Holomorphic families of injections

by

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§ 1. Introduction and statement of results

This paper contains new proofs and extensions of some recent results by Mañé, Sad and Sullivan [11] and by Sullivan and Thurston [15]. It is convenient to begin with the following definition.

Let E be a subset of the Riemann sphere $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$ containing at least 4 points. Let Δ_r denote the open disc $|z|<r$ in $\mathbf{C}$. A map

$$f\colon \Delta_r\times E\to\hat{\mathbf{C}}$$

will be called *admissible* if $f(0,z)=z$ for all $z\in E$, for every fixed $\lambda\in\Delta_r$ the map $f(\lambda,\cdot)\colon E\to\hat{\mathbf{C}}$ is an injection, and for every fixed $z\in E$ the map $f(\cdot,z)\colon\Delta_r\to\hat{\mathbf{C}}$ is holomorphic (i.e., a meromorphic function of λ).

In other words, an admissible map is a family of injections $E\to\hat{\mathbf{C}}$ holomorphically parametrized by a complex parameter λ, $|\lambda|<r$, which reduces to the identity for $\lambda=0$.

We shall often assume that the admissible map considered is *normalized*, that is, that $\{0,1,\infty\}\subset E$ and $f(\lambda,\zeta)=\zeta$ for $\zeta=0,1,\infty$ and $\lambda\in\Delta_r$. This involves no serious loss of generality. Indeed, given an admissible map $f\colon\Delta_r\times E\to\hat{\mathbf{C}}$ and 3 distinct points ζ_1, ζ_2, ζ_3 in E, let α be the Möbius transformation which takes $0,1,\infty$ into ζ_1, ζ_2, ζ_3 and β_λ be the Möbius transformation which takes $f(\lambda,\zeta_1)$, $f(\lambda,\zeta_2)$, $f(\lambda,\zeta_3)$ into $0,1,\infty$. Then $\hat{f}\colon\Delta_r\times\alpha^{-1}(E)\to\mathbf{C}$, where

$$\hat{f}(\lambda,\hat{z})=\beta_\lambda\circ f(\lambda,\alpha(\hat{z}))$$

is admissible and normalized. (If $f\colon\Delta_r\times E\to\hat{\mathbf{C}}$ is normalized and admissible, then, for every fixed $z\in E-\{\infty\}$, the function $f(\cdot,z)$ is holomorphic.)

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(1) This material is based upon work partially supported by the National Science Foundation under Grant No. NSF MCS-78-27119.

(2) This material is based upon work partially supported by the National Science Foundation under Grant No. NSF MCS-83-01379.

The "λ-*lemma*" by Mañé, Sad and Sullivan [11] asserts that an admissible map $f(\lambda, z)$ is, for every fixed λ, uniformly continuous in z (with respect to the spherical metric) and that the continuous extension of $f(\lambda, \cdot)$ to the closure of E (in $\hat{\mathbf{C}}$) has the Pesin property.

By the *Pesin property* we mean the following. Denote the spherical distance in $\hat{\mathbf{C}}$ by δ. Let $A \subset \hat{\mathbf{C}}$ be a set, and let w: $A \to \hat{\mathbf{C}}$ be a map. For $z \in A$ and $\varepsilon > 0$ let $m(z, \varepsilon)$ and $M(z, \varepsilon)$ denote the infinum and the supremum of $\delta(w(z), w(\zeta))$ for $\zeta \in A$ and $\delta(z, \zeta) = \varepsilon$, if there are such ζ, and set $m(z, \varepsilon) = M(z, \varepsilon) = 1$ if there are none. The function w has the Pesin property if the function

$$P(z) = \overline{\lim_{\varepsilon \to 0}} \frac{M(z, \varepsilon)}{m(z, \varepsilon)}$$

is uniformly bounded.

It is known (cf. [10]) that a homeomorphism w of a plane domain is quasiconformal if and only if w has the Pesin property, and that if w is K-quasiconformal, then $P(z) \leq K$ for almost all (but not necessarily all) z in A.

THEOREM 1. *If f: $\Delta_1 \times E \to \hat{\mathbf{C}}$ is admissible, then every $f(\lambda, \cdot)$ is the restriction to E of a quasiconformal self-map F_λ of $\hat{\mathbf{C}}$, of dilatation not exceeding.*

$$K = \frac{1+|\lambda|}{1-|\lambda|}. \tag{1.1}$$

It is easy to see that the bound (1.1) cannot be improved.

From Theorem 1 we derive the following Corollaries:

COROLLARY 1 (Mañé-Sad-Sullivan). *If f: $\Delta_1 \times \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ is admissible, then, for each $\lambda \in \Delta_1$, the map $f(\lambda, \cdot)$ is a quasiconformal homeomorphism of $\hat{\mathbf{C}}$ onto itself.*

COROLLARY 2. *For each $r<1$ there are constants A, α, and B, depending only on r, such that, if f is a normalized-admissible map on $\Delta_1 \times E$, we have*

$$\delta[f(\lambda, z), f(\lambda', z')] \leq A\delta(z, z')^\alpha + B|\lambda - \lambda'|$$

for $z, z' \in E$ *and* $|\lambda|$, $|\lambda'| \leq r$. *Here δ is the spherical metric.*

COROLLARY 3. *Let $\{E_n\}$ be an increasing sequence of subsets of $\hat{\mathbf{C}}, E = \cup E_n$,* and *$\{f_n\}$ a sequence of normalized admissible map on $\Delta_1 \times E_n$. Then there is an admissible map f on $\Delta_1 \times \hat{E}$ and a subsequence $\{f_{n_k}\}$ which converges to f, uniformly on $\Delta_r \times E_n$ for each n and each $r<1$. Here $\hat{E}$ is the closure of E.*

THEOREM 2. *If* $f\colon \Delta_1\times E\to\hat{\mathbf{C}}$ *is admissible and E has a nonempty interior* ω, *then for each* $\lambda\in\Delta_1$ *the map* $f(\lambda,\cdot)|\omega$ *is a K-quasiconformal homeomorphism of* ω *into* $\hat{\mathbf{C}}$ *with* $K=(1+|\lambda|)/(1-|\lambda|)$. *The Beltrami coefficient of* $f(\lambda,\cdot)|\omega$ *given by*

$$\mu(\lambda,z)=\frac{\partial f(\lambda,z)|\omega}{\partial\bar z}\bigg/\frac{\partial f(\lambda,z)|\omega}{\partial z},$$

is a holomorphic function of $\lambda\in\Delta_1$, qua *element of the Banach space* $L_\infty(\omega)$.

Given an admissible map $f\colon\Delta_1\times E\to\hat{\mathbf{C}}$ we may want to find an admissible map $\hat f\colon\Delta_1\times\hat{\mathbf{C}}\to\hat{\mathbf{C}}$ which extends f. This *extension problem* first posed by Mañé and Sullivan, seems difficult. We can state only partial results.

PROPOSITION 1. *If for every finite set* $E_0\subset\hat{\mathbf{C}}$ *(containing at least three points) and for every point* $y\notin E_0$ *every admissible map of* $\Delta_1\times E_0$ *extends to an admissible map of* $\Delta_1\times(E_0\cup\{y\})$, *then the extension problem is solvable for any set E and any admissible map of* $\Delta_1\times E$.

By means of examples we shall establish, among other things, the following

PROPOSITION 2. *There are admissible maps* $f\colon\Delta_1\times E\to\hat{\mathbf{C}}$ *with a unique admissible extension to* $\Delta_1\times\hat{\mathbf{C}}$. *There are admissible maps of* $\Delta_1\times E$ *which have several admissible extensions to* $\Delta_1\times\hat{\mathbf{C}}$ *and such that all extensions coincide on some but not all components of* $\hat{\mathbf{C}}-\hat{E}$.

If E is a set consisting of three points, then every admissible map f on $\Delta_1\times E$ trivially extends to an admissible map $\hat f$ on $\Delta_1\times\hat{\mathbf{C}}$, for we may assume f normalized and take $\hat f(\lambda,z)=z$. The corresponding result for a set of four points is given by Proposition 3 below which is implied by a result of Earle and Kra [6]. For a set E with n points, $n>4$, we do not know whether every admissible map on $\Delta_1\times E$ extends to an admissible map on $\Delta_1\times(E\cup\{y\})$, for a point $y\in\mathbf{C}-E$.

PROPOSITION 3. *Let* $E=\{0,1,\infty,\alpha\}$ *be a set consisting of four points and* $f\colon\Delta_1\times E\to\hat{\mathbf{C}}$ *an admissible map. Then there is an admissible map* $\hat f\colon\Delta_1\times\hat{\mathbf{C}}\to\hat{\mathbf{C}}$ *which extends f.*

The "improved λ-lemma" by Sullivan and Thurston [14] asserts that there is an $r>0$, which they cannot estimate, such that for every admissible map f on $\Delta_1\times E$ there is an admissible map on $\Delta_r\times\hat{\mathbf{C}}$ which extends $f|\Delta_r\times E$.

THEOREM 3. *If* $f: \Delta_1 \times E \to \hat{\mathbf{C}}$ *is an admissible map, then* $f|\Delta_{1/3} \times E$ *has a canonical admissible extension* $\hat{f}: \Delta_{1/3} \times \hat{\mathbf{C}} \to \hat{\mathbf{C}}$.

This extension is characterized by the following property. Let $\mu(\lambda, z)$ *be the Beltrami coefficient of* $z \mapsto \hat{f}(\lambda, z)$ *and S any component of* $\hat{\mathbf{C}} - \hat{E}$, *where* $\hat{E}$ *is the closure of E in* $\hat{\mathbf{C}}$. *Then*

$$\mu(\lambda, z) = \varrho_S(z)^{-2}\,\overline{\psi(\lambda, z)} \quad \textit{for } z \in S,\ \lambda \in \Delta_{1/3}$$

where $\varrho_S(z)|dz|$ *is the Poincaré line element in S and the function* $\psi(\lambda, z)$ *is holomorphic in* $z \in S$, *antiholomorphic in* $\lambda \in \Delta_{1/3}$.

