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Periodic meromorphic functions

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§ 0. Introduction

Let f be a meromorphic function on $\mathbf{C}^n$: then we define the set Γ_f of its periods as $\{\gamma | f(z+\gamma)=f(z)\ \forall z\in\mathbf{C}^n\}$.

(0.1) Γ_f is indeed a closed complex Lie subgroup of $\mathbf{C}^n$, and therefore there exists a complex vector space decomposition $\mathbf{C}^n=A\oplus B$ such that $\Gamma_f=A\oplus\Gamma'$ with Γ' discrete in B.

Definition 0.2. The function f is said to be *non degenerate* if its group of periods Γ_f is discrete. Otherwise we define more generally its *rank* r to be the dimension of the subspace B as above.

By (0.1) one can easily reduce to the study of non degenerate meromorphic functions.

The basic problems in the theory of periodic meromorphic functions are the following ones:

Problem 1. Given a discrete subgroup Γ of $\mathbf{C}^n$, when does there exist a non degenerate (resp. non constant) Γ-periodic meromorphic function? More generally, given a subgroup Γ of $\mathbf{C}^n$, when does there exist a meromorphic function f such that $\Gamma=\Gamma_f$?

Problem 2. Given a discrete subgroup Γ as above, try to completely describe the field of the Γ-periodic meromorphic functions.

The results of the present article give a complete solution to the first problem, and provide key steps towards the solution of problem 2. To be more explicit and precise, we need to set up some notation.

Definition 0.3. Given a discrete subgroup Γ of $\mathbf{C}^n$, the quotient Lie group $X=\mathbf{C}^n/\Gamma$ is said to be a *quasi-torus*. In other words, quasi-tori are exactly the connected Abelian complex Lie groups. X is said to be a torus if it is compact, and more generally a *Cousin-quasi-torus* if $\mathrm{Hol}(X)$, the ring of holomorphic functions on X, consists only of the constants.

For further use, we denote by $\pi:\mathbf{C}^n\to X$ the quotient map.

(0.4) Let $\mathbf{R}\Gamma$ be the real subspace of $\mathbf{C}^n$ generated by Γ. Then $K=\mathbf{R}\Gamma/\Gamma$ is the maximal compact subgroup of X, viewed as a real Lie group.

We let F be the maximal complex subspace contained in $\mathbf{R}\Gamma$; clearly $\pi(F)$ is contained in K and we shall see in § 1.1 that it is indeed dense in K provided X is Cousin (i.e., $\mathrm{Hol}(X)=\mathbf{C}$). We denote, for further use, by m the complex dimension of F.

(0.5) (i) We may assume that Γ spans $\mathbf{C}^n$ as a complex vector space. Otherwise, X would be a product $X\cong\mathbf{C}^r\times X'$.

(ii) In this case the closure of $\pi(F)$ equals K iff (if and only if) X is Cousin (cf. § 1.1).

(iii) There is, in general, a decomposition of X, unique up to isomorphism, as a product

$$X\cong\mathbf{C}^a\times(\mathbf{C}^*)^b\times X', \quad \text{where } X' \text{ is Cousin.}$$

This is the so called theorem of *Remmert–Morimoto* (cf. [Mo 2], also § 1.1).

Therefore, for the problem of existence of non degenerate meromorphic functions, one can easily reduce to the case of a Cousin quasi-torus. Whereas, for problem 2, since every meromorphic function is the quotient of two sections of a line bundle, it is enough, by the Künneth theorem, to study the space of sections of line bundles in the separate cases of $(\mathbf{C}^*)^b$ (this is done in § 1.4) and of a Cousin quasi torus X (cf. § 2.1).

Problem 1 (existence) is completely solved by the following

MAIN THEOREM. *Given a discrete subgroup Γ of $\mathbf{C}^n$, there exists a non degenerate Γ-periodic meromorphic function if and only if the following generalized* Riemann bilinear relations *are satisfied: there exists a Hermitian form H on $\mathbf{C}^n$ such that*

(i) Im(H), *the imaginary part of* H, *takes values in* $\mathbf{Z}$ *on* $\Gamma\times\Gamma$

(ii) $H|_F$, *the restriction of* H *to the maximal complex subspace* F *of the real span of* Γ, *is strictly positive definite.*

Remark. Condition (ii) can be replaced by the following: H is positive definite (cf. Step I of Theorem 2.4).

A brief historical digression seems appropriate at this point.

(1) The result, in the case of a torus (X compact, $F=\mathbf{C}^n$), goes back to Poincaré in 1898 ([Poi]); essential use is made of the hypothesis of compactness, which allows to prove the so called "*Theorem of Appell–Humbert*" (*linearization of the system of exponents*).

(2) The case when $n=2$ was completely settled by Cousin in [Cou 2], although his formulation of the generalized Riemann relations is not so explicit.

(3) The formulation given here of the GRR (Generalized Riemann Relations) is taken from Andreotti–Gherardelli ([A–G1], [A–G2]) who proved in 1972 the sufficiency of the GRR, using only linear systems of exponents.

(4) A weaker result concerning the necessary conditions (ii) being replaced by: Im(H) is not identically zero on $\Gamma\times\Gamma$) was proved, in the special case where the rank of Γ equals $n+1$, by Huckleberry–Margulis in 1983 ([H–M]).

(5) A weaker result in the general case was proved in the Ph.D. thesis of the junior author ([Cap]): (i) holds, and $H|_F$ must be positive semidefinite and not zero.

The noncompact case is more difficult because of the existence of nonlinearizable (*wild*) systems of exponents, which was first observed by Cousin in [Cou 2], and then rediscovered by several authors: Malgrange ([Mal]), who interpreted it as a property of the first cohomology group $H^1(X, O_X)$ of being non Hausdorff, and later by Vogt ([Vo]) and the present authors.

(0.6) To explain what is a wild system of exponents, recall that a meromorphic function on $\mathbf{C}^n$, by a theorem of Poincaré, is a quotient of two entire functions; hence our Γ-periodic meromorphic function is a quotient of two entire functions satisfying a functional equation

$$h(z+\gamma)=k_\gamma(z)\,h(z) \quad (\forall\gamma\in\Gamma)$$

where $k_\gamma(z)$ is a cocycle defining a line bundle L on X.

Writing the cocycle $k_\gamma(z)$ as $e(f_\gamma(z))$, where the symbol $e(t)$ stands for the exponential of $2\pi it$, the functions $f_\gamma(z)$ were classically referred to as a system of exponents.

A given system of exponents is equivalent to another one whenever, chosen an entire function g, one replaces $f_\gamma(z)$ by adding to it the function $g(z+\gamma)-g(z)$.

A cocycle $k_\gamma(z)$ is said to be *tame* (or linearizable) if it is equivalent (cohomologous) to the exponential of a system of exponents which is linear, i.e. given by polynomials of degree at most 1. A cocycle which is not tame is said to be *wild*, and a quasi-torus X is said to be *wild* if it admits some wild cocycle.

Regarding wild quasi-tori, we have the following result:

THEOREM 1. *Wild quasi-tori form a Borel set W which is*
(1) *empty if* m $(=\dim F)=n$ *(i.e., X is a torus)*
(2) *non empty and of Hausdorff codimension* $2m$ *if* $m<n$.

We remark that part (1) is the classical Appell–Humbert theorem, for which we give a new elementary proof; this proof uses only conditions for the convergence of Fourier series which in turn form the basis for the proof of statement (2).

A second new result that we obtain is the characterization of sections of line bundles on $(\mathbf{C}^*)^n$, in terms of a certain Frechêt space of entire functions in one variable which decrease exponentially in the real direction (§ 1.4).

Let us now explain the principal steps of the proof of the main theorem.

The *key step* consists in showing first a *Lefschetz type theorem,* i.e. that the existence of a non degenerate meromorphic function implies the existence of a holomorphic immersion of X into a projective space.

In the classical theory this was proved by Lefschetz using the linearization of the system of exponents; in other terms, in the classical theory the embedding theorems are a consequence of the Riemann bilinear relations.

Here, instead, the basic trick consists in reversing the classical chain of arguments, since there are systems of exponents which cannot be linearized. Thus we deduce the Generalized Riemann Relations (in § 2), from the existence of the above holomorphic immersion, simply by pulling back the Fubini–Study form on projective space and integrating over the maximal compact subgroup K of X.

The main ingredients of proof of the key step are three. The first one is a general lemma which, by use of the group law on X, produces a meromorphic map, generically with injective differential, out of the given non degenerate meromorphic function.

The second ingredient is a normal form for the system of exponents which was first deviced by Cousin in the case of 2 variables, and then was extended to the case of more variables first by Andreotti and Gherardelli ([A–G1]), and then by Vogt ([Vo]). For this result we refer to these two papers.

The main property of the *Cousin normal form* is that the nonlinear part of the system of exponents is given by F-periodic holomorphic functions.

At this step the crucial hypothesis that X is a Cousin group guarantees that $\pi(F)$ is dense in K, and with standard geometrical arguments about generic translates of divisors we can prove a first weak version of a Lefschetz type theorem (in § 3).

It seems worthwhile to mention an important corollary of our main theorem, namely that the quasi-tori admitting a non degenerate meromorphic function have some algebraic group-structure, and therefore coincide with the *quasi-Abelian varieties* introduced by Severi ([Sev]) and later studied by Rosenlicht ([Ro 1]) and Serre ([Ser]).

Finally, we postpone to a future paper the explicit description of the (infinite dimensional) vector spaces of sections of line bundles in the noncompact case. We conjecture here that a sufficient condition in order that this space is non empty is that the first Chern class $c_1(L)$ can be represented by a positive definite Hermitian form.

Modulo this conjecture, in section 3.3, we can give then a strong version of a Lefschetz type theorem, which is almost as good as in the compact case; moreover, we can characterize the Kodaira dimension of a line bundle L in terms of semipositive Hermitian forms H representing the first Chern class $c_1(L)$, and we relate this integer with the rank of meromorphic functions.

The main body of the paper is divided into three paragraphs:

§ 1. Systems of exponents and wild tori
§ 2. Generalized Riemann bilinear Relations
§ 3. Lefschetz type theorems.

Each paragraph is divided into sections whose contents are as follows:

1.1. Tame and wild systems of exponents
1.2. The theorem of Appell–Humbert and conditions for linearizability
1.3. Hausdorff dimension of the set of wild tori
1.4. Line bundles on $(\mathbf{C}^*)^n$

2.1. Sufficiency of the GRR (via theta functions)
2.2. Chern class of a line bundle on X and associated Hermitian forms
2.3. Proof of the Main Theorem: the key step implies GRR
2.4. Conditions for the existence of meromorphic functions

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Added in proof. After the paper was submitted, we were kindly informed by A. Huckleberry that some steps of the proof of our Main Theorem have recently appeared in the following papers:

[1] OELJEKLAUS, K., *Hyperflächen und Geradenbundel auf homogenen komplexen Mannigfaltigkeiten.* Schriftenreihe Math. Inst. Univ. Münster, Ser. 2, 36, 1985 (*Math. Rev.*, 87 h: 32063).
[2] ABE, Y., Holomorphic sections of line bundles over (H, C) groups. *Manuscripta Math.*, 60 (1988), 379–385.

In [1] it is proven that every hypersurface separable Cousin quasi-torus is meromorphically separable, in [2] that a meromorphically separable (H, C), i.e., Cousin quasi-torus, is quasi Abelian.

We also received, through courtesy of the authors, reprints of

[3] OELJEKLAUS, K. & RICHTOFER, W., On the structure of complex solvmanifolds. *J. Differential Geom.*, 27 (1988), 399–421.
[4] GILLIGAN, B., OELJEKLAUS, K. & RICHTOFER, W., Homogeneous complex manifolds with more than one end. *Canad. J. Math.*, 41 (1989), 163–177.

[3] and [4] treat related matters, in particular in [4] the proof of [1] that hypersurface separability implies meromorphical separability is extended to a more general situation.

Finally, by courtesy of the referee whom we would like to thank, we received a copy of a preprint,

[5] ABE, Y., Sur les fonctions periodiques de plusieurs variables,

which contains partial but interesting results related to the subject of our paper, and in particular to Conjecture 3.22.

§ 1. Systems of exponents and wild tori

1.1. Tame and wild systems of exponents

In this section and almost always in the rest of the paper Γ shall be a discrete subgroup of $\mathbf{C}^n$ with the property of spanning $\mathbf{C}^n$ as a complex vector space (cf. (0.5)(i)). Therefore, as in (0.4), we denote by $n+m$ the rank of Γ, and clearly m is at most equal to n. It is easy then to see that the maximal complex subspace F of the real span $\mathbf{R}\Gamma$ of Γ has complex dimension m.

