NECESSARY CONDITIONS FOR LOCAL SOLVABILITY OF HOMOGENEOUS LEFT INVARIANT DIFFERENTIAL OPERATORS ON NILPOTENT LIE GROUPS

BY

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1. Introduction and allegro

A differential operator L is *locally solvable* at a point x_0 if there exists a neighborhood U of x_0 such that

$$Lu(x) = f(x), \quad \text{all } x \in U,$$

has a solution $u \in C^\infty(U)$ for any $f \in C_0^\infty(U)$. We shall give necessary conditions for local solvability for some classes of left invariant differential operators on nilpotent Lie groups.

Let G be a connected, simply connected, nilpotent Lie group which admits a family of dilations δ_r, $r>0$, which are automorphisms. The δ_r extend to automorphisms of the complexified universal enveloping algebra $U(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. The elements of $U(\mathfrak{g})$ may be identified with the left invariant differential operators on G. An element $L \in U(\mathfrak{g})$ is homogeneous of degree d if $\delta_r(L) = r^d L$, all $r>0$. We equip G with a norm, $|\ |$, which is homogeneous in the sense that if $U_s = \{x \in G: |x| < s\}$, then $\delta_r(U_s) = U_{rs}$.

We shall prove two main theorems concerning the local solvability of a homogeneous element $L \in U(\mathfrak{g})$, with transpose L^τ. The first says that L is unsolvable if $\ker L^\tau$ contains a function in $\mathcal{S}(G)$, the Schwartz space of G. The second result uses the first to obtain a representation-theoretic criterion for unsolvability of L. Let $\hat{G}$ be the set of all irreducible unitary representations of G. If there is an open subset of representations π in $\hat{G}$ such that

(1) $\ker \pi(L^\tau)$ contains a nonzero C^∞ vector, and
(2) $\ker \pi(L^\tau)$ varies smoothly with π,

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then L is not locally solvable. We use this theorem to give some new examples of unsolvable operators on the Heisenberg group. Next, we show by example that one cannot weaken (1) to the condition that $\ker \pi(L^\tau)$ is nonzero. Finally, we give an example of an unsolvable operator L such that $\ker \pi(L^\tau)$ is trivial for almost all π.

The idea of studying the kernel of $\pi(L)$ and $\pi(L^\tau)$ for local properties of L is suggested by the following. A differential operator D is *hypoelliptic* in an open set U if $Du=f$ with $f \in C^\infty(U)$ implies $u \in C^\infty(U)$. Helffer and Nourrigat [14] have shown that L is hypoelliptic if and only if

$$\ker \pi(L) = 0 \quad \text{for all } \pi \in \hat{G},\ \pi \text{ nontrivial;} \tag{1.1}$$

here $\hat{G}$ is the set of all irreducible unitary representations of G. Since L^τ hypoelliptic implies L locally solvable, it is reasonable to suppose that the complete failure of (1.1) to hold, with L replaced by L^τ, might imply L is unsolvable. Our second result then shows this is true under some further hypotheses.

The theorem of Helffer and Nourrigat was first conjectured by Rockland [23], who proved a special case. Rockland also conjectured some results on local solvability, parts of which were later proved independently by the second author [24], G. Lions [20] and the first author [2]. A detailed study of local solvability for second order operators on two step groups was made in [25].

Rockland's conjecture was motivated by the work of Folland and Stein [8], in which the sufficiency of (1.1) for hypoellipticity for a class of second order operators on the Heisenberg group was proved by the construction of a fundamental solution. The idea of using homogeneity and a transformed operator to study hypoellipticity was introduced by Grušin [11], to study operators like $D = \partial^2/\partial x^2 + x^2(\partial^2/\partial y^2) + i\alpha(\partial/\partial y)$. Grušin proves that D is hypoelliptic if and only if ker $\hat{D}$ is trivial, where $\hat{D} = d^2/dx^2 - x^2\tau^2 - \alpha\tau$. In a later paper [12] he also studies local solvability.

The first example of an unsolvable differentiable operator was given by Hans Lewy in his study of the boundary values of holomorphic functions. In fact Lewy's operator is a homogeneous element of $U(\mathfrak{h})$, where $\mathfrak{h}$ is the Heisenberg algebra. Greiner, Kohn and Stein [10] studied the Lewy operator L from this point of view and were able to show that $Lu=f$ is solvable in an open set U if and only if the projection of f onto $\ker L^\tau$ is real analytic. Further results were obtained by Geller [9]. Our present results were motivated by these.

A brief overview of the techniques used in this paper is given as follows. The first main result, Theorem 1, Section 2, is based on the fact that a left invariant locally solvable differential operator on a Lie group must possess a local fundamental solution [25]. Using the fundamental solution we prove that if $\varphi \in C_0^\infty(G)$ we may find functions $\psi_m \in C_0^\infty(G)$ which are uniformly bounded by a polynomial in m and which satisfy $L\psi_m(x) = \varphi(x)$ for $|x| \leqslant m$. In connection with Theorem 1 we may note a recent result of Duflo and Wigner which states that *any* left invariant differential operator on a simply connected nilpotent group has no nontrivial compactly supported distributions in its kernel.

The proof of our second theorem amounts to constructing a Schwartz function in $\ker L^\tau$ from representation-theoretic data. (A function f on G is in $\mathcal{S}(G)$ if and only if $f \circ \exp \in \mathcal{S}(\mathfrak{g})$, where exp denotes the exponential map.) The difficulty involved here is in identifying an element φ of the Schwartz space by studying the operators $\pi(\varphi) = \int_G \varphi(g)\pi(g)\,dg$, $\pi \in \hat{G}$. Here we rely on earlier work of Greenleaf and the first author [4].

2. A necessary condition for local solvability of an operator in terms of the kernel of its transpose

Let G denote a simply connected nilpotent group with dilations.

THEOREM 1. *Let L be a left invariant homogeneous differential operator on G and L^τ its transpose. Suppose that L is locally solvable at 0. Then there exists an integer k such that if $L^\tau f = 0$ with $(1+|x|^k)f \in L^2(G)$, then $f=0$.*

COROLLARY. *If L is as above, then L^τ has trivial kernel on the space of Schwartz functions on G.*

We shall show by example in Section 7 that in general k cannot be taken to be zero.

LEMMA 1. *Suppose that L is as in* Theorem 1. *Then there exists an integer $k_1 \geqslant 0$ satisfying the following. For any $\varphi \in C_0^\infty(G)$ there is a constant $C>0$ and a sequence $\{h_n\} \subset C_0^\infty(G)$ such that*

(i) $\operatorname{supp} h_n \subset \{x \in G\colon |x| \leqslant n+1\}$

(ii) $\sup_{x\in G} |Lh_n(x)| \leqslant Cn^{k_1}$

(iii) $Lh_n(x) = \varphi(x)$ if $|x| < n$.

Assuming Lemma 1, we can prove Theorem 1.

