Conformally natural extension of homeomorphisms of the circle

by

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1. Conformal naturality

Let G be the group of all conformal automorphisms of $D=\{z\in\mathbf{C};\ |z|<1\}$, and G_+ the subgroup, of index two in G, of orientation preserving maps. The group G_+ consists of the transformations

$$z\mapsto\lambda\frac{z-a}{1-\bar{a}z}$$

with $|\lambda|=1$ and $|a|<1$. For each such a, the map

$$g_a\colon z\mapsto\frac{z-a}{1-\bar{a}z} \tag{1.1}$$

in G_+ takes a into 0 and 0 into $-a$.

The group G operates on D, on $S^1=\partial D$, on the set $\mathcal{P}(S^1)$ of probability measures on S^1, on the vector space $\mathcal{T}(D)$ of continuous vector fields on D, etc. Explicitly

$$g\cdot z=g(z)\quad \text{if } z\in D\cup S^1,$$
$$(g\cdot\mu)(A)=g_*\mu(A)=\mu(g^{-1}(A))\quad \text{if } \mu\in\mathcal{P}(S^1) \text{ and } A\subset S^1 \text{ is a Borel set},$$
$$(g\cdot v)(g(z))=g_*(v)(g(z))=v(z)\,g'(z)\quad \text{if } v\in\mathcal{T}(D),\ z\in D, \text{ and } g\in G_+,$$
$$(g\cdot v)(g(z))=g_*(v)(g(z))=\bar{v}(z)\,g'_{\bar{z}}(z)\quad \text{if } v\in\mathcal{T}(D),\ z\in D, \text{ and } g\in G\setminus G_+.$$

(We use the notations g'_z and $g'_{\bar{z}}$ for the complex derivatives of the function $g(z)$, and we

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write g' instead of g'_z if g is holomorphic.) The group $G\times G$ operates on the space $\mathscr{C}(\bar{D})$ of continuous maps of $\bar{D}$ into itself, or on $\mathscr{C}(S^1)$, by $(g, h)\cdot\varphi=g\circ\varphi\circ h^{-1}$.

If G operates on X and Y, a map $T: X\to Y$ is called G-equivariant, or *conformally natural*, if $T(g\cdot a)=g\cdot T(a)$ holds for $g\in G$ and $a\in X$. If $G\times G$ operates on X and Y, we say that $T: X\to Y$ is conformally natural if it is $G\times G$-equivariant.

Example. There is a unique conformally natural map from D to $\mathscr{P}(S^1)$. It is the map $z\mapsto\eta_z$, where η_z is the harmonic measure of z:

$$\eta_z(A)=\frac{1}{2\pi}\int_A\frac{1-|z|^2}{|z-\zeta|^2}|d\zeta|$$

if $A\subset S^1$ is a Borel set.

The purpose of this paper is to extend any homeomorphism φ of S^1 to a homeomorphism $\Phi=E(\varphi)$ of $\bar{D}$, in a conformally natural way. This extension will have the property that if φ admits a quasiconformal extension, then Φ is quasiconformal (but not with the best possible dilatation ratio). Moreover Φ depends continuously on φ. However the assignment $\varphi\mapsto\Phi$ is not compatible with composition: i.e., $E(\psi\circ\varphi)\neq E(\psi)\circ E(\varphi)$ in general.

The idea is the following: given φ, to each $z\in D$ we assign the measure $\varphi_*(\eta_z)$ on S^1. Then we define the conformal barycenter $w\in D$ of this measure and set $w=\Phi(z)$. Each of these steps is done in a conformally natural way. The last step is to show that Φ is a homeomorphism.

We develop the general properties of the extension operator $\varphi\mapsto\Phi$ in Sections 2, 3, and 4. After that we concentrate on the quasiconformal case. Our results in Sections 5 and 6 have applications to the theory of Teichmüller spaces, which we give in Section 7. In Sections 8 through 10 we compare the coefficient of quasiconformality K^* of Φ with

$$K(\varphi)=\inf\{K;\varphi \text{ has a } K\text{-quasiconformal extension to } \bar{D}\}.$$

Our results are rather precise when $K(\varphi)$ is close to one (see Corollary 2 to Proposition 5 in Section 9), but they leave something to be desired when $K(\varphi)$ is large.

In Section 11 we briefly discuss the higher dimensional case. Given a homeomorphism φ of S^{n-1} and a point x in B^n, we again define $\Phi(x)$ to be the conformal barycenter of the measure $\varphi_*(\eta_x)$. In general Φ is not a homeomorphism when $n\geqslant 3$, but Pekka Tukia has pointed out to us that Φ is a quasiconformal homeomorphism if φ is quasiconformal with sufficiently small dilatation. We prove that result in Section 11.

Finally, we want to thank Pekka Tukia for a number of helpful suggestions, especially for encouraging us to write Section 11 and to prove in Section 5 that if φ has a quasiconformal extension then in addition to being quasiconformal, Φ and Φ^{-1} are Lipschitz continuous with respect to the Poincaré metric.

2. The conformal barycenter

Our purpose in this section is to assign to every probability measure μ on S^1, with no atoms, a point $B(\mu)\in D$ so that the map $\mu\mapsto B(\mu)$ is conformally natural and satisfies

$$B(\mu)=0 \quad \text{if and only if} \quad \int_{S^1} \zeta d\mu(\zeta)=0. \tag{2.1}$$

There is a unique conformally natural way to assign to each probability measure μ on S^1 a vector field ξ_μ on D such that

$$\xi_\mu(0)=\int_{S^1} \zeta d\mu(\zeta). \tag{2.2}$$

Indeed, formula (2.2) is equivariant with respect to rotations and complex conjugation. For general w in D we must write

$$\xi_\mu(w)=\frac{1}{(g_w)'(w)}\,\xi_{(g_w)_*(\mu)}(0),$$

i.e.

$$\xi_\mu(w)=(1-|w|^2)\int_{S^1}\left(\frac{\zeta-w}{1-\bar{w}\zeta}\right)d\mu(\zeta), \tag{2.3}$$

and that will make the assignment $\mu\mapsto\xi_\mu$ conformally natural. (Here $g_w\colon D\to D$ is defined as in formula (1.1).) It is clear from (2.3) that the vector field ξ_μ is real-analytic.

PROPOSITION 1 and DEFINITION. *Suppose μ has no atoms. Then ξ_μ has a unique zero in D. We call it the conformal barycenter $B(\mu)$ of μ.*

Proof. We compute

$$\begin{aligned}\xi_\mu(w)&=(1-|w|^2)\int_{S^1}(\zeta-w)(1+\bar{w}\zeta)\,d\mu(\zeta)+o(w)\\&=\xi_\mu(0)-w+\bar{w}\int_{S^1}\zeta^2\,d\mu(\zeta)+o(w).\end{aligned}$$

The Jacobian of ξ_μ at $w=0$ is therefore

$$J\xi_\mu(0) = |(\xi_\mu)'_w(0)|^2 - |(\xi_\mu)'_{\bar{w}}(0)|^2$$
$$= 1 - \int\int_{S^1\times S^1} \zeta^2 \bar{z}^2 \, d\mu(\zeta)\times d\mu(z),$$

so

$$J\xi_\mu(0) = \frac{1}{2}\int\int_{S^1\times S^1} |z^2-\zeta^2|^2 \, d\mu(\zeta)\times d\mu(z) > 0. \tag{2.4}$$

If $\xi_\mu(0)=0$, we conclude that $w=0$ is an isolated singular point of index one. The conformal naturality implies that every zero of the vector field ξ_μ in D is an isolated singular point of index one. To complete the proof it therefore suffices to show that for $r\in]-1,1[$ close to 1 the vector field ξ_μ has no zero on the circle

$$C_r = \{w; |w| = r\}$$

and points inward.

LEMMA 1. Re $\xi_\mu(0)>0$ *if* $\mu(\widehat{[e^{-\pi i/4}, e^{+\pi i/4}]})\geq\frac{2}{3}$.

Proof. Re $\xi_\mu(0)=\displaystyle\int_{S^1} \mathrm{Re}\,(\zeta)\, d\mu(\zeta)\geq(-1)\cdot\frac{1}{3}+(\sqrt{2}/2)\cdot\frac{2}{3}>0.$ Q.E.D.

To complete the proof of Proposition 1, take $\alpha>0$ such that $\mu(J)\leq\frac{1}{3}$ for any arc $J\subset S^1$ of length $\leq\alpha$, and take $r_0<1$ such that the arc J_α of length α centered at 1 is seen from r_0 with angle $3\pi/2$ in Poincaré geometry (i.e., $g_{r_0}(J_\alpha)$ has length $3\pi/2$). If $|w|=r\geq r_0$, let g be the conformal map in G^+ that takes w to 0 and $-w/|w|$ to 1, and let $\nu=g_*(\mu)$. Then Re $\xi_\nu(0)>0$ by Lemma 1, so $\xi_\nu(0)$ points into $g(C_r)$, and the conformal naturality implies that $\xi_\mu(w)$ points into C_r. Q.E.D.

Remarks. (1) It follows from the definition that $B(\mu)$ depends in a conformally natural way on μ and satisfies (2.1).

(2) The result still holds if μ has atoms provided none of them has weight$\geq\frac{1}{2}$. (If no atom has weight$\geq\frac{1}{3}$ the proof is unchanged; otherwise modify it slightly.)