PROPOSITION 1.1. *A quasi-torus X is Cousin (i.e., the only holomorphic functions on X are the constants), if and only if $\pi(F)$ is dense in $K=\mathbf{R}\Gamma/\Gamma$. More generally, any quasi-torus X admits a Remmert–Morimoto decomposition $X \cong \mathbf{C}^a \times (\mathbf{C}^*)^b \times X'$, where X' is Cousin.*

Proof. First of all it is enough to prove the theorem when Γ spans $\mathbf{C}^n$ as a complex vector space (cf. (0.5) (i)). Assume then that $\pi(F)$ is not dense in $K=\mathbf{R}\Gamma/\Gamma$, and denote by H the closed subgroup given by the closure of $\pi(F)$: H is connected, $\pi(F)$ being such. H is a real torus, of dimension $(n+m-b)$; if we let V be the (real) vector space $\pi^{-1}(H)$, then V spans a complex subspace U' of dimension $(n-b)$ and Γ splits as a direct sum $\Gamma' \oplus \Gamma''$ where Γ' is a lattice in V and Γ'' has rank b. Let U'' be the complex span of Γ''. Then clearly $X \cong X' \times (\mathbf{C}^*)^b$ and so firstly X is not Cousin, secondly our proof is accomplished if we show the converse, i.e., that X is Cousin whenever $\pi(F)$ is dense in $K=\mathbf{R}\Gamma/\Gamma$. But in this case if f is holomorphic on X, f is bounded on K, hence its pull-back f' is constant on F; by the density of $\pi(F)$, f' is constant on $\mathbf{R}\Gamma$, and therefore on its complex span $\mathbf{C}^n$. Q.E.D.

From now on, we shall assume that X is a Cousin quasi-torus. Since Γ generates $\mathbf{C}^n$, we can change basis in $\mathbf{C}^n$ and assume that we may write

$$\Gamma = \mathbf{Z}^n \oplus \Lambda \tag{1.2}$$

where $\mathrm{Im}(\Lambda)$ has rank m.

Then every holomorphic function on X can be written as a Fourier series

$$f(z) = \sum_{p \in \mathbf{Z}^n} c_p e(\langle p, z \rangle) \tag{1.3}$$

where $\langle \ldots, \ldots \rangle$ denotes the standard scalar product in $\mathbf{C}^n$, and the above expression is equivalent to the $\mathbf{Z}^n$-periodicity of f.

We also have Λ-periodicity if and only if, for each vector $\lambda \in \Lambda$, we have $f(z+\lambda)=f(z)$, i.e., for each $p \in \mathbf{Z}^n$, $c_p e(\langle \lambda, p \rangle)=c_p$. This equation clearly holds if and only if either $c_p=0$, or $\langle \lambda, p \rangle$ is an integer. Therefore, if we set

$$J_{\mathbf{Z}} = \{p \in \mathbf{Z}^n | \langle \lambda, p \rangle \in \mathbf{Z}\ \forall \lambda \in \Lambda\}, \tag{1.4}$$

then $H^0(X, O_X)=\mathrm{Hol}(X)$ consists of the space of Fourier series (1.3) where $c_p=0$ unless $p \in J_{\mathbf{Z}}$.

(1.5) In particular X is a Cousin quasi-torus iff $J_{\mathbf{Z}}=0$.

We turn now to the consideration of meromorphic functions on X. As it is well known, every meromorphic function f is the quotient of two relatively prime sections of a line bundle L on X. The pull back of L to $\mathbf{C}^n$ is a trivial line bundle, and any line bundle L on X arises as the quotient of $\mathbf{C}^n \times \mathbf{C}$ by an action of Γ such that $\gamma \in \Gamma$ acts sending the pair (z, w) to $(z+\gamma, k_\gamma(z)\, w)$, where, for each $\gamma \in \Gamma$, $k_\gamma(z)$ is a nonvanishing holomorphic function of z satisfying the *cocycle condition*

$$k_{\gamma+\gamma'}(z) = k_{\gamma'}(z+\gamma)\, k_\gamma(z). \tag{1.6}$$

As in (0.6) we write also $k_\gamma(z)$ as $e(f_\gamma(z))$, and two systems of exponents give rise to isomorphic line bundles if and only if their difference is congruent modulo $\mathbf{Z}$ to a system of exponents of the form $g(z+\gamma)-g(z)$, for a suitable holomorphic function $g(z)$ on $\mathbf{C}^n$.

In this framework the (integral) Chern class of the line bundle L is the bilinear alternating function

$$A: \Gamma \times \Gamma \to \mathbf{Z} \tag{1.7}$$

obtained as follows: by the cocycle condition (1.6) we have that

$$E(\gamma, \gamma') = f_{\gamma+\gamma'}(z) - f_{\gamma'}(z+\gamma) - f_\gamma(z) \tag{1.8}$$

is an integer, moreover

$$A(\gamma, \gamma') = \tfrac{1}{2}[E(\gamma, \gamma') - E(\gamma', \gamma)] \tag{1.9}$$

is in fact bilinear and alternating.

Definition 1.10. A line bundle L is said to be *tame* or *linearizable* if it can be realized by a linear system of exponents (this means that the $f_\gamma(z)$ are polynomials of degree $\leqslant 1$). Otherwise, L is said to be *wild*.

Remark 1.11. If L is tame, it is not difficult to find, as in the classical case, that there exists a Hermitian form H on $\mathbf{C}^n$ such that the restriction of its imaginary part to $\Gamma\times\Gamma$ coincides with A. Such a form H is not unique in the noncompact case, but two such Hermitian forms differ by a Hermitian form vanishing on $F\times\mathbf{C}^n$, and taking real values on $\mathbf{R}\Gamma\times\mathbf{R}\Gamma$. L admits then a cocycle in the so called *Appell–Humbert normal form*

$$k_\gamma(z) = \varrho(\gamma)e\Big(-\frac{i}{2}\Big[H(z,\gamma)+\frac{1}{2}H(\gamma,\gamma)\Big]\Big) \tag{1.12}$$

where $|\varrho(\gamma)|=1$, and $\varrho(\gamma)$ is a *semicharacter* for the alternating form A, i.e., $\varrho(\gamma+\gamma')=\varrho(\gamma)\varrho(\gamma')e(\frac{1}{2}A(\gamma,\gamma'))$.

In fact, even when L is not tame there exists a Hermitian form H on $\mathbf{C}^n$ such that the restriction of its imaginary part to $\Gamma\times\Gamma$ coincides with A, and we have the following normal form similar to (1.12)

$$k_\gamma(z) = \varrho(\gamma)e\Big(-\frac{i}{2}\Big[H(z,\gamma)+\frac{1}{2}H(\gamma,\gamma)\Big]+f_\gamma(z)\Big) \tag{1.13}$$

where now $f_\gamma(z)$ is an additive cocycle, i.e. $f_{\gamma+\gamma'}(z)=f_{\gamma'}(z+\gamma)+f_\gamma(z)$.

We postpone the proof to section (2.2). Here we observe that L is linearizable iff $f_\gamma(z)$ is cobordant to a constant function, which means that there exists a holomorphic function $g(z)$ such that

$$f_\gamma(z)-g(z+\gamma)+g(z) \quad \text{is a constant.} \tag{1.14}$$

We are therfore led to the study of $H^1(X, O_X)$.

1.2. The theorem of Appell–Humbert and conditions for linearizability

First of all we take coordinates in $\mathbf{C}^n$ as in (1.2), such that we can write $\Gamma=\mathbf{Z}^n\oplus\Lambda$. Then X appears as a quotient of $Y=(\mathbf{C}^*)^n$ by the homomorphic image of Λ. Since $H^1(Y, O_Y)=0$, we have that

$$H^1(X, O_X) = H^1(\Lambda, H^0(Y, O_Y)). \tag{1.15}$$

We can therefore represent the above cohomology classes by cocycles $f_\lambda(z)$, given for all $\lambda \in \Lambda$, which have the following Fourier series expansion

$$f_\lambda(z) = \sum_{p \in \mathbf{Z}^n} c_{p,\lambda} e(\langle p, z \rangle). \tag{1.16}$$

The cocycle condition translates into the following equality:

$$c_{p,\lambda+\lambda'} = c_{p,\lambda}\, e(\langle p, \lambda' \rangle) + c_{p,\lambda'} = c_{p,\lambda'}\, e(\langle p, \lambda \rangle) + c_{p,\lambda} \quad (\forall \lambda, \lambda' \in \Lambda). \tag{1.17}$$

We set for convenience $r(p,\lambda) = e(\langle p, \lambda \rangle) - 1$, and we notice that if p is not 0 (since by our assumption $J_{\mathbf{Z}}=0$), there exists a vector $\lambda(p)$ such that $r(p, \lambda(p))$ is not zero. Moreover we can choose $\lambda(p)$ to be an element of a fixed set of generators of Λ.

Therefore, by (1.17) we have

$$c_{p,\lambda} = c_{p,\lambda(p)}\, r(p,\lambda)\, r(p,\lambda(p))^{-1}. \tag{1.18}$$

Recall now that the cocycle is linearizable if we can find a Fourier series

$$g(z) = \sum_{p \in \mathbf{Z}^n} a_p e(\langle p, z \rangle)$$

such that

$$f_\lambda(z) - g(z+\lambda) + g(z) \quad \text{is constant.} \tag{*}$$

(*) is clearly equivalent to the following equality

$$c_{p,\lambda} = a_p\, r(p,\lambda) \quad (\forall \lambda \in \Lambda,\ p \in \mathbf{Z}^n - \{0\}). \tag{1.19}$$

For each $p \in \mathbf{Z}^n - \{0\}$ we can therefore set

$$a_p = c_{p,\lambda(p)}\, r(p,\lambda(p))^{-1}, \tag{1.20}$$

and (1.19) is then automatically satisfied by virtue of (1.18).

The main problem now is that (1.20) shows that if such a function $g(z)$ exists, then it is unique, but indeed (1.20) only defines $g(z)$ as a formal Fourier series, and we have to see when is it convergent.

Remark 1.21. It is easy to see, by the cocycle condition, that the $f_\lambda(z)$ are convergent Fourier series iff this holds for λ belonging to the set of m vectors which generate Λ.

We now want to write a sufficient condition to ensure the convergence of the (formal) Fourier series $g(z)$ assuming that $f_\lambda(z)$ is convergent.

$$\exists N \in \mathbf{N}-\{0\} \text{ such that } \forall p \in \mathbf{Z}^n-\{0\}\ \exists \lambda(p) \text{ such that}$$

$$\operatorname{dist}[(\langle p, \lambda(p)\rangle), \mathbf{Z}] > N^{-|p|} \tag{1.22}$$

for a suitable element $\lambda(p)$ of a fixed set of generators of Λ.

PROPOSITION 1.23. *If* Λ *satisfies property* (1.22) *then*

$$H^1(X, O_X) = H^1(\Lambda, \mathbf{C}).$$

Proof. It is sufficient to show that the series $g(z)$ defined as in (1.20) is convergent.

In general a Fourier series $\Sigma_{p \in \mathbf{Z}^n} a_p e(\langle p, z\rangle)$ is convergent iff $\Sigma_{p \in \mathbf{Z}^n} |a_p| \mathbf{k}^p$ is convergent for each $\mathbf{k} \in (\mathbf{R}^+)^n$. In our case, therefore, the formal Fourier series $g(z)$ converges iff

$$\sum_{p \in (\mathbf{Z})\varepsilon} |c_{p,\lambda(p)}|\, |r(p, \lambda(p))|^{-1} \mathbf{k}^p$$

is convergent. For each $\varepsilon>0$ let $(\mathbf{Z})\varepsilon = \{p \in \mathbf{Z}^n |\ |r(p, \lambda(p))| > \varepsilon\}$. Surely, the series

$$\sum_{p \in (\mathbf{Z})\varepsilon} |c_{p,\lambda(p)}|\ |r(p, \lambda(p))|^{-1} \mathbf{k}^p$$

is convergent since, by the assumptions we made, the series

$$f_\lambda(z) = \sum_{p \in \mathbf{Z}^n} c_{p,\lambda} e(\langle p, z\rangle)$$

are convergent. So we can consider only the terms which do not belong to $(\mathbf{Z})\varepsilon$. Since the map $z \to e(z)$ is a homeomorphism of $\mathbf{C}/\mathbf{Z}$ onto $\mathbf{C}^*$, there exists, if p does not belong to $(\mathbf{Z})\varepsilon$, a positive constant M such that:

$$\operatorname{dist}[\langle p, \lambda(i)\rangle, \mathbf{Z}] < M |r(p, \lambda(i))|$$

where $\lambda(i)$ $(i=1, \ldots, m)$ are the elements of a fixed set of generators of Λ. Thus we have:

$$\begin{aligned}\sum_{p \notin (\mathbf{Z})\varepsilon} |c_{p,\lambda(p)}|\, |r(p, \lambda(p))|^{-1} \mathbf{k}^p &< \sum_{p \notin (\mathbf{Z})\varepsilon} |c_{p,\lambda(p)}|\, M (\operatorname{dist}[\langle p, \lambda(p)\rangle, \mathbf{Z}])^{-1} \mathbf{k}^p \\ &< \sum_{p \notin (\mathbf{Z})\varepsilon} |c_{p,\lambda(p)}|\, M N^{|p|} \mathbf{k}^p \\ &< M \sum_{i=1,\ldots,m} \sum_{p \in \mathbf{Z}^n} |c_{p,\lambda(i)}|\, ((1+N)\mathbf{k})^p.\end{aligned}$$

The last series converges since the Fourier series which define the $f_\lambda(z)$ are convergent. Q.E.D.