Proof of Theorem 1. Let $k_1 \gg 0$ satisfy Lemma 1 and put $k = 2k_1$. Suppose f satisfies $(1+|x|^k)f \in L^2$ and $L^\tau f = 0$. We shall show that if L were locally solvable, for any $\varphi \in C_0^\infty(G)$,

$$\int_G f(x)\,\varphi(x)\,dx = 0, \tag{2.1}$$

which would prove the theorem. Let $\{h_k\}$ be the sequence defined in Lemma 1. Then for any integer $n>0$,

$$\left|\int_{|x|\leqslant n+1} f(x)\,\varphi(x)\,dx\right| \leqslant \left|\int_{|x|\leqslant n+1} f(x)\,Lh_n(x)\,dx\right| + \left|\int_{|x|\leqslant n+1} f(x)\,(\varphi - Lh_n)\,(x)\,dx\right|. \quad (2.2)$$

Since $\operatorname{supp} h_n \subset \{x\colon |x| \leqslant n+1\}$, integration by parts is justified for the first integral on the right in (2.2) and we obtain

$$\int f(x)\,Lh_n(x)\,dx = \int L^\tau f(x)\,h_n(x)\,dx = 0$$

since $L^\tau f = 0$ by hypothesis. For the second term we use (iii) to obtain

$$\int_{|x|\leqslant n+1} f(x)\,(\varphi - Lh_n)\,(x)\,dx = \int_{n\leqslant|x|\leqslant n+1} f(x)\,(\varphi - Lh_n)\,(x)\,dx.$$

By (ii),

$$\sup |(\varphi - Lh_n)(x)| \leqslant C_\varphi n^{k_1},$$

C_φ a constant depending on φ. Hence for $|x| \leqslant n+1$

$$|f(x)|\,|(\varphi - Lh_n)(x)| \leqslant C_\varphi |f(x)|\,|x|^k \leqslant C'_\varphi |f(x)|\,|x|^{2k_1} \frac{1}{(1+|x|)^{k_1}}. \quad (2.3)$$

By Schwarz' inequality

$$\int_{n\leqslant|x|\leqslant n+1} |f(x)|\,|(\varphi - Lh_n)\,(x)|\,dx \leqslant C'_\varphi \left\{\int_{n\leqslant|x|\leqslant n+1} |f(x)|^2\,|x^k|^2\,dx\right\}^{\frac{1}{2}} \left\{\int_{n\leqslant|x|\leqslant n+1} \frac{dx}{(1+|x|)^k}\right\}^{\frac{1}{2}}. \quad (2.4)$$

As long as k_1 is chosen sufficiently large so that $(1+|x|)^{-k_1} \in L^2$, both terms on the right hand side of (2.4) go to zero as $n \to \infty$. Since n is arbitrary it follows that (2.1) must hold. This proves Theorem 1, modulo Lemma 1.

Proof of Lemma 1. Let α be the homogeneous degree of L. If L is locally solvable at 0, then by [25, Theorem 15.4] there is a neighborhood U of 0 and a distribution σ on G such that σ is a fundamental solution for L in U, i.e.,

$$L\sigma = \delta \quad \text{in } U,$$

where δ denotes the delta distribution at 0. We may take $U = \{x\colon |x| < \varepsilon\}$ for some $\varepsilon > 0$. Now for any function ψ on G, let ψ_1 be defined by

$$\psi_1(x) = \psi(x^{-1}), \quad x \in G$$

and for $y \in G$ let ψ^y be the function defined by

$$\psi^y(x) = \psi(yx), \quad x \in G.$$

Now for $\psi \in C_0^\infty(G)$ let ψ_α be the function given by

$$\psi_\alpha(y) = \sigma((\psi^y \circ \delta_{r_n})_1), \quad \text{for fixed } n,\, r_n.$$

We denote by x the usual function variable, i.e. $\varphi = \varphi(x)$ and write L_x or L_y and σ_x or σ_y to emphasize which variable L or σ is acting on. Let $V \subset G$ be an open set, and suppose that for a given n the functions $x \mapsto (\psi^y \circ \delta_{r_n})_1(x)$ all have supports contained in U for all $y \in V$. Then

$$\begin{aligned} L_y(\psi_\alpha) &= L_y(\sigma_x(\psi^y \circ \delta_{r_n})_1) = \sigma_x(L_y^\tau(\psi^y \circ \delta_{r_n}(x^{-1}))) \\ &= \sigma_x(r_n^{-\alpha} L_x((\psi^y \circ \delta_{r_n})(x^{-1}))) = r_n^{-\alpha} \sigma_x(L_x(\psi_1^y \circ \delta_{r_n})) \\ &= r_n^{-\alpha}(\sigma_x L_x)(\psi_1^y \circ \delta_{r_n}) = r_n^{-\alpha} \psi_1^y \circ \delta_{r_n}(0) = r_n^{-\alpha} \psi(y). \end{aligned} \tag{2.5}$$

Now choose C'' to satisfy

$$|xy| \leqslant C''(|x| + |y|) \tag{2.6}$$

(which is possible by Knapp and Stein [19, § 2, Remark (3)]) and let $C' > C''$. We shall choose $h_n(y) \in C_0^\infty(G)$ so that

$$h_n(y) = \begin{cases} r_n^\alpha \sigma((\varphi^y \circ \delta_{r_n})_1) & \text{for } |y| \leqslant n \\ 0 & |y| > n+1, \end{cases} \tag{2.7}$$

where $r_n = C'(\varepsilon + n)/\varepsilon$. Suppose $h_n(y)$ satisfies (2.7) above. Then

$$\operatorname{supp}\,[x \mapsto (\varphi^y \circ \delta_{r_n})_1(x)] \subset \{|x| < \varepsilon\}, \tag{2.8}$$

for all $|y| < n$. Indeed, $(\varphi^y \circ \delta_{r_n})_1(x) = \varphi(y\delta_{r_n} x^{-1})$. Now by (2.6),

$$|\delta_{r_n} x^{-1}| \leqslant C''(|y^{-1}| + |y\delta_{r_n} x^{-1}|),$$

and it follows that

$$|y\delta_{r_n} x^{-1}| \geqslant \frac{|\delta_{r_n} x^{-1}|}{C''} - |y^{-1}|.$$

If $|x| \geqslant \varepsilon$,

$$\left|\frac{\delta_{r_n} x^{-1}}{C''}\right| \geqslant \frac{r_n \varepsilon}{C''}$$

so that if $|y| = |y^{-1}| \leqslant n$,

$$|y\delta_{r_n} x^{-1}| \geqslant \frac{r_n \varepsilon}{C''} - n > \varepsilon.$$

Hence $y\delta_{r_n}x^{-1} \notin \operatorname{supp} \varphi$ for $|x| \geqslant \varepsilon$, $|y| < n$, which proves (2.8). Hence we may apply (2.5) to obtain

$$L_y(r_n^\alpha \sigma((\varphi^y \circ \delta_{r_n})_1)) = \varphi(y). \tag{2.9}$$