(3) If $\varphi: S^1\to S^1$ is a homeomorphism, then $B(\varphi_*(\eta_0))$ is the unique point $w\in D$ such that the homeomorphism $g_w\circ\varphi: S^1\to S^1$ has mean value zero. Indeed, if $\mu=\varphi_*(\eta_0)$ and $w\in D$, then

$$(1-|w|^2)^{-1}\xi_\mu(w) = \frac{1}{2\pi}\int_{S^1}\frac{\varphi(\zeta)-w}{1-\bar{w}\varphi(\zeta)}|d\zeta|$$

is the mean value of $g_w \circ \varphi$.

(4) There is a second proof of the uniqueness of $B(\mu)$. One can write

$$\xi_\mu(z) = \int_{S^1} \xi_\zeta(z)\, d\mu(\zeta)$$

where $\xi_\zeta = \xi_{\delta_\zeta}$ is the unit vector field pointing toward ζ. The field ξ_ζ is the gradient (in Poincaré geometry) of a function h_ζ whose level lines are the horocycles tangent to S^1 at ζ. (This function is defined up to a constant, and can be chosen so that $h_\zeta(0)=0$.) Thus ξ_μ is the gradient of

$$h_\mu: z \mapsto \int_{S^1} h_\zeta(z)\, d\mu(\zeta).$$

$B(\mu)$ is a critical point of h_μ, and the uniqueness of $B(\mu)$ can be proved by showing that the restriction of $-h_\mu$ to Poincaré geodesics is strictly convex. We chose a proof that relies on formula (2.4) because this formula will be used in Sections 3 and 10. Thurston has remarked that the function $-h_\mu$ can be interpreted as the average distance to S^1. In fact, if $d(z,w)$ is the Poincaré distance from z to w in D, then

$$\begin{aligned} -h_\zeta(z) &= -\tfrac{1}{2}\log\left(\frac{1-|z|^2}{|z-\zeta|^2}\right) \\ &= \lim_{r\to 1^-}[d(z, r\zeta)-d(0,r)]. \end{aligned}$$

3. Extending homeomorphisms of S^1

Given a homeomorphism $\varphi: S^1 \to S^1$, we define an extension $E(\varphi)=\Phi: \bar{D}\to\bar{D}$ by putting $\Phi(z)=\varphi(z)$ if $z \in S^1$ and

$$\Phi(z) = B(\varphi_*(\eta_z)) \quad \text{if } z \in D.$$

Clearly $\varphi \mapsto \Phi$ is conformally natural, i.e.

$$E(g\circ\varphi\circ h) = g\circ E(\varphi)\circ h \quad \text{for all } g \text{ and } h \in G.$$

LEMMA 2. *The map $\Phi=E(\varphi): \bar{D}\to\bar{D}$ is continuous at every point of S^1.*

Proof. For each arc $J \subset S^1$, let $V(J)$ be the set of $z \in D$ such that J is seen from z with an angle $\geq \pi/2$ in Poincaré geometry. The boundary of $V(J)$ is an arc Γ of the circle through the endpoints of J that makes an angle $\pi/4$ with S^1. For $w \in \Gamma$ there is a map $g \in G$ such that $g(w)=0$, $g(J)=\widehat{[e^{-\pi i/4}, e^{+\pi i/4}]}$, and $g(V(J))=D \cap \{z; |z\sqrt{2}-1| \leq 1\}$. It follows from Lemma 1 and conformal naturality that if $\mu(J) \geq \frac{2}{3}$, the vector field ξ_μ points into $V(J)$ on Γ, and therefore $B(\mu) \in V(J)$.

Let $U(J)=\{z \in D; \eta_z(J) \geq \frac{2}{3}\}$. Then $\Phi(U(J)) \subset V(\varphi(J))$. Now if $\zeta \in S^1$, when J ranges among neighborhoods of ζ in S^1, $J \cup U(J)$ is a neighborhood of ζ in $\bar{D}$ and the sets $\varphi(J) \cup (V(\varphi(J))$ span a fundamental system of neighborhoods of $\varphi(\zeta)$ in $\bar{D}$. Therefore Φ is continuous at ζ. Q.E.D.

THEOREM 1. *The map $\Phi=E(\varphi)\colon \bar{D} \to \bar{D}$ is a homeomorphism whose restriction to D is a real-analytic diffeomorphism.*

Proof. By Lemma 2, it suffices to prove that Φ is real-analytic and that its Jacobian is nonzero at every $z \in D$. By the conformal naturality we may assume that $z=0$, $\Phi(0)=0$, and $\varphi\colon S^1 \to S^1$ has degree one.

By definition, if $z \in D$, $\Phi(z)$ is the unique $w \in D$ such that

$$F(z,w)=\frac{1}{2\pi}\int_{S^1}\left(\frac{\varphi(\zeta)-w}{1-\bar{w}\varphi(\zeta)}\right)\frac{(1-|z|^2)}{|z-\zeta|^2}|d\zeta|=0. \tag{3.1}$$

The function F is real-analytic in $D \times D$, and its derivatives at (0,0) are

$$\begin{aligned} F'_z(0,0)&=\frac{1}{2\pi}\int_{S^1}\bar{\zeta}\varphi(\zeta)\,|d\zeta|, \quad F'_{\bar{z}}(0,0)=\frac{1}{2\pi}\int_{S^1}\zeta\varphi(\zeta)\,|d\zeta|,\\ F'_w(0,0)&=-1, \quad F'_{\bar{w}}(0,0)=\frac{1}{2\pi}\int_{S^2}\varphi(\zeta)^2|d\zeta|. \end{aligned} \tag{3.2}$$

Formula (2.4), with $\mu=\varphi_*(\eta_0)$, implies

$$|F'_w(0,0)|^2-|F'_{\bar{w}}(0,0)|^2=\frac{1}{2}\left(\frac{1}{2\pi}\right)^2\int\int_{S^1\times S^1}|\varphi(z)^2-\varphi(\zeta)^2|^2|dz|\times|d\zeta|>0. \tag{3.3}$$

The Implicit function theorem therefore implies that $\Phi(z)$ is a real-analytic function of z near $z=0$. Moreover, implicit differentiation gives the formula

$$|\Phi'_z(0)|^2-|\Phi'_{\bar{z}}(0)|^2=\frac{|F'_z(0,0)|^2-|F'_{\bar{z}}(0,0)|^2}{|F'_w(0,0)|^2-|F'_{\bar{w}}(0,0)|^2}$$

for the Jacobian of Φ at $z=0$. Since $F_z'(0,0)$ and $F_{\bar z}'(0,0)$ are the coefficients c_1 and c_{-1} in the Fourier expansion

$$\varphi(\zeta)=\sum_{n=-\infty}^{\infty} c_n\zeta^n, \tag{3.4}$$

Theorem 1 follows from

LEMMA 3. *If $\varphi: S^1\to S^1$ is a homeomorphism of degree one with Fourier series* (3.4), *then* $|c_1|>|c_{-1}|$.

Although this lemma is well known, we include a proof so that we can make some estimates later. We compute

$$|c_1|^2-|c_{-1}|^2=\left(\frac{1}{2\pi}\right)^2\int\int_{S^1\times S^1}\operatorname{Re}\left[\varphi(\zeta)\,\bar\varphi(z)\,(z\bar\zeta-\bar z\zeta)\right]|d\zeta|\times|dz|.$$

Put $z=e^{is}$, $\zeta=e^{it}$, and $\varphi(e^{iu})=e^{i\psi(u)}$. Here $\psi\colon \mathbf{R}\to\mathbf{R}$ is continuous and strictly increasing, and $\psi(u+2\pi)=\psi(u)+2\pi$. Now

$$\begin{aligned}
|c_1|^2-|c_{-1}|^2&=2\left(\frac{1}{2\pi}\right)^2\int_{s=0}^{2\pi}\int_{t=0}^{2\pi}\sin(s-t)\sin(\psi(s)-\psi(t))\,ds\,dt\\
&=2\left(\frac{1}{2\pi}\right)^2\int_{u=0}^{2\pi}\sin u\int_{t=0}^{2\pi}\sin(\psi(t+u)-\psi(t))\,dt\,du\\
&=2\left(\frac{1}{2\pi}\right)^2\int_{u=0}^{\pi}\sin u\int_{t=0}^{2\pi}\left[\sin(\psi(t+u)-\psi(t))+\sin(\psi(t+2\pi)-\psi(t+u+\pi))\right]dt\,du.
\end{aligned}$$

Therefore

$$|c_1|^2-|c_{-1}|^2=\left(\frac{1}{2\pi}\right)^2\int_{u=0}^{\pi}\sin u\int_{t=0}^{2\pi}H(t,u)\,dt\,du, \tag{3.5}$$

with

$$\begin{aligned}
H(t,u)=\ &\sin(\psi(t+u)-\psi(t))+\sin(\psi(t+2\pi)-\psi(t+u+\pi))\\
&+\sin(\psi(t+\pi+u)-\psi(t+\pi))+\sin(\psi(t+\pi)-\psi(t+u)).
\end{aligned} \tag{3.6}$$

The integral (3.5) is positive because if α_1, α_2, α_3, α_4 are positive numbers whose sum is 2π, then $\Sigma_{j=1}^4 \sin\alpha_j>0$. The proof of Lemma 3 and Theorem 1 is complete.

Remarks. (1) The quantity $|c_1|^2-|c_{-1}|^2$ is the Jacobian at $z=0$ of the harmonic function $u: D \to \mathbf{C}$ with boundary values φ. It has been known for some time (see Choquet [7] and Kneser [12]) that a harmonic function $u: D \to \mathbf{C}$ whose boundary values map S^1 homeomorphically onto a convex curve Γ is a diffeomorphism onto the interior of Γ.

