On the local solvability and the local integrability of systems of vector fields

by

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Introduction

The *local solvability* of a first-order linear partial differential equation depends on whether it satisfies the so-called Condition (P) (see [4]). Suppose that the differential operator under study is a complex vector field L, nowhere zero, in some open subset of $\mathbf{R}^{n+1}$. If L is *locally integrable,* that is to say, if in the vicinity of every point the homogeneous equation $Lh=0$ has n independent, and smooth, solutions, one can use them to formulate (P) (see [5]). In the case $n=1$, i.e., when L is defined in an open subset Ω of the plane, there is essentially only one such solution (if one exists at all), in the sense that the differential of any other one is collinear to its differential. Call Z such a solution, and view it as a map $\Omega \to \mathbf{C}$. Condition (P) is equivalent to the property that, locally speaking, the pre-images of points under the mapping Z are *connected.*

But it must be emphasized that the local integrability of L is by no means automatic. In his *"Lectures on linear partial differential equations"* (Reg. Conf. Series in Math., No 17 Amer. Math. Soc. 1973). L. Nirenberg has given the example of a

[1] During this work the author was partly supported by NSF grant no. 7903545.

vector field in the plane that only annihilates the constant functions. It is a modification of the Mizohata operator

$$L_0 = \frac{\partial}{\partial y} - iy\frac{\partial}{\partial x}.$$

Note that $L_0 Z=0$ if $Z=x+iy^2/2$, and the pre-images of points under the mapping Z are the points $(x_0, \pm y_0)$. The Mizohata operator is the simplest differential operator that does not possess Property (P), and in a sense is the prototype of all nonlocally solvable operators. Nirenberg's construction was inspired by an argument of Grushin [2] describing right-hand sides f such that the inhomogeneous equation $L_0 u=f$ cannot be solved. In [7] it was shown how Grushin's and Nirenberg's constructions were direct consequences of the fact that the "fibers" of the mapping Z (in the case of the Mizohata operator) are not connected.

The present work is a generalization, and an amplification of the previous ones. It studies 'overdetermined systems" of vector fields in an open subset Ω of $\mathbf{R}^{n+1}$, of the kind

$$L_j = \frac{\partial}{\partial t^j} + \lambda_j(t,x)\frac{\partial}{\partial x}, \quad j=1,\ldots,m,$$

having analytic coefficients (the theory of analytic, semi-analytic and subanalytic sets is heavily relied upon; see [2] and the Appendix by B. Teissier). We assume throughout that the vector fields L_j satisfy the Frobenius (or bracket) condition, which, because of the special form of these vector fields, reads here

$$[L_j, L_k] = 0, \quad j,k=1,\ldots,m. \tag{1}$$

For this kind of vector fields the approximation and representation of solutions of the homogeneous equations

$$L_j h = 0, \quad j=1,\ldots,m, \tag{2}$$

established in [1], are now available, and greatly facilitate the analysis. The present work uses them at every turn.

Using then the unique analytic solution of the Cauchy problem

$$L_j Z = 0, j=1,\ldots,m; \quad Z|_{t=0} = x \tag{3}$$

(keep in mind that $t=(t^1,\ldots,t^m)$ is a set of m variables), we define Property (P) at a given point p by saying that p has a basis of neighborhoods in each one of which the fibers of Z are connected. Such a formulation of (P) agrees with the generalizations of (P) (and of (Ψ)) to the models of complexes of pseudodifferential equations introduced in [6].

The main results (Theorems 2.1 and 2.2) of this work are easy to describe: if (P) does not hold at some point p_0 there are right-hand sides $f_1, \ldots, f_m$, defined and C^∞ near that point, which satisfy the so-called compatibility conditions,

$$L_j f_k = L_k f_j, \quad j, k = 1, \ldots, m, \tag{4}$$

such that the inhomogeneous equations

$$L_j u = f_j, \quad j = 1, \ldots, m, \tag{5}$$

do not have any distribution solution. Furthermore, there are simple modifications of the vector fields L_j that also commute pairwise and such that the homogeneous equations analogous to (2) do not have any C^1 solution h such that $dh(p_0) \neq 0$.

Conversely, suppose that Condition (P) holds at every point of some neighborhood of p_0. Then, for any choice of C^∞ right-hand sides f_j, satisfying (4), in some open neighborhood V of p_0, there is a C^∞ solution u satisfying (5) in a possibly smaller open neighborhood W of p_0. This can be regarded as a generalization of the Poincaré lemma (for one-forms).

All proofs are by construction, and we obtain explicit integral representations of the solution u of (5).

I am grateful to Bernard Teissier for having provided the proofs of some of the properties of analytic sets that were needed. The statements of those properties and their proofs can be found in the Appendix (in French) written by Teissier.

1. Basic concepts and notation

We suppose that we are given an *analytic* vector subbundle T' of the complex cotangent bundle $\mathbf{C}T^*\Omega$ of an analytic manifold Ω. Throughout the work "analytic" will mean "real-analytic". When meaning "complex-analytic" we shall say "holomorphic". We assume that the fibre dimension of T' is equal to one; so T' is a complex line bundle over Ω. We shall call $m+1$ the dimension of Ω.

We assume that T' is *locally integrable*. This means that locally T' is generated by the differential of an analytic function. Let U be an open subset of Ω in which T' is generated by the differential dZ of an analytic function Z. We select a point p_0 in U (it will be the "central point" in the forthcoming study) and suppose that $Z(p_0)=0$. After multiplication of Z by an appropriate complex number we may assume that $d(\operatorname{Re} Z)\neq 0$, $d(\operatorname{Im} Z)=0$ at p_0. We can find local coordinates in U (after as many contractions of the latter as deemed useful), denoted by $t^1, \dots, t^m, x$, vanishing at p_0, and such that

$$Z = x + i\Phi(t,x), \tag{1.1}$$

$$\Phi \text{ real-valued}, \quad \Phi(0,0)=0, \quad d_x\Phi(0,0)=0, \tag{1.2}$$

and, of course, Φ analytic in U. Actually we may even assume

$$\Phi(0,x)\equiv 0. \tag{1.3}$$

It is convenient to assume that

$$U = B_r \times J, \tag{1.4}$$

where B_r is the open ball $\{t\in\mathbf{R}^m; |t|<r\}$, and J an open interval in the real line containing the origin (the equality in (1.4) actually stands for the isomorphism defined by the coordinates t^j, x). We shall also assume that the closure of U in Ω, $\operatorname{Cl} U$, is compact.

We shall denote by Z the mapping $(t,x)\mapsto Z(t,x)$ from U to $\mathbf{C}$. Its image, $Z(U)$, is easy to describe: it is the union of a collection of intervals

$$\{x_0\}\times I(x_0), \quad x_0\in J, \tag{1.5}$$

where $I(x_0)$ is the image of B_r via the map $t\mapsto\Phi(t,x_0)$. Of course $I(x_0)$ is always an interval containing zero, but otherwise fairly arbitrary. In particular it is reduced to zero whenever $\Phi(t,x_0)\equiv 0$ (and only then!).

We must now introduce the orthogonal $T'^{\perp}$ of T': it is a vector subbundle of the complex tangent bundle $\mathbf{C}T\Omega$, analytic, whose fibres have dimension m (incidentally we always suppose $m\geq 1$). In U it is generated by m analytic vector fields L_j, $j=1,\dots,m$, such that

$$L_j Z = 0, \quad j=1,\dots,m. \tag{1.6}$$

If we further require

$$L_j t^k = \delta_j^k \text{ (Kronecker's index)}, \quad j,k=1,\dots,m, \tag{1.7}$$

the L_j are uniquely determined, since $dt^1,\dots,dt^m, dZ$ obviously span the whole cotangent space $\mathbf{C}T_p^*\Omega$ at every point p of U. We have

$$L_j = \frac{\partial}{\partial t^j} + \lambda_j(t, x)\frac{\partial}{\partial x}, \quad j = 1, \dots, m. \tag{1.8}$$

Of course,

$$\lambda_j = -Z_{t^j}/Z_x = -i\Phi_{t^j}/(1+i\Phi_x), \tag{1.9}$$

where subscripts mean differentiation. Note that, by (1.6), we have $L_j\bar{Z}=L_j(Z+\bar{Z})=L_j(2x)=2\lambda_j$, i.e.,

$$\lambda_j = \tfrac{1}{2}L_j\bar{Z}. \tag{1.10}$$

It is also convenient to introduce the vector field

$$L_0 = Z_x^{-1}\frac{\partial}{\partial x}. \tag{1.11}$$

Of course we have

$$L_0 t^k = 0, \quad k = 1, \dots, m, \quad L_0 Z = 1. \tag{1.12}$$

Thus $L_0, L_1, \dots, L_m$ is the basis in $\mathbf{C}T_p\Omega$ $(p \in U)$ dual of the basis $dZ, dt^1, \dots, dt^m$ of $\mathbf{C}T_p^*$. We have, in U,

$$[L_j, L_k] = 0, \quad j, k = 0, 1, \dots, m. \tag{1.13}$$

Indeed, $L_jL_k - L_kL_j$ annihilates Z and all t^l.

If F is a C^1 function in U, we have

$$dF = \sum_{j=1}^{m} L_j F\, dt^j + L_0 F\, dZ. \tag{1.14}$$

We shall need the results of [1] relating to the solutions of the homogeneous equations

$$L_j h = 0, \quad j = 1, \dots, m. \tag{1.15}$$

To help the reader we restate here the main theorems of [1]. Set $U''=B_{r'}\times J'$, with $0<r'<r$, and J' an open interval whose compact closure is containted in J. By $\mathrm{Cl}\, U'$ we denote the *closure* of U'.

THEOREM I. *Let h be a continuous solution of* (1.15) *in some open neighborhood of* $\mathrm{Cl}\, U'$. *Then h is the uniform limit, in* $\mathrm{Cl}\, U'$, *of a sequence of polynomials, with complex coefficients, in $Z(t, x)$.*

THEOREM II. *Let h be a distribution solution of* (1.15) *in some open neighborhood of* $\mathrm{Cl}\, U'$. *There are, then, an integer $q \geq 0$ and a C^1 solution of* (1.15) *in a neighborhood of* $\mathrm{Cl}\, U'$, *f, such that $h = L_0^q f$ in U'.*

By combining Theorems I and II we see that *any distribution such as h, in* Theorem II, *is the limit, in the distribution sense, in U', of a sequence of polynomials in* Z. Indeed, if $P(Z)$ is such a polynomial so is $L_0[P(Z)]$.

A remark we shall use is the following one: Let V be an open subset of U' in which $d_t\Phi$ does not vanish. Then in V the system $L=(L_1, \dots, L_m)$ is elliptic (its characteristic set is void), and every distribution solution of (1.15) in V is an analytic function. *If then $\{P_\nu(Z)\}$ is a sequence of polynomials in Z, which converges to the distribution h of Theorem II in $\mathscr{D}'(U')$, in V it necessarily converges to h in the C^∞ sense.* Indeed, on the space of solutions of (1.15) in V, the topologies induced by $\mathscr{D}'$ or by C^∞ are the same (and so are, of course, all the intermediary ones, such as that of uniform convergence on compact sets).

Let us stress an important consequence of Th. I:

COROLLARY. *Let h be a continuous solution of* (1.15) *in the neighborhood of* $\mathrm{Cl}\, U'$. *There is a continuous function $\tilde{h}$ on $Z(\mathrm{Cl}\, U')$, holomorphic in the interior of that set, such that $h = \tilde{h} \circ Z$ in $\mathrm{Cl}\, U'$.*

In particular note that h is constant on the *fibers* of the map Z in U'. If V is any subset of U by a fiber of Z in V we mean a set

$$(t, x) \in V; \quad Z(t, x) = z_0 \quad (z_0 \in \mathbf{C}). \tag{1.16}$$

Because of the peculiar form of the function Z (see (1.1)), the fiber (1.16) is given by

$$(t, x) \in V;\ x = x_0,\ \Phi(t, x_0) = y_0 \quad (z_0 = x_0 + iy_0), \tag{1.17}$$

and can thus be identified to a subset of the ball B_r.

2. Condition (P) and statement of the theorems

We use the notation and concepts introduced in Section 1. In particular, U will have the meaning given to it there. We shall reason as if Ω were an open subset of $\mathbf{R}^{m+1}$; then U is the product set (1.4).

Definition 2.1. We shall say that the system $L=(L_1, \dots, L_m)$ satisfies Condition (P) at a point p of U if there is a basis of neighborhoods of p in U, in each one of which the fibers of Z are connected.

We shall say that L satisfies Condition (P) in U if it satisfies Condition (P) at every point of U.

In Definition 2.1 one may replace Z by any other smooth function whose differential spans T' at each point of U (possibly after the latter set has been contracted about p). This is made evident by the Corollary in Section 1. Thus the validity of (P) at $p \in U$ is truly a property of the system L or, more accurately, of the line bundle T' (or of $T'^{\perp}$).

We shall be concerned with the inhomogeneous equations

$$L_j u = f_j, \quad j = 1, \dots, m, \tag{2.1}$$

where $f_1, \dots, f_m$ are C^∞ functions near p_0 satisfying the compatibility conditions:

$$L_j f_k = L_k f_j, \quad j, k = 1, \dots, m. \tag{2.2}$$

We shall also construct a modification $L_j^{\#}$ of L_j for each j, and consider the homogeneous equations

$$L_j^{\#} h = 0, \quad j = 1, \dots, m. \tag{2.3}$$

THEOREM 2.1. *Suppose that the system $L=(L_1, \dots, L_m)$ does not satisfy Condition* (P) *at the point p_0.*

Then there are two C^∞ functions f, g in an open neighborhood $V \subset U$ of p_0, vanishing of infinite order at p_0 such that the following facts are true:

$$\text{the functions } f_j = \lambda_j f,\ j = 1, \dots, m \text{ (see (1.8)) satisfy the compatibility conditions (2.2) in } V; \tag{2.4}$$

$$\text{the vector fields in } V,\ L_j^{\#} = L_j - \lambda_j g\,\partial/\partial x, \quad j = 1, \dots, m, \text{ commute pairwise.} \tag{2.5}$$

Furthermore, given any open neighborhood $W \subset V$ of p_0, the following is true:

$$\text{no distribution } u \text{ in } W \text{ satisfies (2.1);} \tag{2.6}$$

$$\text{the differential of every function } h \in C^1(W) \text{ that satisfies (2.3) vanishes at } p_0. \tag{2.7}$$

THEOREM 2.2. *Suppose that L satisfies Condition* (P) *in U. Then every open neighborhood $V \subset U$ of p_0 contains another open neighborhood of p_0, W, having the following property:*

$$\text{Given any set of } m\ C^\infty \text{ functions } f_1, \ldots, f_m \text{ in } V, \text{ satisfying the compatibility conditions (2.2) there is a } C^\infty \text{ function } u \text{ in } W \text{ satisfying (2.1) in } W. \tag{2.8}$$

The proofs of Theorems 2.1, 2.2, given in Sections 4, 5, 6, 7 are constructive: we shall give explicit representations of the functions f and g in Theorem 2.1, and of the solution u of (2.1), in Theorem 2.2.

3. About condition (P)

We restate Condition (P) (Definition 2.1) in the following manner:

$$\text{Every open neighborhood } V_p \subset U \text{ of } p \text{ contains another open neighborhood } W_p \text{ of } p \text{ which intersects at most one connected component of every fiber of } Z \text{ in } V_p. \tag{3.1}$$

Indeed, suppose first that V_p contains a neighborhood W'_p of p in which every fiber of Z is connected. Then we can take W_p in (3.1) to be the interior of W'_p. Conversely, suppose that (3.1) holds; call W'_p the union of all the connected components of fibres of Z in V_p which intersect W_p.

