1}}\mathscr{C}_{l,i}^n=\overline{C_r\setminus C_r(\eta)}$$

and

$$\bigcup_{i=1}^{2n}\bigcup_{l=1}^{(q+1)^{n-1}}\partial\mathscr{C}_{l,i}^n=\mathcal{N}_{\mu,\eta}. \tag{30}$$

All these n-dimensional cubes are diffeomorphic to $[-1/2qm, 1/2qm]^n$ by Lipschitz maps $\tilde f_{l,i}$ such that $|\nabla\tilde f_{l,i}|\leq K_{12}$, $|\nabla\tilde f_{l,i}^{-1}|\leq K_{12}$. On $\mathcal{M}_{\mu,\eta}$ (that is on the faces of these cubes which are not included in $\partial C_r(\eta)$ and ∂C_r), we define $w(\mu,\eta)$ by:

$$w(\mu,\eta)=u\circ\pi_r \quad \text{on } \mathcal{M}_{\mu,\eta}. \tag{31}$$

Note that on $\partial C_r(\eta)$, we have $\pi_r=\Phi^{-1}(r,\mu,\eta)$. Since for x in $\mathcal{M}_{\mu,\eta}\cap\partial C_r(\eta)$, $\Phi^{-1}(x)=\pi_r(x)$ is in $\bigcup_{i=1}^{2n}\bigcup_{l=1}^{(q+1)^{n-1}}\partial\mathscr{C}_{l,i}^{n-1}$, $u(x)$ lies in $\bigcup\tilde{B}_\varrho(y_i,2\delta_0)\subset B(y_r,8n\delta_0)$. Thus $w(\mu,\eta)(x)=u\circ\pi_r(x)=\varphi(y_r,8n\delta_0)\circ u\circ\pi_r(x)$. This shows that the definitions (31) and (23) are compatible on $\mathcal{M}_{\mu,\eta}\cap\partial C_r(\eta)$. We see that by (23), (28), (31) we have defined $w(\mu,\eta)$ on the boundaries of the cubes $\mathscr{C}_{l,i}^n$, and that, since the different definitions we gave for $w(\mu,\eta)$ on the different parts are compatible on the points where they intersect, it follows that $w(\mu,\eta)$ is in $W^{1,p}(\bigcup_{i=1}^{2n}\bigcup_{l=1}^{(q+1)^{n-1}}\partial\mathscr{C}_{l,i}^n,N^k)$. Now we are able to extend $w(\mu,\eta)$ to the interior of the cubes $\mathscr{C}_{l,i}^n$ using a standard radial extension of the boundary value, that is:

$$w(\mu,\eta)(x)=w(\mu,\eta)\tilde{f}_{l,i}^{-1}\left(\frac{\tilde{f}_{l,i}(x)}{\|\tilde{f}_{l,i}(x)\|\,2mq}\right) \quad \text{on } \mathscr{C}_{l,i}^n. \tag{32}$$

(Recall that $\mathscr{C}_{l,i}^n\overset{f_{l,i}}{=}[-1/2mq,1/2mq]^n$.) It is easy to verify that (32) defines now $w(\mu,\eta)$ on $C_r\setminus C_r(\eta)$, and that $w(\mu,\eta)$ is in $W^{1,p}(C_r\setminus C_r(\eta),N^k)$. This definition is also compatible with the definition of $w(\mu,\eta)$ on $C_r(\eta)$ given by (23), and thus $w(\mu,\eta)$ is in $W^{1,p}(C_r,N^k)$. Since the value of $w(\mu,\eta)$ on the faces of C_r, $S_{r,i}$ for $i=1$ to $2n$, depends only on the value of u on this faces, defining $w(\mu,\eta)$ on all the cubes C_r in Q_m by the previous method, we see that $w(\mu,\eta)$ is in $W^{1,p}(Q_m,N^k)$. Note moreover that, for each cube C_r (which is not in contact with ∂P_m) in Q_m, $w(\mu,\eta)$ is continuous on $C_r/C_r(\eta/2)$. We are going to estimate now the integral $\int_{Q_m}|\nabla u-\nabla w(m,\mu,\eta)|^p\,dx$.

Estimation of $\int_{Q_m}|\nabla u-\nabla w(m,\mu,\eta)|^p\,dx$. Let C_r be included in Q_m. We consider the set $\mathscr{U}(m,r,\mu)=\{x\in C_r(\mu)\,|\,u(x)\notin\tilde{B}_\varrho(y_r,8n\delta_0)\}$. We *claim* that we have:

$$\int_{Q_m}|\nabla u-\nabla w(m,\mu,\eta)|^p\,dx\le K_{27}\left(\eta E(u)+\int_{\bigcup_{r\in J_m}\mathscr{U}(m,r,\mu)}|\nabla u|^p\,dx+\int_{\bigcup_{r\in J_m}(C_r\setminus C_r(\mu))}|\nabla u|^p\,dx\right). \tag{33}$$

Proof of the claim. For C_r included in Q_m we are going to estimate the integral of $|\nabla u-\nabla w(\mu,\eta)|^p$ on $C_r(\mu)$, $C_r(\eta)-C_r(\mu)$, and on $C_r\setminus C_r(\eta)$.

On $C_r(\mu)$. We have using relation (23) $w(\mu,\eta)=\varphi(y_r,8n\delta_0)\circ u$. This implies:

$$\int_{C_r(\mu)}|\nabla u-\nabla w(\mu,\eta)|^p\,dx=\int_{u\neq w(\mu,\eta)}|\nabla u-\nabla w(\mu,\eta)|^p\,dx\le K_5(1+\tilde{K}(8n\delta_0)^p)\int_{u\neq w(\mu,\eta)}|\nabla u|^p\,dx.$$

Since, by (23), $\{x \in C_r(\mu) | u(x) \neq w(\mu, \eta)(x)\} \subset \mathcal{U}(m, r, \mu)$ we have

$$\int_{C_r(\mu)} |\nabla u - \nabla w(\mu, \eta)|^p \, dx \leqslant K_{17} \int_{\mathcal{U}(m,r,\mu)} |\nabla u|^p \, dx. \tag{34}$$

On $C_r(\eta) \setminus C_r(\mu)$. We have $w(\mu, \eta) = \varphi(y_r, 8n\delta_0) \circ u \circ \Phi^{-1}(r, \mu, \eta)$. Thus

$$\int_{C_r(\eta) \setminus C_r(\mu)} |\nabla u - \nabla w(\mu, \eta)|^p \, dx \leqslant K_5 (1 + \tilde{K}(8n\delta_0)^p) \, |\nabla \Phi^{-1}|_\infty^p |\nabla \Phi|_\infty^p \int_{C_r(\eta) \setminus C_r(\mu)} |\nabla u|^p \, dx.$$

Since we have assumed $\eta/\mu \leqslant \frac{1}{2} |\nabla \Phi^{-1}|_\infty |\nabla \Phi|_\infty$ is uniformly bounded, and thus

$$\int_{C_r(\eta) \setminus C_r(\mu)} |\nabla u - \nabla w(\mu, \eta)|^p \, dx \leqslant K_{18} \int_{C_r \setminus C_r(\mu)} |\nabla u|^p \, dx. \tag{35}$$

On $C_r \setminus C_r(\eta)$. Since

$$\overline{C_r \setminus C_r(\eta)} = \bigcup_{i=1}^{2n} \bigcup_{l=1}^{(q+1)^{n-1}} \mathscr{C}^n_{l,i}$$

we have

$$\int_{C_r \setminus C_r(\mu)} |\nabla w(\mu, \eta)|^p \, dx \leqslant \sum_{i=1}^{2n} \sum_{l=1}^{(q+1)^{n-1}} \int_{\mathscr{C}^n_{l,i}} |\nabla w|^p \, dx. \tag{36}$$

Since, on every $\mathscr{C}^n_{l,i}$ $w(\mu, \eta)$ is a radial extension of the boundary value, given by (32) we have:

$$\int_{\mathscr{C}^n_{l,i}} |\nabla w(\mu, \eta)|^p \, dx \leqslant K_{19} \frac{1}{mq} E(w(\mu, \eta); \partial \mathscr{C}^n_{l,i}) = K_{19} \eta m^{-1} E(w(\mu, \eta); \partial \mathscr{C}^n_{l,i}). \tag{37}$$

By the construction of $w(\mu, \eta)$ on $\partial \mathscr{C}^n_{l,i}$ we have

$$E(w(\mu, \eta); \partial \mathscr{C}^n_{l,i} \cap \partial C_r) = E(w(\mu, \eta); \mathscr{C}^{n-1}_{l,i}) \leqslant 2E(u; \mathscr{C}^{n-1}_{l,i});$$

$$E(w(\mu, \eta); \partial \mathscr{C}^n_{l,i} \cap \partial C_r(\eta)) \leqslant K_{20} E(u; \mathscr{C}^{n-1}_{l,i}); \quad \text{and}$$

$$E(w(\mu, \eta); \partial \mathscr{C}^n_{l,i} \cap \partial \mathscr{M}_{\mu,\eta}) \leqslant K_{21} E(u; \partial \mathscr{C}^{n-1}_{l,i}) \eta m^{-1}.$$

Adding these relations we obtain

$$E(w(\mu, \eta); \partial \mathscr{C}^n_{l,i}) \leqslant K_{22} [E(u; \mathscr{C}^{n-1}_{l,i}) + E(u; \partial \mathscr{C}^{n-1}_{l,i}) \eta m^{-1}]. \tag{38}$$

Combining (36), (37), (38) and (26) we obtain

$$\int_{C_r \setminus C_r(\eta)} |\nabla w(\mu, \eta)|^p \, dx \leqslant K_{23} \frac{\eta}{m} E(u; Z_r). \tag{39}$$

Since $\int_{C_r\setminus C_r(\eta)}|\nabla u-\nabla w(\mu,\eta)|^p\,dx\leq K_5\int(|\nabla u|^p+|\nabla w(\mu,\eta)|^p)\,dx$, (39) gives:

$$\int_{C_r\setminus C_r(\eta)}|\nabla u-\nabla w(\mu,\eta)|^p\,dx\leq K_{24}[E(u;C_r\setminus C_r(\eta))+\eta m^{-1}E(u;\partial C_r)+\eta^2m^{-2}E(u;Z_r)]. \tag{40}$$

Adding (40) for all the cubes C_r included in Q_m, using the relations (17) and (18) and noting that $C_r\setminus C_r(\eta)\subset C_r\setminus C_r(\mu)$, we obtain:

$$\int_{\bigcup_{r\in J_m}C_r\setminus C_r(\eta)}|\nabla u-\nabla w(\mu,\eta)|^p\,dx\leqslant K_{25}\left(\int_{\bigcup_{r\in J_m}C_r\setminus C_r(\eta)}|\nabla u|^p\,dx+\eta E(u)\right) \tag{41}$$

for m large enough. This completes the estimate on $\overline{C_r\setminus C_r(\eta)}$. Adding (41), (35) and (36) we obtain equality (33), and this completes the proof of the claim.

We are going to prove now that (33) implies that $\int_{Q_m}|\nabla u-\nabla w(m,\mu,\eta)|^p\,dx$ tends to zero when n goes to $+\infty,\mu\to0$, $\eta\to0$ and $\eta/\mu\leq1/2$. First we remark that

$$\operatorname{meas}(\bigcup_{r\in J_m}C_r\setminus C_r(\mu))\leqslant2\mu\to0\quad\text{when}\quad\mu\to0.$$

Thus, by the dominated convergence theorem

$$\int_{\bigcup_{r\in J_m}C_r\setminus C_r(\eta)}|\nabla u|^p\,dx\to0\quad\text{when}\quad\mu\to0. \tag{42}$$

Next, we have to show that $\operatorname{meas}(\bigcup_{r\in J_m}\mathcal{U}(m,r,\mu))\to0$ when $m\to+\infty$ and $\mu\to0$. The proof of this convergence is similar to the proof of relation (8) (see Section I.4). Let $\tilde u(m,\mu)$ be a "blow-up" map u, from C'^n to N^k defined by

$$\tilde u(m,\mu)(x)=u\left(\frac{x}{m}(1-\mu)+x_r\right)\quad\text{on }C'^n.$$

We set $A(m,r,\mu)=\{x\in C'^n|\,\tilde u(m,\mu)\notin\tilde B_\varrho(y_r,8n\delta_0)\}$. For a in$[-1/2,1/2]$ we consider the hyperplanes $P(a,1)$ and the subsets of $[-1/2,1/2]$ defined by:

$$B_1=\left\{a\in\left[-\frac12,\frac12\right]\middle|\,\underline{E}(\tilde u(m,\mu);P(a,1))\geqslant\varepsilon_1=\varepsilon_0(\delta_0,n-2,p,N^k)\right\}.$$

$$B_2=\left\{a\in\left[-\frac12,\frac12\right]\middle|\,\underline{E}(\tilde u(m,\mu);\partial P(a,1))\geqslant\varepsilon_1\right\}.$$

Since $\underline{E}(\tilde u(m,\mu);C'^n)\leq\varepsilon m^{-\nu}$,

$$\operatorname{meas}(B_1)\leq\varepsilon\varepsilon_1^{-1}m^{-\nu}.$$

Likewise, since $\underset{\sim}{E}(\tilde{u}(m,\mu);\, \partial C'^n)\leq \varepsilon m^{-\nu}$,

$$\mathrm{meas}(B_2)\leq \varepsilon\varepsilon_1^{-1} m^{-\nu}.$$

Thus $\mathrm{meas}(B_1\cup B_2)\leq 2\varepsilon\varepsilon_1^{-1} m^{-\nu}$. If a is not in $B_1\cup B_2$, the same argument as in Section I.4 (proof of (18)) shows that $u(P(a,1))\subset \tilde{B}_\varrho(y_r, 4\delta_0)\subset \tilde{B}_\varrho(y_r, 8n\delta_0)$. This proves that

$$A(m,r,\mu)\subset \bigcup_{a\in B_1\cup B_2} P(a,1).$$

And thus $\mathrm{meas}\, A(m,r,\mu)\leq 2\varepsilon\varepsilon_1^{-1} m^{-\nu}$. This gives

$$\mathrm{meas}\Big(\bigcup_{r\in J_m} \mathscr{U}_{m,r,\mu}\Big)\leq K_{28}\, m^{-\nu}\to 0 \quad \text{when} \quad m\to 0. \tag{43}$$

Combining (42), (43) and (33), we see that

$$\int_{Q_m} |\nabla u-\nabla w(m,\mu,\eta)|^p\, dx\to 0 \tag{44}$$

when $m\to 0, \mu\to 0, \eta\to 0$ and $\eta/\mu\leq 1/2$. This completes the construction and the estimate of $w(m,\mu,\eta)$ on Q_m. We come now to the construction of w_m on the "bad" cubes, that is on P_m.