The uniqueness statement in Theorem 3 is based on a result which may be of interest in other connections, too (Lemma II in § 5). It gives a sufficient condition for a quasiconformal self-map of a plane domain which is homotopic to the identity modulo the set-theoretical boundary to be so modulo the ideal boundary.

Our proofs make essential use of the theory of quasiconformal maps and of Teichmüller spaces (see [5], [7], [10] and the references given there). For the convenience of the reader some of the necessary results are stated in § 2. In § 7 we describe the connection between the extension problem and a lifting problem in Tecihmüller space.

§ 2. Preliminaries

All results summarized in this section are known. A reader familiar with Teichmüller theory will scan it in order to note our notations.

(A) We assume the basic results on quasiconformal maps, cf., for instance, [2], [10]. A *Beltrami coefficient* μ in a domain $S \subset \hat{\mathbf{C}}$ is an element of the open unit ball in the complex Banach space $L_\infty(S)$. A μ*-conformal map* F of S is a homeomorphic solution of the Beltrami equation

$$\frac{\partial F}{\partial \bar{z}} = \mu \frac{\partial F}{\partial z}$$

in S. Here the derivatives, taken in the sense of distribution theory, are required to be locally square integrable measurable functions. (One says that μ is the Beltrami coefficient of F.)

The smoothness of a μ-conformal map F depends on μ. In particular F is C_∞ or real analytic if μ is. Any μ-conformal map is differentiable a.e. If F_1 and F_2 are two μ-conformal maps of S, then $F_2 \circ F_1^{-1}$ is conformal.

A map is quasiconformal if it is μ-conformal for some Beltrami coefficient μ. The *dilatation* of F is the number

$$K(F) = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}$$

where $\|\mu\|_\infty$ is the essential supremum of $|\mu(z)|$ in S. If $K(F) \leq A$, F is called *A-quasiconformal.*

Inverses and composites of quasiconformal maps are quasiconformal, and the dilatation obeys the rules: $K(F)=1$ if and only if F is conformal, $K(F^{-1})=K(F)$ and $K(F_1 \circ F_2) \leq K(F_1)K(F_2)$. The partial derivatives of F^{-1} and of $F_1 \circ F_2$ are computable (a.e.) by the classical formulas.

(B) Let μ be a Beltrami coefficient in $\mathbf{C}$. There is a unique μ-conformal homeomorphism $z \mapsto w^\mu(z)$ of $\mathbf{C}$ onto itself which fixes 0, 1 (and, therefore, ∞). This w^μ has a Hölder modulus of continuity, with respect to the spherical metric, depending only on $\|\mu\|_\infty$. For every $z \in \mathbf{C}$, the number $w^\mu(z)$ depends holomorphically on $\mu \in L_\infty(\mathbf{C})$.

If $\|\mu_j\|_\infty \leq k < 1$, the sequence $\{w^{\mu_j}\}$ contains a uniformly convergent subsequence, the limit is of the form w^μ with $\|\mu\|_\infty \leq k$. If the sequence $\{\mu_j\}$ has the limit μ_∞ a.e., then $\mu = \mu_\infty$.

Let U denote, here and hereafter, the upper half-plane in $\mathbf{C}$. Every quasiconformal self-map ω of U has a continuous extension to $U \cup \mathbf{R} \cup \{\infty\} = U \cup \hat{\mathbf{R}}$; this extension will be denoted by the same letter.

If μ is a Beltrami coefficient in U, then there is a unique μ-conformal homeomorphism $z \rightarrow w_\mu(z)$ of U onto itself which fixes $0, 1, \infty$. It has a Hölder modulus of continuity, with respect to the spherical metric, depending only on $\|\mu\|_\infty$. For every $z \in U \cup \mathbf{R}$, the number $w_\mu(z)$ depends real-analytically on $\mu \in L_\infty(U)$.

Convergence theorems similar to the ones stated above for w^μ hold for w_μ.

(C) The image of $\hat{\mathbf{R}}$ under a quasiconformal self-map of $\hat{\mathbf{C}}$ is called a *quasicircle*. A Jordan curve C passing through ∞ is a quasicircle if and only if it satisfies the *Ahlfors condition:* there is an $M>0$ such that for any three distinct finite points a, b, c on C, with b on the finite component of $C-\{a, c\}$,

$$|b-a| \leq M|c-a|.$$

If C does not pass through ∞, this inequality must be satisfied whenever b lies on the component of $C-\{a, c\}$ with the smaller Euclidean diameter, cf. [10].

(D) We recall next some facts from the theory of the *Teichmüller space* $T(S)$ of a Riemann surface S which is not conformal to a sphere, a punctured sphere, a twice punctured sphere or a torus. As a matter of fact, we shall need only the case when $S\subset\hat{\mathbf{C}}$; we assume that S has at least 3 boundary points one of which is the point ∞.

For such an S there always exists a holomorphic universal covering by the upper half-plane U,

$$\pi\colon U\to S; \tag{2.1}$$

the covering group G of π is a torsion-free *Fuchsian group* (discrete subgroup of $PSL(2,\mathbf{R})$). Note that π and G are uniquely determined by S, except that they may be replaced by $\pi\circ\alpha$ and $\alpha^{-1}G\alpha, \alpha\in PSL(2,\mathbf{R})$.

The *Poincaré line element* $\varrho_S(\zeta)|d\zeta|, \zeta\in S$, is defined by the relation

$$\varrho_S(\pi(z))|\pi'(z)| = 2|z-\bar{z}|^{-1};$$

$\varrho_S(z)|dz|$ is invariant under all conformal automorphisms of S.

The Poincaré metric on S can be also characterized as the *only* complete Riemannian metric on S which respects the conformal structure of S, i.e., is given by a line element $ds=\sigma(z)|dz|$, and has Gaussian curvature (-1), i.e., satisfies the partial differential equation $\Delta\log\sigma=\sigma^2$.

We note the monotonicity property:

$$\varrho_{S_0}(z)\geqq\varrho_S(z) \quad \text{if } z\in S_0\subset S.$$

(E) The *limit set* Λ of G is the closure of the set of fixed points of parabolic and hyperbolic elements of G. If $\Lambda=\mathbf{R}\cup\{\infty\}=\hat{\mathbf{R}}$, S is said to have no ideal boundary curves. If $\Lambda\neq\mathbf{R}$, each component I of $\hat{\mathbf{R}}-\Lambda$ defines an *ideal boundary curve* C of S:

$$C = I/\mathrm{Stab}_G(I)$$

where the stabilizer of I in G consists either of the identity only or of all powers of a hyperbolic element γ in G which fixes the endpoints of I. For every $\alpha\in G, C$ is identified with $\alpha(I)/\mathrm{Stab}_G(\alpha(I))$.

Let $b(S)$ denote the union of the ideal boundary curves of S; then $S\cup b(S)$ has a natural topology in which S is open and dense.

Every quasiconformal map $F\colon S \to F(S) \subset \mathbf{C}$ extends by continuity to a homeomorphism of $S \cup b(S)$ onto $F(S) \cup b(F(S))$. The extension will be denoted by the same letter.

(F) The *Teichmüller space* $T(S)$ is the set of equivalence classes $[F]$ of quasiconformal mappings

$$F\colon S \to F(S)$$

where $F(S)$ is another domain in $\hat{\mathbf{C}}$. (No generality would be gained by allowing $F(S)$ to be any Riemann surface.) Two such maps, F and F_1, are *equivalent* if there is a conformal map $h\colon F(S) \to F_1(S)$ such that the map

$$F_1^{-1} \circ h \circ F\colon S \to S$$

is homotopic to the identity modulo $b(S)$. An equivalent condition is that there be a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Psi"] \arrow[d, "\pi"'] & U \arrow[d, "\pi"] \\ S \arrow[r, "F_1^{-1}\circ h \circ F"'] & S \end{tikzcd}$$

such that the quasiconformal map Ψ fixes every point of $\mathbf{R}$.

Note that the Beltrami coefficient of F determines $[F]$, but not vice versa.

The space $T(S)$ is a complete metric space under the *Teichmüller distance function*

$$\langle [F_1], [F_2] \rangle = \inf \log K(F)$$

where F runs over all quasiconformal maps equivalent to $F_1 \circ F_2^{-1}$.

If $\hat{\mathbf{C}} - S$ consists of m points, $T(S)$ is homeomorphic to $\mathbf{C}^{m-3}$.

(G) Let L denote the lower half-plane in $\mathbf{C}$, and let $B(L, G)$ be the complex Banach space of holomorphic functions $\varphi(\zeta)$, $\zeta \in L$ with norm

$$\|\varphi\| = \sup \eta^2 |\varphi(\zeta)| < \infty$$

(where $\zeta = \xi + i\eta$) which satisfy the functional equation of quadratic differentials

$$\varphi(g(\zeta))\, g'(\zeta)^2 = \varphi(\zeta), \quad g \in G.$$

There exists a canonical homeomorphic injection

$$T(S) \hookrightarrow B(L, G) \tag{2.2}$$

(onto a bounded domain) defined as follows. Let F be a μ-conformal map of S. Lift μ, via (2.1), to a Beltrami coefficient $\tilde{\mu}(z)$ in U, by setting

$$\mu(\pi(\zeta))\ \overline{\pi'(\zeta)}_{\ast} / \pi'(\zeta) = \tilde{\mu}(\zeta)$$

and set

$$\hat{\mu}(\zeta) = \begin{cases} \tilde{\mu}(\zeta) & \text{for } \zeta \in U \\ 0 & \text{for } \zeta \in L \end{cases}$$

(We note that $\tilde{\mu}(\zeta)\, d\bar{\zeta}/d\zeta$ and $\hat{\mu}(\zeta)\, d\bar{\zeta}/d\zeta$ are G-invariant, and that $w^{\hat{\mu}}|L$ is conformal.) It turns out that the *Schwarzian derivative*

$$\varphi^{\mu} = \{w^{\hat{\mu}}|L, z\},$$

i.e.,

$$\varphi^{\mu}(\zeta) = u'(\zeta) - \frac{1}{2} u(\zeta)^2, \quad u(\zeta) = \frac{d}{d\zeta} \log \frac{dw^{\hat{\mu}}(\zeta)}{d(\zeta)}, \quad \zeta \in L,$$

is determined by and determines $[F]$. Also, $\varphi^{\mu} \in B(L, \mathrm{G})$ and

$$\|\varphi^{\mu}\| < \frac{3}{2}.$$

The map

$$[F] \mapsto \varphi^{\mu} \tag{2.3}$$

is the desired embedding. From now on we *identify* $T(S)$ with its image.