Remark 1.24. Property (1.22) is invariant under changes of basis in the subgroup Λ. In fact let V be a matrix whose columns are the coordinates of a set of generators of Λ. Thus (1.22) is clearly equivalent to the following:

$$\exists N\in \mathbf{N}-\{0\} \text{ such that } \forall p\in \mathbf{Z}^n-\{0\} \ \operatorname{dist}({}^tVp, \mathbf{Z}^m)>N^{-|p|}. \tag{1.25}$$

Note that here and in the following we consider the Euclidean distance, and we shall denote by $|v|$ the Euclidean norm of a vector v in $\mathbf{C}^n$, whereas for a matrix W we shall denote by $\|W\|$ the operator norm.

Changing basis in Λ, amounts to acting on the vector ${}^tVp$ by a fixed invertible integral unimodular linear transformation B and therefore, since B is a product of elementary transformations, to prove our assertion, it suffices to verify that in $\mathbf{R}^2$ the transformation $(x, y)\to(x+y, y)$ affects the distance d of a vector from $\mathbf{Z}^2$ by substituting d with d' such that:

$$\frac{1}{2}(3-\sqrt{5})\,d\leqslant d'\leqslant\frac{1}{2}(3+\sqrt{5}).$$

We can now give as an immediate corollary, an elementary proof of the so called *theorem of Appell–Humbert*.

THEOREM 1.26. *If X is a complex torus, every system of exponents on X is linearizable.*

Proof. We can observe that, if $n=m$, we can choose a basis in $\mathbf{C}^n$ as in (1.2) such that, if V is the matrix in (1.25), $\operatorname{Im} V$ is invertible. Thus

$$\operatorname{dist}({}^tVp, \mathbf{Z}^m)\geqslant|\operatorname{Im}{}^tVp|\geqslant kp$$

where $k^{-1}=\|(\operatorname{Im}{}^tV)^{-1}\|$. Q.E.D.

1.3. Hausdorff dimension of the set of wild tori

Given a discrete subgroup Γ in $\mathbf{C}^n$ of rank $n+m$, as above, we have already seen that Γ can be represented as $\Gamma=\mathbf{Z}^n\oplus\Lambda$ and that Λ is generated by the columns of a matrix V $(n\times m)$ such that $\operatorname{Im} V$ has rank m (cf. 1.25).

Let us consider, in the space $\mathbf{M}(n, m, \mathbf{C}) \cong \mathbf{C}^{nm}$, the open set of the matrices V such that $\operatorname{Im} V$ has maximal rank m. Now let us consider the following equivalence relation in $\mathbf{M}(n, m, \mathbf{C})$:

(1.27) Two such matrices V and V' are equivalent iff there exist a matrix A in $GL(n, \mathbf{C})$ and a matrix M in $GL(n+m, \mathbf{Z})$ such that:

$$(E_n, V') = A(E_n, V)M$$

where E_n is the identity matrix of dimension n, and A gives a linear automorphism in $\mathbf{C}^n$ which sends the subgroup Γ in the subgroup Γ' generated by the columns of the matrix (I, V').

If we write the matrix M in suitable block form:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{1.28}$$

then the relation (1.27) reads out as

$$A^{-1} = M_{11} + VM_{21} \quad \text{and} \quad (M_{11} + VM_{21})V' = M_{12} + VM_{22}.$$

Then it is clear that A is determined by M and then the previous relation of equivalence can be written as follows: two matrices V and V' in $\mathbf{M}(n, m, \mathbf{C})$ are equivalent iff

(1.29) There exists a element M in $GL(n+m, \mathbf{Z})$ such that:
(i) $(M_{11} + VM_{21})$ is invertible;
(ii) $V' = (M_{11} + VM_{21})^{-1}(M_{12} + VM_{22})$.

Consider now the condition on V which is just the contrary of (1.25) (we shall see that it implies that Γ is wild).

(1.30) $\forall N \in \mathbf{N} - \{0\}$ $\exists p \in \mathbf{Z}^n - \{0\}$ and $q \in \mathbf{Z}^m$ such that $\operatorname{dist}({}^tVp, -q) \leqslant N^{-|p|}$.

PROPOSITION 1.31. *If V satisfies the previous property, then the associated quasi-torus is wild.*

Proof. It suffices to show the existence of a (convergent) cocycle

$$f_\lambda(z) = \sum_{p \in \mathbf{Z}^n} c_{p,\lambda}\, e(\langle p, z\rangle)$$

such that the corresponding formal Fourier series

$$g(z)=\sum_{p\in\mathbf{Z}^n} a_p\, e(\langle p, z\rangle),$$

given by (1.20), is not convergent.

By (1.30) $\forall N\in\mathbf{N}-\{0\}$ $\exists p=p(N)\in\mathbf{Z}^n-\{0\}$ such that $\operatorname{dist}({}^tVp, \mathbf{Z}^m)\leq N^{-|p|}$. Since for $N'>N$ we have that $N^{-|p|}>N'^{-|p|}$, we can assume all the $p(N)$ to be different, and we consider the formal Fourier series

$$g(z)=\sum_N e(\langle p(N), z\rangle),$$

which is clearly divergent. The corresponding $f_\lambda(z)$ is, though, converging since for each k we have

$$\sum_N k^{|p(N)|}N^{-|p(N)|}=\sum_N (k/N)^{|p(N)|}<\infty. \qquad \text{Q.E.D.}$$

We define a matrix V' coarsely equivalent to V if V' can be obtained from V by a sequence of operations of the form:

(i) replacing V by M_1V, with $M_1\in GL(n,\mathbf{Z})$;

(ii) replacing $\operatorname{Re}V$ by $\operatorname{Re}VM_2$, with $M_2\in GL(m,\mathbf{Z})$;

(iii) replacing $\operatorname{Im}V$ by $\operatorname{Im}VB$, with $B\in GL(m,\mathbf{R})$.

One may observe that if V, $V=\operatorname{Re}V+i\operatorname{Im}V$, satisfies (1.30), also every V' which is coarsely equivalent to V still satisfies the above property (1.30).

Hence, by a suitable choice of M_1, M_2 and B, we can find a matrix V', coarsely equivalent to V and of the form:

$$(1.32) \qquad {}^tV'=(b_1+ie_1,\dots,b_m+ie_m, a_1+id_1,\dots,a_{n-m}+id_{n-m}).$$

In this expression, $e_1,\dots,e_m$ is the standard basis in $\mathbf{R}^m$ and all the vectors a_i, b_i, d_i, are in $\mathbf{R}^m$.

Then the set $\mathbf{W}$ of matrices which generate wild quasi-tori can be thought of as being given by points of a real space of (real) dimension $m(2n-m)$. Moreover, property (1.30) can be rewritten as:

$$(1.33) \qquad \mathbf{W}=\bigcap_{N\in\mathbf{N}^+}\ \bigcup_{\substack{p\in\mathbf{Z}^n-\{0\}\\ q\in\mathbf{Z}^m}} W_{N,p,q}$$

where we have set:

$$W_{N,p,q} = \{V \mid \|{}^tVp+q\| \leqslant N^{-|p|}\} \quad \text{and} \quad \|x+iy\| = \max\{|x|, |y|\}. \tag{1.34}$$

For further use, we set:

$$W_N = \bigcup_{\substack{p \in \mathbf{Z}^n - \{0\} \\ q \in \mathbf{Z}^m}} W_{N,p,q}$$

$$W_N^{(i)} = \bigcup_{\substack{p \in \mathbf{Z}^n, p_i \neq 0 \\ q \in \mathbf{Z}^m}} W_{N,p,q}$$

$$W^{(i)} = \bigcap_{N \in \mathbf{N}^+} W_N^{(i)}.$$

Remark 1.35. The conditions defining $W_{N,p,q}$ can be expressed as follows:

(i) $|p_1 b_1 + \ldots + p_m b_m + p_{m+1} a_1 + \ldots + p_n a_{n-m} + q| \leqslant N^{-|p|}$

(ii) $|p_1 e_1 + \ldots + p_m e_m + p_{m+1} d_1 + \ldots + p_n d_{n-m}| \leqslant N^{-|p|}$

and then if $p \in \mathbf{Z}^n$ is such that $p_{m+1} = \ldots = p_n = 0$, then $W_{N,p,q}$ is empty if $N > 1$. Thus:

$$W_N = \bigcup_{i=m+1}^{n} W_N^{(i)}. \tag{1.36}$$

Now we can state the main result of this section, and so complete the proof of Theorem 1 stated in the introduction.

THEOREM 1.37. *The set W of the wild quasi-tori is a Borel subset in* $\mathbf{R}^{m(2n-m)}$ *and its Hausdorff codimension equals* $2m$.

Proof. First of all, if we fix $p \in \mathbf{Z}^n$ such that $p_i \neq 0$, for a suitable $i \geqslant m+1$, and $q \in \mathbf{Z}^m$, then W contains Z_{pq}, where

$$Z_{pq} = \bigcap_{N \in \mathbf{N}^+} W_{N,p,q}.$$

Note that Z_{pq} is a non empty affine subspace of real codimension $2m$ and then $\mathbf{W}$ has codimension at most $2m$. Note also that the union of the Z_{pq}, with p and q varying in $\mathbf{Z}^n$, resp. in $\mathbf{Z}^m$, gives the set of quasi-tori with $J_{\mathbf{Z}} \neq 0$.

Also we note that $\mathbf{W}$ is a Borel set by virtue of (1.33) and (1.34). Consider now the subset W' of $\mathbf{W}$ such that the elements of W' are those matrices V such that the real parts of their entries are between 0 and 1. It is easy to verify, using the relation (1.29), that every element in $\mathbf{W}$ is equivalent to an element in W'; then the equality $\dim \mathbf{W} = \dim W'$ holds.

In order to prove the theorem it is sufficient to show that:

$$\forall \varepsilon > 0, \quad \mu_{m(2n-m-2)+\varepsilon}(W') = 0 \tag{1.38}$$

where μ_δ denotes the Hausdorff measure of dimension δ. In fact, in this case we should have $\dim(W') \leq m(2n-m-2)$. Recall that $\mu_\delta = \sup_{\lambda > 0}(\mu_{\delta,\lambda})$ where, $B(x, r)$ denoting the ball with centre x and radius r,

$$\mu_{\delta,\lambda}(A) = \inf_{A \subset \cup_i B(x_i, r_i), r_i \leq \lambda} \sum_i (r_i)^\delta.$$

Since, now:

$$W' = \bigcap_{N \in \mathbf{N}^+} W'_N. \tag{1.39}$$

we have:

$$\mu_{\delta,\lambda}(W') \leq \inf_{N \to \infty} \mu_{\delta,\lambda}(W'_N). \tag{1.40}$$

By virtue of (1.36), we can infer:

$$\mu_{\delta,\lambda}(W'_N) \leq \sum_{i=m+1}^{n} \mu_{\delta,\lambda}(W'^{(i)}_N)$$

and then in order to conclude the proof it is sufficient to show that:

$$\forall \varepsilon > 0, \forall \lambda > 0, \ \forall i = m+1, \ldots, n, \quad \inf_{N \to \infty} \mu_{m(2n-m-2)+\varepsilon,\lambda}(W'^{(i)}_N) = 0. \tag{1.41}$$

Moreover, since, by virtue of (1.29), the matrices in $W'^{(i)}_N$ are all equivalent to elements of $W'^{(m+1)}_N$, it suffices to prove (1.41) in the case $i = m+1$. Let us consider now the following projection:

$$\begin{gathered} \Pi\colon W'^{(m+1)}_N \to [0,1)^{m(n-1)} \times \mathbf{R}^{m(n-m-1)} \quad \text{such that:} \\ \Pi\colon (b_1, \ldots, b_m, a_1, \ldots, a_{n-m}, d_1, \ldots, d_{n-m}) = (b_1, \ldots, b_m, a_2, \ldots, a_{n-m}, d_2, \ldots, d_{n-m}). \end{gathered} \tag{1.42}$$

It is easy to check that Π is surjective. By the general properties of the Hausdorff measure with respect to the product, it is sufficient now to prove that, if F_N is a fiber of Π,

$$\forall\varepsilon>0,\ \forall\lambda>0,\ \inf_{N\to\infty}\mu_{\varepsilon,\lambda}(F_N)=0. \tag{1.43}$$

Set now $|p|=|p_1|+\ldots+|p_n|$ and observe that 1.35 (i) gives that $|q_j|\leq|p|$ for each j. Since we have:

$$F_N\subset\bigcup_{p_{m+1}\neq 0}\ \bigcap_{q,\,|q_j|\leq|p|}F_{N,p,q}$$

then

$$\mu_{\varepsilon,\lambda}(F_N)\leq\sum_p(2|p|)^m\max_q\mu_{\varepsilon,\lambda}(F_{N,p,q}) \tag{1.44}$$

where in the above expression one sums over those elements p of $\mathbf{Z}^n$ such that $p_{m+1}\neq 0$.