We have now shown that if $h_n(y)$ satisfies (2.7) then it satisfies (iii) of Lemma 1. For (ii) we shall prove first that if $h_n(y)$ is defined by (2.7) for $|y| \leqslant n$, then there exists k_2 such that

$$\sup_{\substack{|y|<n \\ |\alpha| \leqslant d}} |D_y^\alpha h_n(y)| \leqslant C n^{k_2} \tag{2.10}$$

for some constant C, depending on φ. Let $Y_1, Y_2, \ldots, Y_N$ be a basis of $\mathfrak{g}$ consisting of homogeneous vector fields. Then since

$$\frac{\partial}{\partial y_j} = \sum_{k=1}^{N} a_{jk}(x)\, Y_k \tag{2.11}$$

where the $a_{jk}(x)$ are polynomials, (2.10) will follow if we can prove there exists k_3 such that

$$\sup_{\substack{k \leqslant \alpha \\ |y|<n}} |Y_{i_1} Y_{i_2} \ldots Y_{i_k} h_n(y)| \leqslant C' n^{k_3}. \tag{2.12}$$

Now

$$\begin{aligned} Y_{i_1} Y_{i_2} \ldots Y_{i_k} h_n(y) &= r_n^{-\alpha} Y_{i_1}^y Y_{i_2}^y \ldots Y_{i_k}^y (\sigma_x(\varphi(y\delta_{r_n} x^{-1}))) \\ &= (-1)^k r_n^{-\alpha} r_n^l \sigma_x(Y_{i_k}^x Y_{i_{k-1}}^x \ldots Y_{i_1}^x(\varphi(y\delta_{r_n} x^{-1}))), \end{aligned} \tag{2.13}$$

where the homogeneous degree of $Y_{i_1} Y_{i_2} \ldots Y_{i_k}$ is l,

$$= (-1)^k r_n^{l-\alpha} \sigma_x(Y_{i_k}^x \ldots Y_{i_1}^x(\varphi^y \circ \delta_{r_n})_1).$$

Finally, since σ is a distribution of compact support, contained in $\{x: |x| < \varepsilon\}$, it is of finite order, so that there exists an integer l' and a constant C_σ such that for any $\chi \in C_0^\infty(G)$,

$$|\sigma(\chi)| \leqslant C_\sigma \sup_{\substack{|x|<\varepsilon \\ |\alpha| \leqslant l'}} |D^\alpha \chi(x)|$$

Now (2.12) follows from (2.11) and (2.13).

The proof of Lemma 1 will be completed if we can extend h_n so that (ii) is still satisfied. Given (2.10) this may be done by standard techniques. Thus the proof of Lemma 1 is completed by the following.

LEMMA 2. *Let $p_n \in C^\infty(\mathbf{R}^N)$ satisfy*

$$\sup_{\substack{|x| \leqslant n+1 \\ |\alpha| \leqslant l}} |D^\alpha p_n(x)| \leqslant C(n+1)^k. \tag{2.14}$$

Then there exists $q_n(x) \in C_0^\infty(\mathbf{R}^N)$ and C', K' such that

$$\operatorname{supp} q_n(x) \subset \{x\colon |x| \leqslant n+1\}, \tag{2.15}$$

$$q_n(x) = p_n(x), \quad |x| \leqslant n, \tag{2.16}$$

and

$$\sup_{|\alpha| \leqslant l} |D^\alpha q_n(x)| \leqslant C' n^{k'}. \tag{2.17}$$

Proof. Define $Q_n(x) \in C_0^\infty(\mathbf{R}^N)$ by

$$Q_n(x) = \begin{cases} 1, & |x| \leqslant n+\frac{1}{2} \\ b_n(x), & n+\frac{1}{2} \leqslant x \leqslant n+1 \\ 0, & |x| \geqslant n+1, \end{cases}$$

where

$$b_n(x) = e^2 \exp\left(-(1/(|x|-(n+1)))\right)\left(1-\exp\left(-(1/(|x|-(n+\tfrac{1}{2})))\right)\right).$$

Put $q_n(x) = p_n(x) Q_n(x)$. Then (2.15), (2.16) and (2.17) are easily checked.

The proof of Theorem 1 is now complete.

3. Generic representations of nilpotent Lie groups

We shall need to extend some results on representations of nilpotent Lie groups that were given in Section 2 of [2]. These results also apply to groups without dilations.

We begin with an account of Kirillov theory; proofs can be found in [18] or [21]. Given $l \in \mathfrak{g}^*$, we let B_l be the bilinear form on $\mathfrak{g}$ given by $B_l(X, Y) = l[X, Y]$, and we set $\mathfrak{R}_l = \operatorname{Rad} B_l = \{X \in \mathfrak{g}\colon l[X, \mathfrak{g}] = 0\}$. Then $\operatorname{codim} \mathfrak{R}_l$ is an even integer $2k$. One can show that there exist subalgebras $\mathfrak{m}_l$ of $\mathfrak{g}$ such that $\operatorname{codim} \mathfrak{m}_l = k$ and $B_l | \mathfrak{m}_l \times \mathfrak{m}_l = 0$; $\mathfrak{m} = \mathfrak{m}_l$ is called *maximal subordinate* or *polarizing*. The condition on B_l shows that $l\colon \mathfrak{m} \to \mathbf{R}$ is a Lie algebra homomorphism, and thus the map $\lambda\colon M = \exp \mathfrak{m} \to S^1$ defined by $\lambda(\exp X) = e^{il(X)}$ is a one dimensional representation of G. Let $\pi_{l,\mathfrak{m}}$ be the unitary representation of G induced from λ.

THEOREM (Kirillov). (1) *Up to unitary equivalence, $\pi_{l,\mathfrak{m}}$ is independent of the choice of $\mathfrak{m}$. (Thus we may write π_l unambiguously.)*

(2) *The representation π_l is irreducible.*

(3) *If σ is any irreducible unitary representation of G, then there is an element $l \in \mathfrak{g}^*$ such that $\sigma \approx \pi_l$.*

(4) *If $l, l' \in \mathfrak{g}$, then $\pi_l \cong \pi_{l'}$ if and only if there exists $x \in G$: $l' = (\mathrm{Ad}^*\, x) l$. (Thus the* Ad* (G) *orbits parametrize the space $\hat{G}$ of equivalence classes of irreducible representations.)*

Let π be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$, and let $X \in \mathfrak{g}$. We may define $\pi(X)$ by

$$\pi(X)\,v = \lim_{t\to 0} t^{-1}(\pi(\exp tX)\,v - v)$$

when the limit exists. It turns out that $\pi(X)$ is densely defined and closed, and that by iteration we can define $\pi(L)$ for all $L \in U(\mathfrak{g})$. These generators have a dense common domain, $\mathcal{H}^\infty(\pi)$, the space of C^∞ vectors for π, i.e. the space of vectors $f \in \mathcal{H}$ for which $g \mapsto \pi(g)f$ is a C^∞ function from G to $\mathcal{H}$. Let $\pi = \pi_l$ and let $\{Y_1, Y_2, ..., Y_n\}$ be a basis of $\mathfrak{g}$ such that for all i, $\mathfrak{h}_i = \text{span}\,\{Y_1, Y_2, ..., Y_i\}$ is a subalgebra, and $\mathfrak{h} = \mathfrak{h}_{n-k}$ is polarizing for l. Then the subset $\{\exp t_1 Y_{n-k+1} \dots \exp t_k Y_n: t_1, t_2, ..., t_k \in \mathbf{R}\}$ is a cross-section for H/G (where $H = \exp \mathfrak{h}$) and this identification lets us realize π_l on $L^2(\mathbf{R}^k)$. In this realization, $\mathcal{H}^\infty(\pi) \cong \mathcal{S}(\mathbf{R}^k)$, the space of Schwartz class functions on $\mathbf{R}^k$. (See [21] or [6].)