II.4. *Construction of the approximation map* $w(m,\mu,\eta)$ *on* P_m. We consider the connected components $P_m(j)$ of P_m, for $j=1,\dots,t(m)$, $t(m)\in \mathbf{N}^*$; $P_m=\bigcup_{j=1}^{t(m)} P_m(j)$. $P_m(j)$ is a union of cubes C_r with $r\in I_m(j)$, and $\partial P_m(j)\subset \partial Q_m\cup \partial C^n$. We are going to adapt the construction of Section I.3 (in particular Lemma 3) to the case considered here. Instead of considering each cube C_r in $P_m(j)$ separately, we apply the slicing method of Lemma 3 to $P_m(j)$ (as a whole). Thus, slicing $P_m(j)$ by hyperplanes $P(a,k)$ we obtain a division of $P_m(j)$ in little cubes that we denote by $C_{j,b}$. Since $w(\mu,\eta)$ has yet been defined on Q_m we have to respect a compatibility condition on $\partial Q_m\cap \partial P_m$. Indeed, if $S_{r,i}$ is a face of a cube C_r in P_m, which is included in ∂P_m, the map $w(\mu,\eta)$ has been defined (see Section I.3) on $S_{r,i}$ in such a way that $w(\mu,\eta)$ approximates u restricted to $S_{r,i}$, and moreover $w(\mu,\eta)=u$ on $\bigcup_{j=1}^{(q+1)^{n-1}} \partial \mathscr{C}_{l,i}^{n-1}$. Thus in order to have compatible definitions we need the following additional condition:

$$\bigcup_{l=1}^{(q+1)^{n-1}} \partial \mathscr{C}_{l,i}^{n-1}\subset \Big[\bigcup Z_{j,b}\Big]\cap S_{r,i} \quad \text{for each face} \quad S_{r,i}\subset \partial P_m(j)\cap \partial Q_m, \tag{45}$$

where $Z_{j,b}$ is the $(n-2)$-dimensional skeleton of the cube $C_{j,b}$ defined in the same way as Z_r for the cube C_r (see Section II.1).

It is easy, but fastidious to prove, that we may adapt the slicing method of Section I.1 to the set $P_m(j)$, and with a number h large enough of slicing planes (in the directions given by an orthogonal basis), we obtain a division of P_m in $z(h)$ cubes $C_{j,b}(h)$, that is $P_m(j)=\bigcup_{b=1}^{z(h)} C_{j,b}(h)$, such that for h large enough, we have the following conditions:

- $z(h)$ (the number of little cubes $C_{j,b}(h)$) verifies

$$K_{29}(\#I_m(j))\,h^n \leq z(h) \leq 2K_{29}(\#I_m(j)\,h^n \tag{46}$$

(where $\#I_m(j)$ is the number of cubes C_r in $P_m(j)$).

- $C_{j,b}(h)$ is diffeomorphic to the standard cube $[-1/2mh, 1/2mh]^n$ by a diffeomorphism $\tilde{f}_b$ such that $|\nabla f_b|\leq K_{12}, |\nabla f_b^{-1}|\leq K_{12}$.
- u restricted to $\partial C_{j,b}(h)$ is in $W^{1,p}(\partial C_j(h), N^k)$ and u restricted to $Z_{j,b}(h)$ is in $W^{1,p}(Z_{j,b}(h), N^k)\hookrightarrow C^0$, and for h large enough, we have:

$$\begin{cases} \displaystyle\sum_{b=1}^{z(h)} \underline{E}(u; \partial C_{j,b}(h)) \leq K_{30}(\#I_m(j))\, mhE(u; P_m(j)); \\ \displaystyle\sum_{b=1}^{z(h)} \underline{E}(u; Z_{j,b}(h)) \leq K_{30}(\#I_m(j))\, m^2h^2E(u; P_m(j)). \end{cases} \tag{47}$$

- Relation (45) holds.

We consider now a face $S_{r,i}$ which is included in $\partial Q_m \cap \partial P_m(j)$; thus $S_{r,i}$ is a face of some cube C'_r in Q_m. In Section II.3, we have constructed the value of $w(\mu, \eta)$ of $S_{r,i}$, in such a way that $w(\mu, \eta)$ is continuous and

$$w(\mu, \eta) = u \ \text{ on } \bigcup_{l=1}^{q+1} \partial \mathscr{C}_{l,i}^{n-1} \quad \text{and} \quad \underline{E}(w(\mu, \eta); S_{r,i}) \leq 2\underline{E}(u; S_{r,i}). \tag{48}$$

Using the same method as in Section II.3, we may, moreover assume that we have constructed $w(\mu, \eta)$ in such a way that

$$w(\mu, \eta) = u \quad \text{on} \quad \bigcup_{b=1}^{z(h)} Z_{j,b}(h) \cap S_{r,i}. \tag{49}$$

Since we have relation (45), relation (48) is automatically satisfied, and the remainder of the construction of $w(\mu, \eta)$, and our estimation on $\int_{C'_r} |\nabla u - \nabla w(\mu, \eta)|^p\, dx$ are left unchanged by this slight modification. On the set $\bigcup_{b=1}^{z(h)} Z_{j,b}(h)$ we define $w(\mu, \eta)$ by

$$w(\mu, \eta) = u \quad \text{on} \quad \bigcup_{b=1}^{z(h)} Z_{j,b}(h), \tag{50}$$

which is of course compatible with (49). On $\bigcup_{b=1}^{z(h)} \partial C_{j,b}(h)$, we set

$$w(\mu,\eta)=\xi, \tag{51}$$

where ξ is a map in $W^{1,p}(\bigcup_{b=1}^{z(h)} \partial C_{j,b}(h); N^k)\cap C^0$, such that $\xi=u$ on $\bigcup_{b=1}^{z(h)} Z_{j,b}(h)$ such that $E(\xi; \bigcup_{b=1}^{z(h)} \partial C_{j,b}(h))\leq 2E(u; \bigcup_{b=1}^{z(h)} \partial C_{j,b}(h))$, and such that ξ agrees with the value of $w(\mu,\eta)$ on $S_{r,i}$ defined by relation (49). The existence of such a map ξ can be proved using Theorem 1 bis of Section I.6, as was done previously in Section II.3 for defining a continuous value of $w(\mu,\eta)$ on $S_{r,i}$ which satisfies (48). On each cube $C_{j,b}(h)$ we extend $w(\mu,\eta)$ by a radial extension of the boundary value:

$$w(\mu,\eta)(x)=\xi\left(f_b^{-1}(h)\frac{f_b(x)}{2mh\|f_b(x)\|}\right) \quad \text{on } C_{j,b}(h). \tag{52}$$

It is then easy to verify that $w(\mu,\eta)$ is in $W^{1,p}(P_m(j), N^k)$, is continuous except on a finite set of points, which are the barycenter of the cubes $C_{j,b}$. Since we have relation (49) holding, our definitions are compatible on $\partial Q_m\cap\partial P_m$. Thus $w(\mu,\eta)$ is in $W^{1,p}(C^n, N^k)$. We estimate now $\int_{P_m}|\nabla w(\mu,\eta)|^p\,dx$: using relation (47) we have

$$E(w(\mu,\eta), C_{j,b}(h))\leq K_{31}\frac{1}{mh}E(\xi,\partial C_{j,b}(h))\leq 2K_{31}E(u;\partial C_{j,b}(h))(mh)^{-1}.$$

Adding the previous inequality for all the cube $C_{j,b}$ and combining with (47) we obtain

$$E(w(\mu,\eta); P_m)\leq K_{32}E(u;P_m). \tag{53}$$

Since $\mathrm{meas}(P_m)$ goes to zero when $m\to+\infty$, $E(w(m,\mu,\eta);P_m)$ goes to zero when $m\to+\infty$. Combining (53) and (44) we see that $E(u-w(m,\mu,\eta))\to 0$ when $m\to+\infty$, $\mu\to 0$, $\eta\to 0$ and $\eta/\mu\leq 1/2$. This shows that $w(m,\mu,\eta)$ is an approximation sequence of u. In order to complete the proof of Theorem 1, when $M^n=C^n$ and $n-2<p<n-1$, we have to show that $w(m,\mu,\eta)$ can be approximated, in $W^{1,p}(C^n, N^k)$ by maps continuous except at most at a finite number of point singularities. (The proof of Theorem 1 in the case considered here then follows from Lemma 2.)

II.5. *Proof of Theorem* 1 *completed* $(n-2<p\leq n-1)$: $w(m,\mu,\eta)$ *can be approximated by maps continuous except on a finite set.* Note first that on P_m, $w(m,\mu,\eta)$ is already continuous except on a finite number of points. In order to approximate $w(m,\mu,\eta)$ on Q_m by maps having only a finite number of point singularities, we are going to adapt the mollifying argument of Section I.5. Let φ be a smooth radial function from $\mathbf{R}^n$ to $\mathbf{R}^+$ such that $\mathrm{supp}(\varphi)\subset B^n(0,1)$ and $\int_{\mathbf{R}^n}\varphi(x)\,dx=1$. We consider the map φ^σ

from $\mathbf{R}^n$ to $\mathbf{R}^+$ defined by $\varphi^\sigma(x)=\sigma^{-n}\varphi(x/\sigma)$. First, we extend $w(\mu,\eta)$ outside C^n by $w(\mu,\eta)(x)=w(\mu,\eta)(x/\|x\|)$ if $x\notin C^n$. Note that, by the constructions of Section II.3 and Section II.4, $w(\mu,\eta)$ is continuous on the boundaries of the cubes C_r. Hence $w(\mu,\eta)$ restricted to ∂C^n is in $W^{1,p}(\partial C^n, N^k)\cap C^0$, and the extension of $w(\mu,\eta)(x)$ to $\mathbf{R}^n\setminus C^n$ defined above is also continuous on $\mathbf{R}^n\setminus C^n$. For x in $\mathbf{R}^n$ we consider the map

$$w^\sigma(x)=\int_{\mathbf{R}^n}\varphi^\sigma(x-z)\,w(\mu,\eta)(z)\,dz.$$

It is well known that w^σ is smooth on $\mathbf{R}^n$, taking value in a bounded domain of $\mathbf{R}^l$, and that w^σ tends to $w(\mu,\eta)$ in $W^{1,p}(C^n,\mathbf{R}^l)$. We consider now a cube C_r included in Q_m. We are going to modify w^σ in such a way that the new map takes value in N^k, and is continuous except at a finite number of point singularities. On $C_r(\eta)$ we have

$$w(\mu,\eta)=\varphi(y_r,8n\delta_0)\circ u\circ\Phi^{-1}.$$

Hence, the image by $w(\mu,\eta)$ of $C_r(\eta)$ is in $\tilde{B}_p(y_r,8n\delta_0)$. Moreover if one considers the way $w(\mu,\eta)$ is defined on $C_r\setminus C_r(\eta)$, it is easy to check that $w(\mu,\eta)$ takes value in $\tilde{B}_\varrho(y_r,8n\delta_0)$ on $C_r\setminus C_r(2\pi/3)$. Thus, for σ small enough w^σ takes value on $C_r\setminus C_r(\eta)$ in the convex closure (in $\mathbf{R}^l$) of $\tilde{B}_\varrho(y_r,8n\delta_0)$ which is in $\mathcal{O}$, for δ_0 small enough. We choose δ_0 in such a way that this assumption holds (note that δ_0 has not yet been determined). On $C_r\setminus C_r(\eta)$ we are going to modify w^σ in such a way that the new map takes value in $\mathcal{O}$, and has only point singularities. Recall that (see Section II.3) $C_r\setminus C_r(\eta)=\bigcup\mathscr{C}^n_{l,i}$. For simplicity, we may consider that $\mathscr{C}^n_{l,i}$ is the cube $C'^n(1/mq)=[-1/2mq,1/2mq]^n$, that

$$\partial\mathscr{C}^n_{l,i}\cap C_r(\eta)=\{-1/2mq\}\times[-1/2mq,\,1/2mq]^{n-1},$$

that

$$\partial\mathscr{C}^n_{l,i}\cap S_{r,i}=\{1/2mq\}\times[-1/2mq,\,1/2mq]^{n-1},$$

and that:

$$w(\mu,\eta)(x)=w(\mu,\eta)\left[\frac{x}{\|x\|2mq}\right] \tag{54}$$

(the real situation can be deduced from the situation considered here, using the diffeomorphism $\tilde{f}_{l,i}$ which maps $\mathscr{C}^n_{l,i}$ onto $[-1/2mq,1/2mq]^n$). We consider now the set $D^n=C'^n(1/mq)^n\setminus C'^n(1/2mq)$. Since, by construction $w(\mu,\eta)$ is continuous on $\partial\mathscr{C}^n_{l,i}\setminus C_r(\eta)$ and takes value in $\tilde{B}_\varrho(y_r,8n\delta_0)$ on $\partial\mathscr{C}^n_{l,i}\cap C_r(\eta)$, it is easy to see, using relation (54) and the properties of mollified maps, that for σ small enough, w^σ takes value, on D^n in $\mathcal{O}$. Moreover since w^σ converges strongly to $w(\mu,\eta)$ in $W^{1,p}(C^n,N^k)$ there is some $\tau\in[1/2,2/3]$ such that: For σ small enough,

$$E\left(w^\sigma;\partial C'^n\left(\frac{\tau}{mq}\right)\right)\leq K_{34}E(w(\mu,\eta);\mathscr{C}^n_{l,i}). \tag{55}$$

On $\mathscr{C}^n_{l,i}$ we define a map $\tilde{w}^\sigma$ by:

$$\begin{cases}\tilde{w}^\sigma=w^\sigma & \text{on } \mathscr{C}^n_{l,i}\setminus C'^n\left(\dfrac{\tau}{mq}\right);\\ \tilde{w}^\sigma=w^\sigma\left(\dfrac{x\tau}{\|x\|2\tau mq}\right) & \text{on } C'^n\left(\dfrac{\tau}{mq}\right).\end{cases} \tag{56}$$

On $\mathscr{C}^n_{l,i}\,\tilde{w}^\sigma$ is in $W^{1,p}(\mathscr{C}^n_{l,i},\mathcal{O})$, continuous except at one point singularity, the barycenter of $\mathscr{C}^n_{l,i}$. Moreover, we have:

$$E(\tilde{w}^\sigma;\mathscr{C}^n_{l,i})\leq K_{34}E(w(\mu,\eta);\mathscr{C}^n_{l,i})\quad\text{for } \sigma \text{ small enough.} \tag{57}$$

We defined $\tilde{w}^\sigma$ in the previous way, on the cubes $\mathscr{C}^n_{l,i}$, for every C_r in Q_m, which is not in contact with ∂P_m. If ∂C_r intersects ∂P_m we adapt the definition of $\tilde{w}^\sigma$ in such a way that $\tilde{w}^\sigma=w(\mu,\eta)$ on $\partial C_r\cap\partial P_m$. This can be done using the same methods as in the proof of Lemma 6 (cf. Appendix). Then we set on P_m, $\tilde{w}^\sigma=w(\mu,\eta)$, and clearly $\tilde{w}^\sigma\in W^{1,p}(C^n,\mathcal{O})$, for σ small enough. Moreover, for σ small, we have using (57):

$$\int_{C^n}|\nabla\tilde{w}^\sigma-\nabla w(\mu,\eta)|^p\,dx\leq K_{35}E(w(\mu,\eta);\cup_{r\in J_m}C_r\setminus C_r(\eta)). \tag{58}$$

Thus when $m\to 0$, $\mu\to 0$, $\eta\to 0$, $\sigma\to 0$ (chosen in a convenient way), $E(\tilde{w}^\sigma-w(\mu,\eta))$ goes to zero. We set:

$$f^\sigma=\pi\circ\tilde{w}^\sigma\quad\text{on } C^n. \tag{59}$$

f^σ is in $W^{1,p}(C^n,N^k)$, continuous except at a finite numbers of point (f^σ has the same singularities as $\tilde{w}^\sigma$). Moreover

$$E(f^\sigma-w(\mu,\eta))\to 0,\quad\text{when}\quad m\to 0,\mu\to 0,\eta\to 0\ (\eta/\mu\leq 1/2)\text{ and }\sigma\to 0. \tag{60}$$

Since $w(\mu,\eta)$ approximates u in $W^{1,p}$, (60) shows that u can be approximated in $W^{1,p}$ by maps, continuous except at a finite number of points. In the case $p=n-1$, since we assume $\pi_{n-1}(N^k)=0$, Theorem 1 follows from Lemma 1, applied to f^σ. In the case $n-2<p<n-1$, Theorem 1 follows from Lemma 2 (the proof of which has not yet been given), applied to f^σ. The next section is devoted to the proof of Lemma 2, and this will complete the proof of Theorem 1 in the case considered in this section.