For a while we shall forget momentarily that the variable x is there: we shall reason in t-space $\mathbf{R}^m$. We denote by B, B', B'' three *open* balls centered at the origin in $\mathbf{R}^m$, such that

$$B'' \subset B' \subsetneqq B. \tag{3.2}$$

We shall look at a real-valued analytic function φ in B. If A is any subset of B and c any real number we write

$$A^+(c) = \{t \in A; \varphi(t) > c\}, \quad A^-(c) = \{t \in A; \varphi(t) < c\}, \tag{3.3}$$

$$A^0(c) = \{t \in A; \varphi(t) = c\}. \tag{3.4}$$

In other words A^0, A^+, A^- are the level, superlevel and sublevel sets, respectively, of the function φ in A.

LEMMA 3.1. *Suppose that*

$$\text{for every real number } c, B''^{0}(c) \text{ is contained in a single connected component of } B'^{0}(c). \tag{3.5}$$

Then the following property holds:

$$\text{for every real } c, B''^{+}(c) \text{ is contained in a single connected component of } B'^{+}(c) \text{ and } B''^{-}(c) \text{ is contained in a single connected component of } B'^{-}(c). \tag{3.6}$$

Proof. Suppose B'' intersected two distinct connected components, A_1 and A_2, of $B'^{+}(c)$ ($c \in \mathbf{R}$). Then for $c^* > c$ sufficiently close to c, $B'' \cap A_j$ ($j=1,2$) would contain a point where $\varphi = c^*$. But this means that A_j contains a connected component of $B'^{0}(c^*)$ which intersects B'', for each $j=1,2$, and thus (3.5) could not be true. Q.E.D.

The converse of Lemma 3.1 is not true, in general, but the following partial converse will suffice for our needs:

LEMMA 3.2. *Suppose that Property* (3.6) *is valid. Then*

$$\text{for every real } c, B''^{0}(c) \text{ is contained in a single connected component of } (\mathrm{Cl}\, B')^{0}(c). \tag{3.7}$$

$\mathrm{Cl}\, B'$ stands for the closure of B'.

Proof. Consider the singular set of φ in the ball B:

$$\{t \in B;\ d\varphi(t) = 0\}. \tag{3.8}$$

Only a finite number of its connected components intersect $\mathrm{Cl}\, B'$. On each of these components φ is constant, therefore the number of *critical values* of φ in a neighborhood of $\mathrm{Cl}\, B'$ is finite.

Suppose first that c is not a critical value of φ in such a neighborhood, and let t_j ($j=0,1$) be two points on the fibre $B''^{0}(c)$. The latter is equal to the intersection of B'' with an analytic hypersurface in a neighborhood of $\mathrm{Cl}\, B'$; necessarily φ changes sign across that hypersurface. For each $j=0,1$, we can find two point t_j^+, t_j^- in B'', arbitrarily close to t_j, such that $\varphi(t_j^-) < c < \varphi(t_j^+)$. By virtue of (3.6) we can find a smooth curve $\tilde{\gamma}^+$, entirely contained in $B'^{+}(c)$, joining t_0^+ to t_1^+ and, likewise, one $\tilde{\gamma}^- \subset B'^{-}(c)$, joining t_0^- to t_1^-. And by selecting $t_j^\pm$ close enough to t_j we can connect t_j^+ to t_j^- by a smooth arc crossing $B'^0(c)$ only at t_j ($j=0, 1$), and there transversally. In such a way we obtain a continuous curve $\gamma\subset B'$, passing through t_0 and t_1, closed, such that the two components of $\gamma\setminus(\{t_0\}\cup\{t_1\})$, which we shall call γ^+ and γ^-, lie entirely in $B'^+(c)$ and $B'^-(c)$ respectively After smoothing we may suppose that γ is C^∞ and diffeomorphic to the unit circle. We may arrange that the diffeomorphism maps γ^+ onto the upper half-circle, γ^- onto the lower one, and maps the point t_0 onto $(-1,0)$, and the point t_1 onto $(1,0)$. Using coordinates (ξ,η) in the plane and the parameter ξ on both γ^+ and γ^- (pulled back from the upper and lower semicircles), call l_ξ the straight-line segment (in t-space) joining t_ξ^- to t_ξ^+, the points on γ^- and γ^+ respectively, corresponding to the value ξ ($0<\xi<1$) of the parameter. After this we map linearly (and so as to preserve the orientation) onto each l_ξ the vertical segment joining the point $(\xi, -\sqrt{1+\xi^2})$ to the point $(\xi, \sqrt{1+\xi^2})$ in the plane. This defines a continuous mapping π of the open unit disk onto the subset

$$\mathfrak{S} = \bigcup_{0<\xi<1} l_\xi$$

of t-space; the mapping π extends continuously as a mapping of the unit circumference onto the curve γ (which is the boundary of $\mathfrak{S}$). We may therefore pull-back the function φ from $\mathfrak{S}\cup\gamma$ to the closed unit disk $\bar{D}$. It will suffice to show that $(1,0)$ and $(-1,0)$ belong to one and the same connected component of the level curve of $\varphi\circ\pi$ that contains those two points. We note that the level curve in question ($\varphi\circ\pi=c$) intersects the boundary of $\bar{D}$ only at $(1,0)$ and $(-1,0)$ and, by virtue of our construction, is the graph of a continuous function of ξ in the vicinity of both those points. If $(1,0)$ and $(-1,0)$ belonged to two different components, A_1 and A_{-1}, of the level curve $\varphi\circ\pi=c$ in $\bar{D}$, it would be possible to draw a smooth closed curve (without self-intersections) in the plane, winding around A_1 and not intersecting at all the set

$$\{(\xi,\eta)\in\bar{D};\ \varphi(\pi(\xi,\eta))=c\}.$$

Such a curve per force would intersect the upper semicircumference, and also the lower one, and therefore one of its halves (the one lying in $\bar{D}$) would join a point on which $\varphi\circ\pi<c$ to one on which $\varphi\circ\pi>c$ without $\varphi\circ\pi$ ever equalling c on it, which is absurd.

Assume now c to be a critical value of φ in $\mathrm{Cl}\,B'$, and that t_0 and t_1 lie on two disjoint connected components of the level set $\varphi=c$ in $\mathrm{Cl}\,B'$, C_0 and C_1. Since C_0 and $(\mathrm{Cl}\,B')^0(c)\setminus C_0$ are compact we can find two disjoint open subsets of B, W_0 and W, containing each one of those sets respectively. Note that $(\mathrm{Cl}\,B')\setminus(W_0\cup W)$ is a compact set K, and $\varphi(t) \neq c$ for every t in K. We can find two points t'_j $(j=0, 1)$ on the straight-line segment joining t_0 to t_1, such that $t'_0 \in W_0$ and $t'_1 \in W$, $\varphi(t'_0)=\varphi(t'_1)=c'$ not a critical value of φ in $\mathrm{Cl}\, B'$, and $c' \notin \varphi(K)$. By the first part of the proof we know that there is a connected analytic hypersurface $M' \subset B'^0(c')$ containing both t'_0 and t'_1. Since W_0 and W are disjoint M' must intersect K, contrary to the fact that $c' \notin \varphi(K)$, whence a contradiction. Q.E.D.

We relate now the properties (3.5), (3.6), (3.7) to the behavior of the function φ along certain curves in B. By a *piecewise analytic curve* in B we mean a continuous map

$$[0, 1] \ni s \mapsto t(s) \in B$$

which is analytic, except possibly at a finite number of points $0 \leq s_0 < s_1 < \ldots < s_\nu \leq 1$. We shall say that the curve *joins* $t(0)$ to $t(1)$. We shall make use of the following important result (for a proof, see [3]):

$$\text{any two points in a connected analytic subset } A \text{ of } \mathbf{R}^n \text{ can be joined by a piecewise analytic curve entirely contained in } A. \tag{3.9}$$

LEMMA 3.3. *Property* (3.6) *is equivalent to the following one:*

$$\textit{any two points in } B'', t_0, t_1, \textit{ can be joined by a piecewise analytic curve in } \mathrm{Cl}\, B' \textit{ on which } \varphi \textit{ is monotone.} \tag{3.10}$$

Proof. Let us first show that (3.6) implies (3.10). If $\varphi(t_j)=c$, $j=0, 1$, (3.7) (Lemma 3.2) tells us that t_0 and t_1 belong to one and the same connected component of $(\mathrm{Cl}\, B')^0(c)$. Call S' the sphere in $\mathbf{R}^{m+1}$ (where the variable is denoted by $(t^0, t^1, \ldots, t^m)$) centered at the origin and having the same radius as the ball $B' \subset \mathbf{R}^m$. Regard φ as a function defined (and analytic) on S'—which happens not to depend on t^0. Call $\hat{t}_j$ the (unique) point in the upper hemisphere which projects onto t_j (via the coordinate projection $(t^0, t^1, \ldots, t^m) \mapsto (t^1, \ldots, t^m)$; $j=0, 1$). Obviously $\hat{t}_0$ and $\hat{t}_1$ belong to the same connected component of $S'^0(c)$, which is an analytic set. We apply (3.9) and thus get a piecewise analytic curve $\hat{\gamma}$, joining $\hat{t}_0$ to $\hat{t}_1$ and entirely contained in $S'^0(c)$. Projecting $\hat{\gamma}$ into $\mathrm{Cl}\, B'$ provides a piecewise analytic curve joining t_0 to t_1 on which $\varphi=c$.

Suppose $\varphi(t_0)<\varphi(t_1)$ and let l denote the straight-line segment joining t_0 to t_1. Call t'_0 the point on l closest to t_1 such that $\varphi(t'_0)=\varphi(t_0)$, and t'_1 the point between t'_0 and t_1 closest to t'_0 such that $\varphi(t'_1)=\varphi(t_1)$. By the first part of the proof we can join t_j to t'_j by a piecewise analytic curve on which φ is constant. We may therefore assume $t_j=t_j'$ for $j=0,1$. If the derivative of φ along l does not change sign our contention is trivial. Suppose it does and let s_0 be the point closest to t_0 where $\varphi|_l$ reaches a local maximum. Note that $s_0 \neq t_0$ for the derivative of φ along l must be positive in some open interval $]t_0, t_0+\delta[$, and that $\varphi(t_0)<\varphi(s_0)<\varphi(t_1)$, otherwise we could not have $t_j'=t_j$ for both $j=0,1$. Of course it suffices to join s_0 to t_1 by a piecewise analytic curve on which φ is monotone. But we may repeat the reasoning just described after substitution of s_0 for t_0. Since $\varphi|_l$ has only a finite number of extrema we reach the desired goal after a finite number of such repetitions.

Let us now prove that (3.10) implies (3.6). Let t_0, t_1 be two points in B'' such that $\varphi(t_0)\geqslant\varphi(t_1)>c$. By (3.10) they are joined by a piecewise analytic curve γ in $\mathrm{Cl}\,B'$ such that, for all t in γ, $\varphi(t_0)\geqslant\varphi(t)\geqslant\varphi(t_1)$. Of course γ might have one or more arcs lying on the sphere $\partial B'$. But by performing (for instance) a contraction $t\mapsto(1-\varepsilon)\,t$ one can bring such arcs inside B' and connect the end-points of the new arcs to portions of γ inside B' in such a way as to obtain a piecewise analytic curve $\gamma'\subset B'^{+}(c)$ joining t_0 to t_1. Q.E.D.

At this stage we re-introduce the variable x. If A is any subset of $U=B_r\times J$ and x, y any pair of real numbers, we write

$$\begin{aligned} A^{+}(x,y) &= \{p\in A;\ x(p)=x,\ \Phi(p)>y\},\\ A^{-}(x,y) &= \{p\in A;\ x(p)=x,\ \Phi(p)<y\}. \end{aligned} \tag{3.11}$$

Of course, $A^{+}(x,y)$ or $A^{-}(x,y)$ might be empty, as when $x\notin J$.

PROPOSITION 3.1. *Property* (3.1) *is equivalent to each one of the following properties:*

Every open neighborhood $V_p\subset U$ of p contains another open neighborhood of p, W_p, such that, given any pair of real numbers x, y, W_p intersects at most one connected component of $V_p^{+}(x,y)$, and at most one of $V_p^{-}(x,y)$. (3.12)

Every open neighborhood $V_p\subset U$ of p contains another open neighborhood of p, W_p, such that any two points in W_p, of the kind (t_0,x), (t_1,x) can be joined by a piecewise analytic curve in V_p on which x is constant and Φ monotone. (3.13)

Proof. Notice that each one of the properties under consideration, (3.1), (3.12) and

(3.13), remains valid if we increase V_p or decrease W_p. We may therefore assume that $V_p=B'\times J'$, $W_p=B''\times J''$, with B', B'' open balls in $\mathbf{R}^m$ centered at $t(p)$, and J', J'' open intervals in $\mathbf{R}^1$ centered at x(p). We introduce an additional open ball B, centered at $t(p)$, with $B'\subsetneqq B$. If then V_p and W_p are as in (3.1) we derive from Lemma 3.1 that they satisfy the condition in (3.12). Conversely, if the latter is true, then, by Lemma 3.2, every fiber of Z in W_p is contained in a single connected component of a fiber of Z in $(\mathrm{Cl}\,B'')\times J'$ (actually, in $(\mathrm{Cl}\,B')\times J''$), and therefore in a single component of a fiber of Z in $B\times J'$. Since B' is arbitrarily small so is B, whence (3.1). The same argument, but based on Lemma 3.3 rather than 3.2, shows the equivalence of (3.12) and (3.13). Q.E.D.

The version (3.12) of Condition (P) is of the same kind as the solvability condition in [6] (see p. 288).

The version (3.13) of (P) generalizes the standard definition of (P) in the case of a single vector field (see [4], [5]), as we now show.

Indeed separate the coefficients λ_j in (1.9) into their real and imaginary parts:

$$\lambda_j = a_j+\sqrt{-1}\,b_j. \tag{3.14}$$

Note that, with this notation, (1.9) reads $-i\Phi_{t^j}=(a_j+ib_j)(1+i\Phi_x)$, whence

$$d_t\Phi = -|Z_x|^2\sum_{j=1}^{m} b_j\,dt^j. \tag{3.15}$$

This shows that the one-form $b=\Sigma_{j=1}^m b_j dt^j$ has a real-valued, analytic and nowhere vanishing integrating factor. At any rate if makes sense to say that b does not change sign along a given piecewise analytic curve in t-space, and therefore also on any curve in U, of that nature, on which x=Constant: it means of course that the scalar product between b and the oriented unit tangent vector to the curve does not change sign along it. It is of course equivalent to the property that the restriction of Φ to the curve is monotone.

4. Proof of Theorem 2.1

We use the notation of Sections 1, 2, 3. Our starting point will be the hypothesis that Condition (P) is not satisfied at the origin (Definition 2.1). Actually it is convenient to make use of the version (3.12) of (P), or rather of its negation. Let us for instance assume that the following property holds:

$$\text{There is an open neighborhood } V \subset U \text{ of the origin, and a sequence of points in } \mathbf{C},\ z_\nu = x_\nu + iy_\nu,\ \nu = 1, 2, \ldots, \text{ converging to zero, such that any neighborhood of the origin, } W \subset V, \text{ intersects two distinct connected components of } V^+(x_\nu, y_\nu) \text{ (see (3.11)) for some } \nu. \tag{4.1}$$

Note that (4.1) remains valid if we *decrease* V. Thus we shall assume that $V \Subset U = B_r \times J$, and that $V = B_{r_0} \times J_0$. Possibly after a change of subscripts $\nu = 1, 2, \ldots$, we select a sequence of open neighborhoods

$$W_\nu = B_{r_\nu} \times J_\nu, \tag{4.2}$$

with $r_0 > r_\nu \searrow +0$, $J_\nu =]-r_\nu, r_\nu[$, such that, for each ν, W_ν intersects at least two distinct connected components of $V^+(x_\nu, y_\nu)$, $C_{1\nu}$ and $C_{2\nu}$.