(2) The extension operator $\varphi \mapsto E(\varphi)=\Phi$ is uniquely determined by the conformal naturality and the property that $\Phi(0)=0$ if φ has mean value zero. Indeed, if $w=B(\varphi_*(\eta_0))$, then $g_w \circ \varphi$ has mean value zero, so $0=E(g_w \circ \varphi)(0)=g_w(\Phi(0))$. Therefore $\Phi(0)=B(\varphi_*(\eta_0))$, and the formula $\Phi(z)=B(\varphi_*(\eta_z))$ follows by conformal naturality.

4. Dependence on φ

To study how $E(\varphi)$ depends on φ, it is convenient to think of the set $\mathcal{H}(S^1)$ of homeomorphisms $\varphi: S^1 \to S^1$ as a subset of the Banach space $\mathcal{C}(S^1, \mathbf{C})$ of complex-valued continuous functions on S^1, with the sup norm. For each φ in $\mathcal{H}(S^1)$ the extension $\Phi=E(\varphi)$ belongs to the group $\mathrm{Diff}(D) \cap \mathcal{H}(\bar{D})$ of C^∞ diffeomorphisms of D with homeomorphic extensions to $\bar{D}$. We regard $\mathrm{Diff}(D)$ and $\mathcal{H}(\bar{D})$ as subsets of the vector spaces $C^\infty(D, \mathbf{C})$ and $\mathcal{C}(\bar{D}, \mathbf{C})$, each with its standard topology, and we give $\mathrm{Diff}(D) \cap \mathcal{H}(\bar{D})$ the topology induced by the diagonal embedding in $\mathrm{Diff}(D) \times \mathcal{H}(\bar{D})$. Both $\mathcal{H}(S^1)$ and $\mathrm{Diff}(D) \cap \mathcal{H}(\bar{D})$ are topological groups.

PROPOSITION 2. *The map* $E: \mathcal{H}(S^1) \to \mathrm{Diff}(D) \cap \mathcal{H}(\bar{D})$ *is continuous.*

In other words the map $h: (z, \varphi) \mapsto E(\varphi)(z)$ of $\bar{D} \times \mathcal{H}(S^1)$ into $\bar{D}$ is continuous, and the partial derivatives of h (of all orders) with respect to z and $\bar{z}$ are continuous maps of $D \times \mathcal{H}(S^1)$ into $\mathbf{C}$. We shall prove that h is continuous at every point (z, φ) with $z \in S^1$, then that on $D \times \mathcal{H}(S^1)$ it is locally induced by an analytic map of an open set W of $\mathbf{C} \times \mathcal{C}(S^1, \mathbf{C})$ into $\mathbf{C}$.

Proof. (a) *Continuity at points of* $S^1 \times \mathcal{H}(S^1)$. Consider a homeomorphism $\varphi_0 \in \mathcal{H}(S^1)$ and a point $z_0 \in S^1$. Let us return to the proof of Lemma 2. Let V_1 be a neighborhood of $\varphi_0(z_0)$ in $\bar{D}$. One can find a neighborhood J_1 of $\varphi_0(z_0)$ in S^1 such that $\overline{V(J_1)} \subset V_1$, and then neighborhoods J_0 of z_0 in S^1 and W_0 of φ_0 in $\mathcal{C}(S^1, \mathbf{C})$ such that $\varphi(J_0) \subset J_1$ for each $\varphi \in W_0$. Then $\overline{U(J_0)}$ is a neighborhood of z_0 in $\bar{D}$, and $\Phi(\overline{U(J_0)}) \subset \overline{V(J_1)}$ for each $\varphi \in W_0$.

(b) *Local analyticity in* $D\times\mathscr{H}(S^1)$. Let Ω be the open set in $D\times\mathbf{C}\times\mathscr{C}(S^1,\mathbf{C})$ defined by

$$\Omega=\{(z,w,\varphi)\in D\times\mathbf{C}\times\mathscr{C}(S^1,\mathbf{C});|w|\cdot\|\varphi\|<1\},$$

and let $F\colon\Omega\to\mathbf{C}$ be the real-analytic function

$$F(z,w,\varphi)=\frac{1}{2\pi}\int_{S^1}\left(\frac{\varphi(\zeta)-w}{1-\bar{w}\varphi(\zeta)}\right)\frac{1-|z|^2}{|z-\zeta|^2}|d\zeta|.$$

Choose a homeomorphism $\varphi_0\colon S^1\to S^1$ and a point $z_0\in D$. Put $w_0=E(\varphi_0)(z_0)$. Then $F(z_0,w_0,\varphi_0)=0$. Moreover, $|F'_w|^2-|F'_{\bar{w}}|^2$ is positive at (z_0,w_0,φ_0) because it is a positive multiple of the Jacobian of the vector field ξ_μ at its unique zero w_0; here μ is the measure $\varphi_*(\eta_{z_0})$ on S^1. The Implicit function theorem therefore implies that all zeros of F near (z_0,w_0,φ_0) are given by a real-analytic function $w=h(z,\varphi)$, defined in a neighborhood of (z_0,φ_0) in $D\times\mathscr{C}(S^1,\mathbf{C})$. In particular $E(\varphi)(z)=h(z,\varphi)$ if (z,φ) in $D\times\mathscr{H}(S^1)$ is close to (z_0,φ_0). Q.E.D.

COROLLARY. *The functions* $\varphi\mapsto E(\varphi)'_z(0)$ *and* $\varphi\mapsto E(\varphi)'_{\bar{z}}(0)$ *on* $\mathscr{H}(S^1)$ *are continuous.*

5. Quasiconformal extensions

THEOREM 2. *If the homeomorphism* $\varphi\colon S^1\to S^1$ *admits a quasiconformal extension to* $\bar{D}$, *then* $\Phi=E(\varphi)$ *is quasiconformal. In fact both* Φ *and* Φ^{-1} *are Lipschitz continuous in the Poincaré metric on* D.

Proof. Let $\mathscr{H}_+(S^1)$ be the set of $\varphi\in\mathscr{H}(S^1)$ that have degree one. For $\varphi\in\mathscr{H}_+(S^1)$ put $\Phi=E(\varphi)$ and define positive functions $\alpha(\varphi)$ and $\beta(\varphi)$ on D by

$$\alpha(\varphi)(z)=\frac{|\Phi'_z(z)|-|\Phi'_{\bar{z}}(z)|}{1-|\Phi(z)|^2}\Big/\frac{1}{1-|z|^2},$$

$$\beta(\varphi)(z)=\frac{|\Phi'_z(z)|+|\Phi'_{\bar{z}}(z)|}{1-|\Phi(z)|^2}\Big/\frac{1}{1-|z|^2}.$$

The Lipschitz continuity of Φ and Φ^{-1} in the Poincaré metric is equivalent to the existence of positive numbers a and b such that

$$a\leqslant\alpha(\varphi)(z)\leqslant\beta(\varphi)(z)\leqslant b\quad\text{for all } z\in D. \tag{5.1}$$

These inequalities in turn imply that Φ is quasiconformal with dilatation ratio $\leq b/a$. We must therefore prove that if φ admits a quasiconformal extension to $\bar{D}$, then (5.1) holds for some positive numbers a and b.

Since G is a group of isometries in the Poincaré metric, the conformal naturality of the map $\varphi \to \Phi$ implies that

$$\alpha(g \circ \varphi \circ h) = \alpha(\varphi) \circ h \quad \text{and} \quad \beta(g \circ \varphi \circ h) = \beta(\varphi) \circ h$$

for all g and h in G_+. Therefore it suffices to prove that

$$a(K) = \inf \{\alpha(\varphi)(0); \varphi \in \mathcal{H}_K(S^1)\}$$

and

$$b(K) = \sup \{\beta(\varphi)(0); \varphi \in \mathcal{H}_K(S^1)\}$$

are finite positive numbers if $\mathcal{H}_K(S^1)$ is the set of $\varphi \in \mathcal{H}_+(S^1)$ that admit a K-quasiconformal extension to $\bar{D}$ and fix the points $1, i$, and -1. That is easy. Theorem 1 implies that the functions $\varphi \mapsto \alpha(\varphi)(0)$ and $\varphi \mapsto \beta(\varphi)(0)$ are positive on $\mathcal{H}_+(S^1)$. They are also continuous, by Proposition 2 and its corollary. Since the set $\mathcal{H}_K(S^1) \subset \mathcal{H}_+(S^1)$ is compact (see § 5 of [13, Chapter II]), we must have $0 < a(K)$ and $b(K) < \infty$. Q.E.D.

Remarks. (1) The proof shows that for each $K \geq 1$ there is a number K^* such that Φ is K^*-quasiconformal if φ has a K-quasiconformal extension. We shall estimate K^* as a function of K in Sections 9 and 10.

(2) The proof used only the fact that the set of $\varphi \in \mathcal{H}_+(S^1)$ admitting a K-quasiconformal extension to $\bar{D}$ is $G_+ \times G_+$ invariant and has compactness properties. The fact that invariance and compactness properties of this kind characterize the $\varphi \in \mathcal{H}_+(S^1)$ with quasiconformal extensions to $\bar{D}$ was proved by Beurling and Ahlfors [6]. They also gave a simple geometric characterization of these φ and defined a quasiconformal extension operator $\varphi \mapsto \Phi$. Their extension operator is not conformally natural, but it can be taken to be $G_\zeta \times G_\zeta$ equivariant if G_ζ is the subgroup of G leaving a given point $\zeta \in S^1$ fixed.