II.6. *Proof of Lemma* 2. We consider a map v in $W^{1,p}(C^n, N^k)$ $(p<n-1)$ continuous except at most at a finite number of singularities. Let A be a singularity of v and $B^n(A,\sigma_0)$ a neighborhood of A such that v has no other singularity than A in $B^n(A,\sigma_0)$. For every $0<\sigma<\sigma_0$, the homotopy class of v restricted to $\partial B^n(A,\sigma)$ is independent on σ: we call it the homotopy class of v at the singularity A. In order to prove Lemma 2, we present a basic construction for "removing" a singularity (a similar construction is given in [Be1]). This construction is stated as Lemma 7 (for the proof of Lemma 2 we shall actually use the more general version Lemma 7 bis).

II.6.1. *A basic construction for removing a singularity*. We consider more generally an open domain W in $\mathbf{R}^n$, such that ∂W is smooth, and a map v in $W^{1,p}(W;N^k)$ $(p<n-1)$ such that v has only one point singularity A, and such that v is in $C^\infty(W\setminus\{A\};N^k)$. We assume furthermore that there is some point B on the boundary of W such that $[AB]$ is included in W, and that there is some neighborhood $B^n(A;\sigma_0)$ in W, of A such that

$$v(x)=v\left(\frac{x-A}{|x-A|\,\sigma_0}+A\right)\quad \text{on } B^n(A;\sigma). \tag{61}$$

Then we have the following lemma:

LEMMA 7. *Let v be as above and $p<n-1$. There is a sequence of smooth maps $v_m\in C^\infty(W;N^k)$ converging strongly to v in $W^{1,p}(W;N^k)$ which coincide with v outside some small neighborhood K_m of $[AB]$, such that* meas $K_m\to 0$ *when $m\to+\infty$.*

Proof of Lemma 7. For simplicity we may assume that ∂W is flat in some neighborhood of B and that ∂W is orthogonal there to $[AB]$ (the general case is technically more involved but the method remains essentially the same). We may choose orthonormal coordinates such that $A=(0,0,0)$ and $B=(0,0,d)$, where $d=|A-B|$, and such that $\partial W\cap B^n(B,r_1)=B^{n-1}(0,r_1)\times\{0\}$, for some r_1 small enough. For $m\in\mathbf{N}^*$ large enough, we set $a_m=d/(2m-2)$ and we consider the subset K_m of W defined by: $K_m=[-a_m,a_m]^{n-1}\times[-a_m,d]$. We are going to construct a map $v'_m\in W^{1,p}(W;N^k)$ such that $v'_m=v$ on $W\setminus K_m$, v'_m converges strongly to v in $W^{1,p}$, and such that v_m is continuous on K_m except at a finite number of point singularities at which the homotopy class of v is trivial. Then, we apply to v'_m Lemma 1 bis of the Appendix, which shows that v'_m can be strongly approximated by smooth maps: this will complete the proof of Lemma 7.

We divide K_m in m n-dimensional cubes $C_{m,j}$ (which are in fact translates of $[-a_m,a_m]^n$ defined by:

$$C_{m,j}=[-a_m,a_m]^{n-1}\times[(-1+j)\,a_m;(1+j)\,a_m],\quad \text{for}\quad j=0,\dots,m-1.$$

Let σ be such that $0<\sigma<\sigma_0/2$. Since v is smooth on $W\setminus B^n(A,\sigma)$ there is some constant $d(\sigma)$ such that $|\nabla v|_\infty<d(\sigma)$ on $W\setminus B^n(A,\sigma)$. For the cubes $C_{m,j}$ which do not intersect $B(A;\sigma)$ we have $|\nabla v|_\infty<d(\sigma)$, and thus

$$E(v;\partial C_{m,j})\leq 2^n n\, d(\sigma)^p a_m^{n-1}. \tag{62}$$

For the cubes $C_{m,j}$ which intersect $B^n(A,\sigma)$, we have relation (61) holding for these cubes, and thus it is easy to verify that $\sup\{|\nabla v(x)|, x\in\partial C_{m,j}\}<K/a_m$, which leads to the inequality

$$E(v;\partial C_{m,j})\leq K_{36}\, a_m^{n-1-p} \quad \text{for} \quad C_{m,j}\cap B^n(A,\sigma)\neq\varnothing. \tag{63}$$

Since we have at most $T_m(\sigma)=\sigma/a_m+1$ cubes $C_{m,j}$ which intersect $B(A,\sigma)$ combining (62) and (63) we obtain

$$\sum_{j=0}^{m-1} E(v;\partial C_{m,j})\leq K_{37}(\sigma a_m^{n-2-p}+d(\sigma)^p\, a_m^{n-1}\, m). \tag{64}$$

In order to complete the proof of Lemma 7, we shall use the following lemma, the proof of which will be postponed after the completion of the proof of Lemma 7.

LEMMA 8. *Let $\mu>0$, $\varepsilon>0$, $p<n-1$, and φ be a smooth map from $\partial C'^n(\mu)$ to N^k. There is some $0<\alpha_0<\mu/2$, depending only on $|\nabla v|_\infty$ and ε such that for every $0<\alpha<\alpha_0$, there is some smooth map $\bar\varphi$ from $\partial C'^n(\mu)$ to N^k having the following properties:*

The homotopy class of $\bar\varphi$ is trivial;

$$\bar\varphi=\varphi \text{ on } \partial C'^n(\mu)\setminus B^{n-1}(0,\alpha)\times\left\{\frac{\mu}{2}\right\}; \tag{65}$$

$$E\left(\bar\varphi;B^{n-1}(0,\alpha)\times\left\{\frac{\mu}{2}\right\}\right)\leq\varepsilon+E\left(\varphi;B^{n-1}(0,\alpha)\times\left\{\frac{\mu}{2}\right\}\right).$$

Proof of Lemma 7 completed. As a first step, we are going to define a smooth map v_m' on $\bigcup_{j=0}^{m-1}\partial C_{m,j}$ such that $v_m'=v$ on ∂K_m and such that the homotopy class of v_m' restricted to each $\partial C_{m,j}$ is trivial (afterwards we will extend v_m' inside each cube $C_{m,j}$).

Definition of v_m on $\bigcup_{j=0}^{m-1}\partial C_{m,j}$. Let $\varepsilon>0$ be small. We first apply Lemma 8 to $C_{m,0}$, to $\varphi=v$ restricted to $\partial C_{m,0}$, and $\alpha=\min(\varepsilon a_m,\alpha_0)$. Lemma 8 provides us with a map $\bar\varphi$ from $\partial C_{m,0}$ to N^k, satisfying (18). On $\partial C_{m,0}$ we define v_m' by

$$v_m'=\bar\varphi \quad \text{on } \partial C_{m,0}.$$

It follows that v_m' has the following properties: The homotopy class of v_m' on $\partial C_{m,0}$ is trivial, $v_m'=v$ on $\partial C_{m,0}\setminus B^{n-1}(0,\alpha)\times\{a_m\}$ and $E(v_m';B^{n-1}(0,\alpha)\times\{a_m\})\leq\varepsilon$. Hence v_m' is equal to v on $\partial C_{m,0}\cap\partial K_m$. We now consider the next cube $C_{m,1}=[-a_m,a_m]^{n-1}\times[a_m,3a_m]$ and the smooth map $\tilde v_m$ from $\partial C_{m,1}$ to N^k defined by

$$\begin{cases}\tilde v_m = v_m' \text{ on } \partial C_{m,0}\cap\partial C_{m,1}, \text{ that is, on the face } [-a_m,a_m]^{n-1}\times\{a_m\}\\ \tilde v_m = v \text{ elsewhere, that is, } \partial C_{m,1}\setminus[-a_m,a_m]^{n-1}\times\{a_m\}.\end{cases} \tag{66}$$

It is easy to see that the homotopy class of $\tilde v_m$ on $\partial C_{m,1}$ is the same as the homotopy class of v restricted to $\partial C_{m,0}$. We apply once more Lemma 3 to $\tilde v_m$ and $\partial C_{m,1}$. Lemma 3 provides us with a new map from $\partial C_{m,1}$ to N^k satisfying (18). We take v_m' equal to this new map. Note that this definition of v_m' on $\partial C_{m,1}$ is compatible with the previous definition of v_m' on $\partial C_{m,0}$. Moreover the homotopy class of v_m' on $\partial C_{m,1}$ is trivial and $v_m'=v$ on $\partial C_{m,1}\cap\partial K_m$. Repeating this argument m times, we define a smooth map v_m' on $\bigcup_{j=0}^{m-1}\partial C_{m,j}$ such that $v_m'=v$ on ∂K_m and such that the homotopy class of v_m' restricted to each boundary $\partial C_{m,j}$ is trivial.

Definition of v_m' on $K_m=\bigcup_{j=0}^{m-1}C_{m,j}$. For each cube $C_{m,j}$ we extend v_m' defined on $\partial C_{m,j}$ on $C_{m,j}$ in the following way:

$$v_m'(x) = v_m'\left(\frac{x-x_j}{\|x-x_j\|}\,a_m+x_j\right) \quad \text{where } x_j \text{ is the barycenter of } C_{m,j}. \tag{67}$$

It is easy to see that $v_m'=v$ on ∂K_m, that v_m' is in $W^{1,p}(K_m;N^k)$ continuous except at the points x_j, where the homotopy class of v_m' is trivial, and for every small neighborhood of the points x_j, Lipschitz outside this neighborhood. Easy calculations, combining (67), (66) and (65) show that

$$E(v_m';C_{m,j})\leq K(E(v;\partial C_{m,j})+\varepsilon)\,a_m,$$

and adding all these inequalities for $j=0,\dots,m-1$ we obtain, using (64)

$$E(v_m';K_m)\leq Ka_m\left(\sum_{j=0}^{m-1}(E(v;\partial C_{m,j})+\varepsilon\right)\leq K_{38}(\sigma a_m^{n-1}+d(\sigma)^p a_m+d\varepsilon). \tag{68}$$

If we let m go to $+\infty$, and ε go to zero, we see that

$$\lim_{n\to+\infty} E(v_m';K_m)=0. \tag{69}$$

Since $v_m'=v$ on ∂K_m, we may extend v_m' to W by $v_m'=v$ on $W\setminus K_m$. Since we have (69), and since v'_m has only point singularities, at which v'_m has a trivial homotopy class, v'_m can be strongly approximated by smooth maps, equal to v'_m and hence to v outside K_m (see Lemma 1 bis of the Appendix). This completes the proof of Lemma 7. Before we present the more general version of Lemma 7, namely Lemma 7 bis, we give in the next section, the proof of Lemma 8, which has been postponed.

Proof of Lemma 8. Since it will be easier to work on spheres rather than on boundaries of cubes, we consider the sphere S^{n-1}, and a bilipschitz map g_0 from $\partial C'^n(\mu)$ to S^{n-1} such that: $|\nabla g_0| \leq K_{39}\mu^{-1}$; $|\nabla g_0^{-1}| \leq K_{39}\mu$, such that $g_0(0, 0, -\mu/2) = (0, \dots, -1) = P_-$, $g_0(0, \dots, 0, +\mu/2) = (0, \dots, 0, +1) = P_+$, and such that g_0 preserves the orientation. Let $\beta > 0$ be small. We set:

$$V(\beta) = S^{n-1} \cap B^n(P_+, \beta) \quad \text{and} \quad W(\beta) = S^{n-1} \setminus V(\beta).$$

Since $V(\beta)$ and $W(\beta)$ are diffeomorphic, there is a map Φ from S^{n-1} to $W(\beta)$ such that

$$\Phi_\beta|_{W_\beta} = \mathrm{Id}|_{W_\beta}. \tag{70}$$

We consider the subsets of $\partial C'^n(\mu)$ defined by: $\mathcal{V}_{m,\beta} = g_0^{-1}(V(\beta))$; $\mathcal{W}_{m,\beta} = g_0^{-1}(W(\beta))$. (Note that $\mathcal{V}_{m,\beta}$ and $\mathcal{W}_{m,\beta}$ are diffeomorphic.) For $\beta > 0$ small enough,

$$\mathcal{V}_{m,\beta} \subset B^{n-1}(0, \alpha) \times \{\mu\}.$$

We consider the map $\tilde{v}_\beta$ defined from $\partial C'^n(\mu)$ to N^k by:

$$\tilde{v}_\beta = v \circ (g_0^{-1} \circ \Phi_\beta \circ g_0).$$

Note that, since Φ_β is homotopic to a constant map, $\tilde{v}_\beta$ is also homotopic to a constant map. (70) shows that $\tilde{v}_\beta = v$ on $\partial \mathcal{V}_{m,\beta}$. In order to complete the proof of Lemma 8, we recall the following result of B. White [W1]:

LEMMA 9. *Let M and N be Riemannian manifolds, with $\partial M \neq \emptyset$. If M and N are two Lipschitz maps from M to N such that $f = g$ on ∂M, and such that f and g are $[p]$-homotopic relatively to ∂M (that means, homotopic, relatively to ∂M, on some $[p]$-skeleton of M). Then for every $\varepsilon > 0$, there is some Lipschitz map f' which coincides with f and g on ∂M homotopic to f relatively to ∂M, and such that $\|f' - g\| \leq \varepsilon$.*

Proof of Lemma 8 *completed.* We apply Lemma 9 to $M = \mathcal{V}_{m,\beta}$, $f = \tilde{v}_\beta$, $g = v$. Since $\mathcal{V}_{m,\beta}$ is diffeomorphic to B^{n-1}, and since $p < n-1$, any $[p]$-skeleton of $\partial \mathcal{V}_{m,\beta}$ is also a $[p]$-skeleton of $\mathcal{V}_{m,\beta}$. This shows that $\tilde{v}_\beta$ and v are clearly $[p]$-homotopic relatively to $\partial \mathcal{V}_{m,\beta}$.

Thus the theorem gives us the existence of a Lipschitz v' homotopic to $\tilde{v}_\beta$ on $\mathscr{V}_{m,\beta}$ relatively to $\mathscr{V}_{m,\beta}$, such that $v'=v$ on $\partial\mathscr{V}_{m,\beta}$ and such that

$$E(v';\mathscr{V}_{m,\beta})\leq E(v;\mathscr{V}_{m,\beta})+\varepsilon$$

we set $\bar{\varphi}=v'$ on $\mathscr{V}_{m,\beta}$ and $\bar{\varphi}=v$, on $\partial C'^n(\mu)$. Then $\bar{\varphi}$ is homotopic to $\tilde{v}_\beta$ on $\partial C'^n(\mu)$ and thus the homotopy class of $\bar{\varphi}$ is trivial. It is easy to verify that $\bar{\varphi}$ satisfies the conditions (65), and this completes the proof of Lemma 8.