Fix x_0 in J. Then the number of critical values of the mapping $Z(t, x)$ in $\mathrm{Cl}\, V$ that lie on the vertical $\mathrm{Re}\, z = x_0$ is finite. Indeed, they are the values of Z on the set of points (t, x) in $\mathrm{Cl}\, V$ such that

$$x = x_0, \quad d_t \Phi(t, x_0) = 0. \tag{4.3}$$

But in the neighborhood of $\mathrm{Cl}\, V$ the equations (4.3) define an analytic set, of which only finitely many connected components intersect the compact set $\mathrm{Cl}\, V$, and Z is constant on each of these components. This implies that, for each ν, there is $y'_\nu > y_\nu$ such that the fibre of Z in $\mathrm{Cl}\, V$,

$$F(z'_\nu) = \{(t, x) \in \mathrm{Cl}\, V;\ Z(t, x) = z'_\nu = x_\nu + iy'_\nu\} \tag{4.4}$$

intersects both $W_\nu \cap C_{1\nu}$ and $W_\nu \cap C_{2\nu}$, and such that z'_ν is not a critical value of Z in $\mathrm{Cl}\, V$. But then of course W_ν must intersect two distinct components of $F(z'_\nu)$. In other words, we may start from the following hypothesis:

$$\text{There is a totally ordered basis of open neighborhoods of the origin, } W_\nu \subset V, \text{ and a sequence of complex numbers } z_\nu, \text{ converging to zero, none of which is a critical value of } Z(t, x) \text{ in } \mathrm{Cl}\, V, \text{ such that, for each } \nu,\ W_\nu \text{ intersects two distinct connected components of the fiber } F(z_\nu). \tag{4.5}$$

For each $\nu = 1, 2, \ldots$, we select a *closed* disk D_ν, centered at z_ν, with radius $d_\nu > 0$. In the argument below we shall decrease d_ν a finite number of times. First of all we select d_ν small enough that the following conditions are fulfilled:

$$\text{for each } \nu, D_\nu \text{ is entirely contained in the (open) set of noncritical values of } Z(t,x) \text{ in } \mathrm{Cl}\, V, \text{ and in the interior of the image } Z(W_\nu); \tag{4.6}$$

$$\text{the projections into the real axis of the } D_\nu \text{ are pairwise disjoint.} \tag{4.7}$$

For each ν, let C_ν^+ and C_ν^- denote two distinct components of $F(z_\nu)$ which intersect W_ν. Possibly after decreasing d_ν we may make the following assumption:

$$\text{there are two analytic submanifolds of dimension two, } \Sigma_\nu^+ \text{ and } \Sigma_\nu^-, \text{ which intersect respectively } C_\nu^+ \text{ and } C_\nu^-, \text{ and whose closures are disjoint compact subsets of } W_\nu, \text{ each mapped diffeomorphically onto } D_\nu \text{ by } Z. \tag{4.8}$$

And possibly after some more decreasing of d_ν we select two open neighborhoods of C_ν^+ and C_ν^- respectively, in U, $\mathfrak{C}_\nu^+$ and $\mathfrak{C}_\nu^-$, endowed with the following properties:

$$(\mathrm{Cl}\,\mathfrak{C}_\nu^+) \cap (\mathrm{Cl}\,\mathfrak{C}_\nu^-) = \varnothing; \tag{4.9}$$

$$\Sigma_\nu^+ \subset \mathfrak{C}_\nu^+,\ \Sigma_\nu^- \subset \mathfrak{C}_\nu^- \text{ and the image via } Z \text{ of } \mathfrak{C}_\nu^+, \text{ as well as that of } \mathfrak{C}_\nu^-, \text{ is exactly equal to } \mathrm{Int}\, D_\nu; \tag{4.10}$$

$$\text{any connected component of a fiber } F(z) \text{ of } Z \text{ in } \mathrm{Cl}\, V \text{ which intersects } \mathfrak{C}_\nu^\pm \text{ is entirely contained in } \mathfrak{C}_\nu^\pm; \tag{4.11}$$

$$\text{no two distinct connected components of the same fiber } F(z) \text{ intersect either } \mathfrak{C}_\nu^+ \text{ or } \mathfrak{C}_\nu^-. \tag{4.12}$$

For each $\nu = 1, 2, \ldots$, let r'_ν be a number such that $r_\nu < r'_\nu < r_{\nu-1}$ and set $W'_\nu = B_{r'_\nu} \times J'_\nu$, $J'_\nu =]-r'_\nu, r'_\nu[$. We consider a distribution u in W'_ν which is a solution of the inhomogeneous equations (2.1). We shall assume that the right-hand sides are continuous functions in V, and of course satisfy (2.2) in V. Furthermore we assume that

$$\mathrm{supp} f_j \subset Z^{-1}(0) \cup \bigcup_{\nu=1}^{+\infty} \mathfrak{C}_\nu^+. \tag{4.13}$$

The reader will easily check that the set at the right has an intersection with Cl V that is closed. Note also that we have

$$L_j u = 0, \quad j = 1, \ldots, m, \tag{4.14}$$

in the set

$$W'_\nu \setminus \mathrm{Cl}\left(\bigcup_{\nu=1}^{+\infty} \mathfrak{C}_\nu^+\right). \tag{4.15}$$

We introduce, for each $\nu=1,2,\ldots$, a closed disk D'_ν, also centered at z_ν, with radius $d'_\nu > d_\nu$, such that the properties analogous to (4.6), (4.7), (4.8) hold. We call A_ν the annulus $D'_\nu \setminus D_\nu$.

Notice that $L=(L_1,\ldots,L_m)$ is an elliptic system in the pre-image of D'_ν via Z, and therefore u is an analytic function in the set

$$\mathfrak{A}_\nu = W_\nu \cap Z^{-1}(A_\nu).$$

The key to the proof of Theorem 2.1 lies in the following assertion:

$$u \text{ is constant on the fibers of } Z \text{ in } \mathfrak{A}_\nu. \tag{4.16}$$

Proof of (4.16): We note that (4.14) holds in the set

$$\{(t,x);\ |t|<r_\nu,\ d_\nu<|x-x_\nu|<d'_\nu\}. \tag{4.17}$$

We apply Theorems I, II (Section 1) taking $U'=B_{r'}\times J'$ to have compact closure contained in (4.17). According to the remarks at the end of Section 1 we conclude that u is the distribution limit in (4.17) of a sequence of polynomials of Z and that it is the C^∞ limit of that sequence in the intersection of (4.17) with $\mathfrak{A}_\nu$. Thus u must be constant on the fibres of Z in that intersection.

Let us call $\mathfrak{O}$ the interior of the subset $\mathfrak{S}$ of $Z(\mathfrak{A}_\nu)$ such that

$$u \text{ is constant on the fibres of } Z \text{ in } Z^{-1}(\mathfrak{S})\cap\mathfrak{A}_\nu. \tag{4.18}$$

We have just shown that $\mathfrak{O}$ contains the set

$$z\in \mathrm{Int}\,A_\nu,\quad |\mathrm{Re}\,z-x_\nu|>d_\nu. \tag{4.19}$$

Suppose now there is a point z^* in the boundary of $\mathfrak{O}$ with respect to A_ν. We apply once again Theorem I, availing ourselves of the fact that u is a solution of (4.14) in an open neighborhood of $S^*=F(z^*)\cap \mathrm{Cl}\,W_\nu$. There is a number $\delta>0$ such that every point $p^*\in S^*$ is the center of an open ball with radius δ in which u is the C^∞ limit of a sequence of polynomials in Z. Note that the sequence in question may change from point to point. We may suppose that the union of all those balls is contained in a compact subset K of $W'_\nu(\supset W_\nu)$. The restriction of Z to K is of course open, and therefore there is a closed disk D^* centered at z^* with the following property:

$$\mathfrak{A}_\nu \cap Z^{-1}(D^*) \subset \{p \in W_\nu;\ \mathrm{dist}\,(p, S^*) \leq \delta\}. \tag{4.20}$$

Let then $p_j \in \mathfrak{A}_\nu$ be such that $Z(p_j)=\zeta \in D^*$ $(j=1,2)$. We can find $p_j^* \in S^*$ such that $|p_j^*-p_j| \leq \delta$, and there is a continuous function $\tilde{u}_j$ in D^* such that $u=\tilde{u}_j \circ Z$ in the ball centered at p_j^* with radius δ. Moreover, since the system L is elliptic in a full neighborhood of S^*, $\tilde{u}_j$ is holomorphic in an open disk $D'^* \subset D^*$, also centered at z^*, and which can be selected independently of the point p_j^* on S^*. But since $\tilde{u}_1=\tilde{u}_2$ in $D'^* \cap \mathfrak{D}$, we must have $\tilde{u}_1=\tilde{u}_2$ in D'^*, and therefore $D'^* \subset \mathfrak{D}$, which contradicts the fact that its center is a boundary point of $\mathfrak{D}$. We must therefore have $\mathfrak{D}=A_\nu$. Q.E.D.

We draw right-away a consequence of (4.16). Because of the validity of (4.8) when D'_ν is substituted for D_ν, we see that

$$Z(\mathfrak{A}_\nu) = A_\nu.$$

Therefore there is a continuous function in A_ν, $\tilde{u}$, holomorphic in the interior of A_ν, such that $u=\tilde{u} \circ Z$ in $\mathfrak{A}_\nu$. We contend that

$$\tilde{u} \text{ extends holomorphically to the interior of } D'_\nu. \tag{4.21}$$

Indeed call $\Sigma_\nu'^-$ the analogue of Σ_ν^- (see (4.8)) when D'_ν is substituted for D_ν. Since u is a solution of the system of equations (4.14) in some open neighborhood of $\Sigma_\nu'^-$, in which that system is elliptic, u is an analytic function there, and its restriction to $\Sigma_\nu'^-$ is analytic. Let $\tilde{\tilde{u}}$ be the push-forward of the restriction of u to $\Sigma_\nu'^-$ via Z; it defines a real-analytic function in $\mathrm{Int}\, D'_\nu$. In some open neighborhood of each point of $\Sigma_\nu'^-$ u is a uniform limit of polynomials with respect to Z, by Theorem I, as a consequence of which we see that $\tilde{\tilde{u}}$ must be holomorphic in the interior of D'_ν. Since $\tilde{\tilde{u}}=\tilde{u}$ in A_ν this proves our assertion.

We can now proceed with the construction of the functions f and g in Theorem 2.1.

For each ν we select arbitrarily a closed disk D_ν^* centered at z_ν with radius $d_\nu^* < d_\nu$. Let then $\tilde{f}$ be a C^∞ function in the plane, vanishing identically in the complement of the union of the disks D_ν^*, and such, moreover, that

$$\text{for every } \nu = 1, 2, \dots,\ \tilde{f} > 0 \text{ in } \mathrm{Int}\, D_\nu^*. \tag{4.22}$$

Then we define

$$f = \tilde{f} \circ Z \text{ in } V \cap \Bigl(\bigcup_{\nu=1} \mathfrak{C}_\nu^+ \Bigr), \quad f \equiv 0 \text{ everywhere else.} \tag{4.23}$$

Clearly f is a C^∞ function in $V \setminus Z^{-1}(0)$, and vanishes of infinite order on $V \cap Z^{-1}(0)$; thus $f \in C^\infty(V)$.

We contend that the assertion (2.4) is correct.

Proof. It suffices to check (2.2) in some neighborhood of an arbitrary point of $V \cap (\bigcup_\nu \mathfrak{C}_\nu^+)$. There $f = \tilde{f} \circ Z$ and therefore

$$L_j f = \left(\frac{\partial \tilde{f}}{\partial \bar{z}} \circ Z\right) L_j \bar{Z}, \quad j = 1, \ldots, m,$$

since $L_j Z = 0$. Therefore, if we apply (1.10), and set $\tilde{f}_1 = 2(\partial \tilde{f}/\partial \bar{z})$, $f_1 = \tilde{f}_1 \circ Z$, we have:

$$L_j f = \lambda_j f_1. \tag{4.24}$$

On the other hand, the commutation relations (1.13) are equivalent to

$$L_j \lambda_k = L_k \lambda_j, \quad j, k = 1, \ldots, m. \tag{4.25}$$

Combining (4.24) and (4.25) yields at once

$$L_j(\lambda_k f) = L_k(\lambda_j f), \quad j, k = 1, \ldots, m. \tag{4.26}$$

Q.E.D.

Next we define the second function in Theorem 2.1, g. For this we need f to be small enough that the following condition be satisfied:

$$Z_x - f \neq 0 \quad \text{everywhere in } V. \tag{4.27}$$

Since $Z_x = 1 + i\Phi_x$ this is easy to achieve. We take then

$$g = \frac{f}{f - Z_x}, \tag{4.28}$$

and let $L_j^\#$ be the vector fields so denoted in Theorem 2.1.

We now prove that the assertion (2.5) is correct, that is,

$$[L_j^\#, L_k^\#] = 0 \quad j, k = 1, \ldots, m. \tag{4.29}$$

Proof. By virtue of (4.24) we have:

$$L_j g = \lambda_j \varphi + f L_j Z_x / (f - Z_x)^2,$$

with $\varphi = -Z_x f_1/(f - Z_x)^2$. Differentiating the equation $L_j Z = 0$ with respect to x yields:

$$L_j Z_x = -\lambda_{jx} Z_x, \tag{4.30}$$

and thus

$$L_j g+\lambda_{jx} g(g-1)=\lambda_j \varphi, \quad j=1, \ldots, m. \tag{4.31}$$

On the other hand, $[L_j^{\#}, L_k^{\#}]=q\partial/\partial x$, where

$$\begin{aligned} q &= L_k(\lambda_j g)-L_j(\lambda_k g)+\lambda_j g(\lambda_k g)_x-\lambda_k g(\lambda_j g)_x+\lambda_k g\lambda_{jx}-\lambda_j g\lambda_{kx} \\ &=\lambda_j A_k-\lambda_k A_j, \end{aligned}$$

where

$$A_j=L_j g+g^2\lambda_{jx}-g\lambda_{jx}=\lambda_j\varphi, \quad j=1,\ldots,m, \text{ by (4.31)}.$$

This means that $q\equiv 0$. Q.E.D.

Next we prove Assertion (2.6).

We shall prove that, given an arbitrary integer $\nu\geq 1$, there is no distribution u satisfying (2.1) in W'_ν.

Observing that the function $\tilde{f}$ used to define f has compact support, set

$$\tilde{v}=\tilde{f}*(1/2\pi z),$$

where $*$ is the convolution of distributions in the plane. We have, in $\mathbf{R}^2$,

$$\frac{\partial\tilde{v}}{\partial\bar{z}}=\tilde{f}/2. \tag{4.32}$$

Set $v=\tilde{v}\circ Z$ in V. We have, in a neighborhood $\mathfrak{U}$ of $(\mathrm{Cl}\,\mathfrak{C}_\nu^+)\cap V$,

$$L_j v=\left(\frac{\partial\tilde{v}}{\partial\bar{z}}\circ Z\right)L_j\bar{Z}=\lambda_j f=f_j, \quad j=1,\ldots,m,$$

by (1.10), and therefore, by (2.1), we have, in $\mathfrak{U}\cap W'_\nu$,

$$L_j(u-v)=0, \quad j=1,\ldots,m, \tag{4.33}$$

We have the right to take $\mathfrak{U}$ such that it contains the surface $\Sigma_\nu'^{+}$ analogous to Σ_ν^+ in (4.8), when D'_ν is substituted for D_ν. Once again by ellipticity we know that $u-v$ is analytic in some neighborhood of $\Sigma_\nu'^{+}$, and its restriction to $\Sigma_\nu'^{+}$ can be pushed forward via Z as a real analytic function $\tilde{w}$ in $\mathrm{Int}\,D'_\nu$. And again, by Theorem I, we know that the latter is a uniform limit of polynomials of z in the neighborhood of each point of $\mathrm{Int}\,D'_\nu$, therefore $\tilde{w}$ is holomorphic in that set. Since $\tilde{u}$ can be extended holomorphically to $\mathrm{Int}\,D'_\nu$, the same is necessarily true of $\tilde{v}$. This demands

$$\int_{\partial D_\nu} \bar{v}\, dz = 0, \tag{4.34}$$

and therefore, by Stokes' theorem,

$$\int_{D_\nu} \bar{f}\, dz \wedge d\bar{z} = 0, \tag{4.35}$$

which contradicts (4.22).