(H) Now let $\varphi \in B(L, G)$ with $\|\varphi\| < \frac{1}{2}$ be given, and set

$$\nu(\zeta) = -2\eta^2 \varphi(\zeta), \quad \zeta \in U.$$

Then $\nu(\zeta)\, d\bar{\zeta}/d\zeta$ is G-invariant and

$$\nu = \tilde{\mu} \tag{2.4}$$

where

$$\mu(z) = \varrho_S^2(z)\, \overline{\psi(z)} \tag{2.5}$$

with $\psi(z), z \in S$, holomorphic; more precisely

$$\psi(\pi(\zeta))\,\pi'(\zeta)^2 = \overline{\varphi(\bar{\zeta})}. \tag{2.6}$$

Finally, by the Ahlfors-Weill lemma [3]

$$\varphi^\mu = \varphi.$$

A Beltrami coefficient μ in S of the form (2.5) will be called *harmonic*. (The name is suggested by the Kodaira-Spencer deformation theory; in [4] these Beltrami coefficients were called canonical.) We note two consequences of what was said above.

(a) *A point φ in $T(S) \subset B(L, G)$ with $\|\varphi\| < \frac{1}{2}$ can be represented as $[F]$ with the Beltrami coefficient μ of F harmonic and given by* (2.5), (2.6). *Thus μ depends holomorphically on φ.*

(b) *If quasiconformal maps F_1 and F_2 have harmonic Beltrami coefficients μ_1, μ_2, and are equivalent, i.e. if $[F_1]=[F_2]$, then $\mu_1=\mu_2$.*

(I) A quasiconformal map F of S is called a Teichmüller map if either F is conformal or F has a Beltrami coefficient of the form

$$\mu = k|\varphi(z)|/\varphi(z)$$

where $\varphi(z)$ is holomorphic in S and $\varphi \in L_1(S)$.

(c) *If F_1 is a Teichmüller map of S and F_2 another map with $[F_2]=[F_1]$, then either $F_2=F_1$ or $K(F_2)>K(F_1)$.*

This is a special case of *Teichmüller's uniqueness theorem*, as extended by Reich and Strebel [12] and by Strebel [14].

Teichmüller's existence theorem implies that if $[F] \in T(S)$ and $\dim T(S) < \infty$ (in our case, if $\hat{\mathbf{C}} - S$ is finite), then F is equivalent to a Teichmüller map.

(J) The *modular group* Mod(S) of $T(S)$ is the group of holomorphic isometries of $T(S)$ of the form

$$[F] \mapsto [F \circ \Phi^{-1}] = \Phi_*([F])$$

where Φ is any quasiconformal self-map of S. If $\dim T(S) < \infty$ (in our case, if $\hat{\mathbf{C}} - S$ is finite), Mod(S) acts properly discontinously.

(K) In every complex manifold M one can define the *Kobayashi pseudometric* as the largest pseudometric with the property: if z_1 and z_2 are two points in U, d the Poincaré distance between z_1 and z_2, and Φ a holomorphic map of U into M, then the Kobayashi distance between $\Phi(z_1)$ and $\Phi(z_2)$ is less than or equal to d. The following results will be used later.

(d) *A holomorphic map of one complex manifold into another does not increase the Kobavashi distance.*

(e) *If $S=\mathbf{C}-E$ where E is finite and contains at least 3 points, the Kobayashi distance in $T(S)$ coincides with the Teichmüller distance.*

Statement (d) follows from the definition, statement (e) by repeating the argument given in [13] for the case when S is a compact Riemann surface. (Cf. also [6].)

§ 3. The finite case

In this section we prove Theorem 1 and Theorem 3 for the case when the set E is finite (in this case Theorem 2 is vacuous). Without loss of generality we assume that

$$E=\{0,1,\infty,\zeta_1,\dots,\zeta_n\},\quad n>0,$$

and that the given admissible map $f\colon \Delta_1\times E\to\hat{\mathbf{C}}$ is normalized.

Let M_n denote the complex manifold of ordered n-tuples of distinct complex numbers $(z_1,\dots,z_n)$ none of which equals 0 or 1.

LEMMA. *There is a holomorphic universal covering*

$$p\colon T(\hat{\mathbf{C}}-E)\to M_n.$$

(The map p is given by the relation (3.3) *below.)*

Proof. Every point τ of $T(\hat{\mathbf{C}}-E)$ is of the form $[F]$ where F is a quasiconformal map of $\hat{\mathbf{C}}-E$ into $\hat{\mathbf{C}}$. Such an F is of the form $\alpha\circ w^{\mu}$ where $\mu\in L_\infty(\mathbf{C})$, $\|\mu\|_\infty<1$, and $\alpha\in PSL(2,\mathbf{C})$, cf. § 2(A). Since $[\alpha\circ w^{\mu}]=[w^{\mu}]$, every τ is of the form $[w^{\mu}]$.

Now, $[w^{\mu_1}]=[w^{\mu_2}]$ if and only if there is a conformal map h of $w^{\mu_1}(\hat{\mathbf{C}}-E)$ onto $w^{\mu_2}(\hat{\mathbf{C}}-E)$ such that $(w^{\mu_2})^{-1}\circ h\circ w^{\mu_1}$ is homotopic to the identity in $\hat{\mathbf{C}}-E$. But such an h must be a Möbius transformation which fixes $0,1,\infty$, hence the identity. Thus $[w^{\mu_1}]=[w^{\mu_2}]$ if and only if

$$(w^{\mu_2})^{-1}\circ w^{\mu_1}|\hat{\mathbf{C}}-E\quad\text{is homotopic to id},$$

which implies that

$$(w^{\mu_2})^{-1} \circ w^{\mu_1} | E = \text{id}.$$

This shows that

$$(w^{\mu}(\zeta_1), \ldots, w^{\mu}(\zeta_n)) \in M_n \tag{3.2}$$

depends only on $[w^{\mu}]$ rather than on the particular choice of μ. It is clear that every point of M_n can be written in the form (3.2) for some $\mu \in L_\infty(\mathbf{C}), \|\mu\|_\infty < 1$, and we conclude that

$$[w^{\mu}] \mapsto p([w^{\mu}]) = ((w^{\mu}(\zeta_1), \ldots, (w^{\mu}(\zeta_n)) \tag{3.3}$$

is a well defined surjection. We claim it is *holomorphic.*

Indeed, let $[w^{\nu}]$ be a point in $T(\hat{\mathbf{C}}-E)$ and let $\sigma_1, \ldots, \sigma_{n-3}$ be a basis of harmonic (in the sense of § 2 (G)) Beltrami coefficients on $w^{\nu}(\hat{\mathbf{C}}-E)$. By the results stated in § 2 (I) the map

$$(t_1, \ldots, t_n) \mapsto [w^{t_1\sigma_1+\ldots+t_{n-3}\sigma_{n-3}} \circ w^{\nu}]$$

is a biholomorphic homeomorphism of a neighborhood of the origin in $\mathbf{C}^{n-3}$ onto a neighborhood of $[w^{\nu}]$ in $T(\hat{\mathbf{C}}-E)$. On the other hand,

$$w^{t_1\sigma_1+\ldots+t_{n-3}\sigma_{n-3}} \circ w^{\nu} = w^{\mu}$$

with

$$\mu = \frac{t_1\hat{\sigma}_1+\ldots+t_{n-3}\hat{\sigma}_{n-3}+\nu}{1+\bar{\nu}(t_1\hat{\sigma}_1+\ldots+t_{n-3}\hat{\sigma}_{n-3})}$$

where

$$\hat{\sigma}_j(z) = \sigma_j(w^{\nu}(z)) \left|\frac{\partial w^{\nu}(z)}{\partial z}\right|^2 \Big/ \left(\frac{\partial w^{\nu}(z)}{\partial z}\right)^2$$

so that μ depends holomorphically on $(t_1, \ldots, t_{n-3})$ and so does the right hand of (3.3). This proves the assertion.

Now let Γ be the subgroup of $\mathrm{Mod}(\hat{\mathbf{C}}-E)$ (cf. § 2 (J)) consisting of all self-maps $[w^{\mu}] \mapsto [w^{\mu} \circ \omega^{-1}]$ induced by quasiconformal self-maps ω of $\hat{\mathbf{C}}-E$ which fix each point of E. Then Γ acts properly discontinuously on $T(\hat{\mathbf{C}}-E)$. We claim that the action is also *free*. Indeed, assume that $\omega_*([w^{\mu}])=[w^{\mu}]$. This means that $(w^{\mu})^{-1} \circ w^{\mu} \circ \omega^{-1}$ is homotopic to the identity in $\hat{\mathbf{C}}-E$, i.e., that ω is homotopic to the identity, i.e., that $\omega_*=\text{id}$.

Now, $[w^\mu]$ and $[w^\nu]$ have the same image under p if and only if $(w^\mu)^{-1} \circ w^\nu$ fixes every point of E, i.e., if and only if $[w^\mu]$ and $[w^\nu]$ are equivalent under Γ. We conclude that (3.3) is a Galois covering. Since $T(\mathbf{C}-E)$ is a cell, it is the universal covering. The lemma is proved.