In the hypothesis of Lemma 7, we have assumed that $v\in C^\infty(W\setminus\{A\};N^k)$ and that (61) is satisfied. In fact this technical assumption can be omitted, and we have the more general result:

LEMMA 7 bis. *Let v be $W^{1,p}(W;N^k)$ $(p<n-1)$, such that $v\in C^0(W\setminus\{A\};N^k)$ (resp. $C^\infty(W\setminus\{A\};N^k)$. There is a sequence of maps v_m in $W^{1,p}(W;N^k)\cap C^0$ (resp. C^∞) converging strongly to v in $W^{1,p}(W;N^k)$ which coincide with v outside some small neighborhood K_m of $(AB]$ and such that $K_m\to 0$ when $m\to+\infty$.*

Proof of Lemma 7 bis. It suffices to prove that v can be approximated by maps in $W^{1,p}\cap C^\infty(W)\setminus\{A\};N^k)$, which verify (61). This can be done using the idea of the proof of Lemma 1 (cf. Appendix, proof of the general case). Using Lemma 7 bis, we are able now to complete the proof of Lemma 2.

II.6.2. *Proof of Lemma* 2 *completed.* Let $(A_i)_{1\leq i\leq k}$ be the point singularities of v. Let $(B_i)_{1\leq i\leq k}$ be points on ∂C^n, chosen in such a way, that there is a tubular neighborhood W_i of $[A_iB_i]$ in C^n such that $W_i\cap W_j=\varnothing$, if $i\neq j$. We apply Lemma 7 bis to v restricted to W_i, for $i=1,\dots,k$. This lemma provides us with a sequence of maps v_m in $W^{1,p}(C^n,N^k)\cap C^0$ and of some small neighborhood $K_{m,i}$ of $[A_iB_i]$ in W_i such that

$$\operatorname{meas}\left(\bigcup_{i=1}^k K_{m,i}\right)\to 0 \quad\text{when}\quad m\to 0,$$

$$\lim_{m\to+\infty} E(v_m;K_{m,i})=0 \quad\text{and}$$

$$v_m=v \quad\text{on}\quad C^n\setminus\bigcup_{i=1}^k K_{m,i}.$$

Thus $v_m\to v$ in $W^{1,p}(C^n,N^k)$. This completes the proof of Lemma 2(i).

Proof of Lemma 2(ii). We may assume without loss of generality that 0 is not a singularity of v. Let $\sigma>0$ be small and $\sigma'<\sigma$, be such that $E(v;\partial B^n(\sigma'))\leq 2E(v;B^n(\sigma))$.

Let B_i be points on $\partial B^n(\sigma)$, such that there is a tubular neighborhood W_i of $[A_iB_i]$ in $C^n \setminus B^n(\sigma)$, such that $W_i \neq W_j$ if $i \neq j$. Lemma 7 bis gives us the existence of a sequence of maps v_m in $W^{1,p}(C^n \setminus B^n(\sigma')) \cap C^0$, and of some small neighborhood $K_{m,i}$ of $[A_iB_i]$ such that $K_{m,i} \cap \partial C^n = \emptyset$, $\lim_{m \to +\infty} E(v_m; K_{m,i}) = 0$, $v_m = v$ on $C^n \setminus (\bigcup_{i=1}^k K_{m,i} \cup B^n(\sigma'))$ (and thus on ∂C^n), and moreover $\lim_{m \to +\infty} E(v_m; \partial B^n(\sigma')) = E(v; \partial B^n(\sigma'))$ (for this last equality, see the construction of v_m given by Lemma 7 near B_i). We extend v_m on $B^n(\sigma')$ by $v_m = v_m(x\sigma'/|x|)$. Then it is easy to see that $v_m \to v$ in $W^{1,p}(C^n, N^k)$ when $m \to +\infty$, and $\sigma \to 0$, moreover v_m is continuous except at the point singularity 0. Since there is some continuous map v' such that $v' = v$ on $\partial C'^n$ the homotopy class of v_m at 0 is trivial. We apply then Lemma 1 bis to v_m and this completes the proof of Lemma 2(ii).

II.7. *Proof of Theorem* 1 *bis when* $M = C^n$ *and* $n-2 < p \leq n-1$. The construction of Section II.5 shows that v can be approximated on $W^{1,p}(C^n, N^k)$ by maps continuous except at a finite number of point singularities, and which agree with v on the boundary (the conclusion then follows from Lemma 2(ii)).

III. Proof of Theorem 1 when $M^n = C^n$ and $1 \leq p \leq n-2$

We introduce first some notations. For $q \leq n \in \mathbf{N}^*$ and for $x = (x_1, x_2, \ldots, x_n)$ in $\mathbf{R}^n$, we set

$$\|x\|_k = \min\{\max_{i \in S} |x_i|, S \text{ is a } k\text{-element subset of } \{1, 2, \ldots, n\}\}.$$

If $\mu > 0$, we define a skeleton $[C'^n(\mu)]_k$ of the cube $C'^n(\mu)$ by:

$$[C'^n(\mu)]_k = \{x \in C'^n(\mu) / \|x\|_k = \mu/2\}.$$

Clearly $[C'^n(\mu)]_n = C'^n(\mu)$, $[C'^n(\mu)]_{n-1} = \partial C'^n(\mu)$ and $[C'^n(\mu)]_{n-2} = Z(\mu)$ as defined in Section II. More generally, for $k \leq n-1$, $[C'^n(\mu)]_k$ is a union of k-dimensional cubes, which are translates of $C'^k(\mu)$, and $[C'^n(\mu)]_{k-1}$ is the union of the boundaries of the k-dimensional cubes composing $[C'^n(\mu)]_k$. Since we have to consider sets C which are diffeomorphic to standard cubes $C'^n(\mu)$ (for instance C_r, $\mathscr{C}^n_{l,i}, \ldots$) we define the s-skeleton $[C]_k$ of C by:

$$[C]_k = f^{-1}([C'^n(\mu)]_k)$$

where f is a diffeomorphism from C to $C'^n(\mu)$.

Throughout Section III, we will assume that $1 < p \leq n-2$, that $\pi_{[p]}(N^k) = 0$. s represents the largest integer strictly less than p; that is $s = [p]$ if p is not an integer, $s = p-1$ if

p is an integer, and we shall consider a map u in $W^{1,p}(C^n, N^k)$. In order to prove Theorem 1 in the case $1 \leq p \leq n-2$, we are going to adapt the proof given in Section II for the case $n-2 \leq p < n-1$, and we mainly follow the same steps.

III.1. *Division of C^n in little cubes C_r.* We may assume without loss of generality that for $s \leq k \leq n-1$, u restricted to $[C^n]_k$ is in $W^{1,p}([C^n]_k, N^k)$. It follows then by the Sobolev embedding theorem that u restricted to $[C^n]_s$ is continuous. Slicing C^n by hyperplanes $P^n(a,k)$ we may obtain a division of C^n in $(m+1)^n$ cubes C_r, which are translates of $[-1/2m, 1/2m]^n$ (except those in contact with the boundary, see Section I.1), such that:

- u restricted to $[C_r]_k$ is in $W^{1,p}$ for every $s \leq k \leq n-1$. Thus u is continuous on $[C_r]_s$.
- The following relation holds, for m large enough:

$$\sum_{r=1}^{(m+1)^n} \underline{E}(u; [C_r]_k) \leq K_7 m^{n-k} E(u), \quad \text{for every} \quad s \leq k \leq n. \tag{71}$$

Next, we shall define the "bad" and the good cubes.

III.2. *Definition of P_m and Q_m.* Let $\varepsilon > 0$ be small, to be determined later. Q_m is the union of cubes C_r such that:

$$\begin{gathered} \tilde{\underline{E}}_m(u; [C_r]_s) \leq \varepsilon; \\ \tilde{\underline{E}}_m(u; [C_r]_k) \leq \varepsilon m^{-\nu} \quad \text{where } \nu > 0 \text{ is small, and for } s \leq k \leq n. \end{gathered} \tag{72}$$

We set $P_m = C^n \setminus Q_m$. We have:

$$\tilde{\underline{E}}_m(u; [C_r]_k) = m^{q-p} \underline{E}(u; [C_r]_k) \text{ for } s \leq k \leq n. \tag{73}$$

(72) and (73) imply that $\operatorname{meas} P_m \to 0$ when $m \to +\infty$. We first define $w(m, \mu, \eta)$ on Q_m.

III.3. *Definition of $w(\mu, \eta)$ on Q_m.* We consider a cube C_r in Q_m such that $\partial C_r \cap \partial P_m = \varnothing$ (the case $\partial C_r \cap \partial P_m \neq \varnothing$ is an adaption of the previous and will be considered in Section III.4). Since $\tilde{E}(u; [C_r]_s) \leq \varepsilon$, and since $[C_r]_s$ is a union of $2n!$ s-dimensional cubes, we may apply Lemma 2 to u restricted to these cubes. Choosing $\varepsilon < \varepsilon_0(\delta_0/2^n n!, s, p, N^k)$, we conclude that the image of $[C_r]_s$ lies in some domain $B(y_r, \delta_0)$ of N^k, for some y_r in N^k. Next we shall define $w(m, \mu, \eta)$ on $C_r(\eta)$ and then on $C_r \setminus C_r(\eta)$. (We choose η such that $\eta = 1/q$, $q \in \mathbb{N}^*$, as in Section II.)

Definition of $w(m, \mu, \eta)$ on $C_r(\eta)$. We set:

$$w(m,\mu,\eta)=\varphi(y_r, 2^{n+1}n!\,\delta_0)\circ u\circ\Phi^{-1}(r,\mu,\eta), \tag{74}$$

thus on $C_r(\mu)$, $w(m,\mu,\eta)=\varphi(y_r,2^{n+1}n!\,\delta_0)\circ u$.

Definition of $w(m,\mu,\eta)$ *on* $\overline{C_r\setminus C_r(\eta)}$. We use the same method as in Section II.3. We divide each face $S_{r,i}$ in $(q+1)^{n-1}$ cubes $\mathscr{C}^{n-1}_{l,i}$, as in Section II.3. We may adapt the method in such a way that we have:

- u restricted to $[\mathscr{C}^{n-1}_{l,i}]_k$ is in $W^{1,p}$ for every $s\leqslant k\leqslant n$. Thus u is continuous on $[C^{n-1}_{l,i}]_s$.

$$\sum_{l=1}^{(m+1)^{n-1}} \underline{E}(u;[\mathscr{C}^{n-1}_{l,i}]_k)\leqslant K_{40}\left(\sum_{t=k}^{n-1}\underline{E}(u_i\,\mathscr{C}^{n-1}_{l,i}[S_{r,i}]_t)\,q^{t-k}\right). \tag{75}$$

- The image of u restricted to $[\mathscr{C}^{n-1}_{l,i}]_s$ lies in $\tilde{B}_\varrho(y_i,\delta_0)$ for some y_i in N^k depending only on the restriction of u on $S_{r,i}$.

On $[\mathscr{C}^{n-1}_{l,i}]_s$ we set:

$$w(\mu,\eta)=u,\quad \text{which is in } W^{1,p}\cap C^0([\mathscr{C}^{n-1}_{l,i}]_s,N^k). \tag{76}$$

Let $\mathscr{N}_{\mu,\eta}(s)$ be the $(s+1)$-dimensional set defined by:

$$\mathscr{N}_{\mu,\eta}(s)=\bigcup_{i=1}^{2n}\ \bigcup_{l=1}^{(q+1)^{n-1}}\ [\mathscr{C}^{n-1}_{l,i}]_{s+1},$$

and $\mathscr{M}_{\mu,\eta}(s)$ be defined by

$$\mathscr{N}_{\mu,\eta}(s)=\overline{\mathscr{N}_{\mu,\eta}(s)\setminus\partial C_r\cup\partial C_r(\eta)}\ =\pi_r^{-1}\left(\bigcup_{i=1}^{2n}\ \bigcup_{l=1}^{(q+1)^{n-1}}\ [\mathscr{C}^{n-1}_{l,i}]_s\right).$$

On $\mathscr{M}_{\mu,\eta}(s)$ we set:

$$w(\mu,\eta)=u(\pi_r(x))\quad\text{on } \mathscr{M}_{\mu,\eta}(s). \tag{77}$$

Note that (76) and (77) are compatible. Combining (74), (76) and (77) we see that $w(\mu,\eta)$ is defined on $[\mathscr{C}^n_{l,i}]_s$. In order to extend $w(\mu,\eta)$ to $\mathscr{C}^n_{l,i}$ we proceed inductively. Suppose that $w(\mu,\eta)$ has yet been defined on $[\mathscr{C}^n_{l,i}]_k$. Let $A^{k+1}_{l,i,j}$ be the $(k+1)$-dimensional cubes composing $[\mathscr{C}^n_{l,i}]_{k+1}$ and suppose $w(\mu,\eta)$ has not yet been defined on some $A^{k+1}_{l,i,j}$. Since $\partial A^{k+1}_{l,i,j}\subset[\mathscr{C}^n_{l,i}]_k$ we may extend $w(m,\mu,\eta)$ to $A^{k+1}_{l,i,j}$ by a radial extension of the boundary value:

$$w(m,\mu,\eta)=w(m,\mu,\eta)\left(f_{l,i,j}^{-1}\left(\frac{f_{l,i,j}(x)}{2\|f_{l,i,j}(x)\|mq}\right)\right) \tag{78}$$

where $f_{l,i,j}$ is a diffeomorphism from $A_{l,i,j}^{k+1}$ to $C'^{k+1}(mq)$ such that $|\nabla f_{l,i,j}|\leqslant K_{12}|\nabla f_{l,i,j}^{-1}|\leqslant K_{12}$. Applying this method to each $A_{l,i,j}^{k+1}$ where $w(m,\mu,\eta)$ has not yet been defined we see that we have defined $w(m,\mu,\eta)$ on $[\mathscr{C}_{l,i}^n]_{k+1}$, and finally on $\mathscr{C}_{l,i}^n$, if we proceed inductively. Moreover, it is easy to check that:

$$w(\mu,\eta) \text{ is in } W^{1,p}(\overline{C_r\setminus C_r(\eta)}) \text{ and coincides with (74) on } \partial C_r(\eta),$$

$$E(w(\mu,\eta);\mathscr{C}_{l,i}^n)\leqslant K_{41}\left(\frac{\eta}{m}E(u;\partial C_r)+\left(\frac{\eta}{m}\right)^{n-S}\underline{E}(u;[\mathscr{C}_{l,i}^n]_s)\right), \tag{79}$$

$$\text{the image of } w(\mu,\eta) \text{ on } c_r \text{ lies in } \tilde{B}_\varrho(y_r,2^{n+1}n\delta_0).$$

III.4. *Definition of* $w(\mu,\eta)$ *on* P_m. As in part II.4 we consider the connected components $P_m(j)$ of P_m, and a division of P_m in $z(h)$ n-dimensional cubes $C_{j,b}(h)$, for $1\leqslant b\leqslant z(h)$, such that the compatibility condition (45) holds on every face $S_{r,i}\subset\partial P_m(j)\cap\partial Q_m$ and such that:

- For $1\leqslant b\leqslant z(h)$, u restricted to $[C_{j,b}(h)]_k$ is in $W^{1,p}$ for every k in $\{s,\dots,n\}$ and thus u restricted to $[C_{j,b}(h)]_s$ is continuous.
- The following inequality holds, for h large enough and $s<k\leqslant n-1$:

$$\sum_{b=1}^{z(h)}\underline{E}(u;[C_{j,b}(h)]_k)\leqslant K_{42}(mh)^{n-k}E(u;P_m(j)). \tag{80}$$

$$\sum_{b=1}^{z(h)}\underline{E}(u;[C_{j,b}(h)]_k)\cap S_{r,i})\leqslant K_{42}(mh)^{n-k-1}E(u;S_{r,i}). \tag{80'}$$

We consider, for $s\leqslant k\leqslant n$ the set $G(h,k)$ defined by:

$$G(h,k)=\bigcup_{b=1}^{z(h)}[C_{j,b}(h)]_k,$$

(clearly $G(h,k)\subset G(h,k+1)$, and $G(h,k)$ is a union of k-dimensional cubes). On $G(h,s)$, u is continuous. For h large enough, we set:

$$w(\mu,\eta)=u \quad \text{on } G(h,s). \tag{81}$$

We are going to extend the value of $w(\mu,\eta)$ defined by (81) to the sets $G(h,s+1)$, $G(h,s+2),\dots,G(h,n-1)$. Let A^{s+1} be one of the $(s+1)$-dimensional cubes composing

$G(h, s+1)$. $\partial A^{s+1} \subset G(h, s)$, and thus u (and $w(\mu, \eta)$) is continuous on ∂A^{s+1}. Applying Corollary 1 of Section I to A^{s+1} and u in the case p is not an integer (then $\pi_{[s]}(N^k)=0$), and the result of Schoen and Uhlenbeck in the case p is an integer, we deduce that u restricted to A^{s+1} can be approximated in $W^{1,p}(A^{s+1}, N^k)$ by maps in $W^{1,p}(A^{s+1}, N^k) \cap C^0$, which agree with u on ∂A^{s+1}. Hence, let $w_s'(\mu, \eta)$ be a map defined on A^{s+1} such that:

$$\begin{cases} w_s'(\mu,\eta) \text{ is in } W^{1,p}(A^{s+1}, N^k) \cap C^0, \quad \text{and} \quad w'(\mu,\eta) = u \text{ on } \partial A^{s+1}; \\ E(w_s'(\mu,\eta); A^{s+1}) \leq 2E(u; A^{s+1}). \end{cases}$$

Constructing $w_s'(\mu, \eta)$ in the previous way on every A^{s+1}, we define $w(\mu, \eta)$ on $G(h, s+1)$, and $w'(\mu, \eta) \in W^{1,p}(G(h, s+1), N^k) \cap C^0$. Next we consider a cube A^{s+2} in $G(h, s+2)$. Since $\partial A^{s+2} \subset G(h, s+1)$ we may extend $w'(\mu, \eta)$ on A^{s+2} by a radial extension of the boundary value:

$$w_s'(\mu,\eta) = w(\mu,\eta)\left(f_A\left(\frac{f_A(x)}{2mh\|f_A(x)\|}\right)\right) \quad \text{on } A^{s+2}, \tag{82}$$

where f_A is a diffeomorphism between A^{s+2} and $[-1/2mh, 1/2mh]^{s+1}$. Note that $w_s'(\mu, \eta)$ is in $W^{1,p}(G(h, s+2), N^k)$, continuous except at a finite number of point singularities. In order to approximate $w'(\mu, \eta)$ by a continuous map, we shall use the following lemma, which is provided in the Appendix (and which is a version of Lemma 2):

LEMMA 10. *Let $\mu>0$. Let $p>1$, let s be the largest integer strictly less than p. Let $d \in \mathbf{N}^*$, $d \geq s+3$. Let v be in $W^{1,p}(\partial C^d(\mu), N^k)$ continuous except at a finite number of point singularities, which are not on $[\partial C^d(\mu)]_s$. Then v can be approximated in $W^{1,p}(\partial C^d(\mu), N^k)$ by continuous maps, which coincide with v on $[\partial C^d(\mu)]_s$.*

Applying Lemma 10 to each cube A^{s+2} in $G(h, s+2)$, it is easy to see that there is a continuous map $w_{s+1}'(\mu, \eta)$ defined on $G(h, s+2)$ such that:

$$\begin{gathered} w_{s+1}'(\mu,\eta) \quad \text{is in} \quad W^{1,p}(G(h, s+2)) \cap C^0; \\ w_{s+1}'(\mu,\eta) = \bar{w}(\mu,\eta) = u \quad \text{on } G(h,s); \\ E(w_{s+1}'(\mu,\eta), G(h, s+2))) \leq 2E(w_s'(\mu,\eta), G(h, s+2)). \end{gathered} \tag{83}$$

Using the same construction as for $w_s'(\mu, \eta)$, we may extend $w_{s+1}'(\mu, \eta)$ to $G(h, s+3)$ by (82), and $w_s'(\mu, \eta)$ is continuous on that set except at a finite number of point singularities. We then define $w_{s+2}'(\mu, \eta)$ using Lemma A9. Repeating this argument $(n-s)$ times we finally define a map $w_{n-1}'(\mu, \eta)$ such that:

• $w'_{n-1}(\mu,\eta)\in W^{1,p}(P_m(j),N^k)$ and is continuous except at a finite number of points,

(84) $w'_{n-1}(\mu,\eta)$ restricted to $G(h,s)$ coincides with u;

(85) $$E(w'_{n-1}(\mu,\eta),P_m(j))\le K_{43}\left(\frac{1}{mh}\right)^{n-2}\underline{E}(u;G(h,s));$$

(85′) $E(w'_{n-1}(\mu,\eta),S_{r,i})\le K_{43}\left(\frac{1}{mh}\right)^{n-s-1}\underline{E}(u;G(h,s))\cap S_{r,i}))$ for every face $S_{r,i}$ on ∂P_m.

On $P_m(j)$ we set $w(\mu,\eta)=w'_{n-1}(\mu,\eta)$.

Combining (80) and (85) we see that (for h large enough)

(86) $$E(w(\mu,\eta),P_m(j))\le K_{44}E(u;P_m(j)).$$

Likewise, combining (80′) and (85′), we have, for every face $S_{r,i}$ in $\partial P_m\cap\partial Q_m$,

(87) $$\underline{E}(w(\mu,\eta);S_{r,i})\le K_{44}\underline{E}(u;S_{r,i}).$$

The relations (85′) and (87) show that our construction of $w(\mu,\eta)$ on P_m is compatible with the construction on Q_m. Thus $w(\mu,\eta)$ is in $W^{1,p}(C^n,N^k)$.

III.5. *Proof of Theorem* 1 *completed in the case* $1\le p\le n-2$. First we shall estimate $\int_{C^n}|\nabla u-\nabla w(\mu,\eta)|^p\,dx$. Let C_r be a cube in Q_m. Using the same argument as in part II.3, (relations (39) and (40)), and combining (79), (78), (75) we obtain:

$$\int_{C_r\setminus C_r(\eta)}|\nabla w(\mu,\eta)|^p\,dx\le K_{45}\frac{\eta}{m}E(u;\partial C_r)+\sum_{t=s}^{n-2}\left(\frac{\eta}{m}\right)^{n-t}E(u;[C_r]_t).$$

Then following the calculations of part II.3 ((33) to (44)) we prove that

(88) $$\int_{Q_m}|\nabla u-\nabla w(\mu,\eta)|^p\,dx\to 0\quad\text{when}\quad m\to+\infty,\ u\to 0,\eta\to 0\text{ and }\frac{\eta}{\mu}<\frac{1}{2}.$$

On the other hand (86) shows that

(89) $$E(w(\mu,\eta),P_m)\to 0,$$

since $\operatorname{meas}P_m\to 0$. Combining (88) and (90) we see that $w(m,\mu,\eta)$ is an approximation sequence of u. Then using the same method as in Section II.5, it can be showed that $w(m,\mu,\eta)$ can be approximated by maps in $W^{1,p}(C^n,N^k)$, continuous except at a finite number of point singularities (note that on P_m, $w(\mu, \eta)$ is continuous except at a finite number of points). We conclude using Lemma 2 as in Section II.5.

III.6. *Proof of Theorem* 1 *bis, in the case* $1<p\leq n-2$. The proof is the same as the proof of Corollary 1 in the case $n-2<p<n-1$.

IV. The case $\pi_{[p]}(N^k)\neq 0$

When $\pi_{[p]}(N^k)\neq 0$ smooth maps are not dense in $W^{1,p}(M^n, N^k)$ as the results of [SU2] and [BZ] show (see also Theorem A0 for an extension of the result to the case M^n is any manifold of dimension n). Nevertheless, we are able to aproximate maps in $W^{1,p}(M^n, N^k)$ by maps which are smooth except on a singular set, of dimension $n-[p]-1$, which has an analytic shape (see the introduction). These results are stated as Theorem 2 and Theorem 2 bis. For simplicity, we assume in Section IV that $M^n=C^n$. The general case will be considered in Section V.

IV.1. *Proof of Theorem* 2. Let u be in $W^{1,p}(C^n, N^k)$. We use the same construction as in part III. On Q_m let $w(m, \mu, \eta)$ be the map given by the construction of Section III.3. We consider each set $P_m(j)$, and we define first $w(\mu, \eta)$ on $G(h, s)$ by:

$$w(\mu, \eta) = u \quad \text{on} \quad G(h, s).$$

Thus $w(\mu, \eta)$ is continuous on $G(h, s)$. Let A^{s+1} be a cube composing $G(h, s+1)$. If p is an integer, $s+1=p$, we may apply the result of Schoen and Uhlenbeck [SU3] to u restricted to A^{s+1}, and define $w(\mu, \eta)$, as a continuous map, which agrees with u on ∂A^{s+1} and such that:

$$\underline{E}(w(\mu, \eta), A^{s+1}) \leq 2\underline{E}(u; A^{s+1}).$$

If p is not an integer, we set

$$w(\mu, \eta) = u\left(f_A^{-1}\left(\frac{f_A(x)}{2mh\|f_A(x)\|}\right)\right) \quad \text{on} \quad A^{s+2}.$$

Hence $w(\mu, \eta)$ is in $W^{1,p}(G(h, s+1), N^k)$, continuous except at a finite number of points. For $s+1\leq k\leq n$, we are going to define $w(\mu, \eta)$ inductively. Suppose that $w(\mu, \eta)$ is defined on $G(h, k-1)$. For each cube A^k composing $G(h, k)$ we set

$$w(\mu, \eta) = w(\mu, \eta)\left(f_A^{-1}\left(\frac{f_A(x)}{2mh\|f_A(x)\|}\right)\right) \quad \text{on} \quad A^k$$

(note that $\partial A^k \subset G(h, k-1)$). For each k, $s+1<k\leq n$, $w(\mu, \eta)$ is hence in $W^{1,p}(G(h,k), N^k)$ continuous except on a singular set Σ_k of dimension $k-[p]-1$, which has an analytic shape, as described in the introduction. Finally, $w(\mu, \eta)$ is in $W^{1,p}(P_m, N^k)$ continuous on P_m except on a singular set Σ of dimension $n-[p]-1$, which has an analytic shape. Moreover $w(m, \mu, \eta)$ is an approximation sequence of u in $W^{1,p}(C^n, N^k)$ (the calculations are the same as in Section III). Thus in order to prove Theorem 2, we only have to prove that $w(m, \mu, \eta)$ can be approximated by maps in R_p^0. The method for doing that, is essentially the same as the method of Section II.5, and this completes the proof of Theorem 2.

IV.2. *Proof of Theorem* 2 *bis*. The proof of Corollary 2 follows essentially the same ideas as the proof of Theorem 2, combined with the ideas of the proof of Corollary 1.

V. Extension of the results to the case M^n is any compact Riemannian manifold of dimension n

Let M^n be any connected compact Riemannian manifold. Let u be in $W^{1,p}(M^n, N^k)$. Following the ideas of B. White [W1] (section 1, remarks p. 129), we may realize a "cubeulation" of M^n that is, we may regard M^n as a union of n-dimensional cubes, which are diffeomorphic to C^n, and such that any two of them are either disjoint or intersect along a lower dimensional face. Using the ideas of [W1] (section 3, lemma p. 135) we may also assume, that, for $s\leq k<n$ the restriction of u to the k-skeleton (as defined in Section III) of the "cubeulation" is in $W^{1,p}$. Assume now that $\pi_{[p]}(N^k)=0$. In order to prove Theorem 1 in the general, we distinguish two different cases:

- $\partial M^n \neq \emptyset$. Adapting the construction of Section III to each cube of the "cubeulation" we are able to show that u can be approximated in $W^{1,p}$ by maps in $W^{1,p}(M^n, N^k)$, which are continuous except at a finite number of points. If $n-1\leq p<n$ we conclude using Lemma 1. If $1<p<n-1$, we use Lemma 2 (i), and we "evacuate" the singularities toward the boundary. This gives our approximation of u in $W^{1,p}(M^n, N^k)$ by smooth maps.
- $\partial M^n = \emptyset$. We set $\tilde{M}=[0,1]\times M^n$, and we define a map $\tilde{u}$ from $\tilde{M}$ to N^k by $\tilde{u}(t,x)=u(x)$, for $t\in[0,1]$, $x\in M^n$. Thus $\tilde{u}$ is in $W^{1,p}(\tilde{M}, N^k)$, and $\partial\tilde{M}=\{0,1\}\times M^n$. Applying the previous case to $\tilde{M}$, we see that $\tilde{u}$ can be approximated by smooth maps in $C^\infty(M^n, N^k)$. It is then easy to conclude that u can also be approximated by smooth maps, and this completes the proof of Theorem 1 in the general case.

Theorem 1 bis, Theorem 2 and 2 bis are proved following the same ideas.

VI. Weak density results

VI.1. *Proof of Theorem* 3. We first have to show that, if p is not an integer, and $\pi_{[p]}(N^k)\neq 0$, then smooth maps are not sequentially dense for the weak topology. For this purpose, we have to produce a map in $W^{1,p}(M^n, N^k)$ which is not the weak limit of a sequence of smooth maps.

VI.1.1 *A map which is not the weak limit of a sequence of smooth maps.* We restrict our attention to the case $M^n=C^n$ (the general case can be treated in the same way, see Theorem A0 of the Appendix). We consider the map $f\in W^{1,p}(C^n, N^k)$ introduced in [BZ], Theorem 2. For proving that this map cannot be approximated, for the weak topology by a bounded sequence of smooth maps, our argument is essentially the same as in [BZ], Theorem 2 (where it is proved that f cannot be approximated by smooth maps for the strong topology), except that we use Theorem 2.1 of [W2] instead of the result of [W1] used in [BZ]. Note that this result holds only if *p is not an integer.* In fact, for instance for $M^n=B^3$ and $N^k=S^2, \pi_2(S^2)\neq 0$, smooth maps are *sequentially weakly dense* in $W^{1,p}(B^3, S^2)$ (see [Bel]). Next we are going to prove the second part of Theorem 3, namely:

VI.1.2. *Weak limits of smooth maps are also strong limits of smooth maps.* We consider only the case $M^n=C^n$ (the general case in technically more involved but the idea remains essentially the same). We consider first the case $n-1<p<n$.