But $F_{N,p,q}$ is the product of two spheres whose radii are $N^{-|p|}|p_{m+1}|^{-1}$. Hence it follows that:

$$\mu_{\varepsilon,\lambda}(F_{N,p,q})\leq K(N^{-|p|}|p_{m+1}|^{-1})^{2m}\lambda^{-2m}\lambda^{\varepsilon},$$

for a suitable positive constant K. Then we can write:

$$\mu_{\varepsilon,\lambda}(F_N)\leq K\sum_{p_{m+1}=1}^{\infty}\sum_{p_1=0}^{\infty}\cdots\sum_{p_n=0}^{\infty}(p_1+\ldots+p_n)^m\lambda^{\varepsilon-2m}N^{-2m(p_1+\ldots+p_n)}p_{m+1}^{-2m}$$

where we stress the fact that, with a suitable new choice of the constant K, in the sum appear now only non negative values of the p_i's.

If the p_i are positive there is a constant K' such that

$$(p_1+\ldots+p_n)^m\leq K'p_1^m\ldots p_n^m$$

and therefore with a new choice of K

$$\mu_{\varepsilon,\lambda}(F_N)\leq K\lambda^{\varepsilon-2m}\sum_{p_{m+1}=1}^{\infty}p_{m+1}^{-m}N^{-2mp_{m+1}}\prod_{\substack{j=1\\ j\neq m+1}}^{n}\left(1+\sum_{p_j=1}^{\infty}N^{-2mp_j}p_j^m\right).$$

Let us introduce the function

$$g_k(z)=\sum_{h=1}^{\infty}h^kz^h. \tag{1.45}$$

This power series' radius of convergence equals 1 and also:

$$\mu_{\varepsilon,\lambda}(F_N) \leqslant K\lambda^{\varepsilon-2m} g_{-m}(N^{-2m})(1+g_m(N^{-2m}))^{n-1}.$$

Thus, since g_k converges to zero when the argument tends to zero, we have:

$$\inf_{N\to\infty} \mu_{\varepsilon,\lambda}(F_N) < K\lambda^{\varepsilon-2m} \lim_{N\to\infty} [g_{-m}(N^{-2m})(1-g_m(N^{-2m}))^{n-1}] = 0$$

and thus we are through with the proof of (1.43) and of the statement of the theorem. Q.E.D.

We want to conclude this section accomplishing the proof of Theorem 1 stated in the introduction via the following result, obtained independently by Vogt (cf. [Vo]).

PROPOSITION 1.46. *For each integer m, $1 \leqslant m \leqslant n-1$, there exist wild quasi-tori $X=\mathbf{C}^n/\Gamma$ without nonconstant holomorphic functions and such that Γ has rank $n+m$.*

Proof. First of all we can consider the space T corresponding to the matrices V as in (1.32) such that $d_1=\ldots=d_{n-m}=0$. The conditions (1.35) give immediately that, if the p_j's with $j \leqslant m$ are not all zero, the intersection of T with $W_{N,p,q}$ is empty.

Otherwise, if $p_1=\ldots=p_m=0$,

$$T \cap Z_{p,q} \subset \{(b_1, \ldots, b_m, a_1, \ldots, a_{n-m}) \mid e_1, \ldots, e_m, a_1, \ldots, a_{n-m} \text{ are } \mathbf{Q}\text{-linearly dependent}\}.$$

Our strategy shall therefore consist in finding some matrix V in $T \cap W$ such that $e_1, \ldots, e_m, a_1, \ldots, a_{n-m}$ are $\mathbf{Q}$-linearly independent. Let us consider any pair of sequences $(q^{(h)})_{h\in N}$ and $(p^{(h)})_{h\in N}$, respectively of vectors of $\mathbf{Z}^m$ and of $\mathbf{Z}^n$, such that $p_i^{(h)}=0$ for $i=1,\ldots,m$ and $p_i^{(h)}=t_h$ for $i=m+1,\ldots,n$ (these clearly exist).

Assume that there do exist a vector a not in $\mathbf{Q}^m$ and a sequence of integers N_h such that:

$$\lim_{h\to\infty} N_h = +\infty, \quad |t_h a - q^{(h)}| < N_h^{-t_h}. \tag{1.47}$$

Then it is easy to verify that we obtain an element of $T \cap (\mathbf{W}-Z_{p,q})$ in the following way: we simply complete the set $a, e_1, \ldots, e_m$ to a system $a, e_1, \ldots, e_m, a_1, \ldots, a_{n-m-1}$ of $\mathbf{Q}$-independent vectors, we set a_{n-m} equal to $a-a_1-\ldots-a_{n-m-1}$, and choose $b_1, \ldots, b_m$ arbitrarily.

It remains to show that we may choose a, and a sequence N_h such that (1.47) holds. Notice for this that if (1.47) holds, then necessarily it must be $a=\lim_{h\to\infty}(q^{(h)}t_h^{-1})$, and therefore the sequence $(q^{(h)}t_h^{-1})$ must be a Cauchy sequence.

It is easy to see that we may choose sequences N_h, $q^{(h)}$ and t_h so that:

(1.48) (i) The sequences N_h and t_h are increasing;
(ii) $q^{(h)}$ and t_h are relatively prime;
(iii) $|(q^{(h)}t_h^{-1})-(q^{(h+1)}t_{h+1}^{-1})|<2^{-h}t_h^{-1}N_h^{-t_h}$.

Then the sequence $(q^{(h)}t_h^{-1})$ is a Cauchy sequence and its limit $a=\lim_{h\to\infty}(q^{(h)}t_h^{-1})$ is a vector not in $\mathbf{Q}^m$, because of (1.48) (ii) and since, as we shall show, (1.47) holds. In fact, (1.47) amounts to $|a-q^{(h)}t_h^{-1}|<t_h^{-1}N_h^{-t_h}$ but indeed

$$|a-(q^{(h)}t_h^{-1})|<\sum_{l=h,\dots,\infty}|(q^{(l)}t_l^{-1})-(q^{(l+1)}t_{l+1}^{-1})|<\sum_{l=h,\dots,\infty}2^{-l}t_h^{-1}N_h^{-t_h} \quad \text{(by 1.48)}$$
$$<\left(\sum_{l=h,\dots,\infty}2^{-l}\right)t_h^{-1}N_h^{-t_h}<t_h^{-1}N_h^{-t_h},$$

and thus (1.47) is satisfied. This argument ends the proof. Q.E.D.

1.4. Line bundles on $(\mathbf{C}^*)^n$

In order to study meromorphic functions on complex abelian Lie groups, the Künneth formula for line bundles on a product manifold allows us to analyze separately the various factors in the Remmert–Morimoto decomposition (0.5) (iii).

In this section we want to examine the case of $Y=(\mathbf{C}^*)^n$. Since $H^i(Y,O_Y)=0$ if $i\geq 1$ we have:

$$H^1(O_Y^*)\cong H^2(Y,\mathbf{Z})=H^2(\mathbf{Z}^n,\mathbf{Z}). \tag{1.49}$$

The previous isomorphism is obtained as follows. Given an alternating matrix $(n\times n)$ $A=(a_{ij})$ with integral entries, we can consider the bilinear alternating form on $\mathbf{Z}^n\times\mathbf{Z}^n$:

$$\alpha(p,p')=\sum a_{ij}p_ip_j',\quad p,p'\in\mathbf{Z}^n. \tag{1.50}$$

We can extend the form (1.50) to a bilinear form on $\mathbf{C}^n$, $\alpha(z,w)=\Sigma a_{ij}z_iw_j$. To this form we associate the element of $H^1(O_Y^*)$ represented by the cocycle

$$(f_p(z))_{p\in\mathbf{Z}^n}\quad\text{such that}\quad f_p(z)=e(\alpha(z,p)). \tag{1.51}$$

Its coboundary is in fact:

$$\frac{1}{2\pi i}[\log(f_{p+p'}(z))-\log(f_p(z+p'))-\log(f_{p'}(z))]$$

$$= \alpha(z,p+p')-\alpha(z+p',p)-\alpha(z,p') = \alpha(p,p').$$

Definition 1.52. If $t\in\mathbf{Z}$ let $L(t)$ be the line bundle on $(\mathbf{C}^*)^2$ determined by the cocycle $f_p(z)=\mathrm{e}(t(p_2z_1-p_1z_2))$ with $p\in\mathbf{Z}^2$ and $z\in\mathbf{C}^2$. Analogously, let L_A be the line bundle on $(\mathbf{C}^*)^n$ defined as in (1.51).

As it is well known, by the theorem of Frobenius, there exists a change of basis in $\mathbf{Z}^n$ such that, in this new basis, the alternating matrix A has the following form:

$$A=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & T\\ 0 & -T & 0\end{pmatrix}$$

where $T=\mathrm{diag}(t_1,\ldots,t_k)$ and $t_i\in\mathbf{N}-\{0\}$, $t_i|t_{i+1}$.

Remark 1.53. It is easy to see that, in this situation, there exists the following isomorphism:

$$((\mathbf{C}^*)^n,L_A)\cong((\mathbf{C}^*)^2,\,L(t_1))\times\ldots\times((\mathbf{C}^*)^2,L(t_k))\times((\mathbf{C}^*)^{n-2k},1)$$

where

$$(X,L_1)\times(Y,L_2)=(X\times Y,\pi_X^*(L_1)\otimes\pi_Y^*(L_2))$$

and $((\mathbf{C}^*)^{q,1})$ is the trivial bundle on $(\mathbf{C}^*)^q$.

By virtue of this remark and of the Künneth formula we have to study only the case $k=1$, $n=2$.

In order to determine the holomorphic sections of a line bundle on $(\mathbf{C}^*)^2$, we can easily see that it suffices to seek for holomorphic functions $g(z,w)$ on $\mathbf{C}^2$ such that:

$$g(z+1,w)=\mathrm{e}(tw)\,g(z,w) \tag{1.54}$$

$$g(z,w+1)=\mathrm{e}(-tz)\,g(z,w). \tag{1.55}$$

Remark 1.56. Given the functional equation:

$$f(z+1)=\mathrm{e}(k)\,f(z),\quad k\in\mathbf{C}, \tag{1.57}$$

we observe that $\mathrm{e}(kz)$ is a solution and every other entire function which is a solution has the following form:

$$f(z) = e(kz)\,F(e(z))$$

(since $f(z)\,e(-kz)$ must be $\mathbf{Z}$-periodic). Therefore, by (1.55), we have

$$g(z, w)\,e(twz) = F(z, e(w)). \tag{1.58}$$

Thus

$$\begin{aligned} F(z+1), e(w)) &= g(z+1, w)\,e(t(z+1)\,w) \\ &= g(z, w)\,e(tw)\,e(tw)\,e(tzw) = F(z, e(w))\,e(2tw). \end{aligned} \tag{1.59}$$

We can write down $F(z, \zeta) = \Sigma_{p\in\mathbf{Z}} f_p(z)\,\zeta^p$ and then the relation (1.59) gives the equality $F(z+1, \zeta) = F(z, \zeta)\,\zeta^{2t}$ and equivalently

$$\sum_{p\in\mathbf{Z}} f_p(z+1)\,\zeta^p = \sum_{q\in\mathbf{Z}} f_q(z)\,\zeta^{q+2t}.$$

In turn the last equality forces the following relation between the coefficients:

$$f_p(z) = f_{p+2t}(z+1). \tag{1.60}$$

Remark 1.61. If we give an arbitrary set $f_0, \ldots, f_{2t-1}$ of entire functions it is possible to define a formal power series which (formally) verifies the conditions to be a section of the line bundle $L(t)$ on $(\mathbf{C}^*)^2$.