6. Dependence on μ

The most important invariant of a quasiconformal map $f: \bar{D} \to \bar{D}$ is its complex dilatation

$$\mu(f) = f'_{\bar{z}}/f'_z.$$

In this section we study how $\mu(\Phi)$ depends on φ if $\Phi=E(\varphi)$ is quasiconformal. We need some notations.

Let M be the open unit ball in the Banach space $L^\infty(D, \mathbf{C})$. For each $\mu\in M$ there is a unique quasiconformal map f^μ of $\bar{D}$ onto itself that fixes the points $1, i$, and -1 and satisfies the Beltrami equation

$$f'_{\bar z}=\mu f'_z$$

in D. Let φ^μ be the restriction of f^μ to S^1. By Theorem 2, $E(\varphi^\mu)\colon \bar{D}\to\bar{D}$ is quasiconformal, so its complex dilatation belongs to M. That determines a map

$$\sigma\colon \mu\mapsto E(\varphi^\mu)'_{\bar z}/E(\varphi^\mu)'_z \tag{6.1}$$

from M to M. Since $E(\varphi^\mu)$ fixes the points $1, i$, and -1, (6.1) implies

$$E(\varphi^\mu)=f^{\sigma(\mu)} \quad \text{for all } \mu\in M. \tag{6.2}$$

PROPOSITION 3. *The map $\sigma\colon M\to M$ defined by* (6.1) *is continuous. In fact, if $0<k<1$, then σ is uniformly continuous on the set*

$$M_k=\{\mu\in M; \|\mu\|\leq k\}.$$

Proof. Fix $k\in]0,1[$. First we shall prove that the function $\mu\mapsto\sigma(\mu)(0)$ is uniformly continuous on M_k. If not, there are sequences (μ_n) and (ν_n) in M_k and a number $\varepsilon>0$ such that $\|\mu_n-\nu_n\|\to 0$ but

$$|\sigma(\mu_n)(0)-\sigma(\nu_n)(0)|>\varepsilon \quad \text{for all } n. \tag{6.3}$$

By passing to a subsequence we may assume that f^{μ_n} converges uniformly in $\bar{D}$ to some f^μ. Since $\|\mu_n-\nu_n\|\to 0$, f^{ν_n} also converges to f^μ uniformly in $\bar{D}$. But then the corollary to Proposition 2 implies that $\sigma(\mu_n)(0)$ and $\sigma(\nu_n)(0)$ converge to the same limit $\sigma(\mu)(0)$. That contradicts (6.3), so $\mu\mapsto\sigma(\mu)(0)$ is uniformly continuous in M_k.

We will use conformal naturality to finish the proof. First we identify M with the set of bounded measurable conformal structures on D by associating the function $\mu\in M$ with the conformal class of the metric

$$ds=|dz+\mu(z)\,d\bar z|. \tag{6.4}$$

We denote by D_μ the disk D with the conformal structure determined by (6.4). Thus, $f^\mu\colon D_\mu\to D_0$ is a conformal map.

The group G acts on M so that $\nu=g_*(\mu)$ if and only if the map $g:D_\mu\to D_\nu$ is conformal. Explicitly,

$$\begin{aligned} \nu=g_*(\mu) &\quad \text{if and only if} \quad \mu=(\nu\circ g)\,\bar{g}'/g' \text{ for } g\in G_+,\\ \nu=g_*(\mu) &\quad \text{if and only if} \quad \bar{\mu}=(\nu\circ g)\,\overline{g_{\bar z}'}/g_{\bar z}' \text{ for } g\in G\smallsetminus G_+. \end{aligned} \tag{6.5}$$

LEMMA 4. *$\nu=g_*(\mu)$ if and only if $f^\nu\circ g\circ(f^\mu)^{-1}\in G$.*

Proof. By definition, $\nu=g_*(\mu)$ if and only if $g:D_\mu\to D_\nu$ is conformal. Since $f^\nu:D_\nu\to D_0$ and $f^\mu:D_\mu\to D_0$ are conformal, $\nu=g_*(\mu)$ if and only if

$$f^\nu\circ g\circ(f^\mu)^{-1}:D_0\to D_0$$

is conformal. Q.E.D.

COROLLARY. *The map $\sigma:M\to M$ is conformally natural.*

Proof. If $g\in G$ and $\nu=g_*(\mu)$, then Lemma 4 gives

$$f^\nu\circ g=h\circ f^\mu$$

for some $h\in G$. Therefore $\varphi^\nu\circ g=h\circ\varphi^\mu$ on S^1, so

$$E(\varphi^\nu)\circ g=h\circ E(\varphi^\mu)$$

in $\bar{D}$. By (6.2), $f^{\sigma(\nu)}\circ g=h\circ f^{\sigma(\mu)}$, so Lemma 4 implies $\sigma(\nu)=g_*(\sigma(\mu))$. Q.E.D.

End of proof of Proposition 3. We have already proved that given $k\in\,]0,1[$ and $\varepsilon>0$ there is $\delta>0$ such that

$$|\sigma(\mu)(0)-\sigma(\nu)(0)|<\varepsilon$$

if $\|\mu-\nu\|<\delta$ and $\mu,\nu\in M_k$. If $g\in G$, then (6.5) implies $\|g_*(\mu)\|=\|\mu\|$ and $\|g_*(\mu)-g_*(\nu)\|=\|\mu-\nu\|$, so (6.5) and the corollary to Lemma 4 give

$$|\sigma(\mu)(g^{-1}(0))-\sigma(\nu)(g^{-1}(0))|=|\sigma(g_*(\mu))(0)-\sigma(g_*(\nu))(0)|<\varepsilon$$

if $\|\mu-\nu\|<\delta$ and $\mu,\nu\in M_k$. But $g^{-1}(0)$ is any point of D. Q.E.D.

Remark. We shall prove in Section 8 that $\sigma:M\to M$ is a real-analytic map.

7. Teichmüller spaces

If Γ is a Fuchsian group (discrete subgroup of G), we define

$$M(\Gamma)=\{\mu\in M;\gamma_*(\mu)=\mu \text{ for all } \gamma\in\Gamma\}. \tag{7.1}$$

Equivalently, by Lemma 4,

$$M(\Gamma)=\{\mu\in M; f^\mu\circ\gamma\circ(f^\mu)^{-1}\in G \text{ for all } \gamma\in\Gamma\}. \tag{7.2}$$

The Teichmüller space $T(\Gamma)$ is defined by

$$T(\Gamma)=\{\varphi\in\mathcal{H}(S^1);\varphi=\varphi^\mu \text{ for some } \mu\in M(\Gamma)\}.$$

We denote by 1 the trivial subgroup of G, so that $M(1)=M$ and $T(1)$ is the set of $\varphi\in\mathcal{H}(S^1)$ that fix the points $1,i$, and -1 and admit a quasiconformal extension to $\bar{D}$.

The conformal naturality of the assignment $\varphi\mapsto E(\varphi)$ leads to a simple proof of the following theorem of Tukia.

PROPOSITION 4 (Tukia [16]). *For any Fuchsian group* Γ,

$$T(\Gamma)=\{\varphi\in T(1);\varphi\circ\gamma\circ\varphi^{-1}\in G \textit{ for all } \gamma\in\Gamma\}.$$

Proof. Put $S=\{\varphi\in T(1);\ \varphi\circ\gamma\circ\varphi^{-1}\in G \text{ for all } \gamma\in\Gamma\}$. Then $\varphi^\mu\in S$ for all $\mu\in M(\Gamma)$, by (7.2), so $T(\Gamma)\subset S$. Conversely, if $\varphi\in S$, then by conformal naturality

$$E(\varphi)\circ\gamma\circ E(\varphi)^{-1}\in G \quad \text{for all } \gamma\in\Gamma.$$

Moreover, by Theorem 2, $E(\varphi)$ is quasiconformal and $E(\varphi)=f^\mu$, where $\mu\in M$ is given by

$$\mu=E(\varphi)'_{\bar{z}}/E(\varphi)'_z.$$

Since $f^\mu\circ\gamma\circ(f^\mu)^{-1}\in G$ for all $\gamma\in\Gamma$, $\mu\in M(\Gamma)$ and $\varphi^\mu=\varphi\in T(\Gamma)$. Q.E.D.

The space $M(\Gamma)$ inherits a topology from $L^\infty(D,\mathbf{C})$, and $T(\Gamma)$ is given the quotient topology induced by the map $\pi\colon M(\Gamma)\to T(\Gamma)$ defined by $\pi(\mu)=\varphi^\mu$. It is clear from (6.5) and (7.1) that $M(\Gamma)$ is a convex, hence contractible, subset of $L^\infty(D,\mathbf{C})$. Our next goal is to prove that $T(\Gamma)$ is also contractible. That will be an easy consequence of

LEMMA 5. *If* Γ *is a Fuchsian group and* $\sigma\colon M\to M$ *is defined by* (6.1), *then*

(a) σ *maps* $M(\Gamma)$ *into itself,*

(b) *there is a continuous map* $s: T(\Gamma)\to M(\Gamma)$ *such that* $s\circ\pi=\sigma: M(\Gamma)\to M(\Gamma)$,

(c) $\pi\circ\sigma=\pi: M(\Gamma)\to M(\Gamma)$.