The case $n-1<p<n$. Let u be in $W^{1,p}(C^n, N^k)$ such that u is the weak limit of a sequence of maps in $C^\infty(M^n, N^k)$, and let u_m be such a sequence in $C^\infty(M^n, N^k)$ converging weakly to u in $W^{1,p}(M^n, N^k)$, and such that $E(u_m)\leqslant C$. We shall prove that u can be approximated, for the strong topology by smooth maps, and for this purpose we are going to adapt the method of Section I.1. Since $u_m\rightharpoonup u$ in $W^{1,p}, u_m\to u$ strongly in L^1, using the Sobolev embedding theorem. For a in $[0,1]$, let $P(a,k)$ be a slicing plane as considered in Section I.1. Let ε_1 be small to be determined later, and let $\gamma>0$ be small. By Egoroff's theorem, there is a subsequence of u_m (which for sake of simplicity we will also denote by u_m) such that

$$\text{meas}\left\{a\in[0,1]\;\middle|\;\int_{P(a,k)}|u-u_m|\,dx\geqslant\frac{\varepsilon_1}{2n}\right\}\leqslant\gamma, \tag{90}$$

Let $m_0\in\mathbf{N}^*$, be large (and fixed for the moment). Using relation (90), and adapting the slicing method of Section I.1, we may divide C^n in $(m_0+1)^n$ cubes C_r in such a way that relation (2) holds, and that moreover there is some m_1 such that

(91) $$E(u_{m_1};\partial C_r)\leqslant K_{45}\, m_0\, C \quad \text{for} \quad r=1,\dots,(m_0+1)^n;$$

(92) $$\int_{\partial C_r} |u-u_{m_1}|\,dx\leqslant \varepsilon_1 \quad \text{for} \quad r=1,\dots,(m_0+1)^n.$$

Note that relation (91) is a simple consequence of relation (2). Relation (92) can be derived from (90) if we choose, for instance, $\gamma\leqslant 1/5(m_0+1)$. We claim that (91) and (92) imply that the restriction of u to every ∂C_r is homotopic to a constant map. Since u_{m_1} is smooth on C_r, the restriction of u_m to ∂C_r is homotopic to a constant map. On the other hand, we have, for every $\eta_1>0$

$$\begin{aligned}\max\{|u(x)-u_{m_1}(x)|, x\in\partial C_r\} &\leqslant \eta_1(\underline{E}((u-u_{m_1});\partial C_r)+K(\eta_1)\int_{\partial C_r}|u-u_m|\,dx\\ &\leqslant \eta_1(\underline{E}(u;\partial C_r)+\underline{E}(u_{m_1};\partial C_r))+C(\eta_1)\,\varepsilon_1\\ &\leqslant \eta_1\, K_{46}\, m_0+C(\eta_1)\,\varepsilon_1;\end{aligned}$$

where $K(\eta_1)$ and $C(\eta_1)$ are a constant depending on η_1 (here we follow the outlines of the proof of Theorem 2.1 of [W2] and use a Morrey-type inequality (see [W2], Theorem 1.1)). Let $\delta_1=d(\mathcal{O},N^k)$. We choose η_1 so small that $\eta_1 K_{46}\, m_0<\delta_1/2$. Then we choose ε_1 such that $C(\eta_1)\,\varepsilon_1<\delta_1/2$. Then we have

(93) $$\max\{|u(x)-u_{m_1}(x)|, x\in\partial C_r\}\leqslant \delta_1.$$

Relation (93) shows that u and u_{m_1} are homotopic on ∂C_r, and thus u restricted to ∂C_r is homotopic to a constant map.

We consider now a cube C_r, which is in P_{m_0}. Using the construction of Lemma 3, Section I.3, we divide C_r in little cubes $C_{r,l}$. Applying the previous method to u restricted to C_r, and since u restricted to ∂C_r is homotopic to some u_{m_1} we see that u restricted to each $\partial C_{r,l}$ is also homotopic to u_{m_1}, and thus to a constant map. This shows that, on P_{m_0}, the map w_{m_0} restricted to P_{m_0} has only point singularities which have a trivial homotopy class, and thus can be approximated by continuous maps in $W^{1,p}(C_r,N^k)$ (see Lemma 1 bis). The proof of Theorem 3 can then be completed, in the case $n-1<p<n$ adapting the methods of Section I.

The case $1<p<n-1$. The proof of Theorem 3 in that case follows essentially the same ideas, though technically more involved (we have to use the construction of $w(m_0,\mu,\eta)$ of Section III instead of the construction of w_{m_0} of Section I).

Remark. When p is not an integer, Theorem 3 (in the case $\pi_{[p]}(N^k)\neq 0$), and Theorem 1 (in the case $\pi_{[p]}(N^k)=0$) show that the strong closure and the sequentially weak closure are always equal.

Adapting the method of the proof of Theorem 3, we may prove the following:

THEOREM 3 bis. *Let M^n be such that $\partial M^n\neq\emptyset$. Let $p<n$ be such that p is not an integer, and $\pi_{[p]}(N^k)\neq 0$. Let u be in $W^{1,p}(M^n,N^k)$ such that u restricted to ∂M^n is in $W^{1,p}(\partial M^n,N^k)\cap C^0$, and is homotopic to a constant map. Let u_m be a sequence in $W^{1,p}(M^n,N^k)\cap C^0$ such that $u_m=u$ on ∂M^n and $u_m \rightharpoonup u$ in $W^{1,p}$. Then u can be approximated for the strong topology in $W^{1,p}$ by maps in $W^{1,p}(M^n,N^k)\cap C^0$, which coincide with u on the boundary.*

In the following section we give an application of Theorem 3 bis, when we wish to minimize, the energy among smooth maps, with a given boundary value.

VI.2. *An application of Theorem* 3 *bis*. We restrict our attention to the case $M^n=B^n$. Let $1<p<n$, such that p is not an integer, and we assume moreover that $\pi_{[p]}(N^k)\neq 0$. We consider a smooth map ζ from ∂B^n to N^k, such that ζ is homotopic to a constant map. Let $W^{1,p}_\zeta$ be the set defined by

$$W^{1,p}_\zeta=\{u\in W^{1,p}(B^n,N^k);\, u|_{\partial B^n}=\zeta\}$$

and let V_ζ be the strong closure of $C^\infty(B^n,N^k)$ in $W^{1,p}(B^n,N^k)$. Clearly $W^{1,p}_\zeta$ and V_ζ are not empty. Theorem 3 bis shows that the infimum of the energy in V_ζ is achieved. Indeed let u_m be some minimizing sequence for $E(u)$ in V_ζ; we may extract some subsequence converging weakly to some map $\bar u_0$, which is thus the weak limit of a sequence of maps in $C^\infty(B^n,N^k)$. By Theorem 3 bis, $\bar u_0$ is in V_ζ and clearly is a minimizer for E in V_ζ. It is not difficult to see that $\bar u_0$ is a weakly p-harmonic map. Indeed consider for every φ in $C^\infty_c(B^n,\mathbf{R}^l)$ the map $u_t=\bar u_0+t\varphi$, for small $|t|$. Clearly if $|t|$ is small enough, $u_t(x)$ is in $\mathcal{O}$ for a.e. x. Thus $\bar u_t=\pi(u(t))$ is in $W^{1,p}(B^n,N^k)$. Moreover since $\bar u_0$ is in $C^\infty_g(B^n,N^k)$, $\bar u_t$ is also in $\bar C^\infty_g(B^n,N^k)$ Hence $E_p(\bar u_t)\geq E_p(\bar u_0)$, and thus $dE_p(\bar u_t)/dt=0$. The latest equality then yields the conclusion. Adapting the regularity theory for minimizers for E in $W^{1,p}_\zeta$, developed by Schoen and Uhlenbeck in the case $p=2$ ([SU1], [SU2]) and by Hardt and Lin [HL], Fuchs [F], and Luckhaus [L] in the general case, we may prove the following:

THEOREM 4. *Assume $\pi_{[p]}(N^k)\neq 0$, p is not an integer and ζ is as above then*

$$\inf\{E(u),\, u\in W^{1,p}_\zeta\cap \bar C^\infty_\zeta(B^n,N^k)\}$$

is achieved by some map $\bar{u}_0$ *(in* $\bar{C}^\infty_\zeta(B^n, N^k)$*) which is weakly p-harmonic. Moreover* $\bar{u}_0$ *is smooth outside a closed singular set Z on zero* $(n-[p]-1)$ *dimensional Hausdorf measure. If* $n-1<p<n$*,* $\bar{u}_0$ *has only a finite number of isolated point singularities, at which* $\bar{u}_0$ *has a trivial homotopy class, and which are not on the boundary.*

The proof of the partial regularity of $\bar{u}_0$ is rather technical and will be given in a forthcoming paper.

VI.2.1. *Remark.* We consider the special case $M^n=B^3$, $N^k=S^2$ and $p=2\alpha$, where $1<\alpha<3/2$. Let g be a smooth map from ∂B^3 to S^2 having degree zero. We set $W^{1,2\alpha}_g(B^3,S^2)=\{u\in W^{1,2\alpha}(B^3,S^2), u=g \text{ on } \partial B^3\}$, and we consider the strong closure of $C^\infty_g(B^3,S^2)=\{u\in C^\infty(B^3,S^2), u=g \text{ on } \partial B^3\}$, in $W^{1,2\alpha}$. Then Theorem 3 bis and Theorem 4 tell us that the infimum of the energy $E_{2\alpha}(u)=\int_{B^3}|\nabla u|^{2\alpha}\,dx$ is achieved in $\bar{C}^\infty_g(B^3,S^2)$ and that the minimizers are weakly 2α-harmonic maps, smooth except at most at a finite number of points, where the degree of these maps is zero. More generally, let q be in $\mathbf{N}^*$, let $A_1,\dots,A_i,\dots,A_q$ be points in B^3, and let $d_1,\dots,d_i,\dots,d_q$ be in $\mathbf{Z}^*$ such that $\Sigma_{i=1}^q d_i=\deg(g)$ (here and in the sequel, we shall not assume that $\deg(g)$ is zero). We consider the subsets $T_g(A_1,\dots,A_q;d_1,\dots,d_q)$ of $W^{1,2\alpha}_g(B^3,S^2)$ defined by:

$$T_g(A_1,\dots,A_q;d_1,\dots,d_q)=\{U\in W^{1,2\alpha}_g(B^3,S^2); u\in C^\infty(B^3\setminus(A_1,\dots,A_q);S^2);$$
$$\deg(u;A_i)=d_i, \text{ for } i=1,\dots,q\}.$$

It is easy to verify that $T_g(A_1,\dots,A_q;d_1,\dots,d_q)$ is not empty. Using the ideas of the proofs of Theorem 3 and Theorem 3 bis it is then possible to show that the strong closure of $T_g(A_1,\dots,A_q;d_1,\dots,d_q)$ is stable under weak convergence. This implies that the infimum of the energy $E_{2\alpha}$ is achieved in $\bar{T}_g(A_1,\dots,A_q;d_1,\dots,d_q)$ by a map which is weakly 2α-harmonic. On the other hand, if q and q' are in $\mathbf{N}^*$ then:

$$\bar{T}_g(A_1,\dots,A_q;d_1,\dots,d_q)\cap\bar{T}_g(A'_{1'},\dots,A'_{q'};d'_{1'},\dots,d'_{q'})=\varnothing$$

if and only if

$$\{(A_1,d_1),(A_2,d_2),\dots,(A_q,d_q)\}\neq\{(A'_{1'},d'_{1'}),\dots,(A'_{q'},d'_{q'})\}.$$

In other words, if we look for minimizers of $E_{2\alpha}$ on each set $\bar{T}_g(A_1,\dots,A_q;d_1,\dots,d_q)$ we find infinitely many (and in fact uncountably infinitely many) different weakly 2α-harmonic maps in $W^{1,2\alpha}_g(B^3,S^2)$. Adapting the previous ideas to the case N^k is any compact Riemannian manifold of dimension k, and considering singular sets of dimension $n-[p]-1$, we may prove the following:

THEOREM 4 bis. *Let $1<p<n$, such that p is not an integer, and such that $\pi_{[p]}(N^k)\neq 0$. Let ξ be a smooth map from ∂B^n to N^k such that $W^{1,p}_{\xi}(B^n, N^k)$ is not empty. There are uncountably infinitely many p-harmonic maps in $W^{1,p}_{\xi}(B^n, N^k)$.*

VI.2.2. *Remark.* Let p and N^k be as above and let ξ be any constant map. Then Theorem 4 bis provides an infinity of weakly p-harmonic maps which are constant on the boundary. On the other hand there are no smooth harmonic maps which are constant on the boundary (see [Wo] and [KW]).

As pointed out, the proof of Theorem 3 and Theorem 3 bis, cannot be extended to the case p is an integer. In this case we have nevertheless a weaker result (Theorem 5).

VI.3. *Proof of Theorem* 5. We consider first the case $p=n-1$.

The case $p=n-1$. In this case, we know that maps, smooth, except at a finite number of point singularities (this is, maps in R^{∞}_{n-1}) are dense in $W^{1,p}(M^n, N^k)$ for the strong topology. In order to prove Theorem 5, it thus suffices to prove that any map in R^{∞}_{n-1} can be approximated, for the weak topology, by smooth maps. For this purpose, we shall use the following lemma:

LEMMA 8 bis. *Let $\mu>0$, $\varepsilon>0$ and $p=n-1$. Let φ be a smooth map from $\partial C'^n(\mu)$ to N^k. There is some $0<\alpha_0<\mu/2$, depending only on $|\nabla v|_\infty$ and ε, and some constant F depending only on the homotopy class of φ, such that for every $0<\alpha<\alpha_0$, there is a smooth map $\tilde\varphi$ from $\partial C'^n(\mu)$ to N^k having the following properties:*

$$\tilde\varphi \textit{ is homotopic to a constant map,}$$

$$\tilde\varphi=\varphi \quad on \quad \partial C'^n(\mu)\setminus B^{n-1}(0,\alpha)\times\left\{\frac{\mu}{2}\right\},$$

$$E\left(\tilde\varphi; B^{n-1}(0,\alpha)\times\left\{\frac{\mu}{2}\right\}\right)\le E\left(\varphi; B^{n-1}(0,\alpha)\times\left\{\frac{\mu}{2}\right\}\right)+\varepsilon+F.$$

We postpone the proof of Lemma 8 bis, and we complete the proof of Theorem 5, in the case $p=n-1$. Let v be in R^{∞}_{n-1}. Let a_i be the singularities of v, for $i=1,\dots,k$, and F_i the constant arising in Lemma 8 bis, corresponding to the homotopy class of v at the singularity a_i. We claim, that there is a sequence of smooth map v_m in $C^\infty(M^n, N^k)$ and some smooth open subset K_m of M^n such that:

(94) $$\operatorname{meas} K_m \to 0 \quad \text{when} \quad m \to +\infty,$$

(95) $$\lim_{n\to+\infty} E(v_m, K_m) \leq \sum_{i=1}^{k} F_i,$$

(96) $$v_m = v \quad \text{on} \quad M^3 \setminus K_m.$$

The construction of this sequence is exactly the same as the constructions given in Lemma 2 and Lemma 7, except that we replace Lemma 8 by Lemma 8 bis, and $\tilde{\varphi}$ by $\bar{\varphi}$. We may easily deduce that $v_m \rightharpoonup v$ in $W^{1,p}$. Indeed, since $E(v_m; M^n)$ is bounded (by (95) and (96)), passing to a subsequence if necessary, v_m converges weakly to some map v' in $W^{1,p}(M^n, N^k)$. Since v_m converges to v almost everywhere (by (94) and (96)), $v'=v$. This completes the proof of Theorem 5, in the case $p=n-1$.