In fact, it suffices to define

$$\begin{aligned} &f_{i+2tm}(z) = f_i(z-m) \quad \text{for} \quad m\in\mathbf{Z} \text{ and } i = 0, \ldots, 2t-1 \\ &F(z, e(w)) = \sum_{q\in\mathbf{Z}} f_q(z)\,e(qw) \end{aligned} \tag{1.62}$$

and finally let

$$g(z, w) = F(z, e(w))\,e(-twz).$$

It easy to check that the so defined $g(z)$, at least formally, verifies the conditions (1.54) and (1.55). Q.E.D.

Clearly g is convergent on $\mathbf{C}^2$ if and only if $F(z, \zeta)$ is convergent on $\mathbf{C}\times\mathbf{C}^*$.

Remark 1.63. If f is an entire function on $\mathbf{C}$, the expression $\Sigma_{p\in\mathbf{Z}} f_p(z+p)\,\zeta^p$ defines a holomorphic function on $\mathbf{C}\times\mathbf{C}^*$ if and only if

(1.64) For each compact subset K of $\mathbf{C}$ and for each $\varepsilon>0$ there exists a positive integer $q_0(K, \varepsilon)$ such that: if $|q|>q_0(K, \varepsilon)$ then $\|f\|_{K+q}<\varepsilon^{|q|}$.

In fact, it easy to see that for each compact K and for each $\varrho>0$ the series $\Sigma_{q\in\mathbf{Z}}|f(z+q)|\varrho^q$ converges uniformly on K.

Remark 1.65. The functions which verify property (1.63) form a non empty space W; for example, $f(z)=\exp(-z^2)$ is an element of W. More generally, every function of the form

$$f(z)=\exp(a_{2n}z^{2n}+a_{2n-1}z^{2n-1}+\ldots+a_0) \tag{1.66}$$

where a_{2n} is a negative real number, belongs to W.

In fact, if K is a compact subset of $\mathbf{C}$, then on the strip $K+\mathbf{R}$, if x denotes the real part of z, $\|f(z)\|=\exp(a_{2n}x^{2n}+O(x^{2n-1}))$ and therefore:

$$\begin{aligned}\|f\|_{K+q}&=\|f(z+q)\|_K=\exp(a_{2n}(x+q)^{2n}+O((x+q)^{2n-1}))\\&=\exp(-|a_{2n}|q^{2n}+O(q^{2n-1}))\end{aligned}$$

(where the constants depend upon K).

Moreover, in the same fashion we see that all the functions obtained as a product $g(z)f(z)$, with $f(z)$ as in (1.66), and $g(z)$ such that $\|g(z+q)\|_K\leq\exp(bq^{2n})$, with $b<|a_{2n}|$, are elements of W. Among the functions $g(z)$ which satisfy this property there are, e.g., polynomials and exponentials of polynomials of sufficiently low degree. Restricting W to the real line, we get a subspace of the Schwartz class.

We can summarize the above discussion as follows

THEOREM 1.67. *The space of sections of the line bundle $L(t)$ on $(\mathbf{C}^*)^2$ defined in* (1.52) *is isomorphic, via the correspondence set up in* (1.61), (1.62), *to the direct sum of $2t$ copies of the space of entire functions W defined in* 1.65.

§2. Generalized Riemann bilinear Relations

2.1. Sufficiency of the GRR (via theta functions)

We have already seen (cf. 1.2) that, if the subgroup Γ spans $\mathbf{C}^n$, then it is possible to choose a suitable system of coordinates on $\mathbf{C}^n$ such that, in these coordinates, Γ is generated by the columns of a matrix of the form (E_n, V) where E_n is the identity matrix and $W=\operatorname{Im} V$ has maximal rank m. Then, acting with a permutation of the standard basis of $\mathbf{C}^n$, we can assume that the square matrix B_2 obtained by taking the last m rows of W is invertible. Let B_1 be the matrix obtained by taking the first $(n-m)$ rows of W.

Let us consider the following further change of coordinates on $\mathbf{C}^n$.

$$u = z' - B_1 B_2^{-1} z'', \quad v = B_2^{-1} z'' \tag{2.1}$$

where ${}^t z' = (z_1, \ldots, z_{n-m})$, ${}^t z'' = (z_{n-m+1}, \ldots, z_n)$ are the old coordinates and

$${}^t u = (u_1, \ldots, u_{n-m}), \quad {}^t v = (v_1, \ldots, v_m) \tag{2.2}$$

are the new ones.

The so defined coordinates (u, v) are said to be *apt coordinates*. We can point out that, in these apt coordinates

(i) the subspace F is defined by the system of linear equations $u_1 = \ldots = u_{n-m} = 0$, i.e., by the vector equation $u=0$;

(ii) the real subspace $\mathbf{R}\Gamma$ of $\mathbf{C}^n$ is defined by the vector equation $\operatorname{Im} u=0$;

(iii) the standard vectors $e_1, \ldots, e_{n-m}$ can be completed to a basis of Γ.

Definition 2.3. A system of coordinates (u, v) satisfying the above properties (i)–(iii), is said to be an *apt system of coordinates*.

We want now to give a quick proof of the sufficiency part of the Main Theorem, essentially along the same lines as in [A–G2], but without appealing to the corresponding statement in the compact case.

THEOREM 2.4. *Given a discrete subgroup Γ of $\mathbf{C}^n$, there exists a non degenerate Γ-periodic meromorphic function if the following* Generalized Riemann bilinear Relations (GRR) *are satisfied:*

there exists a Hermitian form H on $\mathbf{C}^n$ such that

(i) $\operatorname{Im}(H)$, *the imaginary part of H, takes values in $\mathbf{Z}$ on $\Gamma \times \Gamma$;*

(ii) $H|_F$, *the restriction of H to the maximal complex subspace F of the real span of Γ, is strictly positive definite.*

Proof. We shall give the proof through a sequence of steps, and we can obviously assume that Γ spans $\mathbf{C}^n$. Thus we can use a system of apt coordinates as above.

Step I. We can assume that H is positive definite on all of $\mathbf{C}^n$, since (cf. 1.11), we can alter H, keeping (i) and (ii) satisfied, by adding a Hermitian form H' whose imaginary part vanishes on the real span of Γ (in a system of apt coordinates, H' is given by a form in the u variables and with real coefficients). In fact $H+H'$ is positive definite, by the criterion of the principal minors, iff all the determinants of the minors formed by the last i rows and i columns are positive (we are now working in a system of apt coordinates). It is easy to see that this can be achieved with a suitable choice of H' a real diagonal matrix with sufficiently large coefficients.

Step II. We choose a line bundle L together with a cocycle in the Appell–Humbert normal form (1.12), and we show that if L has a non zero section, then this section is represented by a nondegenerate function in the trivialization associated to the chosen cocycle.

In fact, such a section is given by a function in $\mathbf{C}^n$ satisfying a functional equation

$$(2.5)\qquad h(z+\gamma)=k_\gamma(z)\,h(z),\quad \text{where}\quad k_\gamma(z)=\varrho(\gamma)\,e\Big(-\frac{i}{2}\Big[H(z,\gamma)+\frac{1}{2}H(\gamma,\gamma)\Big]\Big).$$

If h were degenerate, after a change of coordinates in $\mathbf{C}^n$ we could assume that $\partial_1(h(z))$ is identically zero, where $\partial_1=\partial/\partial z_1$.

From (2.5) we infer that $\partial_1(k_\gamma(z))$ is also identically zero, and from the explicit form of $k_\gamma(z)$ we deduce that $H(e_1,\gamma)=0$, for each $\gamma\in\Gamma$, hence e_1 belongs to the kernel of H, a contradiction.

Step III. Let $h(z)$ be as in step II. Then (Lefschetz' trick, cf. 3.7) it is easy to see that, by (2.5), for every vector a in $\mathbf{C}^n$, the quotient

(2.6) $\quad f_a(z)=h(z+a)\,h(z-a)\,h(z)^{-2}$ is a Γ-periodic meromorphic function,

and therefore it suffices to show that $f_a(z)$ is non degenerate for some choice of a.

As in step II, otherwise, after a linear change of coordinates, we may assume $\partial_1(f_a(z))$ is identically zero as a function of a and z. This implies that $\partial_1(\log f_a(z))$ is identically zero: using (2.6), we get

$$\partial_1(\log(h(z+a))+\log h(z-a))=2\partial_1(\log h(z)).$$

If the function h is a unit, we obtain that the imaginary part A' of H vanishes on $\Gamma\times\Gamma$, a contradiction. If instead $\{h=0\}$ defines a non empty divisor D, pick a vector a such that $D, D+a, D-a$, have no common components. We again derive a contradiction if $\partial_1(\log h(z))$ is not identically zero, since, in the previous formula, the right hand side has a pole at D, whereas the left hand side does not.

Finally, $\partial_1(\log h(z))$ being identically zero contradicts the assumption that h be non degenerate.

Step IV. By the previous steps, the proof is reduced to the following assertion (cf. [A–G1], [A–G2], [Cap]), which should be valid more generally also in the wild case.

CLAIM 2.7. *A line bundle L on a quasi-torus X, given by a linearized cocycle in the Appell–Humbert normal form* (2.5) *with H positive definite, has some non zero section.*

Proof. The assertion follows immediately from the following Proposition 2.8, implying in particular that we can find a lattice Γ' containing Γ and such that the imaginary part of H, A', takes integral values on $\Gamma' \times \Gamma'$. We have now a torus X' and (extending the semicharacter of L) a line bundle L' on X' which pulls back to L. Since the Riemann Relations hold for L', it suffices to take a classical theta function for L'.
Q.E.D.

PROPOSITION 2.8. *Let H be a positive definite Hermitian form on $\mathbf{C}^n$ whose imaginary part A' takes integral values on $\Gamma \times \Gamma$, where Γ is a discrete subgroup of $\mathbf{C}^n$. Then there do exist lattices (i.e., discrete subgroups of maximal rank) $\Gamma_1, \Gamma_2 \supset \Gamma$, such that $\Gamma_1 \cap \Gamma_2 = \Gamma$, and such that A' takes integral values on $\Gamma_i \times \Gamma_i$, for $i=1,2$.*

Proof. We prove first a very useful auxiliary result.

LEMMA. *Let Γ be as above, and let B be a finite subset of $\mathbf{C}^n$. Then one can find a vector a such that*

(i) *for each element $\gamma \in \Gamma, A'(a, \gamma)$ is an integer;*

(ii) *the subgroup $\Gamma'' = \Gamma \oplus \mathbf{Z} a$ is a discrete (of rank$=n+m+1$);*

(iii) *the vector a does not belong to the $\mathbf{Q}$-span of B.*

Proof. Consider $\mathbf{C}^n$ as a real vector space V. Then the imaginary part A' of H, being non degenerate, defines an isomorphism, which we still denote by A', of V with its dual vector space $V^\vee$. Let i be the inclusion of $\mathbf{R}\Gamma$ in V, and let p be the surjection of $V^\vee$ onto $(\mathbf{R}\Gamma)^\vee$.

Inside $(\mathbf{R}\Gamma)^\vee$ we consider the $\mathbf{Z}$-dual $\Gamma^\vee$ of Γ. We seek for a vector a inside $A'^{-1}(p^{-1}(\Gamma^\vee))$, which does not belong to $\mathbf{R}\Gamma$, and furthermore does not belong to a countable set (the $\mathbf{Q}$-span of B).

Notice that $p(A'(i(\mathbf{R}\Gamma)))$ has codimension equal to the dimension of the intersection $\mathbf{R}\Gamma \cap \mathrm{Ann}(\mathbf{R}\Gamma)$. We have two possibilities:

(1) $\mathbf{R}\Gamma \cap \mathrm{Ann}(\mathbf{R}\Gamma) = 0$: then for any a' in $\Gamma^\vee$, $A'^{-1}(p^{-1}(a'))$ is an affine subspace intersecting $\mathbf{R}\Gamma$ in only one point, and we can choose a different point a not belonging to the countable subset given by the $\mathbf{Q}$-span of B.

(2) $p(A'(i(\mathbf{R}\Gamma)))$ does not coincide with $(\mathbf{R}\Gamma)^\vee$, and we can choose a' in $\Gamma^\vee$ not contained in $p(A'(i(\mathbf{R}\Gamma)))$. Then $A'^{-1}(p^{-1}(a'))$ is an affine subspace not intersecting $\mathbf{R}\Gamma$, and we choose a outside of the $\mathbf{Q}$-span of B. Q.E.D. for the lemma

We can now apply the previous result inductively, constructing discrete subgroups $\Gamma_1', \Gamma_2' \supset \Gamma$, such that $\Gamma_1' \cap \Gamma_2' = \Gamma$, and such that A' takes integral values on $\Gamma_i' \times \Gamma_i'$, for $i=1,2$. Each time we want to replace each Γ_i' by a discrete subgroup of higher rank it suffices to apply the lemma letting B be the union of respective bases of the Γ_i''s. Then the condition $\Gamma_1' \cap \Gamma_2' = \Gamma$ is still preserved. Q.E.D. for 2.8.