The given admissible map $f: \Delta_1 \times E \to \hat{\mathbf{C}}$ may be identified with a holomorphic vector-valued map $\mathbf{f}: \Delta_1 \to M_n$ which takes $\lambda \in \Delta_1$ into

$$\{f(\lambda, \zeta_1), \ldots, f(\lambda, \zeta_n)\} \in M_n.$$

This maps lifts, via (3.3), to a holomorphic map

$$\tilde{\mathbf{f}}: \Delta_1 \to T(\hat{\mathbf{C}}-E) \subset B(L, G)$$

(where G is a torsion-free Fuchsian group with $\hat{\mathbf{C}}-E$ conformal to U/G). The map $\tilde{\mathbf{f}}$ is uniquely determined by the requirement that $\tilde{\mathbf{f}}(0)=[\mathrm{id}]$, i.e. the origin in $B(L, G)$.

In Δ_1 the Kobayashi distance (cf. § 2(K)) between 0 and λ equals the Poincaré distance $\log K$, where

$$K = \frac{1+|\lambda|}{1-|\lambda|}. \tag{3.4}$$

The holomorphic map $\tilde{\mathbf{f}}$ does not increase the Kobayashi distance so that the Teichmüller (=Kobayashi) distance between the points [id] and $\mathbf{f}(\lambda)$ in $T(\hat{\mathbf{C}}-E)$ is at most $\log K$. This means that there exists, for each $\lambda \in \Delta_1$, a $\nu_\lambda \in L_\infty(\mathbf{C})$, with $K(w^{\nu_\lambda}) \leq K$, i.e. with $\|\nu_\lambda\| \leq |\lambda|$ and such that

$$w^{\nu_\lambda}(\zeta_j) = f(\lambda, \zeta_j), \quad j = 1, \ldots, n. \tag{3.5}$$

Theorem 1 follows (for E given by (3.1)).

(Note that we have no reason to assume that ν_λ depends holomorphically on λ. Whether it can be so chosen, for all $|\lambda|<1$, is equivalent to the Mañé-Sullivan problem.)

Next we observe that $\tilde{f}$ maps Δ_1 into the ball $\|\varphi\|<\frac{3}{2}$ in the ((n-3)-dimensional) Banach space $B(L, G)$ cf. § 2(H). By the Schwarz lemma (which is valid for vector-valued functions), $\tilde{f}$ takes the disc $|\lambda|<\frac{1}{3}$ into the ball $\|\varphi\|<\frac{1}{2}$. By § 2 (H) (a) there exists, for each $\lambda \in \Delta_{1/3}$, a harmonic Beltrami coefficient ν_λ in $\hat{\mathbf{C}}-E$, which depends holomorphically on $\tilde{\mathbf{f}}(\lambda) \in B(L, G)$, and hence on λ, and such that (3.5) holds. Since $w^{\nu_\lambda}(z)$ depends holomorphically on λ, the admissible map $\tilde{f}(\lambda, z) = w^{\nu_\lambda}(z)$, $|\lambda|<\frac{1}{3}$, $z \in \hat{\mathbf{C}}$, is the extension of $f|\Delta_{1/3} \times E$ the existence of which is asserted by Theorem 3.

(The uniqueness of this extension follows from statement (b) in § 2(H) and from Lemma II in § 5 below.)

§ 4. Proof of Theorems 1 and 2 and of the corollaries

Let $f: \Delta_1 \times E \to \hat{\mathbf{C}}$ be a normalized admissible map, with E infinite. Choose a sequence of finite sets E_j, $j=1,2,\ldots$ such that $\{0,1,\infty\} \subset E_j \subset E$ for all j and $E_1 \cup E_2 \cup \ldots$ is dense in E. For a fixed $\lambda \in \Delta$, denote by F_j a K-quasiconformal self-map of $\hat{\mathbf{C}}$ such that $F_j | E_j = f(\lambda, \cdot) | E_j$, K being given by (1.1). Such F_j exist, since Theorem 1 holds for finite E. Since all F_j fix $0, 1, \infty$ and are K-quasiconformal, a subsequence converges uniformly (in the spherical metric) to a K-quasiconformal homeomorphism $F: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ with $F = f(\lambda, \cdot)$ on $\bigcup E_j$.

Had we assumed $f(\lambda, \cdot)$ to be continuous, we could have concluded that $F(z) = f(\lambda, z)$ for $z \in E$, but we made no such assumption. However, let c be a point in E. Replacing E_j by $E_j \cup \{c\}$ and repeating the previous construction we obtain a K-quasiconformal self-map F' of $\mathbf{C}$ which coincides with $f(\lambda, \cdot)$ on $\bigcup E_j \cup \{c\}$. But F and F' are continuous everywhere and coincide on $\bigcup E_j$, hence on E, hence $F(c) = F'(c) = f(\lambda, c)$. Since c is arbitrary, $F | E = f(\lambda, \cdot)$. Theorem 1 is proved.

Remark. Theorem 1 with a weaker estimate than (1.1) for the dilatation of F_λ could be derived from the part of Theorem 3 proved in § 3. We omit the details.

Corollary 1 now follows by observing that, if f is an admissible map on $\Delta_1 \times \mathbf{C}$, then $f(\lambda, \cdot)$ has an extension which is a quasiconformal homeomorphism of $\hat{\mathbf{C}}$ onto itself. But the only possible extension of $f(\lambda, \cdot)$ is $f(\lambda, \cdot)$, and so $f(\lambda, \cdot)$ is a quasiconformal homeomorphism of $\hat{\mathbf{C}}$ onto itself.

For the second corollary, let f be a normalized admissible map on $\Delta_1 \times E$. Then for each λ with $|\lambda| \leq r < 1$, the map $f(\lambda, \cdot)$ has a K-quasiconformal extension with $K \leq (1+r)/(1-r)$. Since (cf. § 2 (B)) this extension has a Hölder modulus of continuity depending only on K (and hence only on r), so does $f(\lambda, \cdot)$. Thus there are constants A and α, depending only on r such that

$$\delta[f(\lambda, z), f(\lambda, z')] \leq A \delta(z, z')^{\alpha}$$

for all $|\lambda| \leq r$ and all $z, z' \in E$. For a fixed $z' \in E$ ($z' \neq 0, 1, \infty$) the map $f(\cdot, z')$ is a holomorphic function on Δ, which omits the values 0 and 1. By Schottky's theorem (cf. for instance [8], p. 261) there is a constant B depending only on r so that

$$\delta[f(\lambda, z'), f(\lambda', z')] \leq B |\lambda - \lambda'|$$

for $|\lambda| \leq r$. Corollary 2 now follows by the triangle inequality.

To establish Corollary 3, we assume that f_n is a normalized admissible map on $\Delta_1 \times E_n$. It follows from the uniform equicontinuity expressed in Corollary 2 that a subsequence $\{f_{n_k}\}$ converges uniformly on each $\Delta_r \times E_n$, $r<1$, to a map $g: \Delta_1 \times E \to \hat{\mathbf{C}}$. Then for each $z \in E$ the function $g(\cdot, z)$ is holomorphic. To see that $g(\lambda, \cdot)$ is injective, we use Theorem 1 to find an extension F_k of $f_{n_k}(\lambda, \cdot)$ which is a normalized K-quasiconformal homeomorphism of $\hat{\mathbf{C}}$ onto $\hat{\mathbf{C}}$ with $K \leq (1+|\lambda|)/(1-|\lambda|)$. Since the normalized K-quasiconformal homeomorphisms form a normal family, there is a subsequence which converges to a K-quasiconformal homeomorphism F_λ of $\hat{\mathbf{C}}$ onto $\hat{\mathbf{C}}$. Since F_λ is an extension of $g(\lambda, \cdot)$, we must have $g(\lambda, \cdot)$ injective. Thus g is admissible on $\Delta_1 \times E$. Consequently, it is uniformly continuous on $\Delta_r \times E$ for each $r<1$. From this it follows that g has a continuous extension f to $\Delta_1 \times \hat{E}$ such that $f(\cdot, z)$ is holomorphic for each $z \in \hat{E}$. For each $\lambda \in \Delta_1$ the map $f(\lambda, \cdot)$ is the restriction to $\hat{E}$ of the homeomorphism F_λ. Thus f is admissible on $\Delta_1 \times \hat{E}$, establishing Corollary 3.

We proceed to prove Theorem 2 assuming that E has a non-empty interior ω. The first assertion follows from Theorem 1 (as in the proof of Corollary 1). We now establish the holomorphic dependence on μ_λ on λ.

Since $L_\infty(\omega)$ is the dual of $L_1(\omega)$, it suffices to show that, for every $\alpha \in L_1(\omega)$,

$$\Psi(\lambda) = \iint_\omega \alpha(z)\mu_\lambda(z)\,dx\,dy$$

is holomorphic in Δ_1. A standard argument shows that one may assume α to be of compact support in ω. In this case there is an $\varepsilon > 0$ such that for $z \in \omega, \alpha(z) \neq 0$ and $0<h<\varepsilon$ the point $z+h$ and $z+ih$ lie in ω. Since quasiconformal maps are a.e. differentiable,

$$\begin{aligned}\Psi(\lambda) &= \iint_\omega \alpha(z) \frac{f_x(\lambda, z)+if_y(\lambda, z)}{f_x(\lambda, z)-if_y(\lambda, z)}\,dx\,dy \\ &= \iint_\omega \alpha(z) \lim_{h\downarrow 0} \frac{1+i\sigma_\lambda(z, h)}{1-i\sigma_\lambda(z, h))}\,dx\,dy\end{aligned}$$

where

$$\sigma_\lambda(z, h) = \frac{f(\lambda, z+ih)-f(\lambda, z)}{f(\lambda, z+h)-f(\lambda, z)}.$$

For fixed z ($\neq 0, 1, \infty$) and h, σ_λ is a holomorphic function of $\lambda \in \Delta_1$ which never equals 0 or 1 and equals i for $\lambda=0$. One concludes easily, by Schottky's theorem, that there is a number $r, 0<r<1$, such that for $|\lambda|<r, |\sigma_\lambda(z, h)-i| \leq 1/2$, and therefore

$$\left|\frac{1+i\sigma_\lambda(z, h)}{1-i\sigma_\lambda(z, h)}\right| \leq 9.$$

It follows, by the theorem on dominated convergence, that for $|\lambda|<r$ the sequence of holomorphic functions of λ

$$\Psi_n(\lambda) = \int\int_\omega \alpha(z)\frac{1+i\sigma_\lambda(z,1/n)}{1-i\sigma_\lambda(z,1/n)}\,dx\,dy$$

converges boundedly to $\Psi(\lambda)$ as $n\to\infty$. Thus $\Psi(\lambda)$ is holomorphic in λ for $|\lambda|<r$ and so is $\mu_\lambda \in L_\infty(\omega)$.