Proof. (a) Let $\mu\in M(\Gamma)$. Then $\varphi^\mu\in T(\Gamma)$ and, as we saw in the proof of Proposition 4,

$$E(\varphi^\mu)\circ\gamma\circ E(\varphi^\mu)^{-1}\in G \quad \text{for all } \gamma\in\Gamma.$$

By (6.2), $E(\varphi^\mu)=f^{\sigma(\mu)}$, so $\sigma(\mu)\in M(\Gamma)$.

(b) By definition, if $\pi(\mu)=\pi(\nu)$, then $\varphi^\mu=\varphi^\nu$, so $E(\varphi^\mu)=E(\varphi^\nu)$ and $\sigma(\mu)=\sigma(\nu)$. Hence there is a well defined map $s: T(\Gamma)\to M(\Gamma)$ such that $s\circ\pi=\sigma$ on $M(\Gamma)$. The map s is continuous because σ is, by Proposition 3.

(c) Since $E(\varphi^\mu)=f^{\sigma(\mu)}$, $\varphi^{\sigma(\mu)}$ is the restriction of $E(\varphi^\mu)$ to S^1. Therefore $\varphi^{\sigma(\mu)}=\varphi^\mu$ and $\pi(\sigma(\mu))=\pi(\mu)$. Q.E.D.

THEOREM 3. *The Teichmüller space* $T(\Gamma)$ *of any Fuchsian group* Γ *is contractible.*

Proof. By Lemma 5, $\pi\circ s\circ\pi=\pi\circ\sigma=\pi$, so $\pi\circ s: T(\Gamma)\to T(\Gamma)$ is the identity map. Since $M(\Gamma)$ is contractible, so is $T(\Gamma)$. An explicit contraction is the map $(\varphi, t)\to\pi((1-t)\,s(\varphi))$ from $T(\Gamma)\times[0,1]$ to $T(\Gamma)$. Q.E.D.

Remarks. (1) For more information about Teichmüller spaces see Bers [5] and the literature quoted there.

(2) It is classical that $T(\Gamma)$ is contractible when $T(\Gamma)$ is finite dimensional (i.e. $\Gamma\setminus D$ has finite Poincaré area). The contractibility for all Γ was conjectured by Bers [3, Lecture 1], who introduced the infinite dimensional Teichmüller spaces. Bers' conjecture was proved for $\Gamma=1$ in [11] and announced for finitely generated subgroups of G_+ in [9]. Tukia [15] proved that $T(\Gamma)$ is contractible for many infinitely generated groups Γ, and indeed is homeomorphic to a Banach space in many cases. He also informed the second author in 1983 that the methods of [16] can be extended to prove that all $T(\Gamma)$ are contractible.

(3) If $\Gamma\subset G_+$, Proposition 4 has an equivalent formulation. By results of Bers [4], there is a homeomorphism θ from $T(1)$ onto an open subset Δ of the Banach space B of holomorphic functions f on $\mathbf{C}\setminus\bar{D}$ with norm

$$\|f\|=\sup\{|f(z)|(1-|z|^2)^2; |z|>1\}<\infty.$$

G_+ acts on B so that $g\cdot f=h$ if and only if $f=(h\circ g)(g')^2$. Bers proves that θ maps $T(\Gamma)$ homeomorphically into

$$B(\Gamma) = \{f \in B; \gamma \cdot f = f \quad \text{for all } \gamma \in \Gamma\},$$

so $\theta(T(\Gamma)) \subset B(\Gamma) \cap \Delta$. If

$$S = \{\varphi \in T(1); \varphi \circ \gamma \circ \varphi^{-1} \in G \quad \text{for all } \gamma \in \Gamma\},$$

then the Lemma in [8] says that $\theta(S) = B(\Gamma) \cap \Delta$, so Proposition 4 is equivalent to the statement

$$\theta(T(\Gamma)) = B(\Gamma) \cap \Delta.$$

For further comments on Proposition 4 see Section two of Tukia [16].

8. Analytic dependence on μ

In this section we shall prove that $\sigma: M \to M$ is a real-analytic map. First we need to strengthen the corollary to Proposition 2.

LEMMA 6. *For each $\varphi_0 \in \mathcal{H}_+(S')$ there is a holomorphic function $f: V \to \mathbf{C}$, defined in an open neighborhood V of φ_0 in $\mathscr{C}(S^1, \mathbf{C})$, such that*

$$|f(\varphi)| < 1 \quad \textit{for all } \varphi \in V, \tag{8.1}$$

$$f(\varphi) = E(\varphi)'_{\bar{z}}(0)/E(\varphi)'_z(0) \quad \textit{for all } \varphi \in V \cap \mathcal{H}_+(S^1). \tag{8.2}$$

Proof. The proof of Proposition 2 shows that for each $\varphi_0 \in \mathcal{H}_+(S^1)$ there is a real-analytic function $h(z, \varphi)$, defined for (z, φ) near $(0, \varphi_0)$ in $\mathbf{C} \times \mathscr{C}(S^1, \mathbf{C})$, such that $E(\varphi)(z) = h(z, \varphi)$ if $\varphi \in \mathcal{H}_+(S^1)$ and (z, φ) is in the domain of h. The complex derivatives $h'_z(0, \varphi)$ and $h'_{\bar{z}}(0, \varphi)$ are real-analytic functions of φ, and

$$|h'_{\bar{z}}(0, \varphi_0)| < |h'_z(0, \varphi_0)|,$$

so $f(\varphi) = h'_{\bar{z}}(0, \varphi)/h'_z(0, \varphi)$ is real-analytic and satisfies (8.1) and (8.2) in some open neighborhood V of φ_0.

Now the map $H: \mathscr{C}(S^1, \mathbf{C}) \to \mathscr{C}(S^1, \mathbf{C})$ defined by

$$H(\psi)(\zeta) = \zeta \exp(i\psi(\zeta)) \quad \text{for all } \zeta \in S^1 \text{ and } \psi \in \mathscr{C}(S^1, \mathbf{C})$$

is holomorphic. Choose $\psi_0 \in \mathscr{C}(S^1, \mathbf{C})$ so that $H(\psi_0) = \varphi_0$. By the Inverse function theorem, H maps some open neighborhood W of ψ_0 biholomorphically onto an open neighborhood $H(W)$ of φ_0 in $\mathscr{C}(S^1, \mathbf{C})$; we may assume $H(W) \subset V$. Since the function

$f \circ H$ is real-analytic in W, there is a holomorphic function F, defined in an open neighborhood $W' \subset W$ of ψ_0, such that $|F(\psi)|<1$ for all $\psi \in W'$ and $F=f \circ H$ in $W \cap \mathscr{C}(S^1, \mathbf{R})$. The function $F \circ H^{-1}$ is holomorphic in $H(W')$ and equals f on $H(W') \cap \mathscr{H}_+(S^1)$. Q.E.D.

THEOREM 4. *The map* $\sigma: M \to M$ *defined by* (6.1) *is real-analytic.*

Proof. Let $M(\mathbf{C})$ be the open unit ball in $L^\infty(\mathbf{C}, \mathbf{C})$, and define a conjugate linear involution $\mu \mapsto \mu^*$ of $L^\infty(\mathbf{C}, \mathbf{C})$ onto itself by

$$\mu^*(z) = \bar{\mu}(1/\bar{z})\,(z/\bar{z})^2 \quad \text{for all } z \in \mathbf{C}.$$

Let $M^* = \{\mu \in M(\mathbf{C}); \mu = \mu^*\}$. The map that sends μ to its restriction to D is a real-analytic equivalence of M^* with M, and we shall identify M with M^* for the remainder of this section.

The projection operator $P\mu = (\mu + \mu^*)/2$ has norm one, and so does $I-P$; note that $P(M(\mathbf{C})) = M^*$.

For each $\mu \in M(\mathbf{C})$ there is a unique quasiconformal map f^μ of the extended complex plane onto itself that fixes the points $1, i$, and -1 and satisfies the Beltrami equation

$$f'_{\bar{z}} = \mu f'_z$$

in $\mathbf{C}$. Let φ^μ be the restriction of f^μ to S^1. For $\mu \in M^*$, $f^\mu(D) = D$, so the new definitions of f^μ and φ^μ agree with the old ones.

Now the results of Ahlfors and Bers [2] show that if $0<k'<1$ there is $r'>0$ such that

$$|\varphi^\mu(\zeta)| < 2 \quad \text{if } \zeta \in S^1,\ \|\mu\| < k' \text{ and } \|\mu - P\mu\| < r'.$$

Further, the map $\mu \mapsto \varphi^\mu$ from

$$V(k', r') = \{\mu \in M(\mathbf{C});\ \|\mu\| < k' \text{ and } \|\mu - P\mu\| < r'\}$$

to $\mathscr{C}(S^1, \mathbf{C})$ is holomorphic (and bounded). Since the set $V(k', r')$ is convex, it follows that $\mu \mapsto \varphi^\mu$ is Lipschitz continuous on $V(k, r)$ if $0<k<k'$ and $0<r<r'$. We conclude that given any $k \in]0,1[$ and $\delta>0$, there is $r>0$ such that

$$\|\varphi^\mu - \varphi^\nu\| < \delta \quad \text{if } \mu \text{ and } \nu \in V(k, r) \text{ and } \|\mu - \nu\| < r. \tag{8.3}$$

Now fix $k \in]0,1[$ and put $M_k^* = \{\mu \in M^*; \|\mu\| < k\}$. The set

$$A_k = \{\varphi \in \mathscr{H}_+(S^1);\ \varphi = \varphi^\mu \text{ for some } \mu \in M_k^*\}$$

has compact closure in $\mathscr{C}(S^1, \mathbf{C})$. Therefore, by Lemma 6, there is $\delta>0$ such that for every $\varphi_0 \in A_k$ there is a holomorphic function $f: B(\varphi_0, \delta) \to \mathbf{C}$ that satisfies (8.1) and (8.2) with $V=B(\varphi_0, \delta)$. Given that $\delta>0$, choose $r>0$ so that (8.3) holds.