We next examine what happens when we vary the representations π_l. Let $X_1, ..., X_n$ be a basis of $\mathfrak{g}$ such that for all i, $\mathfrak{g}_i = \text{span}\,\{X_1, ..., X_i\}$ is an ideal of $\mathfrak{g}$. (We call such a basis a *strong Malcev* basis for $\mathfrak{g}$.) Let $l_1, ..., l_n$ be the dual basis of $\mathfrak{g}^*$.

THEOREM 2. *There are complementary subsets S and T of $\{1, ..., n\}$ and subsets U, V of $\mathfrak{g}^*$ such that if $V_1 = \text{span}\,\{l_j: j \in S\}$ and $V_2 = \text{span}\,\{l_j: j \in T\}$, then*

(1) *V is a Zariski-open subset of V_1 and U is a Zariski open subset of $\mathfrak{g}^*$.*

(2) *U is closed under the action of* Ad*.

(3) *Every* Ad**-orbit contained in U intersects V_1 in exactly one point, and $U \cap V_1 = V$. (Thus one can use V to parametrize the* Ad**-orbits in U.)*

(4) *There is a function Q: $V_1 \times V_2 \to V_1$, rational in V_1 and polynomial in V_2, such that if $l \in V$, then* graph $Q(l, \cdot) = \mathcal{O}_l$, *the* Ad**-orbit parametrized by l.*

(This is essentially Theorem 1 of [2].) We say that the Ad*-orbits in U (or the corresponding irreducible representations under the Kirillov correspondence) are "in general position" or "typical", and we denote by Γ the set of corresponding representations of G. It should be noted that whether a representation is or is not in general position depends the choice of Malcev basis (or, more precisely, on the choice of the chain of ideals $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots \subset \mathfrak{g}$). Furthermore, the polynomial function P on $\mathfrak{g}$ such that $l \in U \Leftrightarrow P(l) \neq 0$ can be explicitly described; it is analogous to a Pfaffian. Let $k = \frac{1}{2}$ card (T); k is an integer, $2k$ is the dimension of a typical orbit, and the usual way of describing a typical representation is as acting on $\mathcal{L}^2(\mathbf{R}^k)$. We shall return to this point soon.

For a crucial step in the proof of our second main result (Theorem 3) we shall have to embed $\mathfrak{g}$ as a subalgebra of a larger algebra $\mathfrak{n}$ and lift properties of the representations of

$\mathfrak{g}$ to those of $\mathfrak{n}$. For this, let $\mathfrak{g}^1$ be a nilpotent Lie algebra such that $\mathfrak{g}$ is an ideal of $\mathfrak{g}^1$ of codimension 1. We identify objects corresponding to $\mathfrak{g}^1$ by the superscript^1. We therefore choose a basis $X_1, X_2, ..., X_{n+1}$ of $\mathfrak{g}^1$ such that $X_1, ..., X_n$ span $\mathfrak{g}$; we let $l_1^1, ..., l_{n+1}^1$ be the dual basis in $(\mathfrak{g}^1)^*$, and we let S^1 and T^1 be the complementary subsets of $\{1, ..., n+1\}$ described above (but for $\mathfrak{g}^1$). According to results in Section 3.1 of [21], $k^1=k$ or $k^1=k+1$. In the former case, an analysis of the proof of Theorem 1 shows that $T^1=T$ and $S^1=S\cup\{n+1\}$.

PROPOSITION 1. *Suppose $k^1=k$. Then U^1 is the pre-image of U from the restriction map taking $(\mathfrak{g}^1)^*$ to $\mathfrak{g}^*$. The restriction map is bijective from each orbit in U^1 to its image in U.*

Proof. The second statement follows from the discussion in Sections 3.1 and 3.2 of [21]. For the first, we need to analyze the function P^1. P^1 is an Ad*-invariant polynomial defined on V^1 by

$$P^1(l^1) = \det\,(l^1([X_i, X_j]) \colon i, j \in T^1).$$

But since $T^1=T$, $P^1(l^1)=P(l)$, where $l=l^1|_{\mathfrak{g}}$. Thus V^1 is the pre-image of V, and the first claim follows.

We now consider the representations in general position (still in the case $k^1=k$). It is possible (see [5] or [27]) to find a rational function $l\mapsto\mathfrak{m}_l$ from $\mathfrak{g}^*$ to the subspaces of $\mathfrak{g}$ of codimension k such that for all l in a Zariski-open set, $\mathfrak{m}_l$ is a maximal subordinate (=polarizing) subalgebra of l. The construction in [27] makes it clear that in our case, if $l=l^1|_{\mathfrak{g}}$, then $\mathfrak{m}_l$ is a subalgebra of codimension 1 in $\mathfrak{m}_{l^1}^1$ and $\mathfrak{m}_{l^1}^1 \not\subset \mathfrak{g}$.

Let $Y_1(l^1), ..., Y_{m^1}(l^1)(m^1=n+1-k)$ be a basis of $\mathfrak{m}_{l^1}^1$ such that for each $j\leqslant m^1$, $Y_1(l^1), ..., Y_j(l^1)$ is a subalgebra. Extend this basis to a basis $Y_1(l^1), ..., Y_{m+1}(l^1)$ of $\mathfrak{g}$ with the same property. We may assume that $Y_j(l^1)\in\mathfrak{g}$ for $j\neq m^1$; as noted in [5], we may also assume that the Y_j vary rationally with l^1. Then every element of $G^1=\exp\mathfrak{g}^1$ has a unique expansion $x=\exp\,(x_1 Y_1(l^1)) \dots \exp x_{n+1} Y_{n+1}(l^1))$.

Since $\pi_{l^1}^1$ is induced from a representation of $M_{l^1}^1$ ($=\exp \mathfrak{m}_{l^1}^1$), we may realize $\pi_{l^1}^1$ on $\mathcal{L}^2(\mathbf{R}^k)$ by using

$$\exp \mathbf{R} Y_{m^1+1}(l^1) \dots \exp \mathbf{R} Y_{n+1}(l^1) \cong \mathbf{R}^k$$

as a cross-section for $M_{l^1}^1\diagdown G^1$. Of course, $\pi_{l^1}^1/G=\pi_l$. Finally, we may compute

$$\pi_{l^1}^1(X_{n+1}) = \frac{d}{du}\,\pi_{l^1}^1(\exp uX_{n+1})\bigg|_{u=0}.$$

A tedious but straightforward computation shows:

PROPOSITION 2. *If $k=k^1$, then there is a rational map $l^1 \mapsto D_{l^1}$ of $(\mathfrak{g}^1)^* \to U(\mathfrak{g})$ such that on a Zariski-open set of $\mathfrak{g}^*$,*

$$\pi^1_{l^1}(X_{n+1}) = \pi^1_{l^1}(D_{l^1}) = \pi_l(D_l).$$

Indeed, the same sort of argument shows that for all $D \in U(\mathfrak{g})$, the operator $\pi_l(D)$ is a differential operator with polynomial coefficients which are rational functions of l.