Now let λ_0 be any point in Δ_1 and set $s=1-|\lambda_0|$, $E_0=f(\lambda_0,E)$, $\omega_0=f(\lambda_0,\omega)$ and

$$g(\tau,\zeta)=f(\lambda_0+s\tau,z) \quad \text{where } \zeta=f(\lambda_0,z).$$

Then ω_0 is the interior of E_0 (by Theorem 1) and $g\colon \Delta_1\times E_0\to\hat{\mathbf{C}}$ is admissible. By what was proved above, the Beltrami coefficient of $g(\tau,\cdot)|\omega_0$, which we shall denote by ν_τ, is a holomorphic function of τ for $|\tau|<r$, with values in $L_\infty(\omega_0)$.

Let μ_{λ_0} denote the Beltrami coefficient of $f(\lambda_0,\cdot)|\omega$ and μ_λ, as before, that of $f(\lambda,\cdot)|\omega$. Since

$$f(\lambda,\cdot)|\omega = (g(\tau,\cdot)|\omega_0)\circ f(\lambda_0,\cdot)|\omega, \quad \tau=(\lambda-\lambda_0)/s$$

we obtain

$$\mu_\lambda = \frac{\hat{\nu}_\tau+\mu_{\lambda_0}}{1+\bar{\mu}_{\lambda_0}\hat{\nu}_\tau}$$

where

$$\hat{\nu}_\tau(z) = \nu_\tau(w(z))\frac{|w_z(z)|^2}{w_z(z)^2}, \quad w=w^{\mu_{\lambda_0}}.$$

Since $\hat{\nu}_\tau\in L_\infty(\omega)$ is a holomorphic function of $\nu_\tau\in L_\infty(\omega_0)$ and ν_τ a holomorphic function of $t\in\Delta_r$, the element $\mu_\lambda\in L_\infty(\omega)$ depends holomorphically on λ for $|\lambda-\lambda_0|<sr$. This completes the proof.

§ 5. Proof of Theorem 3

Let $f\colon \Delta_1\times E\to\hat{\mathbf{C}}$ be a given admissible map which we may, and do, assume to be normalized. Let $E_1, E_2, \ldots$ be a sequence of finite sets such that

$$\{0,1,\infty\}\subset E_1\subset E_2\subset\ldots \tag{5.1}$$

and

$$E_1 \cup E_2 \cup \dots \quad \text{is dense in } E \tag{5.2}$$

Let f_j denote the extension of the admissible map $f|\Delta_{1/3}\times E_j$ to $\Delta_{1/3}\times\hat{\mathbf{C}}$ constructed in §3.

By Corollary 2 we may assume (selecting if need be a subsequence) that $\{f_j\}$ converges to an admissible map $\hat{f}$ of $\Delta_{1/3}\times\mathbf{C}$. Since

$$\hat{f}|\Delta_{1/3}\times(E_1 \cup E_2 \cup \dots) = f|\Delta_{1/3}\times(E_1 \cup E_2 \cup \dots),$$

$\hat{f}$ is an admissible extension of $f|\Delta_{1/3}\times E$.

Let $\hat{E}$ denote the closure of E and let S denote, from now on, a component of $\hat{\mathbf{C}}-\hat{E}$. Also, let $\varrho_j(z)|dz|$ denote the Poincaré metric in $\hat{\mathbf{C}}-\hat{E}_j$, and $\varrho_S(z)|dz|$ the Poincaré metric on S. We claim that

$$\lim_{j\to\infty} \varrho_j(z) = \varrho_S(z), \quad z \in S \tag{5.3}$$

(uniformly on compact subsets).

Indeed, by the monotonicity property of the Poincaré metric, cf. §2(D),

$$\varrho_j|S \leqslant \varrho_{j+1}|S \leqslant \varrho_S$$

so that there is a limit

$$\lim_{j\to\infty} \varrho_j(z) = \varrho_\infty(z) \leqslant \varrho_S(z), \quad z \in \omega. \tag{5.4}$$

Since each ϱ_j satisfies the partial differential equation

$$\frac{\partial^2 \log \varrho}{\partial x^2} + \frac{\partial^2 \log \varrho}{\partial y^2} = \varrho^2$$

(expressing the fact that the Gauss curvature of the Poincaré metric is (-1)), standard "elliptic" estimates show that (5.4) holds uniformly on compact subsets of S and that the second partials also converge. Hence ϱ_∞ satisfies the same equation, i.e. the metric $\varrho_\infty(z)|dz|$ has Gaussian curvature (-1).

In order to show that

$$\varrho_\infty = \varrho_S \tag{5.5}$$

it suffices to show that the ϱ_∞ metric is complete, i.e. that for any rectifiable curve C in S, leading to a boundary point ξ of S, we have

$$\int_C \varrho_\infty(z)|dz| = +\infty. \tag{5.6}$$

If $\hat{\zeta} \neq 0, 1, \infty$, there is a sequence $\{\zeta_i\}$, $\zeta_i \in E_i$, with

$$\lim_{i\to\infty} \zeta_i = \hat{\zeta}.$$

Let $\tau(z, \zeta)|dz|$ be the Poincaré metric in $\hat{\mathbf{C}} - \{0, 1, \infty, \zeta\}$. Then

$$\int_C \tau(z, \hat{\zeta})|dz| = +\infty$$

and also

$$\lim_{i\to\infty} \int_C \tau(z, \zeta_i)|dz| = +\infty$$

since $\tau(z, \zeta)$ depends continuously on (z, ζ). By monotonicity of the Poincaré metric, $\tau(z, \zeta_i) \leq \varrho_j(z)$ for j sufficiently large, so that $\varrho_\infty(z) \geq \tau(z, \zeta_i)$ and (5.6) follows. The proof of (5.6) for the cases $\hat{\zeta} = 0, 1, \infty$ is left to the reader. Relation (5.3) is established.

Now we can show that the extension $\hat{f}$ has the characteristic property asserted by Theorem 3, i.e., that *the Beltrami coefficient of $\hat{f}|S$ is harmonic* (for every component S of $\hat{\mathbf{C}} - \hat{E}$).

Indeed, by the construction in § 3 the Beltrami coefficient $\mu_j(\lambda, z)$ of $f_j(\lambda, z)$ is harmonic in $\hat{\mathbf{C}} - E_j$ and depends holomorphically on λ, i.e.,

$$\mu_j(\lambda, z) = \varrho_j(z)^{-2}\, \overline{\psi_j(\lambda, z)}$$

where $\psi_j(\lambda, z)$ is holomorphic in $z \in \hat{\mathbf{C}} - E_j$ and antiholomorphic in $\lambda \in \Delta_{1/3}$. Noting (5.3) and selecting if need be a subsequence we may assume that

$$\lim_{j\to\infty} \mu_j(\lambda, z) = \varrho_S(z)^{-2}\, \overline{\psi(\lambda, z)} \quad \text{for } z \in S,$$

uniformly on compact subsets of $\Delta_{1/3} \times S$. Hence $\varrho_S(z)^{-2}\, \overline{\psi(\lambda, z)}$ is the restriction of the Beltrami coefficient of the map $z \mapsto \hat{f}(\lambda, z)$ to S, and $\psi(\lambda, z)$ is antiholomorphic in λ, holomorphic in z. The existence part of Theorem 3 is proved.

LEMMA I. *Let W be a quasiconformal self-map of U and Γ a curve in U which converges to a point $x_0 \in \hat{R}$ in a Stolz sector. Then $W(\Gamma)$ converges to $W(x_0)$ in a Stolz sector.*

This is known and follows from the results by Agard and Gehring [1], as observed by the referee. We give a proof for the sake of completeness.

We assume that x_0 and $W(x_0)$ lie in R and leave the cases $x_0=\infty$ to the reader. Let Γ be defined by the continuous function $t\mapsto x(t)+iy(t)\in U$, $\tau<t<1$, with $x(t)\to x_0$, $y(t)\to 0$ for $t\to 1$. The hypothesis of Lemma I means that there is an $m>0$ and $\varepsilon>0$ such that

$$|y(t)|\geq m|x(t)-x_0| \quad \text{for } 1-\varepsilon<t<1.$$

To prove the assertion it suffices to show that if Γ_+ and Γ_- denote the lines $y=m(x-x_0)$ and $y=-m(x-x_0)$ in U, then $W(\Gamma_+)$ and $W(\Gamma_-)$ converge to $W(x_0)$ in a Stolz sector. It will suffice to treat $W(\Gamma_+)$.