Finally we prove the assertion (2.7).

We choose f so small that $|g|<1$. Then, if ν is large enough, the system $L^{\#}=(L_1^{\#}, \ldots, L_m^{\#})$ is elliptic in some neighborhood (in V) of $V \cap \mathrm{Cl}\,\mathfrak{C}_\nu^+$. Fixing ν thus we call $\mathfrak{U}$ that neighborhood, and assume that it contains $\Sigma_\nu'^+$, as we did above. Therefore, if $h \in C^1(W_\nu')$ satisfies the homogeneous equations (2.3) in W_ν', h will be a C^∞ function in $\mathfrak{U} \cap W_\nu'$. We rewrite (2.3) as follows:

$$L_j h = \lambda_j g h_x, \quad j = 1, \ldots, m. \tag{4.36}$$

Thus we see that the right-hand sides are C^∞ functions off $Z^{-1}(0)$. We are going to use the following property:

$$gh_x \text{ is locally constant on each fiber of } Z \text{ in } W_\nu' \cap \mathfrak{C}_\nu^+. \tag{4.37}$$

Proof. The fibers of Z in $\mathfrak{C}_\nu^+$ are connected analytic submanifolds of dimension $m-1$ on which $d_t\Phi \neq 0$. It suffices therefore to show that the differential of gh_x along those submanifolds vanishes identically or, which is the same, that at each point in $\mathfrak{C}_\nu^+$, $d_t(gh_x)$ is collinear to $d_t\Phi$ or, again equivalently,

$$d_t(gh_x) \wedge d_t\Phi = 0. \tag{4.38}$$

We derive from (4.36):

$$\begin{aligned} 0 = [L_j, L_k]\, h &= L_j(\lambda_k g h_x) - L_k(\lambda_j g h_x) \\ &= (L_j\lambda_k - L_k\lambda_j)\, g h_x + \lambda_k L_j(gh_x) - \lambda_j L_k(gh_x) \\ &= \lambda_k \frac{\partial}{\partial t^j}(gh_x) - \lambda_j \frac{\partial}{\partial t^k}(gh_x) \end{aligned}$$

by (1.8) and (4.25). Applying then (1.9) yields at once (4.38). Q.E.D.

In particular, according to (4.37), gh_x is constant on the fibers of Z in a suitably small open neighborhood of $\Sigma_\nu'^{+}, \mathfrak{V}$. We shall call $\tilde{q}$ the push-forward via Z of the restriction of gh_x to $\mathfrak{V}$. Since g is a multiple of f, $\tilde{q}$ vanishes identically in $Z(\mathfrak{V})\setminus D_\nu^*$.

On the other hand, since $L_j h=0, j=1, \ldots, m$, in the set (4.15), the conclusions (4.16) and (4.21) are valid when $u=h$.

Thus we find ourselves in the same circumstances as in the proof of (2.6), but now with h playing the role of u and $\tilde{q}$ that of $\tilde{f}$. We may then introduce $\tilde{h}_1=\tilde{q}\ast(1/2\pi z)$ and $h_1=\tilde{h}_1\circ Z$. The reasoning applied to $u-v$ in the remarks that follow (4.33) applies equally well to $h-h_1$ (noting that we reason only in a neighborhood of $\Sigma_\nu'^{+}$). We reach the conclusion analogous to (4.35):

$$\int_{D_\nu} \tilde{q}\, dz \wedge d\bar{z} = 0. \tag{4.39}$$

Suppose we had $h_x \neq 0$ at the origin. Since

$$gh_x = fh_x/(f-Z_x),$$

we see that $\tilde{q}$ is the product of $\tilde{f}$ by a continuous function which is different from zero at the origin. As $\nu \to +\infty$ the argument of this function in D_ν is arbitrarily close to its argument at the origin, while $\tilde{f} \geq 0$ everywhere and $\tilde{f}>0$ in $\operatorname{Int} D_\nu^*$. This precludes that (4.39) be true, and therefore we must have $h_x=0$ at the origin. But then the equations (2.3) (or (4.36)), and the expressions (1.8) of the vector fields L_j, demand that $d_t h$ also be equal to zero at the origin, whence (2.7).

The proof of Theorem 2.1 is complete.

5. Geometric preliminaries to the proof of Theorem 2.2

We shall take the neighborhood V, in the statement of Theorem 2.2, in the form $B' \times J'$, where B' is an open ball in $\mathbf{R}^m$, centered at the origin, and J' an open interval in $\mathbf{R}^1$ (and $V \Subset U = B \times J$).

We shall begin by applying the following result of [3]:

The image $Z(\operatorname{Cl} V)$ is a finite disjoint union of connected analytic submanifolds of $\mathbf{R}^2$, M_i $(1 \leq i \leq \nu)$, having the following property: for each i, $Z^{-1}(M_i)$ is a finite disjoint union of connected analytic submanifolds of $\operatorname{Cl} V, N_{i,j}$ $(j=1, \ldots, \mu_i)$, *such that the restriction of Z to $N_{i,j}$ is an analytic map of constant rank onto M_i, and such furthermore that every $N_{i,j}$ is a subanalytic* ([3], *Definition* 3.1) *subset of* $\operatorname{Cl} V$.

Recalling that $Z=x+\sqrt{-1}\,\Phi(t,x)$, and writing $z=x+\sqrt{-1}\,y$ we apply the above to the *boundary* of $Z(\mathrm{Cl}\,V)$. It consists of two vertical segments, to which we do not pay attention, and of two pieces defined by piecewise analytic equations

$$y=F^{\pm}(x), \quad x\in J'. \tag{5.1}$$

Of course we have

$$F^{+}(x)=\sup_{t\in B'}\Phi(t,x), \quad F^{-}(x)=\inf_{t\in B'}\Phi(t,x). \tag{5.2}$$

In passing note that these are continuous functions of x (in addition to being piecewise analytic).

We may and shall contract J' about zero in such a way that F^{+} and F^{-} are both analytic in $\mathrm{Cl}\,J'$ for $x\neq 0$. Then, still disregarding the vertical portions of the boundary of $Z(\mathrm{Cl}\,V)$, and also the points in that boundary corresponding to $x=0$ (there are one or two such points), we are left with *four* analytic curves. If we call x_1 and x_2 the boundary points of J', the curves on the left (i.e., for $x<0$) are

$$\tilde{C}_l^{\pm}: x_1<x<0, \quad y=F^{\pm}(x),$$

and the ones on the right,

$$\tilde{C}_r^{\pm}: 0<x<x_2, \quad y=F^{\pm}(x).$$

Again by the results of [3], quoted at the beginning, we can select four connected analytic submanifolds of $\mathrm{Cl}\,V$, which are also subanalytic in $\mathrm{Cl}\,V$, $C_l^{\pm}$, $C_r^{\pm}$, such that Z maps each one of them onto the corresponding $\tilde{C}$. We select four points $t_l^{\pm}$, $t_r^{\pm}$ in $\mathrm{Cl}\,B'$, such that $(t_l^{\pm},0)$ belongs to the closure of $C_l^{\pm}$, $(t_r^{\pm},0)$ to that of $C_r^{\pm}$. Then we apply Proposition 3.9 of [3]:

There is an analytic map

$$]-1,1[\,\ni s\to(t(s),x(s))\in\mathbf{R}^{m+1}$$

such that

$$t(0)=t_r^{+},\ x(0)=0, \quad \text{and} \quad (t(s),x(s))\in C_r^{+} \quad \text{for } s\neq 0.$$

Necessarily $x(s)=s^{2k}[c_0+O(s)]$ for some integer $k\geqslant 1$ and some $c_0>0$. Therefore, for some $\eta>0$, we have an inverse of the map $s\to x(s)$,

$$[0,\eta]\in x\mapsto s=\chi(x^{1/2k})\in[0,1[,$$

with χ analytic in an open neighborhood of the closed interval $[0,\eta^{1/2k}]$. We may then define

$$t^+(x)=t(\chi(x^{1/2k})),\quad x\in[0,\eta].$$

Similar reasonings lead to the definitions of $t^+(x)$ in $[-\eta,0]$, and of $t^-(x)$ in $[0,\eta]$ and in $[-\eta,0]$, if necessary after a decrease of η. Note that, as the points t_r^+ and t_l^+ may be different, the limits of $t^+(x)$ as x converges to zero from the right and from the left are not necessarily equal; same remark about $t^-(x)$. At any rate we may state:

There are two analytic maps from $J'\setminus 0$ to $\mathrm{Cl}\,B'$, $x\mapsto t^\pm(x)$, such that, for all (t,x) in V, (5.3)

$$\Phi(t^-(x),x)\leqslant\Phi(t,x)\leqslant\Phi(t^+(x),x). \tag{5.4}$$

Furthermore, there are points $t_r^\pm$, $t_l^\pm$ in $\mathrm{Cl}\,B'$ such that

$$t_r^\pm=\lim_{x\to+0}t^\pm(x),\quad t_l^\pm=\lim_{x\to-0}t^\pm(x), \tag{5.5}$$

and there is an integer $q\geqslant 1$ such that

$$\left|\frac{d}{dx}t^\pm(x)\right|\leqslant\text{const.}\,|x|^{-1+1/q},\quad 0\neq x\in J'. \tag{5.6}$$

Another property of analytic sets we shall need is embodied in the following assertion (Corollary 1, Appendix, § 1):

There is an integer $N_0\geqslant 0$ such that, given any straight line in $\mathbf{R}^{m+1}$, l, on which x is constant, the derivative of Φ in the direction of l changes sign at most N_0 times in $l\cap\mathrm{Cl}\,V$. (5.7)

This follows from the compactness of $\mathrm{Cl}\,V$ and the analyticity of Φ.

Let then $\tilde V=\tilde B'\times\tilde J'\Subset V$, endowed with the following property:

$\tilde V$ intersects at most one connected component of each fiber of Z in V. (5.8)

We are now going to take up anew the proof of Lemma 3.3 and make it "metrically" more precise. Let $x_0\in\tilde J'$, $t_j\in\tilde B'$ $(j=0,1)$. For simplicity let us write $\varphi(t)=\Phi(x_0,t)$. We assume that

$$\varphi(t_0) \leq \varphi(t_1). \tag{5.9}$$

We denote by γ_0 the straight-line segment joining t_0 to t_1 oriented from the former to the latter. Let $J_0=[t_0, s_0]$ be the largest interval of this kind in γ_0 on which $\varphi' \geq 0$ (primes will denote derivation in the direction of γ_0). It might happen that $s_0=t_0$ if φ starts decreasing right-away on starting from t_0. At any rate let s_0' be the closest point to t_1 on γ_0 such that $\varphi(s_0')=\varphi(s_0)$. Clearly s_0' must belong to an interval $[s_0', s_1]$ with $s_1>s_0'$ (for the natural order on γ_0), on which $\varphi' \geq 0$. We take s_1 to be as close to t_1 as possible, and then repeat the argument just presented for s_1 in the place of s_0. We determine thus a sequence of intervals

$$[t_0, s_0], [s_0', s_1], \ldots, [s_{j-1}', s_j], \ldots, [s_N', t_1] \tag{5.10}$$

possibly with $t_0=s_0$ and $s_N'=t_1$) on each one of which $\varphi' \geq 0$, and such that $\varphi(s_j)=\varphi(s_j')$ for every $j=0, 1, \ldots, N$. A moment of thought will convince the reader that $N-1 \leq N_0$, the number in (5.7).

Now we avail ourselves of (5.8). We shall use the following property (Proposition 3, Appendix, § 2):

$$\text{There is a constant } M>0 \text{ such that, if } p_0, p_1 \text{ are any two points in } \tilde{V} \text{ lying on the same fibre of } Z \text{ in } V, F, \text{ there is a piecewise analytic curve joining } p_0 \text{ to } p_1, \text{ lying entirely in } F, \text{ and whose length does not exceed } M. \tag{5.11}$$

For each $j=0, 1, \ldots, N$ let γ_j' be a curve of the kind above, joining s_j to s_j', lying entirely on a fiber of Z in V and having length $\leq M$. We shall then call γ the continuous (piecewise analytic) curve in $\tilde{B}'$,

$$[t_0, s_0]+\gamma_0'+[s_0', s_1]+\ldots+[s_{j-1}', s_j]+\gamma_j'+\ldots+[s_N', t_1]. \tag{5.12}$$

It is clear that

$$\text{length } \gamma \leq |t_0-t_1|+N_0 M, \tag{5.13}$$

and also that φ *is monotone increasing along* γ.

Next we parametrize the points on γ_j' by the arc-length normalized (that is, the total arc length of γ_j' is equal to one) and starting at s_j. We parametrize in the same manner the straight-line segment $[s_j, s_j'] \subset \gamma_0$ and join by a straight-line segment the point in $[s_j, s_j']$ to the one in γ_j' corresponding to the same values of the normalized arclengths. As the latters vary from 0 to 1 these segments make up a two-chain $\mathfrak{c}_j$. We call $\mathfrak{c}$ the two-chain

$$[t_0, s_0]+\mathfrak{c}_0+[s_0', s_1]+\ldots+[s_{j-1}', s_j]+\mathfrak{c}_j+\ldots+[s_N', t_1]. \tag{5.14}$$

In (5.14) the segments are regarded as two-chains: they are identified to rectangles that are infinitely flat (and in what concerns the first and the last one, possibly reduced to a point). We give to the chain $\mathfrak{c}$ the orientation that makes its boundary be equal to $\gamma-\gamma_0$ (both oriented from t_0 to t_1). By Corollary 2, § 2, Appendix, we have

$$\text{the area of } \mathfrak{c} \text{ is bounded independently of } t_0, t_1, x_0. \tag{5.15}$$

6. Proof of Theorem 2.2: construction of L^1 solutions

We deal with m C^∞ functions $f_1, \ldots, f_m$, satisfying the compatibility conditions (2.2) in some open neighborhood $V^*=B^*\times J^*\subset U$ of $\mathrm{Cl}\, V$. It is convenient to introduce the following one-form in the open ball $B^*\subset\mathbf{R}^m$, depending smoothly on $x\in J^*$:

$$f(t,x)=\sum_{j=1}^m f_j(t,x)\,dt^j. \tag{6.1}$$

Let λ_j denote, as usual, the coefficients in L_j (see (1.8)), and set

$$f_2=\sum_{j<k}(\lambda_k f_j-\lambda_j f_k)\,dt^j\wedge dt^k. \tag{6.2}$$

We have, whatever the complex number ζ,

$$d_t(e^{-i\zeta Z}Z_x f)=\frac{\partial}{\partial x}(e^{-i\zeta Z}Z_x f_2). \tag{6.3}$$

Proof of (6.3): Differentiation with respect to x of $L_jZ=0$ yields

$$\frac{\partial}{\partial t^j}(e^{-i\zeta Z}Z_x)+\frac{\partial}{\partial x}(e^{-i\zeta Z}Z_x\lambda_j)=0. \tag{6.4}$$

If we combine this with (2.2) we get:

$$\frac{\partial}{\partial t^j}(e^{-i\zeta Z}Z_x f_k)+\frac{\partial}{\partial x}(e^{-i\zeta Z}Z_x\lambda_j f_k)=\frac{\partial}{\partial t^k}(e^{-i\zeta Z}Z_x f_j)+\frac{\partial}{\partial x}(e^{-i\zeta Z}Z_x\lambda_k f_j),$$

which is precisely (6.3).