By construction, for each $\mu_0 \in M_k^*$ there is a holomorphic function $F(\mu)=f(\varphi^\mu)$, defined in the convex open set $V(k, r) \cap B(\mu_0, r)$, such that

$$|F(\mu)| < 1 \tag{8.4}$$

and

$$F(\mu) = \sigma(\mu)(0) \quad \text{if } \mu \in M^*. \tag{8.5}$$

These open sets cover $V(k, r)$, so analytic continuation produces a holomorphic function $F: V(k, r) \to \mathbf{C}$ that satisfies (8.4) and (8.5).

Again we will use conformal naturality to complete the proof. Formula (6.5) defines an action of G on $L^\infty(\mathbf{C}, \mathbf{C})$, and the map P from $L^\infty(\mathbf{C}, \mathbf{C})$ to itself is conformally natural. Therefore the set $V(k, r)$ is G-invariant, and we can define a map H from $V(k, r)$ to the Banach space $B(D, \mathbf{C})$ of bounded complex valued functions on D by putting

$$H(\mu)(w) = F((g_w)_*(\mu)) \quad \text{for all } \mu \in V(k, r) \text{ and } w \in D.$$

(Here g_w is defined as in formula (1.1).) Since $(g_w)_*$ and F are holomorphic, the function $\mu \mapsto H(\mu)(w)$ is holomorphic for each $w \in D$. Since $|H(\mu)(w)|<1$ for all $w \in D$ and $\mu \in V(k, r)$, H is holomorphic (see for instance Lemma 3.4 in [10]). Finally, (8.5) and the conformal naturality of the map σ imply that $H(\mu)(w)=\sigma(\mu)(w)$ for all $\mu \in M_k^*$ and $w \in D$. Therefore σ is real-analytic in M_k^*. Q.E.D.

9. The derivative of $\sigma(\mu)$ at $\mu=0$

PROPOSITION 5. *The derivative of* $\sigma: M \to M$ *at* $\mu=0$ *is the linear map* $\sigma'(0): L^\infty(D, \mathbf{C}) \to L^\infty(D, \mathbf{C})$ *given by*

$$\sigma'(0)\, \nu(z) = \frac{3}{\pi} \int\int_D \frac{\nu(w)(1-|z|^2)^2}{(1-\bar{z}w)^4}\, du\, dv \quad \text{for all } z \in D \text{ and } \nu \in L^\infty(D, \mathbf{C}). \tag{9.1}$$

Proof. Fix any $\nu \in L^\infty(D, \mathbf{C})$. For $t \in \mathbf{R}$ sufficiently close to zero, Theorem 4 implies that

$$\sigma(t\nu) = t\sigma'(0)\, \nu + o(t).$$

By the results of Ahlfors–Bers [2],

$$\varphi^{t\nu}(\zeta)=\zeta+t\dot{\varphi}(\zeta)+o(t) \quad \text{uniformly for } \zeta\in S^1$$

and

$$\Phi^{t\nu}(z)=f^{\sigma(t\nu)}(z)=z+t\dot{f}(z)+o(t) \quad \text{for all } z\in D.$$

Further, $\dot{f}'_{\bar{z}}=\sigma'(0)\,\nu$.

Now, for $z\in D$, the definition of $\Phi(z)$ gives

$$\begin{aligned}0&=\frac{1}{2\pi}\int_{S^1}\frac{\varphi^{t\nu}(\zeta)-\Phi^{t\nu}(z)}{1-\bar{\Phi}^{t\nu}(z)\,\varphi^{t\nu}(\zeta)}\frac{(1-|z|^2)}{|z-\zeta|^2}|d\zeta|\\&=\frac{1}{2\pi}\int_{S^1}\left[\frac{\zeta-z}{1-\bar{z}\zeta}+t\left\{\frac{\dot{\varphi}(\zeta)-\dot{f}(z)}{1-\bar{z}\zeta}+\frac{(\zeta-z)\,(\zeta\bar{\dot{f}}(z)+\bar{z}\dot{\varphi}(\zeta))}{(1-\bar{z}\zeta)^2}\right\}\right]\frac{(1-|z|^2)}{|z-\zeta|^2}|d\zeta|+o(t).\end{aligned}$$

Therefore

$$0=\frac{1}{2\pi}\int_{S^1}\left[\frac{\dot{\varphi}(\zeta)}{1-\bar{z}\zeta}+\frac{\bar{z}(\zeta-z)\,\dot{\varphi}(\zeta)}{(1-\bar{z}\zeta)^2}-\frac{\dot{f}(z)}{1-\bar{z}\zeta}+\frac{\zeta(\zeta-z)\,\overline{\dot{f}(z)}}{(1-\bar{z}\zeta)^2}\right]\frac{(1-|z|^2)}{|z-\zeta|^2}|d\zeta|,$$

$$\begin{aligned}\dot{f}(z)&=\frac{1}{2\pi}\int_{S^1}\dot{\varphi}(\zeta)\left(\frac{1-\bar{z}z}{1-\bar{z}\zeta}\right)^2\frac{(1-|z|^2)}{|z-\zeta|^2}|d\zeta|\\&=\frac{1}{2\pi i}\int_{S^1}\dot{\varphi}(\zeta)\left(\frac{1-\bar{z}z}{1-\bar{z}\zeta}\right)^3\frac{d\zeta}{\zeta-z},\end{aligned}$$

and

$$\sigma'(0)\,\nu(z)=\dot{f}'_{\bar{z}}(z)=\frac{3}{2\pi i}\int_{S^1}\dot{\varphi}(\zeta)\frac{(1-|z|^2)^2}{(1-\bar{z}\zeta)^4}d\zeta. \tag{9.2}$$

Now the Ahlfors–Bers theory gives

$$\dot{\varphi}(\zeta)=-\frac{1}{\pi}\int\int_D\frac{\nu(w)\,dudv}{w-\zeta}+h(\zeta)$$

where h is continuous in $\bar{D}$ and holomorphic in D. Since

$$\frac{3}{2\pi i}\int_{S^1}h(\zeta)\frac{(1-|z|^2)^2}{(1-\bar{z}\zeta)^4}d\zeta=0 \quad \text{for all } z\in D,$$

by Cauchy's theorem, (9.2) gives

$$\sigma'(0)\,\nu(z)=\frac{3}{2\pi i}\int_{S^1}\left(\frac{1}{\pi}\int\int_D\frac{\nu(w)}{\zeta-w}dudv\right)\frac{(1-|z|^2)^2}{(1-\bar{z}\zeta)^4}d\zeta.$$

An application of Fubini's theorem and Cauchy's formula gives (9.1). Q.E.D.

COROLLARY 1. *$\|\sigma'(0)\,\nu\|\leq 3\|\nu\|$ for all $\nu\in L^\infty(D,\mathbf{C})$.*

Proof. For all $z\in D$,

$$|\sigma'(0)\,\nu(z)|\leq\frac{3\|\nu\|}{\pi}\int\int_D\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}\,du\,dv=3\|\nu\|.\qquad \text{Q.E.D.}$$

COROLLARY 2. *For $\varphi\in\mathcal{H}^+(S^1)$, put*

$$K(\varphi)=\inf\{K;\varphi \text{ has a } K\text{-quasiconformal extension to } \bar{D}\}\tag{9.3}$$

and let $K^*(\varphi)$ be the coefficient of quasiconformality of $\Phi=E(\varphi)$. Given any $\varepsilon>0$ there is $\delta>0$ such that for all $\varphi\in\mathcal{H}_+(S^1)$

$$K^*(\varphi)\leq K(\varphi)^{3+\varepsilon}\quad \text{if } K(\varphi)\leq 1+\delta.$$

Proof. We may assume that $K(\varphi)<\infty$ and, by conformal naturality, that φ fixes $1,i$ and -1. Then there is $\mu\in M$ such that $\varphi=\varphi^\mu$ and

$$K(\varphi)=\frac{1+\|\mu\|}{1-\|\mu\|}.$$

In addition, since $\Phi=f^{\sigma(\mu)}$,

$$K^*(\varphi)=\frac{1+\|\sigma(\mu)\|}{1-\|\sigma(\mu)\|}.$$

By Corollary 1, if $c>3$, then $\|\sigma(\mu)\|\leq c\|\mu\|$ and

$$K^*(\varphi)\leq\frac{1+c\|\mu\|}{1-c\|\mu\|}$$

if μ is close to zero. Furthermore, if $3<c<3+\varepsilon$, then

$$\frac{1+ct}{1-ct}<\left(\frac{1+t}{1-t}\right)^{3+\varepsilon}$$

for small positive numbers t. Q.E.D.

Remark. If $\nu(z)\equiv 1$, then $\sigma'(0)\,\nu(z)=3(1-|z|^2)^2$. Therefore the operator $\sigma'(0)$ has norm three, and the exponent $3+\varepsilon$ in Corollary 2 cannot be replaced by any number less than three.