Observe that W may be extended to a quasiconformal self-map of $\hat{\mathbf{C}}$ by setting $W(\bar{z})=\overline{W(z)}$. Let R_+ and R_- denote the real rays $x\geq x_0$ and $x\leq x_0$ respectively. The Jordan curves $\Gamma_+\cup R_+\cup\{\infty\}$ and $\Gamma_+\cup R_-\cup\{\infty\}$ are both quasicircles and so are their W-images $W(\Gamma_+)\cup W(R_+)\cup\{\infty\}$ and $W(\Gamma_+)\cup W(R_-)\cup\{\infty\}$; note that $W(R_+)$ and $W(R_-)$ are the real rays $x\geq W(x_0)$ and $x\leq W(x_0)$. For $\xi>0$, set

$$c=W(x_0+\xi+im\xi), \quad b=W(x_0), \quad a=\operatorname{Re} c$$

For ξ small enough the Ahlfors condition (cf. § 2 (C)) yields

$$|\operatorname{Re} c-W(x_0)|\leq M|\operatorname{Im} c|$$

(provided $\operatorname{Re} c\neq W(x_0)$, but if $\operatorname{Re} c=W(x_0)$ the above inequality is trivial). Hence the curve $W(\Gamma_+)$ converges to $W(x_0)$ in the Stolz sector

$$|y|\geq\frac{1}{M}|x-W(x_0)|.$$

LEMMA II. *Let $S\subset\hat{\mathbf{C}}$ be a domain whose (set theoretical) boundary ∂S contains at least 3 points. Let I be the interval $-A\leq t\leq A$. Let*

$$w: I\times(S\cup\partial S)\to S\cup\partial S$$

be a continuous map such that

(i) *$w(0,z)=z$ for $z\in S\cup\partial S$,*

(ii) *$w(t,z)=z$ for $t\in I, z\in\partial S$,*

(iii) *for $t\in I, w(t,\cdot)$ is a topological self-map of $S\cup\partial S$, which is*

(iv) *K-quasiconformal on S for some fixed K.*

Then, for $t\in I$, the map $w(t,\cdot)|S$ is equivalent to the identity in the sense of Teichmüller space theory (cf. § 2 (F)).

Proof. Let U be the upper half plane and $\pi: U \to S$ a holomorphic universal covering with covering group G. For every $t \in I$, let $z \to W(t, z)$ be a topological self-map of U such that

$$\pi \circ W(t, \cdot) = w(t, \cdot) \circ \pi \tag{5.7}$$

and the point $W(t, i)$ is a continuous function of t, with $W(0, i) = i$. Then $W(0, \cdot) = \mathrm{id}$, the map $W(t, \cdot)$ is K-quasiconformal and, by known continuity properties of quasiconformal maps, it extends to a continuous self-map of $U \cup \hat{\mathbf{R}}$ (in the spherical metric); we denote this extension by the same letter W.

Now $W(t, z)$ depends continuously on $(t, z) \in I \times U$. The maps $W(t, \cdot)$ have a modulus of continuity (in the spherical metric) depending only on the number K and the compact set $W(I, i)$. We conclude that $W(t, z)$ is continuous in t also for $z \in \hat{\mathbf{R}}$.

To prove the lemma we must show that

$$W(t, x) = x \quad \text{for } t \in I, x \in \mathbf{R} \tag{5.8}$$

Assume first that the group G is of the first kind, i.e., that the closure Λ of the set of attracting fixed points of elements of G coincides with $\hat{\mathbf{R}}$. From (5.7) we conclude that for $g \in G$

$$g_t = W(t, \cdot) \circ g \circ W(t, \cdot)^{-1} \in G.$$

Clearly, g_t depends continuously on t; since G is discrete, $g_t = g_0$. But $g_0 = g$, so that $W(t, \cdot)$ commutes with g. Hence $W(t, \cdot)$ fixes the attracting fixed point of every $g \in G$. Since G is of the first kind, (5.8) follows.

Consider next the case when G is of the second kind, i.e., not of the first kind (this includes the case when S is simply connected, π is bijective and $G = 1$). Now $\hat{\mathbf{R}} - \Lambda$ is open and dense in $\hat{\mathbf{R}}$. If x_0 is a (finite) point in $\hat{\mathbf{R}} - \Lambda$, there is an $\varepsilon > 0$ such that in the intersection of the disc $|z - x_0| < \varepsilon$ with U the function $\pi(z)$ is injective. Hence, in the intersection of U with a disc $|z - x_0| < \varepsilon' < \varepsilon$, the function $\pi(z)$ is the quotient of two bounded holomorphic functions. This implies, in view of the classical theorem by Fatou and by F. and M. Riesz, that there is a subset $\theta \subset \hat{\mathbf{R}} - \Lambda$ of full measure such that at every $x \in \theta$ the function $\pi(z)$ has a sectorial limit $\pi(x)$, and $\pi(x)$ is not constant on any subset of θ of positive measure.

We claim that the map $W(t, \cdot)$ fixes θ, for each $t \in I$, and that

$$\pi(W(t, x)) = \pi(x) \quad \text{if } x \in \theta. \tag{5.9}$$

Indeed, let α be a curve in U which converges to $x \in \theta$ in a Stolz sector. Then the curve $W(t, \alpha)$ in U converges to $W(t, x)$ in a Stolz sector, by Lemma I, and, by the relation (5.7), $\pi \circ W(t, \alpha) = w(t, \pi(\alpha))$. By continuity of $w(t, \cdot)$ on the closure on S we obtain that $W(t, x) \in \theta$ and

$$\pi(W(t, x)) = w(t, \pi(x));$$

since $w(t, \cdot)$ fixes every point on ∂S, and $\pi(x) \in \partial S$, (5.9) follows.

Now $W(t, x)$ is, for x fixed, a continuous function of t which equals to x for $t=0$. Unless $W(t, x)=x$ for $t \in I$, the set $W(I, x) \in \hat{\mathbf{R}}$ would contain an interval I_0 of positive length. By (5.9) the function $\pi(x)$ would be constant on the intersection $I_0 \cap \theta$. Since $I_0 \cap \theta$ could not be a null-set this is impossible. Hence $W(t, x)=x$ for $x \in \theta$ and, since θ is dense in $\hat{\mathbf{R}}$, for all x. Relation (5.8) is proved and so is Lemma II.

We return to the proof of Theorem 3 and proceed to show that if $\hat{f}_1$ and $\hat{f}_2$ are two admissible extensions of $f|\Delta_{1/3} \times E$ to $\Delta_{1/3} \times \hat{\mathbf{C}}$, both having harmonic Beltrami coefficients in each component S of $\hat{\mathbf{C}} - \hat{E}$, then

$$\hat{f}_1 = \hat{f}_2. \tag{5.10}$$

We observe first that

$$\hat{f}_1(\lambda, S) = \hat{f}_2(\lambda, S) \tag{5.11}$$

for all $\lambda \in \Delta_{1/3}$.

Indeed, noting the continuity properties of admissible maps stated in Corollary 2, as well as the fact that $\hat{f}_1(0, \cdot) = \hat{f}_2(0, \cdot) = \mathrm{id}$, we conclude first that (5.11) holds for sufficiently small $|\lambda|$. The same argument shows that the set Θ of those λ for which (5.11) holds is open. But if (5.11) is false, for some $\lambda = \lambda_1 \in \Delta_{1/3}$, then $\hat{f}_1(\lambda_1, S) = \hat{f}_2(\lambda_1, S_1)$ where S_1 is a component of $\hat{\mathbf{C}} - \hat{E}$ distinct from S. Hence the set $\Delta_{1/3} - \Theta$ is also open. Therefore, $\Theta = \Delta_{1/3}$. Q.E.D.

Now let $\nu_j(\lambda, z)$ be the Beltrami coefficients of $\hat{f}_j(\lambda, z), j=1,2$. By Theorem 2, $\nu_j(\lambda, \cdot)$ depends holomorphically on $\lambda \in \Delta_{1/3}$. In particular, $|\nu_j(\lambda, z)| \leq k = k(\varepsilon) < 1$ if $|\lambda| < \frac{1}{3} - \varepsilon$, for every sufficiently small $\varepsilon > 0$. We may assume that f is normalized (cf. § 1); in this case so are the maps $\hat{f}_j$ and

$$\hat{f}_j(\lambda, z) = w^{\nu_j}, \quad \nu_j = \nu_j(\lambda, \cdot), \quad j = 1, 2.$$

Set

$$W(\lambda, \cdot) = \hat{f}_2(\lambda, \cdot)^{-1} \circ \hat{f}_1(\lambda, \cdot).$$

This function is certainly not holomorphic in λ but is easily seen to be continuous in that variable. In every component S of $\mathbf{C}-\hat{E}$ and for $A=\frac{1}{3}-\varepsilon, \varepsilon>0$ and small, and for every real α, the function

$$W(te^{i\alpha}, z), \quad -A \leqslant t \leqslant A, z \in S \cup \partial S$$

satisfies the hypotheses and hence the conclusion of Lemma II. Therefore $\hat{f}_1(\lambda, \cdot)|S$ and $\hat{f}_2(\lambda, \cdot)|S$ are equivalent in the sense of Teichmüller space theory. But by hypotheses $\hat{f}_1(\lambda, \cdot)|S$ and $\hat{f}_2(\lambda, \cdot)|S$ have harmonic Beltrami coefficients. Hence, by § 2 (H) (b), these coefficients coincide. Therefore the map

$$\hat{f}_2(\lambda, \cdot)^{-1} \circ \hat{f}_1(\lambda, \cdot)|S$$

is holomorphic. Since it fixes every point of ∂S it is the identity. Thus (5.11) holds on S and therefore on $\hat{\mathbf{C}}-\hat{E}$. Since this relation holds on $\hat{E}$ by hypothesis, it is valid everywhere. Theorem 3 is established.

§ 6. Proofs of Propositions 1, 2, and 3

We begin with the proof of Proposition 1. Assume, therefore, that for each finite set E_0 and each $y \notin E_0$ and every admissible f on $\Delta_1 \times E_0$ there is an admissible extension to $\Delta_1 \times (E_0 \cup \{y\})$. Let E be an infinite set (containing 0, 1, and ∞), $y \notin E$, and f an admissible map on $\Delta_1 \times E$. We proceed to show that f can be extended to an admissible map on $\Delta_1 \times (E \cup \{y\})$. It suffices to consider the case when f is normalized. Let $\{E_n\}$ be an increasing sequence of finite sets whose union D is dense in E. By assumption $f|\Delta_1 \times E_n$ has an admissible extension f_n to $\Delta_1 \times (E_n \cup \{y\})$. By Corollary 3 of Theorem 1 there is an admissible map $\tilde{f}$ on $\Delta_1 \times (E \cup \{y\})$ and a subsequence of $\{f_n\}$ which converges to $\tilde{f}$ pointwise on $\Delta_1 \times D$. For $m \geqslant n$ and $z \in F_n$ we have

$$f_m(\lambda, z) = f(\lambda, z),$$

and so for $z \in D$

$$\tilde{f}(\lambda, z) = f(\lambda, z).$$

Since $\tilde{f}$ and f are continuous, we must have $\tilde{f}(\lambda, z) = f(\lambda, z)$ for all $z \in E$, whence $\tilde{f}$ is an admissible extension of f to $\Delta_1 \times (E \cup \{y\})$.