The case $p<n-1$. In this case the proof of Theorem 5 is technically more involved. The main idea is essentially the same, but we have to combine it with the constructions given in Section III.

Warning. We are only able to prove that smooth maps are dense for the weak topology, when p is an integer, but we are not able to prove that smooth maps are *sequentially* dense for the weak topology, which is a more difficult question.

We come now to the proof of Lemma 8 bis.

Proof of Lemma 8 bis. The notations are the same as in the proof of Lemma 8. We consider the map $\bar{v}=v\circ g_0^{-1}$ from S^{n-1} to N^k. Set

$$F_0 = \inf\{E(\xi; S^{n-1}) \mid \xi \in C^\infty(S^{n-1}, N^k) \text{ is homotopic to } v\}.$$

(Note that it is not known, in general, whether this infimum is achieved or not.) We consider a map $\bar{v}_0$ from S^{n-1} to N^k, homotopic to $\bar{v}$ such that:

$$E(\bar{v}_0, S^{n-1}) \leq F_0 + \frac{\varepsilon}{2}.$$

We set $e_0=\bar{v}_0(0,0,\ldots,-1)$ ($e_0 \in N^k$). It is easy to construct a map $\bar{v}_1$, homotopic to $\bar{v}$ such that $\bar{v}_1=e_0$ in some small neighborhood of $(0,0,\ldots,-1)$, and such that

$$E(\bar{v}_1, S^{n-1}) \leq E(\bar{v}_0, S^{n-1}) + \frac{\varepsilon}{4}.$$

Let π_S be the stereographic projection from S^{n-1} to $\mathbf{R}^{n-1}$. We consider the map Φ_1

from $\mathbf{R}^{n-1}$ to N^k defined by:

$$\Phi_1 = \bar{v}_1 \circ \pi_S^{-1}.$$

Since $p=n-1$, and π_S is a conformal map, we have $E(\Phi_1, \mathbf{R}^{n-1})=E(\bar{v}_1, S^{n-1})$ and for some large ball $B^{n-1}(C)$ $(C>0)$, $\Phi_1(x)=e_0$ on $\mathbf{R}^{n-1}\setminus B^{n-1}(C)$. Since the energy is invariant by scaling $(p=n-1)$, we may always assume that $C=1$, and thus $\Phi_1(x)=e_0$ on ∂B^{n-1}.

We consider now the cube $C'^n(\mu)$. Set $e_1=v(0,0,\dots,-\mu/2)$ $(e_1\in N^k)$. We consider a geodesic between e_0 and e_1, and a parameterization of that geodesic ζ_{e_1} from $[1/2, 1]$ to N^k, such that $\zeta_{e_1}(1/2)=e_0$, $\zeta_{e_1}(1)=e_1$. We may choose ζ_{e_1} in such a way, that there is a constant F_1 depending only on N^k such that $\max_{x\in[1/2,1]}\{|\nabla\zeta_{e_1}(x)|\}<F_1$, for every $e_1\in N^k$. For $\alpha<\mu/2$ small enough, we consider the map $\tilde{\varphi}$ from $\partial C'^n(\mu)$ to N^k defined by:

$$\tilde{\varphi}(x) = v(x) \quad \text{on} \quad C'^n(\mu)\setminus B^{n-1}(0;\alpha)\times\left\{\frac{\mu}{2}\right\} \quad (x=(x_1,x_2,\dots,x_n)),$$

$$\tilde{\varphi}(x) = \Phi_1\left(\frac{2x'}{\alpha}\right) \quad \text{on} \quad B^{n-1}\left(0;\frac{\alpha}{2}\right)\times\left\{\frac{\mu}{2}\right\}, \quad \text{where} \quad x'=(x_1,x_2,\dots,x_{n-1})\in\mathbf{R}^{n-1},$$

$$\tilde{\varphi}(x) = \zeta_{e_1}\left(\frac{x'}{2}\right) \quad \text{on} \quad B^{n-1}(0,\alpha)\setminus B^{n-1}\left(0,\frac{\alpha}{2}\right)\times\left\{\frac{\mu}{2}\right\}.$$

$\tilde{\varphi}$ is Lipschitz on $\partial C'^n(\mu)$ and it is easy to see that $\tilde{\varphi}$ is homotopic to a constant map, and that

$$E\left(\tilde{\varphi}; B^{n-1}(0;\alpha)\times\left\{\frac{\mu}{2}\right\}\right) \leqslant F_0+K_{47}F_1+\varepsilon \leqslant F+\varepsilon.$$

This shows that $\tilde{\varphi}$ satisfies the conditions of Lemma 8 bis, and completes the proof.

In the special case, where p is an integer and $N^k=S^p$, we are able to prove that smooth maps are sequentially dense in $W^{1,p}(B^n, S^p)$ for the weak topology (Theorem 6).

VI.4. *Proof of Theorem* 6. *The case* $p=n-1$ has been treated in [Be1]. For the general case $p<n-1$, the proof will be given in a forthcoming paper.

Remark. In [Be1] we were able to characterize the maps $u\in W^{1,p}(B^{p+1}, S^p)$, $p\in\mathbf{N}^*$, which can be approximated by smooth maps: let D be the standard volume form on S^p and D^* the pull-back of this volume form by u (when u is in $W^{1,p}(B^{p+1}, S^p)$, the coefficients of D^* are in L^1); u can be approximated by smooth maps if and only if $dD^*=0$ (in a distributional sense). Does this result hold for the space $W^{1,p}(B^n, S^p)$, for $n>p+1$? More generally what would be the equivalent result, for the space $W^{1,p}(B^n, S^p)$ with $k\neq p$?

In the next section, we extend some of our result to the Sobolev spaces $W^{r,p}$.

VII. Extension of the results to the Sobolev spaces $W^{r,p}(M^n, N^k)$

We have the following

THEOREM 7. *Let $r \in \mathbf{N}^*$, $n \in \mathbf{N}^*$ and $p>1$ be such that $rp<n$. Smooth maps from M^n to N^k are dense in $W^{1,p}(M^n, N^k)$ if and only if $\pi_{[rp]}(N^k)=0$.*

The fact that this condition is necessary, was proved by Escobedo in [E]. The reverse, namely that the condition $\pi_{[rp]}(N^k)=0$ is sufficient for smooth maps to be dense in $W^{r,p}(M^n, N^k)$, can be proved by adapting the proof of Theorem 1 with some slight changes. (Note that when $rp \geqslant n$ smooth maps are always dense in $W^{r,p}(M^n, N^k)$.)

When r is not an integer, one may conjecture that the result of Theorem 7 still holds. This is a more difficult question, and we are only able to extend the result of Theorem 7 in the case $r=1-1/p$, that is in the case of trace spaces. The idea of the proof is to use a lifting (in $\mathbf{R}^l$) of the map, and to adapt the proof of Theorem 1 (details will be given in a forthcoming paper).

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Appendix

In this appendix $K_1', K_2' \dots$ will represent absolute constants, depending only on N^k, n and p. We have the following result, which is used in the proof of Theorem 1.

THEOREM A0. *Assume $\pi_{[p]}(N^k) \neq 0$. There is some map in $W^{1,p}(M^n, N^k)$ which cannot be approximated by smooth maps in $W^{1,p}(M^n, N^k)$.*

A.1. *Proof of Theorem* A0. In the special case $M^n=B^n$, the proof of Theorem A0 has yet been given in [BZ]. We are going to show how this proof can be extended to the case M^n is any manifold. In order to do this, we need the following result.

Claim. There is a map $\tilde{f}$ in $W^{1,p}(B^n, N^k)$, such that f restricted to a neighborhood of ∂B^n is a constant map, and f cannot be approximated by smooth maps.

Before we give a proof of this claim, we first show how Theorem A0 follows from the claim. Let x_0 be some point in M^n, and for δ_0 small enough, consider the geodesic ball $B^n_\varrho(x_0, \delta_0)$ centered at x_0 of radius δ_0. Let Φ be a diffeomorphism from $B^n_\varrho(x_0, \delta_0)$ to B_n. On $B^n_\varrho(x_0, \delta_0)$ we consider the map g defined by

$$g(x) = \tilde{f} \circ \Phi(x) \quad \text{for} \quad x_0 \in B^n_\varrho(x_0, \delta_0).$$

Since $\tilde{f}$ is a constant map on ∂B^n, g is a constant map on $\partial B^n_\varrho(x_0, \delta_0)$. Let a be the value of g on ∂B^n_ϱ. We extend g to M^n by

$$g(x) = a \quad \text{for} \quad x \notin M^n \setminus B^n_\varrho(x_0, \delta_0).$$

Clearly g is in $W^{1,p}(M^n, N^k)$. Since $\tilde{f}$ cannot be approximated by smooth maps in $C^\infty(B^n, N^k)$, g cannot be approximated by smooth maps in $C^\infty(M^n, N^k)$, and this completes the proof of Theorem A0.

Proof of the claim. The case $n-1<p<n$. Here $[p]=n-1$, and we assume $\pi_{n-1}(N^k)\neq 0$. Let φ be a smooth map from S^{n-1} to N^k, such that φ is not homotopic to a constant map, and φ is constant on $V^{n-1}=S^{n-1}\cap\{(x_1, \ldots, x_n)\in \mathbf{R}^n | x_0 \leq 1/2\}$. We set

$$E^n_- = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_n \leq 0\}, \quad E^n_+ = \{(x_1, \ldots, x_n) \in \mathbf{R}^n | x_n \geq 0\},$$

$$P_- = \left(0, 0, \ldots, -\frac{1}{2}\right) \quad \text{and} \quad P_+ = \left(0, 0, \ldots, +\frac{1}{2}\right).$$

We define a map f from $\mathbf{R}^n$ to N^k by

$$f(x) = \varphi\left(\frac{x-P_-}{|x-P_-|}\right) \quad \text{for } x \text{ in } E^n_-;$$

$$f(x) = \varphi\left(\frac{x-P_+}{|x-P_+|}\right) \quad \text{for } x \text{ in } E^n_+;$$

It is easy to verify that f is constant outside $\Omega=(B^n(P_-;1)\cap E^n_-)\cup(B^n(P_+;1)\cap E^n_+)$, and that f is in $W^{1,p}(\mathbf{R}^n, N^k)$. Thus for $r\geq 3$, f is constant on $\partial B^n(r)$, and one can show that f cannot be approximated by smooth maps, using the same argument as in [BZ], Theorem 2 (note that f has two point singularities P_+ and P_-). The map $\tilde{f}$ is then obtained by

$$\tilde{f}(x) = f(3x) \quad \text{for} \quad x \in B^n.$$

The general case: $n-1<p$. We set $q=n-[p]$, and we are going to argue inductively on q (the case $q=1$ has already been settled in the previous paragraph). Assume that we have found a map $\tilde{f}_q$ in $W^{1,p}(B^n, N^k)$ such that $\tilde{f}_q=a$ (a constant) on ∂B^n ($a \in N^k$) and $\tilde{f}_q$

cannot be approximated by smooth maps. We extend $\tilde{f}_q$ to $\mathbf{R}^n$ by $\tilde{f}_q=a$ on $\mathbf{R}^n \setminus B^n$. In $\mathbf{R}^{n+1}$ we consider the domain $\Omega=\Omega_1 \cup \Omega_2 \cup \Omega_3$ where

$$\Omega_1 = C'^n(2)\times\left[-\frac{1}{2},\frac{1}{2}\right];$$

$$\Omega_2 = C'^n(2)\times\left[-\frac{1}{2},\frac{5}{2}\right] = C'^{n+1}(A^+;2),$$

$$\Omega_3 = C'^n(2)\times\left[-\frac{1}{2},-\frac{5}{2}\right] = C'^{n+1}(A^-;2),$$

$$\Omega = C'^n(2)\times\left[-\frac{5}{2},\frac{5}{2}\right],$$

(we have set $A^+=(0,0,\dots,3/2)$, $A^-=(0,0,\dots,-3/2)$). We define a map f_{q+1} on $\mathbf{R}^{n+1}$ with value in N^k in the following way. On Ω_1 we set

$$f_{q+1}(x_1,x_2,\dots,x_n,x_{n+1}) = \tilde{f}_q(x_1,\dots,x_n).$$

On $\partial\Omega_2 \setminus \partial\Omega_1$ we set

$$f_{q+1}(x) = a \quad \text{for} \quad x\in\partial\Omega_2,\ x\notin\partial\Omega_1$$

and likewise on $\partial\Omega_3 \setminus \partial\Omega_1$, we set

$$f_{q+1}(x) = a \quad \text{for} \quad x\in\partial\Omega_3,\ x\in\partial\Omega_1.$$

Finally, on Ω_2 we define $\tilde{f}_{q+1}$ by:

$$f_{q+1}(x) = \tilde{f}_{q+1}\left(2\frac{x-A^+}{\|x-A^+\|}+A^+\right)$$

(note that $2(x-A^+)/\|x-A^+\|+A^+ \in \partial\Omega_2$), and likewise on Ω_3, we define $\tilde{f}_{q+1}$ by

$$f_{q+1}(x) = \tilde{f}_{q+1}\left(2\frac{x-A^-}{\|x-A^-\|}+A^-\right).$$

It is easy to verify that $\tilde{f}_{q+1}$ is in $W^{1,p}(\Omega, N^k)$ and that the value of f_{q+1} on the boundary $\partial\Omega$ is constant, namely a. Moreover using the same method as in [BZ], it is easy to see that f_{q+1} cannot be approximated by maps in $C^\infty(\Omega, N^k)$. We extend f_{q+1} to $\mathbf{R}^n$ by setting:

$$f_{q+1}(x) = a \quad \text{if} \quad x\notin\Omega$$

for $r\geqslant 4$, we see that $f_{q+1}=a$ on $\partial B^{n+1}(r)$. Thus if we set

$$\tilde{f}_{q+1}=f(4x) \quad \text{for} \quad x\in B^{n+1}$$

we clearly see that $\tilde{f}_{q+1}$ is a map in $W^{1,p}(B^{n+1},N^k)$ such that $f=a$ on ∂B^n, and that cannot be approximated by smooth maps. This completes the inductive argument, and the proof of the claim.

Remark. The singular set of $\tilde{f}$ is actually homomorphic to $S^{n-[p]-1}$.