We now turn to the case where $k^1 = k+1$. In this case, the proof of Theorem 1 shows that $n+1 \in T^1$ and that $S^1 \subset S$. For all representations π_l of N parametrized by orbits in a Zariski-open subset of U, $\mathrm{Ind}_{G \to G_1}(\pi_l)$ is irreducible, and $\mathrm{Ind}_{G \to G_1}(\pi_l) \cong \pi^1_{l^1}$, where l^1 is any element of $\mathfrak{g}_1^*$ whose restriction to $\mathfrak{g}_1$ is l. (From now on, we restrict attention to these l.) The representations π_{l_1} and π_{l_2} of G induce to equivalent representations of G^1 if and only if there is a $t \in \mathbf{R}$: Ad^* $(\exp tX_{n+1})(l_1) \in O_{l_2}$, the orbit of l_2. (Note that for $x \in G^1$, Ad x takes $\mathfrak{g}$ to $\mathfrak{g}$; we call the contragredient Ad^* even if $x \notin G$.) All these facts are proved in Section 3.1 of [21].

Let $Y_1(l), \ldots, Y_m(l)$ be a basis for $\mathfrak{m}(l)$, a maximal subordinate subalgebra of l; we may assume that the Y_j vary rationally with l. We may complete this basis to a rationally varying basis $Y_1(l), \ldots, Y_n(l)$ of $\mathfrak{g}$, and, as discussed above, we may model π_l on $\mathbf{R}^k$ by using $\exp \mathbf{R} Y_{m+1}(l) \ldots \exp \mathbf{R} Y_n(l)$ as a cross-section for $M(l) \diagdown G$. If l^1 is any extension of l, then $\mathfrak{m}(l)$ is also maximal subordinate for l^1, and we may use $Y_1(l), \ldots, Y_m(l), X_{n+1}$ as the basis for constructing representations. The next proposition is proved in [21, Part II, Chapter II, Section 5].

PPOPOSITION 3. *If $k^1 = k+1$, and if l is as above, let $l(u) = \mathrm{Ad}^*$ $(\exp uX_{n+1})\, l$. Suppose that l^1 extends l, and that $\pi^1_{l^1}$ is modeled as described above. Then if $f = f(t,u) = f_u(t)$ $(t \in \mathbf{R}^k,\ u \in \mathbf{R})$ is a function in $\mathcal{S}(\mathbf{R}^{k+1})$, we have*

$$\pi^1_{l^1}(X_{n+1})\, f = \frac{\partial f}{\partial u};$$

$$\pi^1_{l^1}(Y) f(t,u) = \pi_{l(u)}(Y) f_u(t), \quad \forall Y \in \mathfrak{g}.$$

Note. The restriction $f \in \mathcal{S}(\mathbf{R}^{k+1})$ is simply to insure that f is in the domain of the various unbounded operators. The same formulas apply to any f in the domain of the given operators.

Note also that if $l \in U$, then $l(u) \in U$. For $l(u)$ is certainly in general position with respect to the dual basis to Ad $(\exp uX_{n+1})(X_1), \ldots,$ Ad $(\exp uX_{n+1})(X_{n+1})$, and this basis gives rise to the same chain of ideals as the original basis.

4. Representation-theoretic criteria for unsolvability

We continue with the notation of the previous section. Our first task is to make precise the notion of vectors varying smoothly with respect to representations.

Consider the elements of $\mathfrak{g}^*$ in general position and the corresponding set Γ of representations of G; thus $\Gamma \leftrightarrow V$. (We shall sometimes restrict to a Zariski-open set of these elements in $\mathfrak{g}^*$ and to the corresponding subset of Γ; we shall still refer to these elements as in general position and to the set of representations as Γ.) As we have seen, we may choose rationally varying polarizing subalgebras of $\mathfrak{g}$ for the elements of Γ. It is now easy to check the following:

PROPOSITION 4. *Let notations be as in the previous section. For each $l \in V$, we can choose an explicit realization of π_l on $\mathfrak{L}^2(\mathbf{R}^k)$ such that if φ: $V \to \mathfrak{S}(\mathbf{R}^k)$ is a C^∞ map, then for every $D \in U(\mathfrak{g})$, the map*

$$l \mapsto \pi_l(D)\varphi(l)$$

is C^∞.

We describe φ (or the functions $\varphi(l)$) as *smoothly varying*. Note that the choice of realization for the proposition is far from unique; we can conjugate each π_l by a unitary operator U_l, where $l \mapsto U_l$ is a C^∞ map.

PROPOSITION 5. *Let G be a connected normal subgroup of G^1 of codimension 1, and let $L \in U(\mathfrak{g})$. Suppose that there is a nonzero smoothly varying family φ of vectors for G such that $\varphi(l) \in \ker \pi_l(L)$, $\forall l \in V$. Then there is a nonzero smoothly varying family φ^1 of vectors for G^1 such that $\varphi^1(l^1) \in \ker \pi^1_{l^1}(L)$, $\forall l^1 \in V^1$.*

Proof. We may assume that φ has compact support in l. Let k and k^1 be as in Section 4. There are two cases to consider.

Case 1. $k = k^1$. In this case, any representation $\pi^1_{l^1}$, $l^1 \in V^1$, has the property that $\pi^1_{l^1}|_G = \pi_l$, $l = l^1|_{\mathfrak{g}}$. We define $\varphi^1(l^1) = \varphi(l)$. Then $\varphi^1(l^1) \in \ker \pi^1_{l^1}(L)$ because $\pi^1_{l^1}(L) = \pi_l(L)$, and $\varphi^1(l^1)$ is smoothly varying because of Proposition 2.

Case 2. $k^1 = k+1$. We may (perhaps by reducing $\operatorname{supp}\varphi$ further) assume that if $l \in \operatorname{supp}\varphi$, then π_l induces to an irreducible representation of G^1. For $l \in V$, define $l(u)$ as in Proposition 3 and φ^1 by

$$\varphi^1(l^1)(t, u) = \varphi(l(u))(t), \quad l = l^1|_{\mathfrak{g}}.$$

It is easy to see from Proposition 3 that φ^1 meets the requirements of the proposition.

We require a consequence of Theorem 2.1 in [4]; it may be convenient to have a specific statement of what we need.