We now suppose f is an admissible map on $\Delta_1 \times E$ and choose a countable set $D = \{y_n\}, y_n \notin E$, which is dense in $\hat{\mathbf{C}}-E$. Set $E_0 = E$ and $E_n = E \cup \{y_1, \ldots, y_n\}$. By the preceding paragraph we may define admissible maps f_n on $\Delta_1 \times E_n$ recursively so that $f_0 = f$ and f_n is an extension of f_{n-1} from $\Delta_1 \times E_{n-1}$ to $\Delta_1 \times E_n$. Since the closure of $\bigcup E_n$ is $\hat{\mathbf{C}}$, Corollary 3 asserts that there is an admissible map $\hat{f}$ on $\Delta_1 \times \mathbf{C}$ and a subsequence of $\{f_n\}$ which converges to $\hat{f}$ pointwise on $E_0 = E$. But the restriction of f_n to $\Delta_1 \times E$ is f and hence the restriction of $\hat{f}$ to $\Delta_1 \times E$ is f, i.e. $\hat{f}$ is an admissible extension of f to $\Delta_1 \times \hat{\mathbf{C}}$. This establishes Proposition 1.

We now construct some examples.

Example 1. Let E be the unit circumference $|z|=1$, and let $f: \Delta_1 \times E \to \hat{\mathbf{C}}$ be given by

$$f(\lambda, z) = z + \lambda z^{-1}. \tag{6.1}$$

Then f maps E onto an ellipse with semi-axes $1+|\lambda|$ and $1-|\lambda|$. The map $\hat{f}$ defined by

$$\hat{f}(\lambda, z) = z + \lambda \bar{z}$$

is an admissible extension of f to $\Delta_1 \times (\Delta_1 \cup E)$. For each $\lambda \in \Delta_1$ the map $\hat{f}(\lambda, \cdot)$ is K-quasiconformal with $K = (1+|\lambda|)/(1-|\lambda|)$. Teichmüller's uniqueness theorem implies that $\hat{f}(\lambda, \cdot)$ is the only $(1+|\lambda|)/(1-|\lambda|)$ quasiconformal extension of $f(\lambda, \cdot)$. But the first assertion of Theorem 2 is that any admissible extension $\tilde{f}$ of f to $\Delta_1 \times (\Delta_1 \cup E)$ must have the property that $\tilde{f}(\lambda, \cdot)$ is $(1+|\lambda|)/(1-|\lambda|)$ quasiconformal. Therefore $\tilde{f}(\lambda, \cdot) = \hat{f}(\lambda, \cdot)$, and so $\hat{f}$ is the only admissible extension of f to $\Delta_1 \times (\Delta_1 \cup E)$.

For each real $\alpha, 0 \leqslant \alpha \leqslant 1$, the map $\tilde{f}_\alpha$ defined by

$$\tilde{f}(\lambda, z) = \hat{f}(\lambda, z) \quad \text{for } |z| \leqslant 1$$

$$\tilde{f}_\alpha(\lambda, z) = z + \lambda(\alpha z^{-1} + (1-\alpha)\bar{z}) \quad \text{for } |z| > 1$$

is an admissible extension of f to $\Delta \times \hat{\mathbf{C}}$.

On the other hand, if E is the set $|z| \geqslant 1$ and the map $f: \Delta_1 \cup E$ is defined by (6.1), then, by the reasoning above, f has a unique extension to $\Delta_1 \times \hat{\mathbf{C}}$.

This example establishes the assertion of Proposition 2.

Example 2. Let $E = \{0, 1, \infty, \zeta_1, \ldots, \zeta_n\}$ and let φ be a holomorphic function in $L_1(\hat{\mathbf{C}} - E)$, i.e. φ is a rational function regular on $\hat{\mathbf{C}} - E$, having at most simple poles at the points $0, 1, \zeta_1, \ldots, \zeta_n$, and vanishing to at least third order at ∞. Set

$$\mu_\lambda = \lambda |\varphi| / \varphi.$$

We define f on $\Delta_1 \times \hat{\mathbf{C}}$ by setting

$$\hat{f}(\lambda, z) = w^{\mu_\lambda}(z),$$

where w^{μ_λ} has the usual meaning, cf. § 2 (B). Then $\hat{f}$ is an admissible map of $\Delta_1 \times \hat{\mathbf{C}}$. Set

$$f = \hat{f} | \Delta_1 \times E.$$

Thus $\hat{f}$ is an admissible extension of f. For each $\lambda \in \Delta_1$, Teichmüller's uniqueness theorem (cf. § 2 (I)) asserts that $\hat{f}(\lambda, \cdot)$ is the only $(1+|\lambda|)/(1-|\lambda|)$ quasiconformal extension of $f(\lambda, \cdot)$. Thus $\hat{f}(\lambda, \cdot)$ is the only admissible extension of $f(\lambda, \cdot)$ by Theorem 2. This establishes once more the first assertion of Proposition 2.

It should be noted that now $\hat{f} | \Delta_{1/3} \times \hat{\mathbf{C}}$ is not the canonical extension of f described by Theorem 3. Hence the canonical extension of f to $\Delta_{1/3} \times \hat{\mathbf{C}}$ can not be extended to an admissible map of $\Delta_1 \times \hat{\mathbf{C}}$ to $\hat{\mathbf{C}}$.

The two preceding examples depend on Theorem 1 to obtain strong restrictions on the possible admissible extensions. The following curious example is of a somewhat different nature.

Example 3. Let E be the unit circumference $|z|=1$ and g the function on $\Delta_1 \times E$ defined by

$$g(\lambda, z) = z + \lambda^2 z^{-1}$$

Thus $g(\lambda, z) = f(\lambda^2, z)$, where f is the map used in Example 1. For each $\lambda \in \Delta_1$ the function $g(\lambda, \cdot)$ maps E onto an ellipse whose major axis is the segment from

$$-\frac{\lambda}{|\lambda|}(1+|\lambda|^2)$$

to

$$\frac{\lambda}{|\lambda|}(1+|\lambda|^2),$$

and whose minor axis has length $2(1-|\lambda|)^2$. If $z_0 \in \Delta_1$ and $\hat{g}$ is any admissible extension of g to $\Delta_1 \times (E \cup \{z_0\})$, then

$$\operatorname{Im} \frac{\hat{g}(\lambda, z_0)}{\lambda} \to 0$$

as $|\lambda| \to 1$. Thus

$$\frac{\hat{g}(\lambda, z_0)}{\lambda} = A\lambda^{-1} + \bar{A}\lambda + B,$$

where B is real. Consequently,

$$\hat{g}(\lambda, z_0) = A+B\lambda+\bar{A}\lambda^2.$$

Since $\hat{g}(0, z_0) = z_0$, we have

$$\hat{g}(\lambda, z_0) = z_0+B\lambda+\lambda^2\bar{z}_0.$$

From the fact that $|g(\lambda, z_0)|\leq 2$ for $\lambda \in \Delta_1$, we see that $|B|\leq 2(1-|z_0|)$.

If $\hat{g}$ is an admissible extension of g to $\Delta_1\times(\Delta_1 \cup E)$, then $\hat{g}$ must have the form

$$\hat{g}(\lambda, z) = z+B(z)\lambda+\lambda^2\bar{z}. \tag{6.1}$$

From the continuity and quasiconformality of $\hat{g}(\lambda, \cdot)$ it follows that B is continuous on $\Delta_1 \cup E$ and $B(z)=0$ for $|z|=1$. Differentiating (6.1), we obtain

$$\begin{aligned} \partial\hat{g}/\partial z &= 1+\beta_z\lambda, \\ \partial\hat{g}/\partial\bar{z} &= \lambda B_{\bar{z}}+\lambda_2. \end{aligned} \tag{6.2}$$

Since B is real, $B_z=\overline{B_{\bar{z}}}$, and the Beltrami coefficient μ of $\hat{g}$ is given by

$$\mu = \lambda\frac{\lambda+B_{\bar{z}}}{1+\lambda\overline{B_{\bar{z}}}}. \tag{6.3}$$

Because $|\mu|\leq|\lambda|$, we must have $|B_{\bar{z}}|\leq 1$.

Conversely, if $\hat{g}$ has the form (6.1) with B real, $B(z)=0$ for $|z|=1$, and $|B_{\bar{z}}|\leq 1$, then (6.3) shows that $|\mu|\leq|\lambda|<1$. This together with (6.2) shows that $\hat{g}(\lambda, 0)$ is a local homeomorphism. Since $\hat{g}$ is the identity on $|z|=1$, it is a homeomorphism of $|z|\leq 1$ onto itself.

We conclude that a function $\hat{g}$ on $\Delta\times\Delta$ is an admissible extension of g if and only if it has the form (6.1) with $B(z)$ real, $B(z)=0$ for $|z|=1$ and $|B_{\bar{z}}|\leq 1$.

Observe that $\hat{g}$ is strongly restricted in its dependence on λ but only mildly in its dependence on z.

Now we prove Proposition 3, essentially following Earle and Kra [6]. Let $E=\{0, 1, \infty, \alpha\}$ and set $\varphi(z)=[z(z-1)(z-\alpha)]^{-1}$. For each $\zeta \in \Delta_1$ let μ_ζ be the Beltrami differential

$$\mu_\zeta = \zeta|\varphi(z)|/\varphi(z)$$

on $\mathbf{C}-E$. Define h on $\Delta_1\times\hat{\mathbf{C}}$ by

$$h(\zeta, z) = w^{\mu_\zeta}(z),$$

Then h is an admissible map on $\Delta_1 \times \hat{\mathbf{C}}$.