A.2. LEMMA A0. (i) *Let u be in $W^{1,p}(C'^n,N^k)$ $(p<n)$. Then u can be approximated in $W^{1,p}(C'^n,N^k)$ by maps such that their restriction to $\partial C'^n$ is in $W^{1,p}(\partial C'^n,N^k)$.*

(ii) *u can be approximated in $W^{1,p}(C'^n,N^k)$ by maps in $W^{1,p}(C'^n,N^k)$ such that their restriction to $[C'^n]_k$ is in $W^{1,p}([C']^n_k,N^k)$ for every $s<k\leq n-1$, where s is the largest integer strictly less than p.*

Proof of Lemma A0. (i) For $\mu>0$ we consider the cube $C'^n(1-\mu)$ and its boundary $\partial C'^n(1-\mu)$. For almost every μ, u restricted to $\partial C'^n(1-\mu)$ is in $W^{1,p}$ and we have

$$\int_0^\mu \underset{\sim}{E}(u;\partial C'^n(1-\mu))\,d\mu\leq K_1' E(u;C'^n\setminus C'^n(1-\mu)).$$

Thus there is some α in $[0,\mu]$ such that u restricted to $C'^n(1-\alpha)$ is in $W^{1,p}$ and

$$\underset{\sim}{E}(u;\partial C'^n(1-\alpha))\leq \frac{1}{\mu}K_1' E(u;C'^n(1-\mu)$$

then we consider the map $v'(\mu)$ defined on C'^n by

$$v'(\mu)=u \quad \text{on} \quad C'^n\setminus C'^n(1-\alpha)$$
$$v'(\mu)=u\left(\frac{x}{\|x\|}(1-\alpha)\right) \quad \text{on} \quad C'^n\setminus C'^n(1-\alpha).$$

Clearly $v'(\mu)$ is in $W^{1,p}(C'^n,N^k)$, the restriction of $v'(\mu)$ to $\partial C'^n$ is also in $W^{1,p}$ and it is easy to see that $v'(\mu)\to u$ in $W^{1,p}$. This completes the proof of Lemma A0(i). The proof of Lemma A0(ii) is technically slightly more involved but follows essentially the same idea.

Next we give the proof of Lemma 1.

A3. *Proof of Lemma* 1. Since the problem is mainly local (as later considerations will show), we may assume that u has only one point singularity centered at zero. So we assume that $u\in W^{1,p}\cap C^0(B^n\setminus\{0\},N^k)$. We shall first treat the simpler case $u\in C^\infty(B^n\setminus\{0\},N^k)$. Afterwards, we will consider the general case.

First case: $u\in C^\infty(B^n\setminus\{0\},N^k)$. Let $r<1$ be small. We consider the set:

$$\mathcal{M}(r)=\{\varphi\in \operatorname{Lip}(B^n,N^k),\ \varphi=u \ \text{ on } B^n\setminus B^n(0,r)\}.$$

Since $\pi_{n-1}(N^k)=0$, $\mathcal{M}(r)$ is not empty. We set:

$$\mu(r)=\inf_{\varphi\in\mathcal{M}} E(\varphi;B^n(0;r)).$$

We claim that:

$$\mu(r)\leqslant\frac{3^p+1}{1-3^{p-n}}E(u;B^n(0;r)). \tag{1'}$$

Proof of the claim. Let $\varphi\in\mathcal{M}(r)$. We consider the map $\tilde\varphi$ defined by:

$$\tilde\varphi(x)=\varphi(3x) \quad\text{if}\quad |x|\leqslant\frac{r}{3},$$

$$\tilde\varphi(x)=\varphi\left(\frac{x}{|x|}\left(\frac{4}{3}-\frac{|x|}{r}\right)\right)\quad\text{if}\quad \frac{r}{3}\leqslant|x|\leqslant\frac{2r}{3},$$

$$\tilde\varphi(x)=(x)\quad\text{if}\quad |x|\geqslant\frac{2r}{3}.$$

Clearly $\tilde\varphi\in\mathcal{M}$. Easy calculations show that

$$E(\tilde\varphi;B^n(0;r))\leqslant 3^{p-n}E(\varphi;B^n(0;r))+(3^p+1)\,E(u;B^n(0;r)).$$

Taking a sequence $\varphi_n\in\mathcal{M}$ such that $E(\varphi_n;B^n(0;r))\to\mu(r)$ we obtain

$$\mu\leqslant 3^{p-n}\mu+(3^p+1)\,E(u;B^n(0;r))$$

which leads to (1′), and completes the proof of the claim.

For $r<1$ small, we consider a map $u_r\in\mathcal{M}_r$ such that

$$E(u_r;B^n(0;r))\leqslant 2\frac{3^p+1}{1-3^{p-n}}E(u;B^n(0;r)).$$

It is then easy to show, using (1′) that u_r converges to u in $W^{1,p}$ when r goes to zero. This completes the proof of Lemma 1 in the first case.

The general case: $u\in C^0(B^n\setminus\{0\},N^k)\cap W^{1,p}$. It suffices to construct a sequence of maps $u_n\in C^\infty(B^n\setminus\{0\},N^k)\cap W^{1,p}$ which converges to u in $W^{1,p}$ (then, we may apply the first case). In order to construct u_n, we extend u to $B^n(0;2)$ by $u(x)=u(x/|x|)$, and we consider a mollifier ξ from $\mathbf{R}^n\to\mathbf{R}^+$ such that $\int_{\mathbf{R}^n}\xi(x)\,dx=1$ and $\operatorname{supp}(\xi)\subset B^n(0,1)$. For

$\sigma>0$ small we set $\xi^\sigma(x)=\sigma^{-n}\xi(x/\sigma)$. For x in B^n, and σ small, we consider the map u^σ defined by:

$$u^\sigma(x)=\int_{\mathbf{R}^n}\xi^\sigma(x-z)\,u(z)\,dz \quad \text{for} \quad x\in B^n.$$

$u^\sigma\in C^\infty(B^n;\mathbf{R}^l)$ and u^σ converges to u in $W^{1,p}$, and uniformly on every compact subset of $B^n\setminus\{0\}$. Let $0<r<1$ be small. Choose σ small such that:

$$E\left(u^\sigma;B^n(0;r)\setminus B^n\left(0;\frac{r}{2}\right)\right)\leqslant 2E\left(u;B^n(0;r)\setminus B^n\left(0;\frac{r}{2}\right)\right) \quad \text{and}$$

$$\max\left\{|u^\sigma(x)-u(x)|;x\in B^n\setminus B^n\left(0;\frac{r}{2}\right)\right\}\leqslant r.$$

Choose $r_0\in[r/2,r]$ such that:

$$E(u^\sigma;S_{r_0})\leqslant\frac{2}{r}E\left(u^\sigma;B^n(0;r)\setminus B^n\left(0;\frac{r}{2}\right)\right)\leqslant\frac{4}{r}E(u;B(0;r)).$$

We consider the map u_r defined by:

$$u_r=\pi\circ u^\sigma \quad \text{on} \quad B^n\setminus B^n(0;r_0)$$

$$u_r=\pi\circ u^\sigma\left(\frac{xr_0}{|x|}\right) \quad \text{on} \quad B^n(0;r_0).$$

u_r is in $C^\infty(B^n\setminus\{0\};N^k)\cap W^{1,p}$, and it is easy to see that $u_r\to u$ in $W^{1,p}$, then r goes to zero. This completes the proof of Lemma 1 in the general case.

Remark. Using exactly the same method, we may prove the following (which is used in the proof of Lemma 7):

A4. LEMMA 1 bis. *Let $u\in W^{1,p}(M^n,N^k)$ $(p<n)$ be continuous except at most at a finite number of points. Assume that the homotopy class of u at each singularity is trivial (we do not assume that $\pi_{n-1}(N^k)$ is trivial). Then u can be approximated in $W^{1,p}$ by smooth maps between M^n and N^k.*

A.5. *Proof of Lemma* 5. We consider first the ball $B^l(y,\delta)$ in $\mathbf{R}^l$ (recall that $N^k\subset\mathbf{R}^l$ and the smooth map $\xi_{(y,\delta)}$ from $\mathbf{R}^l$ to $B^l(y,\delta)$ defined by:

$$\xi_{(y,\delta)}(z)=z \quad \text{if } z \text{ is in } B^l(y,\delta);$$

$$\xi_{(y,\delta)}(z)=\frac{z-y}{|z-y|}\delta \quad \text{if } z \text{ is not in } B^l(y,\delta).$$

Note that $\xi_{(y,\delta)}(z) \in \partial B^l(y,\delta)$ if $z \notin B^l(y,\delta)$. Next we claim that there is a smooth map $P_{(y,\delta)}$ from $B^l(y,\delta)$ to $\tilde{B}_\varrho(y,\delta)=B^l(y,\delta)\cap N^k$ such that:

$$P_{(y,\delta)}(z)=z \quad \text{if} \quad z\in \tilde{B}_\varrho(y,\delta)=B^l(y,\delta)\cap N^k;$$

$$|\nabla P_{(y,\delta)}|_\infty \leqslant 2 \quad \text{if } \delta \text{ is small enough.}$$

Indeed, in the case $\tilde{B}_\varrho(y,\delta)$ is linear, $P_{(y,\delta)}$ can be easily constructed using an orthogonal projection onto $\tilde{B}_\varrho(y,\delta)$. The general case follows by linearization. We set:

$$\varphi(y,\delta)(x)=P_{(y,\delta)}\circ \xi_{(y,\delta)}(x) \quad \text{for} \quad x\in N^k.$$

It is then easy to show that $\varphi(y,\delta)$ satisfies the conditions of Lemma 5.

A.6. *Proof of Lemma* 6. For simplicity we replace $C^n(\mu)$ by B^n. For $r>0$ small we consider the map $\tilde{u}_r$ defined from $B^n(2)$ to $\tilde{B}_\varrho(y,\delta)$ by:

$$(2') \qquad \begin{aligned} u_r(x)&=u(x) \quad \text{if} \quad 0\leqslant |x|\leqslant 1-2r;\\ u_r(x)&=u\left(\frac{x}{|x|}\right) \quad \text{if} \quad 1-r\leqslant |x|\leqslant 2;\\ \tilde{u}_r(x)&=u\left(2x+(2r-1)\frac{x}{|x|}\right) \quad \text{if} \quad 1-2r\leqslant |x|\leqslant 1-r. \end{aligned}$$

It is easy to verify that $\tilde{u}_r\in W^{1,p}(B^n,\tilde{B}_\varrho(y,\delta))$ and that $\tilde{u}_r\to u$ in $W^{1,p}$ when $r\to 0$. We consider a mollifier ξ from $\mathbf{R}^n$ to $\mathbf{R}^+$ such that $\int_{\mathbf{R}^n}\xi(x)\,dx=1$ and $\operatorname{supp}(\xi)\subset B^n$. For $\sigma>0$ we set $\xi^\sigma=\sigma^{-n}\xi(\xi/\sigma)$ and we consider on B^n the map $\tilde{u}_{r,\sigma}$ defined by

$$\tilde{u}_{r,\sigma}=\int_{\mathbf{R}^n}\xi^\sigma(x-z)\,\tilde{u}_r(z)\,dz.$$

$\tilde{u}_{r,\sigma}$ is in $C^\infty(B^n;B^l(y,\delta))$ and $\tilde{u}_{r,\sigma}\to u_r$ in $W^{1,p}(B^n,\mathbf{R}^l)$ when σ goes to zero. Moreover since $\tilde{u}_r$ is continuous on the set $1-r\leqslant|x|\leqslant 2$ (as (2′) shows) and equal to $u(x/|x|)$, the restriction of $\tilde{u}_{r,\sigma}$ to ∂B^n tends uniformly to the restriction of u to ∂B^n, when $\sigma\to 0$. We consider now the map $u_{r,\sigma}$ defined by:

$$u_{r,\sigma}(x)=\pi\circ\tilde{u}_{r,\sigma}(x) \quad \text{if} \quad 0\leqslant|x|\leqslant 1-2r;$$

$$u_{r,\sigma}(x)=\pi\circ\tilde{u}_{r,\sigma}\left(2x+(2r-1)\frac{x}{|x|}\right) \quad \text{if} \quad 1-2r\leqslant|x|\leqslant 1-r;$$

$$u_{r,\sigma}(x)=\pi\left[\frac{1}{r}(1-|x|)\,\tilde{u}_{r,\sigma}\left(\frac{x}{|x|}\right)+\frac{1}{r}(|x|-(1-r))\,u\left(\frac{x}{|x|}\right)\right] \quad \text{if} \quad |x|\geqslant 1-r.$$

$u_{r,\sigma}$ is in $C^0(B^n, \tilde{B}_\varrho(y, \delta)) \cap W^{1,p}$. Moreover easy calculations show that $\tilde{u}_{r,\sigma} \to u$ in $W^{1,p}$ when $r \to 0$, $\sigma \to 0$. This completes the proof of Lemma 6.

Combining the method of the proof of Lemma 6, with the proof of the approximation result of Schoen and Uhlenbeck ([SU2], [SU3]), we may also prove the following, which is the equivalent of Theorem 1 bis for the case $p=n$:

LEMMA 6 bis. *Assume $p=n$, and let v be in $W^{1,p}(C^n, N^k)$ such that v retricted to ∂C^n is in $W^{1,p} \hookrightarrow C^0$. Then v can be approximated by maps in $W^{1,p} \cap C^0(C^n, N^k)$ which coincide with v on the boundary ∂C^n.*

A.7. *Proof of Lemma* 10. For simplicity, we may assume that $\mu=1$ and work on C'^d. In order to prove Lemma 10, we need the following:

LEMMA A11. *Let $p>1$, and let $d>p$. Let v be in $W^{1,p}(\partial C'^d, N^k)$ continuous except at* one *point singularity A then the homotopy class of v at A is trivial.*

The proof of Lemma A11 is straightforward.

Proof of Lemma 10 *completed.* We are going to use a construction similar to the construction of the proof of Lemma 2(ii). Without loss of generality, we may assume that $P^+=(0,0,\dots,1/2)$ is not a singularity of v. Let $\sigma>0$ be small, and $0<\sigma'<\sigma$ be such that $\underline{E}(v; \partial B^{d-1}(\sigma') \times \{1/2\}) \leq 2E(v; B^{d-1}(\sigma) \times \{1/2\})$. We set $V(\sigma') = B^{d-1}(\sigma') \times \{1/2\}, W(\sigma') = C'^d \setminus V(\sigma')$. Let $(A_i)_{1 \leq i \leq k}$ be the point singularities of v. For two points A and B in $W(\sigma')$ we note $[A, B]_g$ a geodesic line joining A and B. Let $(B_i)_{1 \leq i \leq k}$ be points on $\partial W(\sigma')$, such that $[A_i, B_i]_g$ does not intersect $[\partial C'^d]_s$, and does not intersect $[A_j, B_j]_g$ if $j \neq i$ (it is always possible to find such points $(B_i)_{1 \leq i \leq k}$, since the codimension of $[\partial C'^d]_s$ in $\partial C'^d$ is at least 2). As in Lemma 2(ii), we apply Lemma 7 bis, which gives us the existence of maps v_m in $W^{1,p}(C^d \setminus B^d(\sigma')) \cap C^0$, and of small neighborhoods $K_{m,i}$ of $[A_i, B_i]_g$ such that $K_{m,i} \cap K_{m,j} = \emptyset$, if $i \neq j$, $K_{m,i} \cap [\partial C'^d]_s = \emptyset$, $\lim_{m \to \infty} E(v_m; K_{m,i}) = 0$, $v_m = v$ on $W(\sigma') \setminus \bigcup_{i=1}^k K_{m,i}$ (and thus on $[\partial C'^d_s]$). We extend v_m on $B_n(\sigma')$ by $v_m = v_m(x\sigma'/|x|)$. Then $v_m \in W^{1,p}(C'^d, N^k)$, $v_m = v$ on $[\partial C'^d_s]$, and v_m has only one point singularity at P^+. Moreover it is easy to see that $v_m \to v$ in $W^{1,p}(C'^d, N^k)$ when $m \to +\infty$, and $\sigma \to 0$. The conclusion then readily follows from Lemma A11 and Lemma 1.

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