10. Estimating $K^*(\varphi)$

We shall give an explicit upper bound for the coefficient of quasiconformality $K^*(\varphi)$ of $\Phi=E(\varphi)$ if φ admits a K-quasiconformal extension to $\bar{D}$. The estimates here provide a second proof of Theorem 2.

PROPOSITION 6. *Suppose $\varphi \in H_+(S^1)$ admits a K-quasiconformal extension to $\bar{D}$. If $\Phi=E(\varphi)$ fixes $0 \in D$, then for all ζ_1 and $\zeta_2 \in S^1$*

$$a(K)^{-1}\left(\frac{|\zeta_1-\zeta_2|}{16}\right)^K \leq |\varphi(\zeta_1)-\varphi(\zeta_2)| \leq 16\, a(K)\, |\zeta_1-\zeta_2|^{1/K} \tag{10.1}$$

where

$$a(K)=4(1+\sqrt{2})\,(16/\sqrt{3})^K. \tag{10.2}$$

Proof. Let $\psi: D \to D$ be a K-quasiconformal extension of φ, let $w=\psi(0)$, and put $\tilde{\psi}=g_w \circ \psi$. Then $\tilde{\psi}(0)=0$, so the boundary values $\tilde{\varphi}=g_w \circ \varphi$ of $\tilde{\psi}$ satisfy the Hölder inequalities

$$\left(\frac{|\zeta_1-\zeta_2|}{16}\right)^K \leq |\tilde{\varphi}(\zeta_1)-\tilde{\varphi}(\zeta_2)| \leq 16\, |\zeta_1-\zeta_2|^{1/K} \quad \text{for all } \zeta_1 \text{ and } \zeta_2 \in S^1 \tag{10.3}$$

(see [13, p. 66]). In addition $E(\tilde{\varphi})(0)=g_w(0)=-w$. We shall estimate $|w|$.

If $J=\widehat{[\alpha,\beta]} \subset S^1$ is any arc with $|\alpha-\beta| \leq c=(\sqrt{3}/16)^K$, then (10.3) implies that $\tilde{\varphi}_*(\eta_0)(J) \leq 1/3$. Choose $r \in]0,1[$ so that the arc $J_1=\widehat{[\bar{\alpha}_1,\alpha_1]}$ with $|\alpha_1-\bar{\alpha}_1|=c$ is seen from r with an angle $3\pi/2$ in Poincaré geometry. As in the proof of Proposition 1, Lemma 1 and conformal naturality imply that $\xi_{\tilde{\varphi}_*(\eta_0)}$ points inward on C_r. Thus $|w|=|E(\tilde{\varphi})(0)|<r$, and

$$\left(\frac{1-r}{1+r}\right)|\zeta_1-\zeta_2| \leq |g_{-w}(\zeta_1)-g_{-w}(\zeta_2)| \leq \left(\frac{1+r}{1-r}\right)|\zeta_1-\zeta_2| \tag{10.4}$$

for all ζ_1 and $\zeta_2 \in S^1$. Since $\varphi=g_{-w}\circ\tilde{\varphi}$, (10.3) and (10.4) imply (10.1) with $a(K)=(1+r)/(1-r)$.

It remains to show that $(1+r)/(1-r)$ is bounded by the right hand side of (10.2). Put $\alpha_1=e^{it}$, where $0<t<\pi/2$ and $|\alpha_1-\bar{\alpha}_1|=2\sin t=c$. The defining property of $r \in]0,1[$ is that $g_r(\alpha_1)=e^{3\pi i/4}$. That implies

$$r=\frac{2+\sqrt{2}(\cos t-\sin t)}{2\cos t+\sqrt{2}}=\frac{c+(4-c^2)^{1/2}}{2+c\sqrt{2}},$$

so

$$\frac{1+r}{1-r}=\frac{(1+\sqrt{2})(2+(4-c^2)^{1/2})}{c}<\frac{4(1+\sqrt{2})}{c}. \qquad \text{Q.E.D.}$$

PROPOSITION 7. *There are positive numbers $A<4\times 10^8$ and $B<35$ such that*

$$K^*(\varphi)\leqslant A\ \exp(BK(\varphi)) \quad \textit{for all } \varphi\in\mathscr{H}_+(S^1). \tag{10.5}$$

Here $K^*(\varphi)$ is the coefficient of quasiconformality of $\Phi=E(\varphi)$, and $K(\varphi)$ is defined by (9.3).

Proof. Assume that $K=K(\varphi)<\infty$, and put $\Phi=E(\varphi)$. Suppose that $\Phi(0)=0$, so that φ satisfies the Hölder inequalities (10.1). Implicit differentiation yields the formula

$$1-\frac{|\Phi'_{\bar z}(0)|^2}{|\Phi'_z(0)|^2}=\frac{(|F'_z(0,0)|^2-|F'_{\bar z}(0,0)|^2)(|F'_w(0,0)|^2-|F'_{\bar w}(0,0)|^2)}{|F'_{\bar w}(0,0)\,F'_{\bar z}(0,0)-F'_w(0,0)\,F'_z(0,0)|^2}. \tag{10.6}$$

Here $F(z,w)$ and its derivatives at (0,0) are given by (3.1) and (3.2). We must estimate the right side of (10.6).

The inequality

$$|F'_{\bar w}(0,0)\,\overline{F'_{\bar z}(0,0)}-\overline{F'_w(0,0)}\,F'_z(0,0)|^2\leqslant 4$$

is immediate from (3.2). Moreover, (3.5) implies that

$$|F'_z(0,0)|^2-|F'_{\bar z}(0,0)|^2\geqslant\left(\frac{1}{2\pi}\right)^2\int_{t=0}^{2\pi}\int_{u=\pi/3}^{2\pi/3}H(t,u)\sin u\ du\,dt\geqslant\frac{\varepsilon}{2\pi}$$

if $H(t,u)\geqslant\varepsilon$ in $[0,2\pi]\times[\pi/3,2\pi/3]$. According to (3.6), $H(t,u)$ is the sum of four terms

$$\sin(\psi(t')-\psi(t'')),$$

and each increment $(t'-t'')\in[\pi/3,2\pi/3]$ if $u\in[\pi/3,2\pi/3]$. Therefore

$$|e^{it'}-e^{it''}|\geqslant 1,$$

and (10.1) gives

$$\begin{aligned}|e^{i\psi(t')}-e^{i\psi(t'')}|&=|\varphi(e^{it'})-\varphi(e^{it''})|\\&\geqslant(16^K a(K))^{-1}=\delta(K)>0.\end{aligned}$$

Hence $\psi(t')-\psi(t'')\geqslant\delta(K)$, and $H(t,u)$ is bounded below on $[0,2\pi]\times[\pi/3,2\pi/3]$ by

$$\varepsilon(K) = \min\{\sum_{j=1}^{4} \sin\alpha_j; \sum_{j=1}^{4} \alpha_j = 2\pi \text{ and } \alpha_j \geq \delta(K) \text{ if } 1 \leq j \leq 4\}$$
$$= 3\sin\delta(K) - \sin 3\delta(K) > 3.99\delta(K)^3.$$

Therefore $|F'_z(0,0)|^2 - |F'_{\bar z}(0,0)|^2 > 3.99\delta(K)^3/2\pi$.

Next, (3.3) gives

$$|F'_w(0,0)|^2 - |F'_{\bar w}(0,0)|^2 = \frac{1}{2\pi}\int_{S^1} \lambda(z)\,|dz|,$$

with

$$\lambda(z) = \frac{1}{4\pi}\int_{S^1} |\varphi(\zeta)^2 - \varphi(z)^2|^2 |d\zeta|.$$

Given $z \in S^1$, find z' so that $\varphi(z') = -\varphi(z)$. Then

$$|\varphi(\zeta)^2 - \varphi(z)^2| = |(\varphi(\zeta)-\varphi(z))\,(\varphi(\zeta)-\varphi(z'))|.$$

The inequality (10.1) and Hölder's inequality imply that

$$4\pi\lambda(z) \geq \delta(K)^4 \int_{S^1} |(\zeta-z)\,(\zeta-z')\,|^{2K} |d\zeta|$$
$$\geq \delta(K)^4 (2\pi)^{1-K}\left(\int_{S^1} |(\zeta-z)\,(\zeta-z')\,|^2 |d\zeta|\right)^K$$
$$\geq \delta(K)^4 2^{K+1}\pi,$$

where $\delta(K) = (16^K a(K))^{-1}$ as before. Therefore

$$|F'_w(0,0)|^2 - |F'_{\bar w}(0,0)|^2 > 2^{K-1}\delta(K)^4$$

and (10.6) gives the inequality

$$1 - \frac{|\Phi'_{\bar z}(z)|^2}{|\Phi'_z(z)|^2} > 3.99 \times 2^K \delta(K)^7/16\pi, \tag{10.7}$$

first when $z = \Phi(z) = 0$, then in general, by conformal naturality.

If $k^* = \sup\{|\Phi'_{\bar z}(z)/\Phi'_z(z)|; z \in D\}$ (<1), then

$$K^*(\varphi) = \frac{1+k^*}{1-k^*} < \frac{4}{1-(k^*)^2}.$$

Therefore (10.7) and the definition of $\delta(K)$ imply that

$$K^*(\varphi) < 64\pi \times 2^{27K} a(K)^7/3.99,$$

with $a(K)$ given by (10.2). Q.E.D.