2.2. Chern class of a line bundle on X and associated Hermitian forms

In this section X shall be any quasi-torus, and L shall be a line bundle on X. We have already represented (1.7) the Chern class of L as an element in $H^2(X, \mathbf{Z}) \cong H^2(\Gamma, \mathbf{Z})$ given by a bilinear alternating function $A: \Gamma \times \Gamma \to \mathbf{Z}$. Now, using the de Rham isomorphism, the Chern class of L is also represented by a closed differentiable 2-form on X which is of type (1,1), and, given a metric h on the line bundle L, is obtained as follows:

$$(*) \qquad \omega = -\frac{1}{2\pi i}\partial\bar{\partial}(\log h).$$

On the other hand, we have

PROPOSITION 2.9. *The operator* av *defined on the space of differential forms on X and defined by averaging with respect to a Haar measure on the maximal compact subgroup K of X has the properties*:

(i) *if $\omega' = \mathrm{av}(\omega)$, then ω' has K-invariant coefficients, where K is acting on X by translation*;

(ii) *if ω is a closed differential form, ω is cohomologous to ω'*;

(iii) *if ω is of type (p, q), then also ω' is of type (p, q)*;

(iv) *the restriction of ω' to $V \times V \times \ldots \times V$, where V is the tangent space ($=\mathbf{R}\Gamma$) to K, has constant coefficients.*

Proof. Parts (i) and (iii) are trivial. Furthermore it is easy to see that ω' is a closed form if ω is closed. In fact

$$\omega' = \int_K \omega(z+k)\, dk$$

hence

$$d\omega' = \int_K d\omega(z+k)\,dk$$

and ω' is a closed form.

In order to prove that ω and ω' are cohomologous one can use de Rham's theorem (it suffices to know it for $\mathbf{R}$, S^1, and products of these manifolds). In fact ω and ω' are cohomologous iff, r being the total degree of ω ($r=p+q$) for every parallelotope P spanned by r vectors in Γ, we have that

$$\int_P \omega = \int_P \omega'.$$

We have

$$\int_P \omega' = \int_P \int_K \omega(z+k)\,dk = \int_K dk \int_{P+k} \omega = \int_P \omega$$

since the parallelotopes $P+k$ are all homotopic to each other.

Alternatively, as the referee points out, since the inclusion of K in X is a homotopy equivalence, in order to verify that two closed differential forms are cohomologous, it suffices to take their restriction to K; we are then reduced to the case of a torus where the average is just the harmonic part.

We can observe now that from the differentiable viewpoint X is diffeomorphic to $K \times \mathbf{R}^{n-m}$ and we can use in $\mathbf{C}^n$ the apt coordinates (u, v); setting $y = \operatorname{Im} u$, $u = w + iy$, $x = (w, v)$, ω' can be written, being K-invariant, as follows:

$$\omega' = \sum f_{IJ}(y)\, dx^{\wedge I} \wedge dy^{\wedge J}$$

and therefore its differential is:

$$d\omega' = \sum \frac{\partial f_{IJ}}{\partial y_h} dy_h \wedge dx^{\wedge I} \wedge dy^{\wedge J}$$

In turn, ω' being closed, we see, by inspecting in the above formula for $d\omega'$ the terms where the differentials dy appear with degree 1, that the coefficients f_{IJ} where the set J is empty are constant. This is exactly statement (iv). Q.E.D.

Proposition 2.10. *Every closed differential form ω of type (p, q) is cohomologous to a form $\omega^\wedge$ of type (p, q) with constant coefficients.*

Any such $\omega^\wedge$ can be simply obtained just by evaluating the coefficients of ω' (cf. 2.9) *at any given point of X: therefore we can achieve that $\omega^\wedge$ and ω' have the same restriction to $V \times V \times \ldots \times V$ ($V = \mathbf{R}\Gamma$ as in* 2.9).

Proof. It suffices to show that, fixing a point in X, e.g. the 0 of the group law, and constructing $\omega^\wedge$ as above ($\omega^\wedge$ is closed having constant coefficients), then the difference $\omega' - \omega^\wedge = \omega''$ is cohomologous to zero. Using the same type of coordinates as in the proof of Proposition 2.9, we can write

$$\omega'' = \sum f_{IJ}(y)\, dx^{\wedge I} \wedge dy^{\wedge J},$$

where now $f_{IJ}(y) = 0$ if the set J is empty. Since X is diffeomorphic to $K \times \mathbf{R}^{n-m}$ it follows that the differential forms $f_{IJ}(y)\, dy^{\wedge J}$ are exact, and then we easily obtain that also ω'' is exact. Q.E.D.

COROLLARY 2.11 (First Riemann bilinear Relation). *Let L be a line bundle on X and let $A: \Gamma \times \Gamma \to \mathbf{Z}$ be the bilinear alternating function representing the Chern class of L.*

Then there exists a Hermitian form H on $\mathbf{C}^n$ such that the restriction of its imaginary part $\mathrm{Im}(H)$ *to $\Gamma \times \Gamma$ coincides with A.*

Proof. It suffices to apply the previous Proposition 2.10 to a form of type $(1,1)$ representing the Chern class $c_1(L)$ of L as in (2.8): we obtain thus a form of type $(1,1)$ and with constant coefficients which also represents $c_1(L)$. To this last form is naturally associated a Hermitian form H on $\mathbf{C}^n$, and the desired assertion follows from the above mentioned isomorphism $H^2(X, \mathbf{Z}) \cong H^2(\Gamma, \mathbf{Z})$. Q.E.D.

Remark 2.12. The same argument as in (1.11) gives that the Hermitian form H is not unique, but indeed the restriction H'' of H to $F \times F$ is uniquely determined.

In fact if

$$\omega = -\frac{1}{2\pi i}\partial\bar{\partial}(\log h)$$

is a Chern form for L, and ω', as in (2.2), is equal to $\mathrm{av}(\omega)$, then the proof of (2.11) shows that the restriction to $F \times F$ of ω' is a $(1,1)$ form with constant coefficients whose associated Hermitian form is precisely H''.

2.3. Proof of the Main Theorem: the key step implies the GRR

In this section X shall be a quasi-torus which admits a non degenerate meromorphic function f. The following result shall be proven in section 3.3 (Theorem 3.9):

Key step. If a non degenerate meromorphic function f is obtained as the quotient of two relatively prime sections of a line bundle L, then there exists a holomorphic map $\Phi: X \to \mathbf{P}^n$, given by sections of a suitable power $L'^{\otimes m}$ of a line bundle L' with $c_1(L')=c_1(L)$, which is an immersion.

We state again the Main Theorem about the existence of meromorphic functions:

MAIN THEOREM. *Given a discrete subgroup Γ of $\mathbf{C}^n$, there exists a non degenerate Γ-periodic meromorphic function if and only if the following* Generalized Riemann bilinear Relations (GRR) *are satisfied: there exists a Hermitian form H on $\mathbf{C}^n$ such that*

(i) $\mathrm{Im}(H)$, *the imaginary part of H, takes values in $\mathbf{Z}$ on $\Gamma \times \Gamma$;*

(ii) $H|_F$, *the restriction of H to the maximal complex subspace F of the real span of Γ, is strictly positive definite.*

Proof. First of all the sufficiency of GRR was shown in section 2.1. In order to prove the necessity of GRR, we shall use the immersion $\Phi: X \to \mathbf{P}^n$ provided by the key step. Using this, we can consider, as a Chern form for L, the pull-back of the Fubini-Study form on $\mathbf{P}^n$ (divided by m). That is,

$$\omega = \frac{1}{2\pi i} \partial\bar{\partial}(\log |\Phi|).$$

With this choice, since Φ is an immersion, the Chern form ω gives a Hermitian form on the tangent bundle of $\mathbf{C}^n$ which is positive definite at each point of $\mathbf{C}^n$.

Basic remark. The set of positive definite Hermitian forms is a convex cone, therefore an average of positive definite Hermitian forms is still positive definite.

The above remark shows that the $(1, 1)$ form $\omega' = \mathrm{av}(\omega)$ (cf. Proposition 2.9) also provides a Hermitian form positive definite at each point of $\mathbf{C}^n$. The rest of the proof follows now immediately from Corollary 2.4 and from Remark 2.12. Q.E.D.

2.4. Conditions for the existence of meromorphic functions

In the previous section, we have established the Main Theorem, which gives necessary and sufficient conditons for the existence of a non degenerate meromorphic function on X.

Here we want to generalize this result, concerning the existence of meromorphic functions of a given rank r.

First of all we make an important observation:

Remark 2.13. If there is on X a non constant meromorphic function f of rank equal to r, then, as we saw in (0.1), there is a complex vector space decomposition $\mathbf{C}^n=A\oplus B$ such that $\Gamma_f=A\oplus\Gamma''$, with Γ'' discrete in B. Therefore Γ fits into an exact sequence $0\to\Gamma^*\to\Gamma\to\Gamma'\to 0$ where $\Gamma^*=\Gamma\cap A$, and Γ is the projection of Γ in B. We have thus an exact sequence of quasi-tori

$$(2.14)\qquad 0\to X^*\to X\xrightarrow{p} X'\to 0,\quad \text{where}\quad X^*=A/\Gamma^*,\quad X'=B/\Gamma'.$$

We can rephrase the equality $\Gamma_f=A\oplus\Gamma''$ by saying that f is the pull-back under p of a nondegenerate function f' on X'. In particular:

(2.15) If f is given as the quotient of two relatively prime sections of a line bundle L on X, then L is isomorphic to the pull-back under p of a line bundle L' on X'.

We can also observe that if X is Cousin, then necessarily X' is Cousin, too.

PROPOSITION 2.16. *There exists a meromorphic function on X of rank r, if and only if there is an exact sequence of quasi-tori as in* (2.14), *with* $\dim(X')=r$ *and a Hermitian form H on $\mathbf{C}^n$ such that*

(i) *the restriction of its imaginary part* $\mathrm{Im}(H)$ *to* $\Gamma\times\Gamma$ *takes integral values, and its kernel contains* $\Gamma^*=\Gamma\cap A$

(ii) *H is positive semidefinite, has r positive eigenvalues, and has as its kernel exactly the complex subspace A.*

Proof. Giving such a Hermitian form H as above is equivalent to giving a Hermitian form H' on $B=\mathbf{C}^n/A$ satisfying the Generalized bilinear Relations for the quasi-torus X'. We can thus apply the Main Theorem and step I of Theorem 2.4. Q.E.D.

Remark 2.17. Let us consider the set of positive semidefinite Hermitian forms H on $\mathbf{C}^n$ such that:

(i) the restriction of the imaginary part $\mathrm{Im}(H)$ to $\Gamma\times\Gamma$ takes integral values;

(ii) the image Γ' of Γ into $\mathbf{C}^n/\ker H$ is discrete.

This set is indeed an Abelian semigroup: in fact it suffices to show that property (ii) holds for a sum $H=H_1+H_2$ if H_1, H_2 are as above. But then $\ker(H_1+H_2)=$

$\ker H_1 \cap \ker H_2$, and $\mathbf{C}^n/\ker H$ naturally is a subspace of $\mathbf{C}^n/\ker H_1 \oplus \mathbf{C}^n/\ker H_2$, so that Γ' is discrete, being a subgroup of $\Gamma_1' \oplus \Gamma_2'$.

Definition 2.18. Let A be the intersection of all the subspaces $\ker H$, with H as in 2.17, or equivalently the smallest such subspace. Then the image $\Gamma^{\#}$ of Γ into $\mathbf{C}^n/\ker H$ is discrete, and we define

$$X^{\#} = (\mathbf{C}^n/\ker H)/\Gamma^{\#}, \tag{2.19}$$

the *quasi-Abelian (or meromorphic) reduction of* X.

THEOREM 2.20. *Every meromorphic function f on X is a pull-back of a meromorphic function $f^{\#}$ on the quasi-Abelian reduction $X^{\#}$ of X. In particular the maximal rank of a meromorphic function on X equals the dimension* $\dim(X^{\#})$ *of the quasi-Abelian reduction of X.*

Proof. By Proposition 2.16 it is clear that there exists a non degenerate meromorphic function $f^{\#}$ on $X^{\#}$, hence the second assertion follows from the first. But if f is non constant on X there exists, cf. Remark 2.13, an exact sequence

$$0 \to X^* \to X \xrightarrow{p} X' \to 0$$

as in (2.14) such that f is the pull-back of a non degenerate function on X', and we can take a Hermitian form H as in 2.16 whose kernel equals A. Therefore the projection $p: X \to X'$ factors through the projection of X onto $X^{\#}$, and we are done. Q.E.D.