PROPOSITION 6. *Suppose that* $\mathfrak{g}$ *contains an Abelian ideal* $\mathfrak{m}$ *such that for all* $l \in U$, $\mathfrak{m}$ *is a polarizing subalgebra for* l. *Let* φ *be a smoothly varying function on* V, *and let* $\alpha \in C_0^\infty(V)$ *satisfy* supp $\alpha \subset$ supp φ. *Then there is a function* $f \in \mathcal{S}(G)$ *such that*

$$\pi_l(f) = \alpha(l) P_l, \quad \forall l \in V$$

where P_l *is the projection on the space spanned by* $\varphi(l)$. (*For* $l \notin$ supp α, $\pi_l(f) = 0$.)

Proof. Theorem 2.1 of [4] says that under the given hypotheses on G, we can identify G with $\mathbf{R}^n$ and V with $\mathbf{R}^{n-2k}$ so as to make $\mathcal{S}(G) = \mathcal{S}(\mathbf{R}^n)$ and to arrange that if $l \in V$, then $\pi_l(f)$ is an integral operator on $\mathbf{R}^k$ with kernel given by

$$K_{l,f}(x, t) = (\mathfrak{F}_0 f \circ A)(x, t, l)$$

where $\mathfrak{F}_0 f$ is a partial Fourier transform and A is a rational map with no singularities on V. Conversely, if $K_l(x, t) = K(x, t, l)$ is such that $K \circ A^{-1}$ extends to a Schwartz function g, then $K_l = K_{l,f}$, $f = \mathfrak{F}^{-1} g$. In our case, we may assume that $\|\varphi(l)\| = 1$ for all $l \in$ supp α. Then $K_l(x, t) = \varphi(l)(x)\overline{\varphi(l)(t)}\alpha(l) \in C_c^\infty(\mathbf{R}^n)$ and $K \circ A^{-1} \in C_c^\infty(\mathbf{R}^n)$; the proposition follows.

THEOREM 3. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra with dilations, and let* L *be a homogeneous differential operator in* $U(\mathfrak{g})$. *Suppose that there is a nonzero smoothly varying function* φ *on the representations of* G *such that* $\pi_l(L^\tau)\varphi(l) = 0$ *for all* l. *Then* ker $L^\tau \cap \mathcal{S}$ *is nontrivial and* L *is not locally solvable.*

Proof. In view of Theorem 1, it suffices to show ker $L^\tau \cap \mathcal{S}$ is nontrivial. Since

$$L^\tau f = f * L = (L^\tau * f^\tau)^\tau,$$

we need to find a function $g \in \mathcal{S}(G)$ such that $\pi_l(L^\tau)\pi_l(g) = 0$ for all representations π_l in general position. For then $\pi_l(L^\tau * g) = 0$, $\forall l$, and the Plancherel theorem says that $L^\tau * g = 0$.

Suppose first that $\mathfrak{g} = \mathfrak{n}_{2n}$, the Lie algebra of all upper triangular $(2n) \times (2n)$ matrices. ($\mathfrak{n}_{2n}$ can be given dilations in a variety of ways.) Let $\mathfrak{m}$ be the subspace of all matrices of the form

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where each entry stands for an $n \times n$ matrix. Then $\mathfrak{m}$ is maximal subordinate for all l in general position, as noted in Section 9 of [18]. Let C be a compact subset in $V \cap$ supp φ, and let α be a nonzero C^∞ function with support $\subseteq C$. Now Proposition 6 applies; we can find a function $g \in \mathcal{S}(G)$ such that

$$\pi_l(g) = \alpha(l) P_l, \quad P_l = \text{projection on span } \varphi(l).$$

Moreover, $\pi_l^\tau(L)\pi_l(g)=0$, $\forall l$, since $\pi_l^\tau(L)\varphi(l)=0$. Thus the theorem holds in this special case. A similar proof shows that the theorem holds if $\mathfrak{g}=\mathfrak{n}_{2n+1}$, the algebra of all $(2n+1)\times(2n+1)$ upper triangular matrices.

In general, we can imbed any nilpotent Lie algebra $\mathfrak{g}$ in some N_n. (The proof given in [1] is easily adapted to give an imbedding such that the dilations extend to dilations of N_n. But see the note after the proof.) Proposition 5 and induction show that there exists a nonzero smoothly varying function φ_n on the representations of N_n such that $\pi_l(L^\tau)\varphi_n(l)=0$ for all representations π_l of N_n in general position. Hence we can find a nonzero Schwartz function g_n on N_n such that $L^\tau g_n=0$. Define g on G by $g(x)=g_n(yx)$, where y is so chosen that $g\not\equiv 0$. Then $L^\tau g=0$, and the theorem follows.

Note. The homogeneity of L and the existence of dilations of $\mathfrak{g}$ were used only in applying Theorem 1, and not in constructing the function g. Thus the fact that $\mathfrak{g}$ can be imbedded in some $\mathfrak{n}_n$ so that the dilations extend is not necessary for this proof.

5. Unsolvable operators on certain stratified groups

The nilpotent Lie algebra $\mathfrak{g}$ (or the corresponding group, G) is called *stratified* if it can be written as a (vector space) direct sum,

$$\mathfrak{g}=\sum_{j=1}^{s}\mathfrak{g}_j,$$

such that

$$[\mathfrak{g}_i,\mathfrak{g}_j]\subseteq\mathfrak{g}_{i+j}\quad(\text{when } i+j>s,\ \mathfrak{g}_{i+j}=(0))$$

and $\delta_r|_{\mathfrak{g}_j}=r^j$. We assume that $\mathfrak{g}_s\neq(0)$. Let $\mathfrak{g}'=\mathfrak{g}/\mathfrak{g}_s$, and let G' be the corresponding group. We identify $(\mathfrak{g}')^*$ with $\{l\in\mathfrak{g}=l|_{\mathfrak{g}_s}=0\}$ and $\hat{G}'$ with the set of representations π_l, $l\in(\mathfrak{g}')^*$, up to equivalence. Thus $\hat{G}'$ is a subset of $\hat{G}$, of Plancherel measure 0.

As in [14], we define Sobolev spaces $\mathcal{H}^m(\pi)$, $m\in\mathbf{N}$, corresponding to each unitary representation π, by completing the space $\mathcal{H}^\infty(\pi)$ of C^∞ vectors with respect to the norm

$$\|v\|_{m,\pi}^2=\sum_P\|\pi(P)v\|^2,\tag{5.1}$$

where P runs over a basis for the elements of degree $\leqslant m$ in $U(\mathfrak{g})$. By [14], if A is any homogeneous, left invariant hypoelliptic differential operator on G of degree m, with $m\geqslant s^s$, m divisible by $s!$, then for every nontrivial irreducible representation π of G there is a constant $C=C_{P,\pi}$ such that

$$\|\pi(P)v\|\leqslant C\|\pi(A)v\|,\quad v\in\mathcal{H}^\infty(\pi).\tag{5.2}$$

Thus the norm given by $\|\pi(A)v\|$ is equivalent to the norm on $\mathcal{H}^m(\pi)$.