Remark. For purposes of comparison, we note that if $h: \mathbf{R} \to \mathbf{R}$ has a K-quasiconformal extension to $\mathbf{C}$. then it has a Beurling–Ahlfors extension $w: \mathbf{C} \to \mathbf{C}$ with coefficient of quasiconformality

$$K(w) < \frac{1}{8} e^{\pi K}. \tag{10.8}$$

Indeed the assumption on h implies that h satisfies a "ϱ-condition" with

$$\varrho(h) < \frac{1}{16} e^{\pi K}.$$

(For a proof see p. 65 of [1].) This in turn implies that h has a Beurling–Ahlfors extension w satisfying (10.8), by results of M. Lehtinen (see [14]).

11. The higher dimensional case

Let $\varphi: S^{n-1} \to S^{n-1}$ be a homeomorphism, $n \geq 3$. The methods of Sections 2 and 3 generalize to extend φ to a continuous map $\Phi: \bar{B}^n \to \bar{B}^n$. First we must define the conformal barycenter of a probability measure μ on S^{n-1} with no atoms. As in Section 2, Remark 4, let

$$h_\mu(x) = \frac{1}{2} \int_{S^{n-1}} \log \frac{1-|x|^2}{|x-u|^2}\, d\mu(u), \quad x \in B^n,$$

and let ξ_μ be the gradient of h_μ in Poincaré (hyperbolic) geometry. The proofs of Proposition 1 and Lemma 1 generalize to show that ξ_μ has a unique zero in B^n. By definition, that zero is the conformal barycenter $B(\mu)$ of μ. The map $\mu \mapsto B(\mu)$ is conformally natural (with respect to the group G of all Möbius transformations that map $\bar{B}^n$ onto itself).

For x in B^n, the (hyperbolic) harmonic measure η_x on S^{n-1} is defined using the hyperbolic Poisson kernel:

$$\eta_x(E) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \left(\frac{1-|x|^2}{|x-u|^2} \right)^{n-1} d\omega(u).$$

Here $d\omega(u)$ is the $(n-1)$-dimensional Hausdorff measure on S^{n-1}, and ω_{n-1} is the total measure of S^{n-1}. Now, as in Section 3, we extend the homeomorphism $\varphi: S^{n-1} \to S^{n-1}$ to $\bar{B}^n$ by putting $\Phi(x) = B(\varphi_*(\eta_x))$ if $x \in B^n$. The proof of Lemma 2 generalizes to show that $\Phi: \bar{B}^n \to \bar{B}^n$ is continuous. The map $\varphi \mapsto \Phi$ is conformally natural.

The proof of Proposition 2 in Section 4 also generalizes, but the statement must be modified because in general Φ is not a homeomorphism. The general statement is

PROPOSITION 2′. *The assignment $\varphi \mapsto \Phi$ defines a continuous map of $\mathscr{H}(S^{n-1})$ into $\mathscr{C}^\infty(B^n, \mathbf{R}^n) \cap \mathscr{C}(\bar{B}^n, \mathbf{R}^n)$.*

Here $\mathscr{H}(S^{n-1})$ and $\mathscr{C}(\bar{B}^n, \mathbf{R}^n)$ have the compact-open topology, $\mathscr{C}^\infty(B^n, \mathbf{R}^n)$ has the $\mathscr{C}^\infty$ topology, and $\mathscr{C}^\infty(B^n, \mathbf{R}^n) \cap \mathscr{C}(\bar{B}^n, \mathbf{R}^n)$ has the topology induced by the diagonal embedding in $\mathscr{C}^\infty(B^n, \mathbf{R}^n) \times \mathscr{C}(\bar{B}^n, \mathbf{R}^n)$.

Given these preliminaries we can prove the following theorem about quasiconformal extensions, which was pointed out to us by Pekka Tukia.

THEOREM 5 (Tukia). *Given any $M>1$ there is a number $K>1$, depending only on M and n, such that if $\varphi: S^{n-1} \to S^{n-1}$ is K-quasiconformal, then $\Phi: \bar{B}^n \to \bar{B}^n$ is a quasiconformal homeomorphism and*

$$M^{-1} d(x, y) \leq d(\Phi(x), \Phi(y)) \leq M d(x, y) \quad \textit{for all } x, y \in B^n. \tag{11.1}$$

Here d is the Poincaré distance in B^n.

Proof. We imitate the proof of Theorem 2. Given $\varphi \in \mathscr{H}(S^{n-1})$ and $x \in B^n$, put

$$\alpha(\varphi)(x) = \inf\left\{\frac{(1-\|x\|^2)\|\Phi'(x)\,u\|}{1-\|\Phi(x)\|^2};\ u \in S^{n-1}\right\},$$

$$\beta(\varphi)(x) = \sup\left\{\frac{(1-\|x\|^2)\|\Phi'(x)\,u\|}{1-\|\Phi(x)\|^2};\ u \in S^{n-1}\right\}.$$

LEMMA 7. *Given any $M>1$ there is $K>1$, depending only on M and n, such that if $\varphi: S^{n-1} \to S^{n-1}$ is K-quasiconformal, then*

$$M^{-1} \leq \alpha(\varphi)(x) \leq \beta(\varphi)(x) \leq M \quad \textit{for all } x \in B^n. \tag{11.2}$$

Proof. Since G is the group of isometries of B^n in the Poincaré metric, the conformal naturality of the map $\varphi \mapsto \Phi$ implies that

$$\alpha(g \circ \varphi \circ h) = \alpha(\varphi) \circ h \quad \text{and} \quad \beta(g \circ \varphi \circ h) = \beta(\varphi) \circ h$$

for all g and h in G. Therefore it suffices to prove the existence of $K>1$ such that

$$M^{-1} \leq \alpha(\varphi)(0) \leq \beta(\varphi)(0) \leq M$$

if $\varphi: S^{n-1} \to S^{n-1}$ is K-quasiconformal and fixes the points e_1, $-e_1$, and e_n. The proof is by contradiction. If no such K exists, a compactness argument produces a sequence (φ_k) of quasiconformal maps and an element $g \in G$ such that $\varphi_k \to g$ in $\mathcal{H}(S^{n-1})$ and, for each k, either $\alpha(\varphi_k)(0) < M^{-1}$ or $\beta(\varphi_k)(0) > M$. Now Proposition 2′ implies that the functions $\varphi \mapsto \alpha(\varphi)(0)$ and $\varphi \mapsto \beta(\varphi)(0)$ are continuous on $\mathcal{H}(S^{n-1})$. Since $\alpha(g)(0) = \beta(g)(0) = 1$ we have reached the required contradiction. Q.E.D.

End of proof of Theorem 5. If $M>1$, let $K>1$ be given by Lemma 7. If $\varphi: S^{n-1} \to S^{n-1}$ is K-quasiconformal, the left hand inequality in (11.2) implies that the Jacobian of Φ is never zero, so $\Phi: B^n \to B^n$ is a local homeomorphism. This in turn implies that $\Phi: \bar{B}^n \to \bar{B}^n$ is a homeomorphism, and (11.2) then implies both that Φ is quasiconformal and that inequality (11.1) holds. Q.E.D.

References

[1] Ahlfors, L. V., *Lectures on quasiconformal mappings*. Van Nostrand-Reinhold, Princeton, 1966.

[2] Ahlfors, L. V. & Bers, L., Riemann's mapping theorem for variable metrics. *Ann. of Math.*, 72 (1960), 385–404.

[3] Bers, L., *On moduli of Riemann surfaces*. Lecture notes, E.T.H., Zürich, 1964.

[4] — Automorphic forms and general Teichmüller spaces, in *Proceedings of the Conference on Complex Analysis* (Minneapolis 1964), pp. 109–113. Springer, Berlin, 1965.

[5] — Finite dimensional Teichmüller spaces and generalizations. *Bull. Amer. Math. Soc.*, 5 (1981), 131–172.

[6] Beurling, A. & Ahlfors, L. V., The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96 (1956), 125–142.

[7] Choquet, G., Sur un type de transformation analytique généralisant la répresentation conforme et définie au moyen de fonctions harmoniques. *Bull. Sci. Math.* (2), 69 (1945), 156–165.

[8] Earle, C. J., The Teichmüller space of an arbitrary Fuchsian group. *Bull. Amer. Math. Soc.*, 70 (1964), 699–701.

[9] — The contractability of certain Teichmüller spaces. *Bull. Amer. Math. Soc.*, 73 (1967), 434–437.

[10] — On quasiconformal extensions of the Beurling–Ahlfors type, in *Contributions to Analysis*, pp. 99–105. Academic Press, New York, 1974.

[11] Earle, C. J. & Eells, J., On the differential geometry of Teichmüller spaces. *J. Analyse Math.*, 19 (1967), 35–52.

[12] Kneser, H., Lösung der Aufgabe 41. *Jahresber. Deutsch. Math.-Verein.*, 35 (1926), 123–124.

[13] Lehto, O. & Virtanen, K. I., *Quasiconformal mappings in the plane*. Springer, Berlin and New York, 1973.

[14] LEHTINEN, M., Remarks on the maximal dilatation of the Beurling–Ahlfors extension. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 9 (1984), 133–139.
[15] TUKIA, P., On infinite dimensional Teichmüller spaces. *Ann. Acad. Sci. Fenn Ser. A I Math.*, 3 (1977), 343–372.
[16] — Quasiconformal extension of quasisymmetric mappings compatible with a Möbius group. *Acta Math.*, 154 (1985), 153–193.

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