§ 3. Lefschetz type theorems

3.1. Generic immersion lemma

In this section we shall prove, using the group structure on X, that the existence of a non degenerate meromorphic function f on X implies the existence of a meromorphic map $F: X \to \mathbf{P}^n$ whose differential is of rank equal to n on an open dense set.

We can actually prove a slightly more general statement if we recall the concept of rank of a meromorphic function.

Definition 3.1. Let f be a meromorphic function on a complex Lie group G: then the rank of f is defined to be the difference between the (complex) dimension of G and the dimension of the subgroup Γ_f of periods of f.

The proof of the following result is almost obvious:

PROPOSITION 3.2. *Let f be a meromorphic function on* $\mathbf{C}^n$ *(or, more generally, on a complex Lie group G). Then if r is the rank of f, then* $(n-r)$ *is the dimension of the vector space of homogeneous first order differential operators with constant coefficients which annihilate f.*

We come now to the main result of this section:

THEOREM 3.3. *Let f be a meromorphic function on* $\mathbf{C}^n$ *of rank r. Then r is the maximal integer k such that there do exist vectors* $a_1, \ldots, a_k$ in $\mathbf{C}^n$ *such that, setting*

$$(\#) \qquad F(z) = (f(z+a_1), \ldots, f(z+a_k)) \in \mathbf{C}^k,$$

the meromorphic map F has differential of maximal rank $=k$ *on an open dense set of* $\mathbf{C}^n$.

Proof. Let k be the maximal integer as above. It is immediate to verify that k is $\leq r$, since the rank of the differential of F is given by the rank of the matrix

$$B = (f_i(z+a_j))_{\substack{i=1,\ldots,n,\\ j=1,\ldots,k}}$$

where the symbol f_i stands for $\partial f/\partial z_i$.

We denote, for further use, by $J(i_1, \ldots, i_k)$ the determinant of the minor of the matrix B obtained by choosing the rows $i_1, \ldots, i_k$.

We can clearly assume that $J = J(1, \ldots, k)$ be not identically zero. By the maximality of k it follows that, for each vector a in $\mathbf{C}^n$, the matrix B' obtained from B by adding the column $(f_i(z+a))$, has also rank equal to k. We choose now $z^*, a_1^*, \ldots, a_k^*$, vectors such that $J^* = J(z^*, a_1^*, \ldots, a_k^*)$ is not zero, we set $w = z^* + a$, and we consider the matrix B^* obtained from B' replacing $z, a_1, \ldots, a_k$, respectively by the constants $z^*, a_1^*, \ldots, a_k^*$. We observe that thus the first k columns of B^* have constant entries, whereas the last column is given by $f_j(w)$. Also B^* has rank equal to k, therefore all the $(k+1)\times(k+1)$ minors of B^* obtained by choosing the first k rows and the ith one $(i=k+1, \ldots, n)$ have zero determinant. Expanding these determinants by Laplace's rule according to the last column, we obtain, for each $i=k+1, \ldots, n$, a relation of linear dependence with constant coefficients among the partial derivatives $f_1(w), \ldots, f_k(w)$, and $f_i(w)$, where the coefficient of $f_i(w)$ equals J^* which is not zero. Therefore we have shown that r is $\leq k$, hence $k=r$, and the proof is accomplished. Q.E.D.

COROLLARY 3.4 (Generic immersion lemma). *Let f be a meromorphic function of rank r on a quasi-torus X. Assume that f is the quotient of two relatively prime sections of a line bundle L. Then there exists a line bundle L', whose Chern class $c_1(L')$ is r times the Chern class $c_1(L)$ of L, and $(r+1)$ independent sections of L' giving a meromorphic map $\Phi: X \to \mathbf{P}^r$, whose differential is of maximal rank on an open dense set of X.*

In particular, if f is nondegenerate, then we obtain a meromorphic map $\Phi: X \to \mathbf{P}^n$ which is a generic immersion.

Proof. It suffices to consider the meromorphic map F defined in the previous Theorem 3.3. Follwoing standard notation, given an element a of the group X, we denote by $T_a: X \to X$ the translation by a (i.e., $T_a(z)=z+a$), and by L_a the pull-back of the line bundle L by T_a. It follows immediately by (1.8) that the Chern class of L_a equals the Chern class of L. We can see moreover rather easily that the meromorphic map F is given by $(r+1)$ sections of the line bundle

$$L' = L_{a_1} \otimes \ldots \otimes L_{a_r}.$$

From the previous remark we see that $c_1(L')=rc_1(L)$. Q.E.D.

3.2. Cousin normal form for the system of exponents

In this section we shall consider a given quasi-torus $X=\mathbf{C}^n/\Gamma$, and a system of *apt coordinates* (u, v) in $\mathbf{C}^n$ for Γ (cf. section 2.1). We recall that in these coordinates F is defined by $u=0$, whereas $\mathbf{R}\Gamma$ is the real subspace where $\mathrm{Im}(u)=0$.

As mentioned in the introduction, we defer the reader to [A–G1] or to [Vo] for the proof of the following result:

THEOREM 3.5. *Let L be a line bundle on the quasi torus X, let $A: \Gamma \times \Gamma \to \mathbf{Z}$ be the bilinear alternating function representing the Chern class of L, and let H be* (*cf.* Theorem 2.4) *a Hermitian form on $\mathbf{C}^n$ such that the restriction of its imaginary part* $\mathrm{Im}(H)$ *to $\Gamma \times \Gamma$ coincides with A. Let (u, v) be a system of apt coordinates: then L can be represented by a cocycle in the following* Cousin normal form:

$$k_\gamma(z) = \varrho(\gamma)\, \mathrm{e}\left(-\frac{i}{2}\left[H(z, \gamma)+\frac{1}{2}H(\gamma, \gamma)\right]+\psi_\gamma(u)\right), \tag{3.6}$$

where, as in (1.12), *$\varrho(\gamma)$ is a semicharacter for the alternating form A, and $\psi_\gamma(u)$, as the* *expression suggests, is a $F+\mathbf{Z}^n$-periodic holomorphic function (and so it is given as a Fourier series in the u variables).*

COROLLARY 3.7 (Lefschetz' trick). *Let L be a line bundle on X, and let $a_1, \ldots, a_r$ be vectors in the subspace F such that $\Sigma_{i=1,\ldots,r} a_i=0$. Then $L^{\otimes r}$ is isomorphic to $L'=L_{a_1}\otimes\ldots\otimes L_{a_r}$. (Here, by abuse of language, given a vector a in $\mathbf{C}^n$ we still denote by L_a the line bundle obtained as the pull-back of L by the translation on X induced by a.)*

Proof. Since a_i is a vector in F, the u coordinate of z equals the u coordinate of $(z+a_i)$. Therefore $k_\gamma(z+a_i)=k_\gamma(z)\,\mathrm{e}(-i/2H(a_i,\gamma))$. Hence

$$\prod_{i=1,\ldots,r} k_\gamma(z+a_i)=k_\gamma(z)^r,$$

as wanted. Q.E.D.

3.3. Lefschetz type theorems and the key step

In this section, on the one hand we shall generalize the classical theorems of Lefschetz about linear systems on Abelian varieties to the case of quasi-tori.

On the other hand, we shall finally prove the key step, and provide also the tools we invoked in section 2.1.

THEOREM 3.8. *If the line bundle L admits a non zero section s, then for $r\geq 2$ the linear system $|L^{\otimes r}|$ has no base points (in other terms, $L^{\otimes r}$ is generated by global sections).*

Proof. Using the Remmert–Morimoto decomposition (Proposition 1.1), and by the Künneth formula, it suffices to prove the result for X a Cousin quasi-torus or in the case when L is linearizable (cf. 1.4).

If X is Cousin, we choose vectors $a_1,\ldots,a_r$ in the subspace F such that $\Sigma_{i=1,\ldots,r} a_i=0$. Let a_i^* be $\pi(a_i)\in X$; by Lefschetz's trick (Corollary 3.7) $\Pi_{i=1,\ldots,r} s(x+a_i^*)$ is a section of $L^{\otimes r}$.

We just have to prove that all these sections do not have common zeros. Otherwise, there would be a point x in X such that $s(x+a^*)=0$ for each vector in F (fixing x and pulling back to the F^{r-1} given by $\Sigma_{i=1,\ldots,r} a_i=0$ we see that some $s(x+a_i^*)$ has to be identically zero in a_i^*, and one concludes since a_i can be an arbitrary vector in F). We know that the closure of $\pi(F)$ equals K, hence s vanishes on $x+K$. Hence the pull-back of s vanishes on a translate of $\mathbf{R}\Gamma$, and therefore on its complex span, and we obtain that s has to be identically zero, a contradiction.

In the case when L is linearizable, we can take arbitrary vectors $a_1, \ldots, a_r$, and the proof is even simpler. Q.E.D.

THEOREM 3.9 (key step). *Let f be a meromorphic function of rank r on a quasi-torus X, and assume as usual that f is the quotient of two relatively prime sections of a line bundle L. Then there is an integer $d \leq 2r(n+1)$, a line bundle L'' with $c_1(L'')=d\,c_1(L)$, and a space U of sections of L'' giving a holomorphic map $\Phi: X \to \mathbf{P}^N$ whose differential has everywhere rank at least r. In particular, if f is non degenerate, we obtain a holomorphic immersion $\Phi: X \to \mathbf{P}^N$.*

Proof. First of all we apply Corollary 3.4, obtaining L' with $c_1(L')=rc_1(L)$, and such that its sections give a meromorphic map with differential generically of rank r. Then we apply Theorem 3.8, obtaining as new bundle $L^\wedge = L'^{\otimes 2}$, some sections of which give a holomorphic map in some projective space with differential of rank $r' \geq r$ on the complement of some hypersurface Δ in X. Choose general points $b_1, \ldots, b_{n+1}$, such that the $(n+1)$ hypersurfaces $\Delta - b_1, \ldots, \Delta - b_{n+1}$, have empty intersection and finally choose

$$L'' = L^\wedge_{b_1} \otimes \ldots \otimes L^\wedge_{b_{n+1}}.$$

We can conclude observing that the tensor product of two line bundles L_1 and L_2 generated by global sections is also generated by global sections, and that the set of quotients of sections of the tensor product contains the two subsets given by the quotients of sections of L_1, resp. L_2. Hence L'' is generated by global sections and for each point x of X we can find $(r+1)$ sections of L'' giving a map which is holomorphic and with differential of rank r at x. It is easy now to find a space U as required. Q.E.D.

Remarks 3.10. The preceding theorem gives a much weaker result than the theorem of Lefschetz in the compact case.

In fact in the compact case we have that if H is positive definite, then

(3.11) $L^{\otimes d}$ gives an embedding for $d \geq 3$, and $L^{\otimes 2}$ gives a generically injective holomorphic map unless L has only one section.

(3.12) There exists a nondegenerate function f whose group of periods coincides with Γ.

(3.13) There exists a Γ-periodic function f of rank $\geqslant r$ iff the transcendence degree of the field Mer(X) of meromorphic functions on X is at least r.

(3.13) is the only statement which cannot have an analogue in the noncompact case: in fact Hefez, in [Hef], proves that if X is not compact either Mer(X) consists only of the contants, or its transcendence degree is infinite.

The correct analogue of (3.13) is thus, the following

PROPOSITION 3.14. *There exists a Γ-periodic function f of rank $\geqslant r$ iff there is a meromorphic map of X into a projective space such that its image has dimension at least r (or, equivalently, cf. what follows, if there is a line bundle L on X with Kodaira dimension at least r).*

Proof. By Theorem 2.20 the maximal rank of a meromorphic function on X equals the dimension of the quasi-Abelian reduction $X^{\#}$ of X; moreover, by Corollary 3.4, $X^{\#}$ admits a generic immersion in a projective space. Conversely, if we have a meromorphic map $\Phi: X \to \mathbf{P}^r$, whose differential is generically of rank m, we can pick a point x in X where $\Phi: X \to \mathbf{P}^r$ is holomorphic and with differential of rank $=m$. Hence there are meromorphic functions $f_1, \dots, f_m$ which can be completed to a set of holomorphic coordinates centered at x: if we choose $f = f_1^2 + \dots + f_m^2$, then the Hessian matrix of f at x has rank equal to m: this implies a fortiori that the rank of f is at least equal to m (notice that x is a critical point for f, hence the rank of the Hessian matrix does not depend upon the choice of a system of coordinates). Q.E.D.

We recall some by now standard notation:

Let L be a line bundle on a complex manifold X. Then the associated graded ring $\mathcal{R}(X, L)$ is the direct sum

$$\mathcal{R}(X, L) = \oplus_{m \in \mathbf{N}} H^0(X, L^{\otimes m}) \tag{3.15}$$

and we further define $Q(X, L)$ to be the field of the homogeneous fractions of degree zero of the ring $\mathcal{R}(X, L)$ ($Q(X, L)$ is a subfield of Mer(X)).