We are going to study quite extensively integrals of the following kind

$$I^\varepsilon(t,x)=\frac{1}{2\pi}\int\int\int e^{i\xi[Z(t,x)-Z(s,y)]-\varepsilon\xi^2}g(y)\,Z_y(s,y)\,f(s,y)\,dy\,d\xi.$$

Let us explain what the ingredients are: first of all ε is a number >0 which will eventually tend to zero; $g\in C_c^\infty(J')$ is a suitable cut-off function. The integration with respect to y is performed over $\mathbf{R}^1$. That with respect to ξ is performed either over the half-line $\mathbf{R}_+$: $\xi>0$, or else over $\mathbf{R}_-$. Mostly we shall reason over $\mathbf{R}_+$, and to stress this fact we write ϱ instead of ξ. All the arguments in this case will have an obvious analogue when $\xi<0$. The integrand in I^ε is a one-form in s-space, which we integrate over a piecewise linear curve l contained in $\mathrm{Cl}\,B'$ and joining a certain point t_0 to the variable point t. The point t_0 and the curve l are chosen below.

We specify now how to select V^* and V (W will be specified later). First of all, taking advantage of the fact that $\Phi_x(0,0)=0$, we may assume that

$$|\Phi(t,x)-\Phi(t,y)|\leq\tfrac{1}{4}|x-y|,\quad t\in B^*,\ x,\ y\in J^*. \tag{6.5}$$

We shall require that the closure of J' be contained in J^*, but otherwise we shall keep J' unchanged. Recalling that $J'=]-\eta,\eta[$ we shall require that B^* be small enough that

$$|\Phi(t,x)-\Phi(s,x)|\leq\eta/8,\quad s,\ t\in B^*,\ x\in J^*. \tag{6.6}$$

Next we avail ourselves of Property (P), specifically of the fact that (P) holds at every point $p\in U$ of the kind $(0,x)$, $x\in J$. Because J' is relatively compact in J^* we can find an open ball B_1 such that

$$B'\subset \mathrm{B}_1\Subset B^*,$$

and such that, if $V_1=B_1\times J^*$ (recalling that $V=B'\times J'$), the following holds:

Given any pair $x,y\in\mathbf{R}$, $\mathrm{Cl}\,V$ intersects at most one connected component of $V_1^+(x,y)$ (see (3.11)), and at most one of $V_1^-(x,y)$. (6.7)

We come now to the choice of t_0. When the integration with respect to ξ is performed over $\mathbf{R}_+$ we take $t_0=t^-(x)$, the point in (5.3). When that integration is performed over $\mathbf{R}_-$ we take $t_0=t^+(x)$. In both cases, we have, by virtue of (5.4):

$$\xi\Phi(t_0,x)\leq\xi\Phi(t,x),\quad (t,x)\in V. \tag{6.8}$$

We apply then the properties listed at the end of Section 5. For each $(t,x)\in V$ we select a piecewise analytic curve $\gamma=\gamma(t,x)$, joining t_0 to t, having the following properties:

$$\gamma \text{ is entirely contained in } \mathrm{Cl}\, B_1, \tag{6.9}$$

$$\xi\Phi(t',x)\leq \xi\Phi(t,x), \quad \forall t'\in\gamma, \tag{6.10}$$

$$\text{the length of } \gamma \text{ is bounded independently of } (t,x)\in V. \tag{6.11}$$

Let us call $\gamma_0=\gamma_0(t,x)$ the straight-line segment joining t_0 to t. There is a two-chain $\mathfrak{c}=\mathfrak{c}(t,x)$ whose boundary is equal to $\gamma-\gamma_0$ and whose area is bounded independently of (t,x), entirely contained in $\mathrm{Cl}\, B_1$.

We then describe our choice of the path l: it consists of the straight-line segment joining t_0 to the origin of t-space, followed by the straight-line segment joining 0 to t.

Let us call I^ε_γ (resp., $I^\varepsilon_{\gamma_0}$) the same integral as I^ε except that the integration with respect ot s is performed over the curve γ (resp., over the straight-line segment γ_0) instead of the curve l. Until otherwise specified we limit our attention to the case $\xi=\varrho>0$. The case $\xi=-\varrho<0$ is dealt with in a similar fashion. We apply Stokes' theorem:

$$2\pi(I^\varepsilon_\gamma-I^\varepsilon_{\gamma_0})=\int\!\!\int\!\!\int_{\mathfrak{c}} e^{i\varrho Z(t,x)-\varepsilon\varrho^2} g(y)\, d_s\big[e^{-i\varrho Z(s,y)} Z_y(s,y)\, f(s,y)\big]\, dy\, d\varrho.$$

We apply (6.3) and perform an integration by parts with respect to y:

$$I^\varepsilon_{\gamma_0}=I^\varepsilon_\gamma+\frac{1}{2\pi}\int\!\!\int\!\!\int_{\mathfrak{c}} e^{i\varrho[Z(t,x)-Z(s,y)]-\varepsilon\varrho^2} g'(y)\, Z_y(s,y)\, f_2(s,y)\, dy\, d\varrho. \tag{6.14}$$

Let us introduce the vector field L_0 of (1.11). Note that

$$P(L_0)(e^{i\zeta Z})=P(i\zeta)\, e^{i\zeta Z}, \tag{6.15}$$

whatever the polynomial with complex coefficients, in one variable, P, and the complex number ζ. Denote by L_0' the transpose of L_0:

$$L_0' v=-\frac{\partial}{\partial x}(Z_x^{-1} v). \tag{6.16}$$

Note that

$$L_0'(Z_x v)=-Z_x L_0 v, \tag{6.17}$$

and therefore

$$P(L_0')(Z_x v) = Z_x P(-L_0)v. \tag{6.18}$$

Thus we get, after an integration by parts,

$$I_\gamma^\varepsilon = \frac{1}{2\pi}\int\int\int_\gamma (1+i\varrho)^{-N} e^{i\varrho[Z(t,x)-Z(s,y)]-\varepsilon\varrho^2} Z_y(s,y)\left(1+iZ_y^{-1}\frac{\partial}{\partial y}\right)^N [g(y)f(s,y)]\,dy\,d\varrho. \tag{6.19}$$

In this integral we write

$$\begin{aligned} Z(t,x)-Z(s,y) &= i[\Phi(t,x)-\Phi(s,x)]+Z(s,x)-Z(s,y)\\ &= i[\Phi(t,x)-\Phi(s,x)]+(x-y)\,[1+i\Phi_1(s,x,y)], \end{aligned}$$

and we derive, from (6.5):

$$|\Phi_1(s,x,y)| \leq \tfrac{1}{4}, \quad s\in B^*,\ x,\ y\in J^*. \tag{6.21}$$

We shall then deform the domain of ξ-integration from $\xi>0$ to the one-chain in $\mathbf{C}$,

$$\zeta = \varrho\left(1+\frac{i}{2}\,\frac{x-y}{|x-y|}\right), \quad \varrho\in\mathbf{R}_+. \tag{6.22}$$

We have

$$\mathrm{Re}\,\{i\zeta[Z(t,x)-Z(s,y)]-\varepsilon\zeta^2\} = -\varrho\{[\Phi(t,x)-\Phi(s,x)]-\tfrac{1}{2}|x-y|-(x-y)\,\Phi_1(s,x,y)\}-\tfrac{3}{4}\varepsilon\varrho^2.$$

We derive from (6.10) and (6.21):

$$\mathrm{Re}\,\{i\zeta[Z(t,x)-Z(s,y)]-\varepsilon\xi^2\} \leq -\tfrac{1}{4}|x-y|\,\varrho-\tfrac{3}{4}\varepsilon\varrho^2. \tag{6.23}$$

By applying (6.11) and taking $N\geq 2$ in (6.19) we obtain at once that

if $\varepsilon\to+0$, $I_\gamma^\varepsilon(t,x)$ converges uniformly in $B'\times J'$ (and a fortiori in $B'\times J_1'$). (6.24)

By taking N in (6.19) as large as needed we reach a similar conclusion for any derivative of I_γ^ε provided we restrict it to compact subsets of $B'\times(J_1'\setminus\{0\})$.

Next we look at the second term, in the right-hand side of (6.14). But now we choose more carefully the cut-off function g. We require

$$g(x)=1 \quad \text{for } |x|<\tfrac{3}{4}\eta. \tag{6.25}$$

We shall now restrict the variation of x to the interval

$$J_1' =]-\tfrac{1}{4}\eta, \tfrac{1}{4}\eta[.$$

Thus, in the integral under consideration, we have

$$|x-y| \geqslant \tfrac{1}{2}\eta. \tag{6.26}$$

We deform the ξ-integration from $\mathbf{R}_+$ to the chain (6.22), but now we take advantage of (6.6), as well as of (6.21). We obtain here, in lieu of (6.23):

$$\operatorname{Re}\{i\zeta[Z(t,x)-Z(s,y)]-\varepsilon\zeta^2\} \leqslant -\tfrac{1}{4}\eta\varrho-\tfrac{3}{4}\varepsilon\varrho^2. \tag{6.27}$$

Using then the fact that $\mathfrak{c}\subset \mathrm{Cl}\, B_1$ and that the area of $\mathfrak{c}$ is bounded independently of (t,x), we conclude that

$$I_\gamma^\varepsilon - I_{\gamma_0}^\varepsilon \text{ converges uniformly, in } B'\times J_1'. \tag{6.28}$$

Next we make use of the analogue of (6.14) when l is substituted for γ. In that case the two-chain $\mathfrak{c}$ must be replaced by a two-chain $\mathfrak{c}_0$ whose boundary is equal to $l-\gamma_0$ and which can be taken piecewise "planar", and with an area that is bounded independently of t and of t_0. The proof that has led us to (6.28) applies also here and we conclude that

$$I^\varepsilon - I_{\gamma_0}^\varepsilon \text{ converges uniformly, in } B'\times J_1'. \tag{6.29}$$

By combining (6.24), (6.28) and (6.29) we obtain:

$$\begin{gathered}\text{When } \varepsilon\to +0,\ I^\varepsilon(t,x) \text{ converges uniformly, in } B'\times J_1',\\ \text{to a function } I(t,x).\end{gathered} \tag{6.30}$$

In both (6.28) and (6.29), if we restrict to $B'\times(J_1'\setminus\{0\})$ the convergence is valid in the C^∞ sense (we are tacitly making use of (5.3)). Therefore, by the remark following (5.24), and by (5.5), the preceding argument shows:

$$\begin{gathered}\text{In } B'\times(J_1'\setminus\{0\}) \text{ the convergence of } I^\varepsilon \text{ to } I \text{ is valid in the } C^\infty \text{ sense;}\\ \text{moreover, } I(t,x) \text{ has finite limits } I(t,+0) \text{ and } I(t,-0), \text{ as } x\to +0 \text{ and}\\ x\to -0 \text{ respectively.}\end{gathered} \tag{6.31}$$

Next we compute $L_j I^\varepsilon$. To do this it is convenient to introduce the integrals $\mathscr{I}^\varepsilon(t_*)$,

$t_* \in B'$: It is the same integral as I^ε except that the path of s-integration is the straight-line segment joining 0 to t_*. In this notation,

$$I^\varepsilon = \mathscr{I}^\varepsilon(t) - \mathscr{I}^\varepsilon(t_0). \tag{6.32}$$

At first we focus on $\mathscr{I}^\varepsilon(t)$. Let us write

$$s = \theta t,\ 0 \leqslant \theta \leqslant 1; \quad f(s, y) = F(\theta, t, y)\, d\theta;$$
$$F(\theta, t, y) = \sum_{k=1}^{m} t^k f_k(\theta t, y).$$

Thus

$$\frac{\partial}{\partial t^j} F(\theta, t, y) = f_j(\theta t, y) + \theta \sum_{k=1}^{m} t^k \frac{\partial f_k}{\partial s^j}(\theta t, y),$$

and therefore, on the straight-line segment joining 0 to t,

$$\frac{\partial}{\partial t^j}\left[e^{-i\xi Z(s,y)} Z_y(s, y) F(\theta, t, y)\right]$$
$$= e^{-i\xi Z(s,y)} Z_y(s, y) f_j(s, y) + \theta \sum_{k=1}^{m} t^k \frac{\partial}{\partial s^j}\left[e^{-i\xi Z(s,y)} Z_y(s, y) f_k(s, y)\right].$$

We take (6.3) into account: the factor of θ, in the right-hand side of the preceding expression, is seen to be equal to

$$\sum_{k=1}^{m} t^k \frac{\partial}{\partial s^k}\left[e^{-i\xi Z(s,y)} Z_y(s, y) f_j(s, y)\right] + \frac{\partial}{\partial y}\left\{e^{-i\xi Z(s,y)} Z_y(s, y) \sum_{k=1}^{m} t^k(\lambda_k f_j - \lambda_j f_k)(s, y)\right\}.$$

Recalling that $\partial/\partial\theta = \Sigma_{k=1}^{m} t^k(\partial/\partial s^k)$ on the straight-line passing through 0 and t, we obtain

$$\frac{\partial}{\partial t^j}\left[e^{-i\xi Z(s,y)} Z_y(s, y) F(\theta, t, y)\right]$$
$$= \frac{\partial}{\partial \theta}\left[\theta\, e^{-i\xi Z(s,y)} Z_y(s, y) f_j(s, y)\right] + \frac{\partial}{\partial y}\left[e^{-i\xi Z(s,y)} Z_y(s, y) \sum_{k=1}^{m} t^k(\lambda_k f_j - \lambda_j f_k)(s, y)\right]. \tag{6.33}$$

It is convenient to introduce the following one-forms:

$$F_j(t, x) = \sum_{k=1}^{m} (\lambda_k f_j - \lambda_j f_k)(t, x)\, dt^k, \quad j = 1, \ldots, m.$$

Once again we limit ourselves to the case $\xi=\varrho>0$. We have:

$$L_j[\mathscr{I}^\varepsilon(t)] = \frac{1}{2\pi}\int\int\int e^{i\varrho Z(t,x)-\varepsilon\varrho^2} g(y)\frac{\partial}{\partial t^j}[e^{-i\varrho Z(s,y)} Z_y(s,y)\,F(\theta,t,y)]\,dy\,d\varrho\,d\theta,$$

where the integration with respect to θ is performed over $[0,1]$. If we take (6.33) into account we get:

$$L_j[\mathscr{I}^\varepsilon(t)] = \frac{1}{2\pi}\int_0^{+\infty}\int_{y\in\mathbf{R}^1} e^{i\varrho[Z(t,x)-Z(t,y)]-\varepsilon\varrho^2} g(y)\,Z_y(t,y)\,f_j(t,y)\,dy\,d\varrho$$
$$-\frac{1}{2\pi}\int\int\int e^{i\varrho[Z(t,x)-Z(s,y)]-\varepsilon\varrho^2} g'(y)\,Z_y(s,y)\,F_j(s,y)\,dy\,d\varrho. \tag{6.34}$$