The map $\zeta \mapsto [w^{\mu_\zeta}]$ is a holomorphic map of Δ_1 into the Teichmüller space $T(\hat{\mathbf{C}}-E)$. Teichmüller's uniqueness theorem asserts it is injective, while Teichmüller's existence theorem asserts it is onto, since φ is (apart from a constant multiple) the only holomorphic function in $L_1(\mathbf{C}-E)$. Thus $T(\mathbf{C}-E)$ is biholomorphically equivalent to Δ_1, and $\zeta \mapsto [w^{\mu_\zeta}]$ is trivially a covering map. Thus by the Lemma of Section 3 the map $h(\cdot, a)$ is a covering map of $M_1 = \mathbf{C} - \{0, 1, \infty\}$.

Let f: $\Delta_1 \times E \to \hat{\mathbf{C}}$ by any normalized admissible map. Then the map $f(\cdot, a)$: $\Delta_1 \to M_1$ lifts to a holomorphic map $\mathbf{f}$: $\Lambda_1 \to \Lambda_1$ so that

$$h(\mathbf{f}(\lambda), a) = f(\lambda, a).$$

The map $\hat{f}$ defined by

$$\hat{f}(\lambda, z) = h(\mathbf{f}(\lambda), z)$$

is thus an extension of f to $\Delta_1 \times \hat{\mathbf{C}}$. Since h is admissible and $\mathbf{f}$ holomorphic, $\hat{f}$ is an admissible extension of f to $\Delta_1 \times \hat{\mathbf{C}}$. This establishes Proposition 3.

Using elliptic functions, we can give a reasonably explicit representation of a function $\hat{f}$ whose existence is asserted by Proposition 3: Let $P(\zeta) = P(\zeta, \omega)$ be the elliptic function with periods 1 and ω which has a double pole at $\zeta = 0$ and is normalized by $P(\frac{1}{2} + \frac{1}{2}\omega) = 0$ and $P(\frac{1}{2}\omega) = 1$. This function is related to the Weirestrass $\wp$ function by

$$P(\zeta) = \frac{\wp(\zeta) - e_3}{e_2 - e_3},$$

and satisfies the differential equation

$$4(P')^2 = (e_2 - e_3) P(P-1)(P-a),$$

where

$$a = a(\omega) = P(\tfrac{1}{2}, \omega).$$

The function a maps the region $0 \leq \operatorname{Re} \omega \leq 1, |\omega - \frac{1}{2}| \geq 1$ univalently onto the upper half-plane, with $0, 1$ and ∞ going into $0, 1$, and ∞, respectively. Thus a is the covering map of the upper half-plane onto $\mathbf{C} - \{0, 1\}$.

The function $P(\zeta)$ maps the triangle T_ω with vertices at $0, 1$, and ω onto $\mathbf{C}$ and is one-to-one in the interior and two-to-one on the edges. If we identify points on each edge which are symmetric about the midpoint of the edge, then T_ω becomes a tetrahedron with vertices corresponding to points congruent to $0, \frac{1}{2}, \frac{1}{2}\omega$ and $\frac{1}{2}+\frac{1}{2}\omega$. The function P maps this tetrahedron univalently onto $\mathbf{C}$ with the vertices going to $\infty, \alpha(\omega), 1,$ and 0.

Let $\zeta=\xi+i\eta$. Then the function

$$\zeta^* = \xi+\omega\eta$$

maps the tetrahedron T_i quasi-conformally and one-to-one onto the tetrahedron T_ω. The Beltrami coefficient of this map is

$$\mu = \frac{i-\omega}{i+\omega}.$$

Hence this map is the extremal quasiconformal map between T_i and T_ω taking corresponding vertices into corresponding vertices.

Thus the function

$$\Phi(\omega, \zeta) = P(\xi+\omega\eta, \omega)$$

is holomorphic with respect to ω for each $\zeta \in \tau_i$, and univalent in ζ for each ω in the upper half-plane.

Let f be a normalized admissible mapping on $\Delta_1 \times \{0, 1, \infty, \alpha\}$, where we denote $f(\lambda, \alpha)$ by $\theta(\lambda)$. Since the mapping $\alpha(\omega)$ is a covering mapping of the upper half-plane onto $\mathbf{C}-\{0,1\}$ and θ maps Δ_1 into $\mathbf{C}-\{0,1\}$, there is, by the monodromy lifting theorem, a holomorphic map $\omega=\psi(\lambda)$ from Δ_1 to the upper half plane such that

$$\theta(\lambda) = \alpha[\psi(\lambda)].$$

Set

$$\hat{f}(\lambda, z) = P(\xi+\psi(\lambda)\eta, \psi(\lambda)),$$

where ζ is chosen so that

$$P(\xi+\psi(0)\eta, \psi(0))\, z.$$

Then the univalence of $\hat{f}$ for a fixed λ follows from the fact that for a fixed ω the map P is univalent from T_ω to $\mathbf{C}$. The function $\hat{f}$ is clearly holomorphic in λ, and hence admissible on $\Delta_1 \times \mathbf{C}$. We also have $\hat{f}(0, z) \equiv z$, and $\hat{f}(\lambda, \theta(0)) = \theta(\lambda)$. Thus $\hat{f}$ is the desired admissible extension of f.

§ 7. Lifting problems in Teichmüller spaces

In the present section we give an interpretation of our results in terms of the possibility of lifting holomorphic maps of Δ_1 into the Teichmüller space $T_{0,n}$ of the n-punctured sphere. It will be convenient for our description to choose a suitable base point P_n in each $T_{0,n}$. Let $\{\zeta_n\}$ be a sequence of points in $\mathbf{C}$ with $\zeta_n \neq \zeta_m$ for $m \neq n$ and $\zeta_n \neq 0, 1, \infty$ and set $E_n = \{0, 1, \infty, \zeta_1, \dots, \zeta_{n-3}\}, n \geq 4$. Recall (§2(E)) that the Teichmüller space $T_{0,n} = T(\hat{\mathbf{C}} - E_n)$ can be realized as the set of equivalence classes $[w^\mu]$ of normalized quasiconformal maps of $\hat{\mathbf{C}}$ onto $\hat{\mathbf{C}}$, the equivalence being defined by $w^\mu \sim w^\nu$ if $w^\mu | E_n = w^\nu | E_n$ and $(w^\mu)^{-1} \circ w^\nu$ is homotopic to the identity in $\mathbf{C} - E_n$.

Since there is a one-to-one correspondence between the normalized quasiconformal maps w^μ and their Beltrami coefficients, we may also consider $T_{0,n}$ to be the unit ball β of Beltrami coefficients in $\hat{\mathbf{C}} - E$ module the equivalence $\mu \sim \nu$ if $w^\mu \sim w^\nu$. The unit ball β of Beltrami coefficients on $\hat{\mathbf{C}} - E_n$ is also the unit ball of Beltrami coefficients in $\hat{\mathbf{C}}$, the difference between $T_{0,n}$ and $T_{0,m}$ for $m > n$ being that the equivalence relation for $T_{0,m}$ is more restrictive than that for $T_{0,n}$. Thus we have a natural projection $\pi_{n,m}$ of $T_{0,m}$ onto $T_{0,n}$. We have also the natural projections $\pi_n: \beta \to T_{0,n}$ which takes each μ into $[w^\mu]$. All these projections are holomorphic, and we have $\pi_{n,m} = \pi_{n,k} \circ \pi_{k,m}$ for $n \leq k \leq m$, and $\pi_n = \pi_{n,m} \pi_m$ for $m \geq n$. If we choose as base point p_n in $T_{0,n}$ the point $p_n = \pi_n(0)$, then $\pi_{n,m}(p_m) = p_n$.

As we saw in §3, each admissible map $f: \Delta_r \times E_n \to \hat{\mathbf{C}}$ corresponds to a unique holomorphic map $\mathbf{f}: \Delta_r \to T_{0,n}$ with $\mathbf{f}(0) = p_n$, and conversely. If $\mathbf{f}: \Delta_r \to T_{0,n}$ with $\mathbf{f}(0) = p_n$ and $\mathbf{g}: \Delta_r \to T_{0,m}$ with $p_m = \mathbf{g}(0)$, then the admissible map corresponding to $\mathbf{g}$ will be an extension of the admissible map corresponding to $\mathbf{f}$ if and only if $\mathbf{f} = \Pi_{n,m} \circ \mathbf{g}$.

We say that a holomorphic map $\mathbf{f}: \Delta_1 \to T_{0,n}$ with $\mathbf{f} = p_n$ can be lifted to a map of Δ_1 into $T_{0,m}$ if there is a holomorphic map $\mathbf{g}: \Delta_1 \to T_{0,m}$ with $\mathbf{g}(0) = p_m$ and $\mathbf{f} = \pi_{n,m} \circ \mathbf{g}$. Since the sequence $\{\zeta_n\}$ can be chosen arbitrarily, we see that the hypothesis of Proposition 1 (the finite extension property) is equivalent to the statement that each holomorphic map $\Delta_1 \to T_{0,n}$ can be lifted to a map into $T_{0,m}$. This observation gives us the following proposition:

PROPOSITION 4. *The hypothesis of Proposition* 1 *is true if and only if for each* n *every holomorphic map* $\Delta_1 \to T_{0,n}$ *can be lifted to a holomorphic map of* Δ_1 *into* $T_{0,n+1}$.

This lifting problem for holomorphic maps of Δ_1 into $T_{0,n}$ is a difficult open problem. We note that lifting from $T_{0,n}$ to $T_{0,n+1}$ is not always possible for maps

$\varphi: D \rightarrow T_{0,n}$ where D is a domain in $\mathbf{C}^p$. Indeed, let $D=T_{0,n}$ and φ the identity map. Hubbard [9] has shown that there is no lift of φ into $T_{0,n+1}$.

We also note that Proposition 1 and Theorem 2 imply that, if for each n every holomorphic map for Δ_1 to $T_{0,n}$ can be lifted to $T_{0,n+1}$, then every holomorphic map from Δ_1 to $T_{0,n}$ can be lifted to a holomorphic map from Δ_1 to the ball β of the (relevant) Beltrami differentials.

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Received August 20, 1984
Received in revised form May 8, 1985