We say that the stratified Lie group G has a *locally uniform differential structure* on representations if for each element $l_0 \in V$, there is a neighborhood U of l_0 such that for each homogeneous element $P \in U(\mathfrak{g})$ of degree k, we can write

$$\pi_l(P) = \pi_{l_0}(P) + \sum_{j=0}^{k} \pi_{l_0}(P_j(l)), \quad l \in U, \tag{5.3}$$

where $P_j(l)$ is a homogeneous element of degree j in $U(\mathfrak{g})$ depending rationally on l (and $P_j(l_0) = 0$). For instance, any two-step nilpotent group $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ has a locally uniform differential structure on representations. Here is a sketch of a proof. If $X \in \mathfrak{g}_1$, then

$$\pi_l(X) = \sum_{j=1}^{k} c_j(l)\, x_j + \sum_{j=1}^{k} d_j(l)\, D_{x_j},$$

and Theorem 7.1 of [18] implies that we can write $\pi_l(X) = \pi_{l_0}(X) + \pi_{l_0}(X(l))$, where $X(l)$ is an element of $\mathfrak{g}_1$ depending rationally on l. Since π_l is scalar on $\mathfrak{g}_2$, a similar claim holds for $X \in \mathfrak{g}_2$.

Another example of a group with a locally uniform differential structure on representations is N_4, the group of 4×4 upper triangular matrices with 1's on the diagonal. The verification is straightforward.

Suppose that the stratified group G has a locally uniform differentiable structure. Then (5.3) shows that the space $\mathcal{H}^m(\pi_l)$ is independent of l locally, and hence on components of V. We shall therefore write simply $\mathcal{H}^m$ for $\mathcal{H}^m(\pi_l)$.

Let $V_{1,\mathbf{C}}$ be the complexification of V_1. For $l \in V_{1,\mathbf{C}}$ sufficiently close to V, we can use (5.3) to define $\pi_l(P)$ for all $P \in U(\mathfrak{g})$. Furthermore, the Sobolev norms for π_l, (as defined by (5.1)) are the same as those for π_{l_0}, l_0 real (and near l), again by (5.2). Let $V_{\mathbf{C}}$ be an open set in $V_{1,\mathbf{C}}$ containing V on which these statements are true and which satisfies $\overline{V_{\mathbf{C}}} = V_{\mathbf{C}}$ (where $\overline{}$ = complex conjugate). Notice that if P is self-adjoint, then

$$\pi_l(P)^* = \pi_{\bar{l}}(P); \tag{5.4}$$

this is essentially the Schwarz reflection principle.

We can now state and prove the main result of this section.

THEOREM 4. *Let G be a stratified nilpotent Lie group with a locally uniform differentiable structure on representations, and let L be a homogeneous left invariant differentiable operator on G. Suppose that*

$$\ker \pi_l(L^\tau) = (0), \quad \forall \pi_l \in \hat{G}' \quad \text{with } l \neq 0, \tag{5.5}$$

and that $\pi_l(L^\tau)$ has a nontrivial kernel for all l in an open subset of $\mathfrak{g}^*$. Then L is unsolvable.

Proof. Let $L_0=(LL^*)^{m_0}$, where $m_0 \geqslant s^s$ and m_0 is divisible by s! Then L_0 satisfies (5.5), and $\ker L_0 \cap \mathcal{S}(G) \neq (0)$.

Let A be a self-adjoint homogeneous left invariant differential operator on G which is hypoelliptic. (Examples are given in [14].) Furthermore we may assume that A and L_0 are of the same degree, m.

From [14] and the hypothesis of local uniformity, we know that for all $l_0 \in V$, there is a neighborhood U of l_0 in $V_{\mathbf{C}}$ such that if $\deg P \leqslant \deg A$, then there is a constant C_p satisfying

$$\|\pi_l(P)v\| \leqslant C_p \|\pi_{l_0}(A)v\|, \quad \forall v \in \mathcal{S}(\mathbf{R}^k) \quad \text{and } \forall l \in U. \tag{5.6}$$

Moreover, since L_0 satisfies the "Ro dégeneré" condition of [14], there is a constant C_p' for every $P \in U(\mathfrak{g})$ with $\deg P \leqslant \deg L_0$ such that

$$\|\pi_l(P)v\|^2 \leqslant C_p'(\|v\|^2 + \|\pi_{l_0}(L_0)v\|^2), \quad \forall v \in \mathcal{S}(\mathbf{R}^k). \tag{5.7}$$

From (5.5), we see that $\pi_l(A)$ and $\pi_l(A)^* = \pi_{\bar{l}}(A)$ are bounded below (let P = identity), and hence that $\pi_l(A)$ is invertible, for $l \in V_{\mathbf{C}}$.

LEMMA 3. (a) *The map $l \mapsto \pi_l(A)^{-j}$, $j \geqslant 1$, is a holomorphic function from $V_{\mathbf{C}}$ to $B(\mathcal{L}^2, \mathcal{H}^{mj})$, the set of bounded operators from $\mathcal{L}^2(\mathbf{R}^k)$ to $\mathcal{H}^{mj}$, in the sense of* [17, § VII.1].

(b) *The map $l \mapsto \pi_l(A)^{j-1}\pi_l(L_0)$ is a holomorphic function from $V_{\mathbf{C}}$ to the operators from $\mathcal{H}^{mj}$ to $\mathcal{L}^2(\mathbf{R}^k)$ for all j.*

Proof (for $j \geqslant 1$). Formula (5.5) shows that $\pi_l(A)^j$ and $\pi_l(A)^{j-1}\pi_l(L_0)$ are bounded from $\mathcal{H}^{mj}$ to $\mathcal{L}^2$, and (5.2) implies directly that $l \mapsto \langle \pi_l(A)^j v, w\rangle$ and $l \mapsto \langle \pi_l(A)^{j-1}\pi_l(L_0)v, w\rangle$ are both holomorphic functions for $v, w \in \mathcal{S}(\mathbf{R}^n)$. Hence $l \mapsto \pi_l(A)^j$ and $l \mapsto \pi_l(A)^{j-1}\pi_l(L_0)$ are holomorphic functions (to $B(\mathcal{H}^{mj}, \mathcal{L}^2(\mathbf{R}^k))$. That proves (b). As the inverse of a holomorphic family is holomorphic (see § VII.1 of [17]), (a) also holds. The case $j=0$ is similar.

COROLLARY. *The map $l \mapsto \pi_l(A)^{-1}\pi_l(L_0)$ is a holomorphic family of bounded operators on $\mathcal{H}^{mj}$ for all j.*

Proof. From the lemma, $l \mapsto \pi_l(A)^{j-1}\pi_l(L_0)\pi_l(A)^{-j}$ is holomorphic from $\mathcal{H}$ to $\mathcal{H}$ (for all j), and $\pi_l(A)^j$ is an isomorphism of $\mathcal{H}^{mj}$ with $\mathcal{H}$.