At this stage we start distinguishing more carefully between those integrals in which the ξ-integration is carried out over $\mathbf{R}_+$ and those in which it is carried out over $\mathbf{R}_-$. We shall label them with superscripts + and − respectively. According to (6.34) we may thus write:

$$L_j[\mathscr{I}^{\varepsilon+}(t)+\mathscr{I}^{\varepsilon-}(t)] = \frac{1}{2\pi}\int\int e^{i\xi[Z(t,x)-Z(t,y)]-\varepsilon\xi^2} g(y)\,Z_y(t,y)\,f_j(t,y)\,dy\,d\xi$$
$$-\frac{1}{2\pi}\int\int\int e^{i\xi[Z(t,x)-Z(s,y)]-\varepsilon\xi^2} g'(y)\,Z_y(s,y)\,F_j(s,y)\,dy\,d\xi,$$

where the integrations with respect to y and to ξ are both performed over $\mathbf{R}^1$. We have

$$\frac{1}{2\pi}\int\int e^{i\xi[Z(t,x)-Z(t,y)]-\varepsilon\xi^2} g(y)\,Z_y(t,y)\,f_j(t,y)\,dy\,d\xi$$
$$= (4\pi\varepsilon)^{-1/2}\int e^{[Z(t,x)-Z(t,y)]^2/4\varepsilon} g(y)\,Z_y(t,y)\,f_j(t,y)\,dy,$$

and it is well-known that the latter integral converges to $g(x)f_j(t,x)$ in the C^∞ sense, as $\varepsilon\to+0$. We also have:

$$\frac{1}{2\pi}\int\int e^{i\xi[Z(t,x)-Z(s,y)]-\varepsilon\xi^2} g'(y)\,Z_y(s,y)\,F_j(s,y)\,dy\,d\xi$$
$$= (4\pi\varepsilon)^{-1/2}\int\int e^{-[Z(t,x)-Z(s,y)]^2/4\varepsilon} g'(y)\,Z_y(t,y)\,F_j(t,y)\,dy.$$

Note that

$$\operatorname{Re}[Z(t,x)-Z(s,y)]^2 = |x-y|^2-[\Phi(t,x)-\Phi(s,y)]^2$$
$$\geqslant |x-y|^2-2([\Phi(t,x)-\Phi(s,x)]^2+[\Phi(s,x)-\Phi(s,y)]^2).$$

If we restrict the variation of x to J_1', as we shall, (6.26) holds. So do (6.5) and (6.6). Consequently, in the above integral,

$$\operatorname{Re}[Z(t,x)-Z(s,y)]^2 \geqslant \eta^2/8.$$

This shows that the preceding integral converges to zero in $C^\infty(B'\times J_1')$. Thus we conclude

$$L_j[\mathcal{I}^{\varepsilon+}(t)+\mathcal{I}^{\varepsilon-}(t)] \text{ converges to } f_j \text{ in } C^\infty(B'\times J_1'). \tag{6.35}$$

We wish now to find the limit of $L_j\mathcal{I}^\varepsilon(t_0)$ as $\varepsilon\to+0$ (see (6.32)). Unlike $\mathcal{I}^\varepsilon(t)$ it may have a discontinuity at $x=0$ originating from the discontinuity, if there is one, of $t_0=t^\pm(x)$. We must therefore apply the well-known formula for the distribution derivative of a function such as $\mathcal{I}^\varepsilon(t_0)$:

$$L_j[\mathcal{I}^\varepsilon(t_0)] = \{L_j[\mathcal{I}^\varepsilon(t_0)]\} + \lambda_j\vartheta^\varepsilon\delta(x), \tag{6.36}$$

where the first term in the right-hand side stands for the function (integrable, as we shall see) which is equal to $L_j[I^\varepsilon(t_0)]$ when $x\neq0$, and where ϑ^ε is the jump of $I^\varepsilon(t_0)$ at $x=0$, and $\delta(x)$ is the Dirac distribution.

When $x\neq0$ we have

$$L_j[\mathcal{I}^\varepsilon(t_0)] = \lambda_j\sum_{k=1}^m \frac{\partial t_0^k}{\partial x}\frac{\partial}{\partial t_0^k} I^\varepsilon(t_0). \tag{6.37}$$

We note that (6.33) holds if we replace everywhere t by t_0 (and thus read $s=\theta t_0$). We obtain

$$\frac{\partial}{\partial t_0^k}\mathcal{I}^\varepsilon(t_0) = \frac{1}{2\pi}\int_0^{+\infty}\int_{y\in\mathbf{R}^1} e^{i\varrho[Z(t,x)-Z(t_0,y)]-\varepsilon\varrho^2} g(y)\,Z_y(t_0,y)\,f_k(t_0,y)\,dy\,d\varrho$$
$$-\frac{1}{2\pi}\iiint e^{i\varrho[Z(t,x)-Z(s,y)]-\varepsilon\varrho^2} g'(y)\,Z_y(s,y)\,F_k(s,y)\,dy\,d\varrho, \tag{6.38}$$

where the s-integration is now performed over the straight-line segment joining 0 to t_0.

In (6.38) we take $t_0=t^-(x)$ (see (5.3)). It is then possible to prove that the right-hand side converges uniformly, in $B'\times J_1'$, *exactly* in the same manner as (6.24) and (6.28) were established. Call $Q_k^+(t,x)$ its limit as $\varepsilon\to+0$. Likewise call $Q_k^-(t,x)$ the limit of the similar integral when ϱ is replaced by $-\varrho$ and t^- by t^+. It is clear that $Q_k^\pm$ have the property analogous to (6.31). On the other hand notice that the right-hand side in (6.38) is a function of $Z(t,x)$ and $t_0=t^\pm(x)$ only, and therefore is constant on the fibres of the map Z. The same is true of its limit. If we take advantage of (5.6) we may state:

When $\varepsilon \to +0$ the restriction of $L_j[\mathscr{I}^{\varepsilon+}(t^-(x))+\mathscr{I}^{\varepsilon-}(t^+(x))]$ to $B'\times(J_1'\setminus\{0\})$ converges, both in L^1 and in C^∞, to $\lambda_j Q^0$, where (6.39)

$$Q^0=\sum_{k=1}^m \frac{\partial(t^-)^k}{\partial x}Q_k^+ + \frac{\partial(t^+)^k}{\partial x}Q_k^-$$

is constant on the fibres of the mapping $Z: B'\times(J_1'\setminus\{0\})\to \mathbf{C}$.

Returning to (6.36) we want now to find the limits of the jumps ϑ^ε. Once again let us look at the case where the ξ-integration is performed over $\mathbf{R}_+$. We have

$$\vartheta^{\varepsilon+}=\frac{1}{2\pi}\iiint e^{i\varrho[Z(t,0)-Z(s,y)]-\varepsilon\varrho^2}g(y)\,Z_y(s,y)\,f(s,y)\,dy\,d\varrho,$$

where the integration is performed over the curve l_0^- defined as follows: l_0^- consists of the straight-line segment joining $t_l^-=t^-(-0)$ to the origin followed by the straight-line segment joining the origin to $t_r=t^-(+0)$. In other words it is similar to the integral I^ε where we have put $x=0$ and $t_0=t_l^-$, $t=t_r^-$, except that we still have $Z(t,0)$ where we ought to have $Z(t^-,0)$. However the argument that led to (6.24) works equally well here. This is due to the fact that $\Phi(s,0)\leq\Phi(t,0)$ for all s on any curve joining t_l^- to t_r^- which is entirely contained in the level set of $\Phi(\cdot,0)$ in $\mathrm{Cl}\,B_1$ (in which those two points lie). For this reason the inequalities (6.23) and (6.27) have analogues here. Note also that $\vartheta^{\varepsilon+}$ is independent of x (and thus we do not have to deal with discontinuities) and that $\vartheta^{\varepsilon+}$ depends on t solely through $Z(t,0)$. As a matter of fact (and this is quite important in what follows), by the analogue of (6.19) we can see that

$$\vartheta^{\varepsilon+}(t)=\tilde{\vartheta}^{\varepsilon+}(Z(t,0)),$$

where $\tilde{\vartheta}^{\varepsilon+}$ is a C^∞ function on the (imaginary) interval $\tilde{B}'$ which is the image of B' under the map $t\mapsto Z(t,0)$. Note that $\tilde{B}'$ might be open, closed or only contain one of its boundary points. In any case $\tilde{\vartheta}^{\varepsilon+}$ is C^∞ up to any point of the boundary of $\tilde{B}'$ when that point belongs to $\tilde{B}'$. This is seen by differentiation under the integral signs in the integrals analogous to (6.19) and to the second term in (6.14). Moreover, as $\varepsilon\to+0$, $\tilde{\vartheta}^{\varepsilon+}$ converges to a C^∞ function $\tilde{\vartheta}^+$ in the C^∞ sense specified above.

By combining what we just said with (6.32), (6.35), (6.39), and by calling I^+ (resp., I^-) the limit of $I^{\varepsilon+}$ (resp., $I^{\varepsilon-}$; see (6.30) and (6.31)), we reach the conclusion that, in $B'\times J_1'$,

$$L_j(I^++I^-)=f_j+\lambda_j Q^0+\lambda_j\vartheta\delta(x), \tag{6.40}$$

where

$$\vartheta = \vartheta(t) = \tilde{\vartheta}(Z(t, 0)),$$

with $\tilde{\vartheta}$ a C^∞ function on the interval $\tilde{B}'$ (in the sense specified above). Let then B_1' be an open ball centered at the origin in t-space, whose closure is contained in B', $\tilde{B}_1'$ the image of B_1' under the mapping $t \to Z(t, 0)$. The function $\tilde{\vartheta}$ is C^∞ in the closed interval $\mathrm{Cl}\, B_1'$ and can be extended as a C^∞ function (also denoted by $\tilde{\vartheta}$) in the whole complex plane. Set then

$$\chi(t, x) = \tilde{\vartheta}(Z(t, x))\, \sigma(x),$$

where $\sigma(x) = \frac{1}{2}x/|x|$ when $x \neq 0$. We have

$$L_j \chi = L_j(\tilde{\vartheta} \circ Z)\, \sigma(x) + \lambda_j \tilde{\vartheta}(Z(t, 0))\, \delta(x),$$

and

$$L_j(\tilde{\vartheta} \circ Z) = (\tilde{\vartheta}_{\bar{z}} \circ Z)\, L_j \bar{Z} = 2\lambda_j(\tilde{\vartheta}_{\bar{z}} \circ Z).$$

Thus, by (6.40), we see that, now in $W_1 = B_1' \times J_1'$,

$$L_j(I^+ + I^- - \chi) = f_j + \lambda_j Q, \tag{6.41}$$

where

$$Q = Q^0 - 2(\tilde{\vartheta}_{\bar{z}} \circ Z)\, \sigma(x). \tag{6.42}$$

We know that Q^0 is constant on the fibres of Z in W_1 and therefore the same is true of Q. By push forward via Z we obtain a function $\tilde{Q}$ in $Z(W_1)$ such that $Q = \tilde{Q} \circ Z$ in W_1. Furthermore we have, by (5.6),

$$|\tilde{Q}(z)| \leq C|x|^{1-1/q} \quad (z = x + iy). \tag{6.43}$$

We shall assume, below, that $\tilde{Q}$ has been extended by zero in $\mathbf{C}^1 \setminus Z(W_1)$, and thus $\tilde{Q} \in L^1$. We define

$$\tilde{w} = \tilde{Q} * \left(\frac{1}{2\pi z} e^{z^2}\right), \quad w = \tilde{w} \circ Z.$$

We have (cf. equation following (4.32)):

$$L_j w = \lambda_j Q, \quad j = 1, \ldots, m, \tag{6.44}$$

in the distribution sense, in W_1. Let us show that $w \in L^2(W_1)$.

Availing ourselves of (6.43) we obtain

$$\begin{aligned}
|w(t,x)| &= \frac{1}{2\pi}\left|\iint_{Z(W_1)} e^{[Z(t,x)-z']^2}\frac{\tilde{Q}(z')}{Z(t,x)-z'}\,dx'dy'\right| \\
&\leqslant C'\int_{x'\in J_1''}\int_{y'\in\mathbf{R}^1} e^{-|y'|^2}\frac{dx'dy'}{|x'|^{1-1/q}|x-x'+iy'|} \\
&\leqslant C''\int_{x'\in J_1''}\left(\int_{\mathbf{R}^1}\frac{dy'}{y'^2+|x-x'|^2}\right)^{1/2}\frac{dx'}{|x'|^{1-1/q}} \\
&\leqslant C'''\int\frac{dx'}{|x-x'|^{1/2}|x'|^{1-1/q}}\in L^2(W_1).
\end{aligned}$$

(We have applied the Cauchy-Schwarz inequality to the integration with respect to y' to go from the second to the third line.)

Define now

$$v = I^+ + I^- - \chi - w.$$

By (6.41) and (6.44) we have, in W_1,

$$L_j v = f_j, \quad j = 1, \dots, m. \tag{6.45}$$

By (6.31) and the obvious properties of χ, $I^+ + I^- - \chi$ is an L^2 function. We have just seen that the same is true of w, and thus $v \in L^2(W_1)$. In the next section we construct a C^∞ solution u in a perhaps smaller neighborhood of the origin.

7. End of proof of Theorem 2.2: construction of C^∞ solutions

Let N be an arbitrary integer $\geqslant 1$. We solve, in $W_1 = B_1' \times J_1'$,

$$L_j v_N = L_0^N f_j, \quad j = 1, \dots, m, \quad v_N \in L^2(W_1), \tag{7.1}$$

where L_0 is the vector field (1.11). Note that $L_0 f_1, \dots, L_0 f_m$ satisfy the compatibility conditions (2.2), since $L_0 L_j = L_j L_0$ for all j. Let J_2' an open interval centered at zero, whose closure is contained in J_1'. And let $\psi \in C_c^\infty(J_1')$ be equal to one in J_2'. We have, in B_1',

$$L_j[\psi(x)\, v_N(t,x)] = \psi(x)\, L_0^N f_j(t,x) + \lambda_j(t,x)\,\psi'(x)\, v_N(t,x), \quad j = 1, \dots, m. \tag{7.2}$$

Let us denote by E_a the space of locally-L^1 functions of x in the real line whose support is contained in the half-line $x \geqslant a$. We shall make use of the following linear operator on E_a, depending smoothly on $t \in B$:

$$(L_0^{-1} f)(t, x) = \int_{-\infty}^{x} Z_y(t, y) f(y) \, dy. \tag{7.3}$$

Evidently we have

$$L_0 L_0^{-1} = L_0^{-1} L_0 = \text{Identity of } E_a, \tag{7.4}$$

and, as a consequence, when acting on E_a,

$$[L_j, L_0^{-1}] = 0, \quad j = 1, \dots, m, \tag{7.5}$$

since $[L_j, L_0] = 0$.