We recall that the Kodaira dimension of L is defined to be -1 if the ring $\mathcal{R}(X, L)$ consists only of the constants, and is defined otherwise to be the maximal dimension of the images of X under the maps into projective space given by the finite dimensional vector subspaces of the $H^0(X, L^{\otimes m})$'s.

In the compact case, obviously the Kodaira dimension, when it is non negative, coincides with the transcendence degree of the field $Q(X, L)$. In our case we can compare the Kodaira dimension of L with another integer, which we now define.

Definition 3.16. Let L be a line bundle on a complex Lie group X: then the *rank* of L is defined to be the maximum rank of a function f in $Q(X, L)$.

Remark 3.17. Unlikely the case of a torus, when the rank of a nonconstant f in $Q(X, L)$ equals the rank of the unique Hermitian form H representing $c_1(L)$, in the case of a quasi-torus such rank can vary with the choice of f, as can be easily seen with the case where L is linearized (pick for example X to be quasi-Abelian, fibering with noncompact fibres onto an Abelian variety X', and let L be the pull-back of a line bundle L' on the Abelian variety X': then we can consider on the one hand the pull-back of a meromorphic function on X', but also on the other hand a nondegenerate f in $Q(X, L)$ whose existence is shown in Theorem 2.4 (notice, concerning the proof of 2.4, that the rank of a section of a line bundle is not an intrinsic notion, since it depends upon the chosen trivialization)).

Given the pair (X, L) we can define, in an analogous fashion to what we did in 2.18, its *reduction* (X', L').

Definition 3.18. The reduction of the pair (X, L) is a triple (X', L', p) with p a surjective group homomorphism $p: X \to X'$, such that

(i) L is isomorphic to $p^*(L')$;

(ii) under the above isomorphism $\mathcal{R}(X, L) = p^*(\mathcal{R}(X', L'))$;

(iii) (X', L', p) is universal in the following sense: for every other such triple (X'', L'', p'') there exists $p': X'' \to X'$, such that L'' is isomorphic to $p'^*(L')$, and p factors as p'' followed by p'.

Definition 3.19. Given a line bundle L, define the *numerical rank* of L to be the maximal rank of a semi-positive definite Hermitian form H on $\mathbf{C}^n$ such tht the restriction of the imaginary part $\mathrm{Im}(H)$ to $\Gamma \times \Gamma$ represents $c_1(L)$ and the image Γ' of Γ into $\mathbf{C}^n/\ker H$ is discrete.

Remark. Considering the set S of all the Hermitian forms as in 3.19, we see, as we did in 2.17, 2.18, that this set has an element H such that $\ker H$ is the intersection of all the subspaces $\ker H'$ for H' in S. Indeed given H_1, H_2 in S, if we set $H'' = \frac{1}{2}(H_1 + H_2)$, then H'' is in S since $\ker(H_1 + H_2) = \ker H_1 \cap \ker H_2$, and $\mathbf{C}^n/\ker H''$ being naturally a subspace of $\mathbf{C}^n/\ker H_1 \oplus \mathbf{C}^n/\ker H_2$, the image Γ' of Γ into $\mathbf{C}^n/\ker H''$ is discrete (being a subgroup of $\Gamma_1' \oplus \Gamma_2'$). Proceeding by induction we obtain the desired H.

THEOREM 3.20. *Let (X, L) be a pair consisting of a quasi torus X and of a line bundle L. Then we have the following inequalities:*

$$\text{numerical rank of } L \geqslant \text{rank of } L \geqslant \text{Kodaira dimension of } L.$$

Moreover, if L has a non zero section, then there exists a reduction (X', L', p) *of* (X, L).

Proof. We notice first of all that we can reduce to the case when X is a Cousin quasi-torus, by the theorem of Remmert–Morimoto and the Künneth formula (the above 3 integers equal the dimension in the cases of $\mathbf{C}^n$ and of $(\mathbf{C}^*)^m$, cases in which the pair concides with its reduction). Furthermore, since a line bundle with a section either is trivial or it has positive Kodaira dimension, we can easily assume that the Kodaira dimension of L (and hence also the rank of L) is at least 1. As in the case of the quasi Abelian reduction of X, we consider the numerical reduction of L.

I.e., among the semi-positive definite Hermitian forms H on $\mathbf{C}^n$ such that the restriction of the imaginary part $\mathrm{Im}(H)$ to $\Gamma \times \Gamma$ represents $c_1(L)$ and the image Γ' of Γ into $\mathbf{C}^n/\ker H$ is discrete, we choose one such H with $\ker H$ smallest and we let X'' be the quotient quasi-torus $\mathbf{C}^n/\ker H/\Gamma'$.

Next, let G be the intersection of all the subgroups of X occurring as the group of periods of a function f in $Q(X, L)$: G is clearly a closed subgroup, and we define X' to be the quotient $X' = X/G$.

By the Main Theorem, the projection of X onto X' factors through the projection onto X'': in fact, if G_f is the group of periods of f, there is a H' representing $c_1(L)$ which is positive definite on $\mathbf{C}^n/G_f$, hence $G_f \supset H' \supset \ker H$, and $G \supset \ker H$.

If F is a holomorphic map of X to a projective space given by sections of some multiple of L, and with differential generically of rank $\mathrm{Kod}(L)$, then we saw that for each point x there is a function f in $Q(X, L)$ of rank greater or equal to $\mathrm{Kod}(L)$, hence the desired inequalities follow. Next, we construct the reduction of (X, L), in the case where L has a nonzero section.

Let s be a non zero section of L: by Theorem 3.8 and since the Kodaira dimension of L is at least 1, we can pick an integer r and 2 relatively prime sections s_1, s_2 of $L^{\otimes r}$ such that s^r is also relatively prime with s_1, s_2. Set $f = s^r/s_1$: then f is the pull-back of a meromorphic function f' on X', and if we express f' as the quotient of two relatively prime sections of a line bundle L'' on X', the divisor of zeros of f' is r times an effective divisor, hence we find a line bundle L' on X' such that L is isomorphic to the pull-back of L', and, more precisely, a section s' of L' on X' such that s differs by the pull-back of s' by an invertible holomorphic function on X.

Finally let t by any (holomorphic) section of some power $L^{\otimes d}$ of L: then $t/s^d=g$ is a meromorphic function on X'. We have a meromorphic section of $L'^{\otimes d}$ given by $t'=g(s')^d$, whose pull-back is holomorphic on X, and thus t' itself is holomorphic. In fact, t differs by the pull-back of t' by an invertible holomorphic function on X, i.e., a non zero constant, since we reduced to the case where X is Cousin. Therefore we have shown that $\mathcal{R}(X,L)=p^*(\mathcal{R}(X,L'))$, and thus that (X',L') is the reduction of (X,L).

Q.E.D.

Remarks 3.21. (i) The hypothesis that L has a non zero section is necessary in the above theorem in order to guarantee the existence of a reduction of the pair (X,L), as it is easily seen in the case where X is compact, and L is non trivial but is a torsion bundle in $\mathrm{Pic}(X)$.

(ii) It is easy to see that every meromorphic map to projective space given by sections of a multiple of L factors through the projection of X onto X'.

The result of the above theorem can be improved to yield equality of num. rank, rank, Kodaira dim., provided one can show that some multiple of L yields an embedding of X'', since then we would have $X'=X''$.

In turn, it is sufficient for this purpose to show that when H is strictly positive definite, then a multiple of L gives an embedding of X.

We shall show that this statement does indeed follow, provided the following conjecture holds true:

CONJECTURE 3.22. *A line bundle L with $c_1(L)$ representable by a positive definite Hermitian form has a section.*

We believe that the answer to this question should be positive, and we plan to return on this matter in the future. We notice that in the case $n=2$ a positive answer is provided by the results of Cousin [Cou2].

We are going now to show some stronger Lefschetz type theorems than 3.9, one of which is still conditional, since it depends upon the validity of Conjecture 3.22.

THEOREM 3.23. *If L is a tame line bundle on a quasi-torus X, with $c_1(L)$ being positive definite on the subspace F, then $L^{\otimes d}$ gives an embedding of X for $d\geqslant 3$.*

Proof. Fix first of all a Hermitian form H which is positive definite on all of $\mathbf{C}^n$ and is such that its imaginary part $A'=\operatorname{Im} H$ restricted to $\Gamma\times\Gamma$ equals $c_1(L)$.

By Proposition 2.8 we can find two compact quotients X_1, resp. X_2, of X such that X embeds in the product of X_1 and X_2.

Since moreover the line bundle L on X is a pull-back of a line bundle L_i on the torus X_i, can apply the classical Lefschetz theorem for these two complex tori $\mathbf{C}^n/\Gamma_i$ (cf. [Cor], [Mum]), obtaining a finite dimensional subspace of $H^0(X, L^{\otimes d})$ yielding a holomorphic embedding of X, as it is easy to see. Q.E.D.

THEOREM 3.24. *Let Γ be a subgroup of $\mathbf{C}^n$ for which the generalized Riemann relations hold. Then there is a meromorphic function f on $\mathbf{C}^n$ such that Γ coincides with the subgroup Γ_f of periods of f.*

Proof. We consider a tame line bundle L with $c_1(L)$ being positive definite on the subspace F (this exists by the GRR), and again by 2.8 we have two lattices Γ_1 and Γ_2, with intersection equal to Γ and such that, denoting by X_1, X_2 the respective quotient tori, the line bundle L is the pull back of respective line bundles L_1, L_2.

It suffices to choose, for $i=1,2$ a section s_i of L_i on X_i whose divisor of zeros is irreducible and has no periods on X_i (both assertions are known in the compact case, the first one being Bertini's theorem, the other holding since the period group of the sections can only vary in a finite set and for each such we have a proper subspace of the space of sections $H^0(X_i, L_i)$); finally we set $f=s_1/s_2$, which is Γ-periodic, but cannot have a larger group of periods Γ', otherwise also the divisor of zeros and the divisor of poles of f would be Γ'-periodic. Q.E.D.

COROLLARY 3.25. *A subgroup Γ of $\mathbf{C}^n$ is the subgroup of periods of a meromorphic functions if and only if there exists a positive semi-definite Hermitian form H on $\mathbf{C}^n$ such that:*

(i) *the restriction of* $\mathrm{Im}(H)$ *to $\Gamma\times\Gamma$ takes integer values*

(ii) *the kernel of H equals the connected component of the identity in Γ.*

CONDITIONAL THEOREM 3.26. *Let L be a line bundle on a quasi-torus X with $c_1(L)$ representable by the imaginary part of a positive definite Hermitian form H. Assume that Conjecture 3.22 holds.*

Then $L^{\otimes d}$ gives an embedding of X for $d\geq 5$.

Proof. We can assume that L is given by a cocycle in Cousin normal form $\varrho(\gamma)\, e(-\frac{i}{2}[H(z,\gamma)+\frac{1}{2}H(\gamma,\gamma)]+f_\gamma(z))$. Let L', L'' be respectively given by the following cocycles:

$$\varrho(\gamma)\, e\left(-\frac{i}{2}\left[H(z,\gamma)+\frac{1}{2}H(\gamma,\gamma)\right]+\frac{d}{2}f_\gamma(z)\right)$$

$$\varrho(\gamma)\, e\left(-\frac{i}{2}\left[H(z,\gamma)+\frac{1}{2}H(\gamma,\gamma)\right]\right).$$

Then $L^{\otimes d} \cong (L')^2 \otimes (L'')^{d-2}$; by virtue of Theorems 3.23 and 3.8, the assertions follow from the following well known lemma (cf. e.g. Mumford's book [Mum]), whose proof we omit.

LEMMA 3.27. *Let X be a complex manifold, and let L, M be line bundles such that M is generated by global sections, and the sections of L give an embedding of X: then the sections of $L \otimes M$ give an embedding of X.*

Q.E.D. for Theorem 3.26.

Remark 3.28. After showing that the Kodaira dimension is the correct analogue in the noncompact case of the transcendence degree of $Q(X, L)$, we observe that the transcendence degree of Mer(X) is uncountable. In fact Pothering ([Pot]) showed even that the subfield $Q(X, L)$ of meromorphic functions obtained as quotients of sections of powers of a fixed tame line bundle L has uncountable transcendence degree.

Some interesting questions are in our opinion:

(3.29) Which is, in the hypotheses of 3.26, the smallest integer d such that $L^{\otimes d}$ gives an embedding of X?

(3.30) Can the results about the index of a line bundle (i.e., the characterization of the set S of integers i such that $H^i(X, L)$ is non zero, cf. [Mum]) be generalized in the noncompact case? For instance, if p is the number of positive eigenvalues of H on the complex subspace F, and r is the number of negative ones, is it true that S is contained in the interval $[r, m-p]$ (where m is the dimension of the complex subspace F)?

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