Let us now rewrite (7.2) in the form

$$L_j(\psi v_N) = L_0^N(\psi f_j) + S_j, \tag{7.6}$$

where $S_j = 0$ if $x \in J_2''$. Applying L_0^{-N} to both sides, and availing ourselves of (7.4), (7.5), yields:

$$L_j L_0^{-N}(\psi v_N) = \psi f_j + L_0^{-N} S_j. \tag{7.7}$$

Observe then that

$$L_0^N(L_0^{-N} S_j) = 0 \quad \text{in } B_1' \times J_2'. \tag{7.8}$$

This implies that, in $B_1' \times J_2'$,

$$L_0^{-N} S_j(t, x) = \sum_{k=0}^{N-1} \sigma_{j,k}(t) Z(t, x)^k. \tag{7.9}$$

But in the same set we have

$$L_0^{-N} S_j = L_j L_0^{-N}(\psi v_N) - f_j, \tag{7.10}$$

hence

$$L_{j'}(L_0^{-N} S_j) = L_j(L_0^{-N} S_{j'}), \quad j, j' = 1, \dots, m. \tag{7.11}$$

This implies at once that there is a distribution σ_k in B_1' such that

$$\sigma_{j,k} = \frac{\partial}{\partial t^j} \sigma_k, \quad j = 1, \dots, m. \tag{7.12}$$

Actually one derives from (7.9) and (7.10) that $\sigma_{j,k}$ are sums of derivatives of locally-L^1 functions; the same is therefore true of σ_k. This property will be used below. Let us then set

$$T_N = \sum_{K=0}^{N-1} \sigma_k(t)\, Z(t,x)^k. \tag{7.13}$$

We have

$$L_j T_N = L_0^{-N} S_j, \quad j=1,\dots,m, \tag{7.14}$$

and therefore, by (7.7), in $B_1' \times J_2'$,

$$L_j u_N = f_j, \quad j=1,\dots,m, \tag{7.15}$$

where

$$u_N = L_0^{-N}(\psi v_N) - T_N. \tag{7.16}$$

Of course the action of L_0^{-1} increases the regularity with respect to x (and does not modify that with respect to t). Thus we see that $L_0^{-N}(\psi v_N)$ is an L^1 function of t in B_1', valued in the space of C^{N-1} functions of x (in **R**). On the other hand, T_N is a sum of derivatives of L^1 functions of t in B_1', valued in the space of C^∞ functions of x in J. This means that u_N is a sum of derivatives of L^1 functions of t in B_1' valued in the space of C^{N-1} functions of x in J_2'. At this point we use the equations (7.15) to trade differentiability with respect to x for differentiability with respect to t. Indeed, in the notation of one-forms (see (6.1)), (7.15) reads (in $B_1' \times J_1'$):

$$d_t u_N = f - i(L_0 u_N)\, d_t \Phi. \tag{7.17}$$

(By (1.9) and (1.11) we have $L_j = \partial/\partial t^j - i\Phi_{t_j} L_0$.) We reach easily the conclusion that, given any integer $\nu \geq 0$, we can find N large enough that $u_N \in C^\nu(B_1' \times J_2')$.

In what follows we suppose that the subscripts N have been selected in such a way that the solution u_N of (7.15) belongs to $C^N(B_1' \times J_2')$.

At last we select $W = B'' \times J''$. We simply require that B'' be an open ball centered at the origin with closure contained in B_1', and J'' an open interval centered at zero with closure contained in J_2'. We apply the C^N version of Theorem I (Section 1):

Every solution $h \in C^N(B_1' \times J_2')$ of the homogeneous equations $L_j h = 0$ ($j=1,\dots,m$) is the limit, in $C^N(\mathrm{Cl}\, W)$, of a sequence of polynomials with respect to $Z(t,x)$.

(We leave the proof of this assertion to the reader: apply Theorem I to $L_0^k h$ instead of h, for $k=0, 1, \dots, N$, use cut-off functions and the operator L_0^{-1}.)

Let us denote by $|f|_N$ the natural norm in $C^N(\mathrm{Cl}\, W)$. For each $N=0, 1, \dots,$ we select a polynomial $P_N \in \mathbf{C}[z]$ such that

$$|u_{N+1}-u_N-P_N(Z)|_N \leq 2^{-N}. \tag{7.18}$$

We set then

$$u_{(0)}=u_0, \quad u_{(N)}=u_N-P_0(Z)-\dots-P_{N-1}(Z) \quad \text{for } N\geq 1.$$

We derive from (7.18):

$$|u_{(N+1)}-u_{(N)}|_N \leq 2^{-N}. \tag{7.19}$$

This shows that the sequence $(u_{(N)})_{N\geq\nu}$ converges to an element of $C^\nu(\mathrm{Cl}\, W)$, of course independent of ν, and therefore belonging to $C^\infty(\mathrm{Cl}\, W)$. Since all $u_{(N)}$ satisfy (2.1) in W so does their limit.

The proof of Theorem 2.2 is complete.

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Appendice: Sur trois questions de finitude en géométrie analytique réelle

by

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Introduction

Soient φ: $Y \to X$ un morphisme (sous-)analytique réel entre espaces analytiques réels (cf. [H1] et [Ha1]) et K un sous-ensemble sous-analytique (cf. loc. cit.) compact de Y. D'après le premier théorème d'isotopie de Thom (cf. [Ma]) et l'existence de stratifications de morphismes propres (cf. loc. cit.), il existe un sous-ensemble sous-analytique Z de $\varphi(K)$, de dimension strictement inférieure à celle de $\varphi(K)$, tel que tout point $x_0 \in \varphi(K) \setminus Z$ possède un voisinage ouvert U tel que $\varphi^{-1}(U) \cap K \to U$ soit une fibration topologique localement triviale. R. M. Hardt a même prouvé (cf. [Ha2]) le remarquable résultat selon lequel on peut choisir Z de telle manière que l'on puisse prendre pour U la composante connexe de $\varphi(K) \setminus Z$ qui contient x_0, et l'on a un homéomorphisme à graphe sous-analytique $(\varphi^{-1}(x_0) \cap K) \times U \xrightarrow{\sim} \varphi^{-1}(U) \cap K$. Même si au-dessus des points de Z la géométrie des fibres $\varphi^{-1}(x) \cap K$, et en particulier leur dimension, saute, on s'attend du fait de l'analyticité à ce que ces changements de géométrie se fassent d'une manière non-sauvage. Voici des façons de préciser cette idée inspirées par les questions de Trèves qui ont motivé cette rédaction. Dans ce qui suit on suppose fixé un plongement $Y \subset \mathbf{R}^m$.

(1) Tout point $x_0 \in \varphi(K)$ possède un voisinage ouvert U dans Y tel qu'il existe un entier N tel que pour tout $x \in U$, l'ensemble sous-analytique $\varphi^{-1}(x) \cap K$ puisse être triangulé avec moins de N simplexes (cf. [H2] et [Ha3] pour la triangulation des sous-analytiques).

(2) La condition 1) est réalisée et de plus les triangulations sont telles qu'il existe une constante V telle que la somme des volumes i-dimensionnels dans $\mathbf{R}^m$ des i-simplexes de la triangulation de $\varphi^{-1}(x) \cap K$ soit inférieure à V pour chaque i et chaque $x \in U$.

(3) Tout point $x_0 \in \varphi(K)$ possède un voisinage ouvert U dans Y tel qu'il existe une constante $\gamma > 0$ telle que, pour tout $x \in U$ et tout couple (a, b) de points appartenant à la

même composante connexe de $\varphi^{-1}(x)\cap K$, il existe un chemin sous-analytique (i.e. analytique par morceaux) contenu dans $\varphi^{-1}(x)\cap K$, joignant a et b et de longueur inférieure à γ.

Pour prouver que (1) est toujours réalisée, il suffit de se souvenir que d'après ([H1], 8.2) l'ensemble $\varphi(K)\setminus Z$ n'a qu'un nombre fini de composantes connexes, d'invoquer le théorème de Goresky sur la triangulation des ensembles stratifiés ([G]) et le théorème de Hardt, pour trouver des triangulations simultanées de toutes les fibres $\varphi^{-1}(x)\cap K$, pour $x\in U$, où U est une composante connexe de $\varphi(K)\setminus Z$. On conclut par restriction de φ à $\varphi^{-1}(Z)$ et récurrence sur la dimension de $\varphi(K)$. Je ne sais prouver ni (2) ni (3) en général (mais voir [B1] et [B2] pour (2)). Je vais m'intéresser ici à des résultats qui sont conséquences de (1) et (3) respectivement.

§ 1. Finitude de changements de signe

PROPOSITION 1. *Soit $p: E\to X$ un morphisme analytique réel entre espaces analytiques réels, dont toutes les fibres sont de dimension algébrique égale à un. Soit $f: E\to \mathbf{R}$ une fonction analytique réelle.*

Dans ces conditions, pour tout compact $K_1\subset X$ et tout compact $K_2\subset E$ il existe un entier $N=N(K_1,K_2)$ tel que, pour tout $x\in K_1$, la restriction de f à $p^{-1}(x)$ change de signe au plus N fois dans $p^{-1}(x)\cap K_2$.

Démonstration. Posons $Y=f^{-1}(0)$, $\varphi=p|Y: Y\to X$ et $K=K_2\subset E$. Puisque K_1 est compact et que les changements de signe correspondent à des zéros isolés de $f|p^{-1}(x)$, il suffit de prouver que tout point $x\in K_1$ possède un voisinage ouvert U tel qu'il existe une constante N_U ayant la propriété que pour tout $x\in U$ le nombre des points de $\varphi^{-1}(x)\cap K$ qui sont isolés dans $\varphi^{-1}(x)$ est inférieur à N_U. Ceci est une conséquence immédiate de (1) ci-dessus appliqué à φ. On recouvre ensuite K_1 par un nombre fini de tels ouverts, disons $K_1=\bigcup_i U_i$ et l'on prend $N=\operatorname{Sup}_i N_{U_i}$. Q.E.D.

On peut donner de la Proposition 1 une autre démonstration, qui a l'avantage de contenir un lemme de finitude qui semble pouvoir s'étendre à la géométrie analytique p-adique où des résultats de cette sorte sont aussi utiles. Cette démonstration est presque identique à celle de D. Barlet (cf. [B1], [B2]) pour des résultats du type du (2) de l'introduction, dans le cas analytique complexe propre. Nous devons utiliser ici le théorème d'aplatissement *local* parce que le complexifié d'un morphisme analytique réel propre n'est pas propre en général.

Avant de donner cette autre démonstration de la Proposition 1 nous rappellons quelques définitions et résultats.

Soit $\varrho: Z \to W$ un morphisme d'espaces analytiques *complexes*, et soit $\eta: W' \to W$ un éclatement local, c'est-à-dire le morphisme composé de l'éclatement d'un sous-espace analytique fermé B d'un ouvert U de W et de l'inclusion $(U, \mathcal{O}_W|_U) \hookrightarrow (W, \mathcal{O}_W)$ d'espaces analytiques. Il existe un sous-espace analytique fermé $Z' \subset W' \times_W Z$ tel que, dans le diagramme naturel

$$\begin{array}{ccc} W' \times_W Z \supset Z' & \xrightarrow{\eta'} & Z \\ \varrho' \downarrow & & \downarrow \varrho \\ W' & \xrightarrow{\eta} & W \end{array}$$

le morphisme η' soit l'éclatement de $\varrho^{-1}(B)$ dans $\varrho^{-1}(U)$ composé avec l'inclusion $\varrho^{-1}(U) \hookrightarrow Z$, et ϱ' soit l'unique morphisme dû à la propriété universelle de l'éclatement.

On peut aussi définir Z' comme étant le sous-espace fermé de $W' \times_U Z$ défini par l'idéal cohérent engendré par les éléments annulés par le composé avec la première projection d'une puissance de l'idéal définissant le diviseur exceptionnel $\eta^{-1}(B) \subset W'$. Le morphisme $\varrho': Z' \to W'$ est appelé *transformé strict* de ϱ par η. On peut ensuite définir le transformé strict d'un morphisme ϱ par une suite finie d'éclatement locaux. On a alors :

THÉORÈME (Hironaka, Lejeune et Teissier, cf. [H1], [H2]). *Soient $\varrho: Z \to W$ un morphisme d'espaces analytiques complexes, w un point de W et L un sous-ensemble compact de $\varrho^{-1}(w)$. Il existe un nombre fini de suites finies $(S_\alpha)_{\alpha \in A}$ d'éclatements locaux de W telles que les énoncés suivants soient vrais :*

(1) *Pour chaque α, le centre de chacun des éclatements locaux apparaissant dans S_α est rare dans son espace ambiant.*

(2) *Notant $\pi_\alpha: W_\alpha \to W$ le morphisme composé des éclatements locaux de S_α, il existe un voisinage ouvert U de w dans W tel que, pour tout compact $K \subset U$, il existe pour chaque α un compact $K_\alpha \subset W_\alpha$ de telle manière que*

$$K \subset \bigcup_{\alpha \in A} \pi_\alpha(K_\alpha).$$

(3) *Pour chaque $\alpha \in A$, le morphisme transformé strict $\varrho_\alpha: Z_\alpha \to W_\alpha$ de ϱ par π_α (i.e., par S_α) est plat en tout point de Z_α dont l'image par le morphisme naturel $Z_\alpha \to Z$ appartient à L.*

Démonstration de la Proposition 1. Posons encore $Y=f^{-1}(0)$, $\varphi=p|Y\colon Y\to X$. Notons $\pi\colon F\to W$ le morphisme complexifié de p. D'après l'hypothèse les fibres de π sont encore de dimension 1, et un zéro de $f|p^{-1}(x)$ isolé dans $p^{-1}(x)$ donne un zéro de la restriction à $\pi^{-1}(x)$ de la fonction $\eta\colon F\to\mathbf{C}$ complexifiée de f, qui est encore isolé dans $\pi^{-1}(x)$.

D'après la remarque déjà faite, pour prouver la Proposition 1, il suffit de vérifier que :

$(*)$ Tout point $x_0\in X$ possède un voisinage ouvert U tel qu'il existe un entier n_U tel que pour tout $x\in U$, notant $\mathrm{Is}(x)$ l'ensemble des points de $\varphi^{-1}(x)\cap K$ qui sont isolés dans $p^{-1}(x)$, on a l'inégalité

$$\sum_{y\in \mathrm{Is}(x)} \dim_{\mathbf{R}} (\mathscr{A}_{p^{-1}(x),y}/(f_y)) \leq n_U$$

où $\mathscr{A}_{p^{-1}(x),y}$ désigne l'algèbre des germes en y de fonctions analytiques réelles sur $p^{-1}(x)$ et f_y le germe en y de $f|p^{-1}(x)$. Nous allons donc nous placer au voisinage d'un point $x_0\in X$ fixé. Remarquons que pour prouver l'énoncé $(*)$, il suffit de le prouver pour le morphisme complexifié, en remplaçant $\mathscr{A}$ par l'algèbre des fonctions analytiques complexes et f par son compléxifié η, c'est-à-dire de prouver l'énoncé suivant :

LEMME. $(**)$ *Soient $\varrho\colon Z\to W$ un morphisme analytique complexe à fibres de dimension ≤ 1, et $K\subset Z$ un sous-ensemble compact. Tout point $w_0\in W$ possède un voisinage ouvert U tel qu'il existe un entier $n_U(K)$ tel que pour tout $w\in U$, notant* $\mathrm{Is}(w)$ *l'ensemble des points $z\in\varrho^{-1}(w)\cap K$ tels que* $\dim_z\varrho^{-1}(w)=0$, *on ait l'inégalité*

$$\sum_{z\in \mathrm{Is}(w)} \dim_{\mathbf{C}} \mathcal{O}_{\varrho^{-1}(w),z} \leq n_U(K).$$

ou $\mathcal{O}$ désigne l'algèbre des fonctions holomorphes. En effet, dans notre cas, nous aurons $\mathcal{O}_{\varrho^{-1}(w),z}=\mathcal{O}_{\pi^{-1}(w),z}/(\eta_z)$ où $\varrho\colon Z\to W$ est le complexifié du morphisme φ, et donc $Z=\eta^{-1}(0)\subset F$.

Démonstration du lemme. On se place au voisinage de $w_0\in W$. En examinant la décomposition en composantes irréductibles de Z au voisinage de K, on se ramène aussitôt au cas où il existe un ouvert analytique dense de W, tel que $|\varrho^{-1}(w)\cap K|$ soit fini pour tout w appartenant à cet ouvert.

On applique alors le Théorème ci-dessus en un point w_0 avec $L=\varrho^{-1}(w_0)\cap K$. On obtient des diagrammes commutatifs :

$$\begin{tikzcd} W_\alpha \times_W Z & Z_\alpha \arrow[l, hook'] \arrow[r, "\psi_\alpha"] \arrow[d, "\varrho_\alpha"] \arrow[dl, phantom] & Z \arrow[d, "\varrho"] \\ & W_\alpha \arrow[r, "\pi_\alpha"'] & U \hookrightarrow W \end{tikzcd}$$

$$\mathrm{pr}_1 \searrow$$

où ϱ_α est plat en tout point de $\varphi_\alpha^{-1}(L)$. Par conséquent le morphisme ϱ_α est fini et plat; soit n_α son degré. Nous allons démontrer le Lemme par récurrence sur la dimension de W. Le résultat est évident si $\dim W=0$. Soit $D_\alpha \subset W_\alpha$ le sous-espace analytique fermé de codimension 1 réunion des images inverses dans W_α des centres éclatements locaux constituant la suite S_α. Nous avons

$$\varrho_\alpha^{-1}(W_\alpha \setminus D_\alpha) \simeq (W_\alpha \setminus D_\alpha) \times_W Z,$$

et par consequent les seuls points w_α tels que

$$\varrho_\alpha^{-1}(W_\alpha) \neq \{w_\alpha\} \times \varrho^{-1}(\pi_\alpha(w_\alpha))$$

sont les points $w_\alpha \in D_\alpha$. D'après l'hypothèse de récurrence, pour tout compact K et tout système $\{K_\alpha\}$ comme dans la partie (2) du Théorème, le morphisme induit $D_\alpha \times_W Z \to D_\alpha$ satisfait le Lemme relativement au compact $K_\alpha \times_W K_2$ et donc, en utilisant un recouvrement fini de K_α, pour tout point $w_\alpha \in D_\alpha \cap K_\alpha$, puisque $W_\alpha \times_W Z = Z_\alpha \cup \mathrm{pr}_1^{-1}(D_\alpha)$, on a l'inégalité

$$\sum_{z_\alpha \in \mathrm{Is}(W_\alpha)} \dim_{\mathbf{C}} \mathcal{O}_{V_\alpha, z_\alpha} \leq n_\alpha + N_\alpha,$$

où N_α est le supremum sur les ouverts $U_{\alpha\beta}$ d'un recouvrement de K_2 des $n_{U_{\alpha\beta}}(K_\alpha \times_W K_2)$ associés au morphisme $D_\alpha \times_W Z \to D_\alpha$.

Puisque $V_\alpha = \varrho^{-1}(\pi_\alpha(w_\alpha))$ on en déduit que pour tout $w \in \pi_\alpha(K_\alpha)$ on a

$$\sum_{z \in \mathrm{Is}(w)} \dim_{\mathbf{C}} \mathcal{O}_{\varrho^{-1}(w), z} \leq n_\alpha + N_\alpha.$$

Puisque $K \subset \bigcup_\alpha \pi_\alpha(K_\alpha)$ on a le résultat cherché avec $N_K = \sup_\alpha (n_\alpha + N_\alpha)$. Q.E.D.

La Proposition 1 nous permet de démontrer l'énoncé (5.7) de [T]:

COROLLAIRE 1. *Soit* $\Phi: \Omega \to \mathbf{R}$ *une fonction analytique réelle définie sur un ouvert* Ω *de* $\mathbf{R}^n$, *et soit* B *un sous-ensemble convexe compact de* $\mathbf{R}^n$ *contenu dans* Ω. *Il existe* *un entier N tel que, pour tout segment de droite l contenu dans B, la dérivée de la restriction de Φ à l change de signe au plus N fois.*

Démonstration. Considérons le fibré en droites affines $E \xrightarrow{p} \Omega \times S^{n-1}$ dont les fibres sont $p^{-1}(x, v) = \{x+tv; t \in \mathbf{R}\}$. La dérivée de Φ dans la direction v au point $x+tv$ définit clairement une fonction analytique réelle $f: E \to \mathbf{R}$, et il suffit d'appliquer la Proposition 1 avec $K_1 = B \times S^{n-1}$ et $K_2 = \{x+tv \in E; x \in B, x+tv \in B\}$. Q.E.D.

§ 2. Distance entre deux points sur une hypersurface de niveau

La proposition suivante, qui est un cas particulier du (3) de l'Introduction, entraine l'énoncé (5.11) dans [T]:

PROPOSITION 3. *Soit $\Phi: \Omega \to \mathbf{R}$ une fonction analytique réelle définie sur un ouvert Ω de $\mathbf{R}^n$, et soit K un sous-ensemble compact de Ω. On suppose que les fibres $\Phi^{-1}(t) \cap K$ sont connexes. Il existe un nombre réel $C>0$ tel que, pour tout $t \in \Phi(K)$, chaque couple de points $x, y \in \Phi^{-1}(t) \cap K$ puisse être relié par un arc, analytique par morceaux, contenu dans $\Phi^{-1}(t) \cap K$ et de longueur $\leq C$.*

Démonstration. Remarquons d'abord que la question est locale sur $\mathbf{R}$, en ce sens qu'il suffit de vérifier qu'un point arbitraire de $\Phi(K)$ possède un voisinage V tel que tout couple de points $x, y \in \Phi^{-1}(t) \cap K$, $t \in V$, peut être relié par un arc contenu dans $\Phi^{-1}(t) \cap K$ et de longueur $\leq C_V$.

Remarquons ensuite que la questions est aussi locale « en haut » : il suffit de montrer que tout point $x \in K$ possède un voisinage ouvert $U \subset K$ tel que deux points quelconques de U appartenant à la même fibre de Φ puissent être joints par un arc contenu dans $\Phi^{-1}(t) \cap U$ et de longueur $\leq C_U$. Soit alors V un voisinage fermé d'un point arbitraire de $\Phi(K)$. On recouvre $\Phi^{-1}(V) \cap K$ par un nombre fini d'ouverts du genre de $U, U_1, \ldots, U_r$. Par l'hypothèse de connexité des fibres $\Phi^{-1}(t) \cap K$ on aura la propriété suivante : si $x, y \in \Phi^{-1}(V) \cap K$ appartiennent à la même fibre de Φ, il existe une suite finie $x = u_1, u_2, \ldots, u_s = y$ de points de cette même fibre, tels que deux points successifs u_j, u_{j+1} appartiennent à l'un des ouverts U_i. D'autre part, si u_j et $u_{j+\nu}$ avec $\nu \geq 2$, appartiennent au même ouvert on peut supprimer tous les points intermédiaires dans la suite et de cette façon se ramener au cas où la longueur de la suite ne dépasse pas r. On peut alors prendre $C_V = \Sigma_{1 \leq j \leq r} C_j$.

Enfin, rappelons le théorème de désingularisation suivant :

THÉORÈME (Hironaka, cf. [H1]). *Soit X un espace analytique réel non singulier et soit $\mathcal{T}$ un faisceau cohérent d'idéaux du faisceau structural $\mathcal{A}_X$. Pour tout point $x \in X$,* *il existe un voisinage ouvert U de x dans X et un morphisme $\pi: X' \to U$ entre espaces analytiques réels, ayant les propriétés suivantes :*

(1) *Le morphisme π est propre, surjectif et algébrique, en ce sens qu'il existe un diagramme commutatif :*

$$\begin{array}{ccc} X' & \xrightarrow{i} & U \times \mathbf{R}^N \\ & \pi \searrow \quad \swarrow \mathrm{pr}_1 & \\ & U & \end{array}$$

où i est une immersion fermée et $i(X')$ est défini par un idéal engendré par des polynômes en les coordonnées de $\mathbf{R}^N$, à coefficients analytiques sur U.

(2) *L'espace analytique X' est non singulier.*

(3) *Si Y est le sous-espace de X défini par $\mathscr{T}$, l'ouvert $X' \setminus \pi^{-1}(Y)$ est dense dans X', et π induit un isomorphisme*

$$X' \setminus \pi^{-1}(Y) \to X \setminus Y.$$

(4) *Pour tout point $x' \in X'$, il existe un système de coordonnées locales $(z'_1, \ldots, z'_n)$ dans X', centré en x', tel que l'ideal $\mathscr{T}.\mathscr{A}_{X',x'}$ soit engendré, dans $\mathscr{A}_{X',x'} \cong \mathbf{R}\{z'_1, \ldots, z'_n\}$, par des monômes*

$$z_1'^{a_1} \ldots z_n'^{a_n}, \quad a = (a_1, \ldots, a_n) \in \mathbf{Z}_+^n.$$

Revenons-en à la démonstration de la Proposition 3. Soit $x_0 \in K$; posons $t_0 = \Phi(x_0)$ et soit $\mathscr{T}$ l'idéal de $\mathscr{A}_\Omega$ engendré par $\Phi - t_0$. D'aprés le théorème précédent il existe un voisinage ouvert U de x_0 dans K, un morphisme propre $\pi: X' \to U$ tel que X' soit non singulier et que, pour tout point x' de X', il existe un système de coordonnées locales $x'_1, \ldots, x'_n$ tel que, dans un voisinage ouvert U' de x', on ait

$$(\Phi - t_0) \circ (\pi|_{U'}) = u(x'_1, \ldots, x'_n)\, x_1'^{a_1} \ldots x_n'^{a_n},$$

avec $a \in \mathbf{Z}_+^n$ et $u \neq 0$ dans U'.

Puisque le morphisme π est propre, on voit, par le même raisonnement de localisation que plus haut, qu'il suffit de montrer que tout point $x' \in X'$ possède un voisinage ouvert W' tel que deux points de W' appartenant à la même fibre de $\Phi \circ \pi$ puissent être joints par un arc contenu dans $\pi^{-1}(\Phi^{-1}(t) \cap K) \cap W'$ et de longueur $\leq C_{W'}$. En effet, on recouvrira $\pi^{-1}(\Phi^{-1}(t_0) \cap K)$ par un nombre fini r de tels ouverts, dont la réunion contient $\pi^{-1}(\Phi^{-1}(V) \cap K)$ (V : voisinage de t_0 dans $\mathbf{R}$). On saura joindre deux points x' et y' de $\pi^{-1}(\Phi^{-1}(t_0)\cap K)$ par la réunion d'un nombre d'arcs ne dépassant pas r, dont la longueur totale est $\leq\Sigma_{1\leq j\leq r}C_{W_j'}$. La réunion des images par π de ces arcs fournira un chemin joignant $\pi(x')$ à $\pi(y')$, de longueur $\leq\Delta\,\Sigma_{1\leq j\leq r}C_{W_j'}$. Ici Δ est une constante, puisqu'il s'agit de borner la longueur d'arcs images par π d'arcs de longueur bornée dans X'. On peut prendre pour Δ n fois le supremum sur $\pi^{-1}(\Phi^{-1}(V)\cap K)$ des valeurs absolues des dérivées partielles des fonctions décrivant π dans des cartes locales.

Finalement on est ramené à prouver :

LEMME. *Soit* $\Phi=u(z_1,\dots,z_n)\,z_1^{a_1}\dots z_n^{a_n}$ *avec* $a\in\mathbf{Z}_+^n$ *et* $u\neq0$ *dans un voisinage ouvert* U' *de* 0 *dans* $\mathbf{R}^n$. *Il existe un voisinage ouvert* $W'\subset U'$ *de* 0 *et une constante* $C_{W'}>0$ *tels que deux points quelconques de* W' *appartenant à la même composante connexe d'une fibre de* Φ *puissent être joints par un arc contenu dans* U' *sur lequel* Φ *est constante, de longueur* $\leq C_{W'}$.

Démonstration. Après contraction de U' et changement de variables on peut supposer que $\Phi=z_1^{a_1}\dots z_n^{a_n}$. On supposera désormais que U' est une boule ouverte contrée à l'origine, de rayon ϱ. Il est clair aussi qu'on peut se ramener au cas où tous les a_j sont ≤1. En effet, admettons que certains de ces exposants soient nuls; après un changement d'indices des variables on peut supposer que $\Phi=z_1^{a_1}\dots z_\nu^{a_\nu}$ avec $a_j\geq1$ pour $j=1,\dots,\nu$. Les arcs recherchés pourront être alors réunion d'un arc sur lequel Φ et $z''=(z_{\nu+1},\dots,z_n)$ sont constants, et d'un segment de droite dans l'espace z'', sur lequel $z'=(z_1,\dots,z_\nu)$ et donc Φ sont constants.

Soient alors $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ deux points de U' appartenant à la même composante connexe d'une fibre de Φ. On supposera $\Phi(x)=\Phi(y)\neq0$, et même que pour chaque $i=1,\dots,n$, x_i et y_i aient le même signe (puisque les a_j sont tout non nuls il en est de même des x_j et des y_j). Posons alors, pour $0\leq t\leq1$,

$$Z_j=x_j|x_j|^{-t}|y_j|^t,\quad j=1,\dots,n.$$

Il est clair que $\Phi(Z)=\Phi(x)=\Phi(y)$. De plus,

$$\|Z\|=(Z_1^2+\dots+Z_n^2)^{1/2}\leq\sum_{i=1}^n|x_j|^{1-t}|y_j|^t\leq n\varrho.$$

Il suffira désormais de faire varier x et y dans une boule W' centrée à l'origine, de rayon $\varrho'\leq\varrho/n$, pour être sûr que Z reste dans U'. Lorsque t varie de 0 à 1 le point $Z(t)$ parcourt un chemin de x à y. La longueur de de chemin est majorée par

$$\sum_{j=1}^{n} \int_0^1 |\dot{Z}_j(t)|\, dt \leq \sum_{j=1}^{n} |x_j - y_j| \leq 2n\varrho'.$$

Puisque cette majoration est uniforme, et que par ailleurs deux points de $\Phi^{-1}(0) \cap U'$ peuvent être joints par un arc de longueur $\leq 2\varrho$, le lemme est démontré.

Remarque : L'hypothèse de connexité les fibres de $\Phi^{-1}(t) \cap K$ est superflue : au vu de la propriété (1) de l'introduction, le nombre des composantes connexes des $\Phi^{-1}(t) \cap K$ est uniformément borné, et nous démontrons en fait ci-dessus qu'il existe $C>0$ tel que pour $t \in \Phi^{-1}(K)$, deux points x, y appartenant à la même composante connexe de $\Phi^{-1}(t) \cap K$ puissent être joints par un chemin contenu dans $\Phi^{-1}(t) \cap K$ et de longueur $\leq C$.

Nous pouvons déduire de la Proposition 3 l'énoncé (5.15) de [T]. Notons $\gamma(x, y)$ la courbe qui joint x à y dans $\Phi^{-1}(t) \cap K$ et qui a été construite dans la démonstration de la Proposition 3. Supposons K *convexe* et notons $l(x, y)$ le segment de droite qui joint x à y. Paramétrons ces deux courbes de façon qu'elles aient la même longueur (et qu'elles soient toutes deux orientées de x à y). Soit alors $S(x, y)$ la surface réglée engendrée par les segments de droite joignant les points sur $\gamma(x, y)$ et $l(x, y)$ qui correspondent à la même valeur du paramètre.

COROLLAIRE 2. *Il existe une constante $C'>0$ qui majore l'aire de $S(x, y)$ quels que soient $x, y \in K$.*

Le Corollaire 2 résulte de la Proposition 3 et de l'énoncé suivant :

LEMME DU LIMON(1). *L'aire de $S(x, y)$ est majorée par le produit de la longueur de chemin $\gamma(x, y)$ et de la longueur maximum des segments de droite qui engendrent $S(x, y)$.*

La preuve de ce lemme est un exercice de calcul différentiel. On pourra prendre la constante C' dans le Corollaire 2 égale à C diam K, où C est la constante obtenue dans la Proposition 3.

Ajouté sur épreuves. Récemment, R. Hardt a répondu affirmativement à la question (2) de l'introduction; voir « Some analytic bounds for subanalytic sets » in « Differential-geometric control theory », Progress in Math., n° 27, Birkhäuser.

(1) On appelle « limon » la courbe engendrée par le bord extrême des marches dans un escalier à vis. Le lemme du limon énonce donc que la surface d'un escalier à vis est majorée par la longueur du limon que multiplie la longueur de la plus longue marche. La terminologie et l'énoncé sont dus à Douady.

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Received March 24, 1982
Received in revised form May 10, 1982