LEMMA 4. *For $l \in V$, suppose that $\pi_l(L_0)$ has a nontrivial kernel. Then the kernel consists entirely of elements of $\mathcal{H}^\infty(\pi_l)$, and 0 is an isolated point in the spectrum of $\pi_l(A)^{-1}\pi_l(L_0)$.*

Proof. Since $mj \geqslant s^s$ the injection of $\mathcal{H}^{mj}$ into $\mathcal{L}_2$ is compact (see [14]). Now Peetre's lemma (see [13] and [14]) implies that $\pi_l(L_0)$: $\mathcal{H}^m \to \mathcal{H}$ has a right inverse when restricted to

$\ker \pi_l(L_0)^\perp$. Hence 0 is an isolated point in the spectrum (since $\pi_l(L_0)$ is self-adjoint). The claim about $\mathcal{H}^\infty$ is proved in [13].

We proceed with the proof of Theorem 4. Choose l_0 such that $\pi_l(L_0)$ has a nontrivial kernel when l is in a neighborhood of l_0 and such that $\dim \ker \pi_{l_0}(L_0) \leqslant \dim \ker \pi_l(L_0)$ for l in that neighborhood (and l real). Let $B_l = \pi_l(A)^{-1}\pi_l(L_0)$. From Lemma 4, we can choose $\varepsilon > 0$ such that for $l \in V$ near l_0, $B_l - \lambda I$ on $\mathcal{H}$ is invertible if $|\lambda| = \varepsilon$, and such that $B_{l_0} - \lambda I$ is invertible if $0 < |\lambda| < \varepsilon$. As noted in § 7.1.1 of [17], for each λ_0 satisfying $|\lambda_0| = \varepsilon$ there is an open neighborhood U_{λ_0} of l_0 in $V_{\mathbf{C}}$ such that $B_l - \lambda_0 I$ is invertible on U_{λ_0}. By shrinking U_{λ_0} slightly if necessary, we may bound $\{\|(B_l - \lambda I)^{-1}\| : l \in U_{\lambda_0}\}$. Now the standard Neumann series argument shows that there is a neighborhood O_{λ_0} of λ_0 such that if $\lambda \in O_{\lambda_0}$, then $B_l - \lambda I$ is invertible on U_{λ_0}. Compactness now gives a neighborhood U of l_0 such that $B_l - \lambda I$ is invertible for all $|\lambda| = \varepsilon$ and all $l \in U$.

Define

$$P(l) = \frac{1}{2\pi i} \int_{|\lambda| = \varepsilon} (B_l - \lambda I)^{-1} d\lambda.$$

Theorem VII, 1.7 of [17] states that $P(l)$ is a (not necessarily orthogonal) projection onto the part of $\mathcal{H}$ associated with the piece of the spectrum in the circle $|\lambda| < \varepsilon$, and that $P(l)$ varies holomorphically with l. When l is real, $\pi_l(L_0)$ is self-adjoint, and therefore $P(l_0)$ is the projection onto $\ker \pi_{l_0}(L_0)$. For $l \in U \cap V$ (and l sufficiently close to l_0), $P(l)$ is a projection onto a space containing $\ker \pi_l(L_0)$, and $\dim \operatorname{range} P(l) = \dim \operatorname{range} P(l_0)$. Hence $P(l)$ must be a projection onto $\ker \pi_l(L_0)$. From the Corollary to Lemma 3, $P(l)$ varies holomorphically on all the spaces $\mathcal{H}^{mj}$. Let $v \in \ker \pi_{l_0}(L_0)$, $v \neq 0$, and let $v(l) = P(l)v$. Then $v(l)$ varies holomorphically with l in all the $\mathcal{H}^{mj}$, and $v(l) \in \mathcal{S}(\mathbf{R}^k)$ for each l.

To complete the proof, it suffices to show that $v(l)(x)$ is a C^∞ function in l and x together, since Theorem 3 then applies. For simplicity of notation, we assume that $k = 1$. Let h_i be the ith Hermite function. Recall that

$$\|h_j\|_\infty = O(j), \quad h_j' = O(\sqrt{j})\,h_{j-1} + O(\sqrt{j})\,h_{j+1},$$

and that a function $\sum_{j=0}^\infty a_j h_j \in \mathcal{L}^2(\mathbf{R})$ is in $\mathcal{S}(\mathbf{R}) \Leftrightarrow \sum_{j=0}^\infty (j^2+1)^n a_j$ converges for all j. Now set

$$v(l)(x) = \sum_{s=0}^{\infty} a_s(l)\, h_s(x), \tag{5.8}$$

and define $\|v(l)\|_{(t)}^2 = \sum_0^\infty (s^2+1)^t |a_s(l)|^2$. For each t, there is a j such that the $\|\ \|_{(t)}$ norm is weaker than the norm on $\mathcal{H}^{mj}$. Since the map $l \mapsto \langle v(l), w\rangle_{mj}$ is holomorphic for all w

(the mj indicates that the inner product is in $\mathcal{H}^{mj}$) the vectors $D_l^\alpha v(l)$ are bounded in some neighborhood of l for every multi-index α and every j. That is,

$$\sum_{s=0}^{\infty} (s^2+1)^t |D_l^\alpha(a_s(l))|^2$$

is uniformly bounded near l_0 for every t and every multi-index α. This shows that (5.7) can be differentiated termwise, first with respect to l arbitrarily often and then with respect to x arbitrarily often, and that the resulting series converges uniformly. Hence all the partial derivatives of $v(l)(x)$ exist and are continuous, and Theorem 4 is proved.

The following answers a question raised in [25].

THEOREM 5. *Suppose* $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2$ *with* $[\mathfrak{g}_1, \mathfrak{g}_2]=\mathfrak{g}_2$ *and* $X_1, X_2, ..., X_p$ *is a basis of* $\mathfrak{g}_1$. *Let* $(a_{i_1 i_2 \ldots i_d})$ *be a positive definite d-form on* $\mathbf{R}^p$ *and put*

$$L=\sum a_{i_1 i_2 \ldots i_d} X_{i_1} X_{i_2} \ldots X_{i_d}. \tag{5.9}$$

Then if $\pi_l(L^\tau)$ *has a nontrivial kernel for all* $l \in U$, *an open subset of* $\mathfrak{g}^*$, *then* L^τ *has a nontrival kernel on* $\mathcal{S}(G)$ *and* L *is unsolvable.*

Proof. The positive definiteness of the form $(a_{i_1 i_2 \ldots i_d})$ is exactly the Ro dégeneré condition (5.4) of Theorem 4. Hence Theorem 5 is an immediate consequence of Theorem 4.

Remark. The converse of Theorem 5 has been proved recently by D. Tartakoff and the second author for a special class of 2-step groups, including the Heisenberg groups.

6. Some examples of unsolvable differential operators on the Heisenberg group

Let H^n be the Heisenberg group of dimension $2n+1$. As is well known, see e.g. [21, Chapitre II, § 1], there is a 1–1 correspondence $\lambda \mapsto \pi_\lambda$ between $\mathbf{R}-\{0\}$ and the set of infinite dimensional irreducible unitary representations of H^n. Furthermore, the measure $|\lambda|^n d\lambda$ on $\mathbf{R}-\{0\}$ is the Plancherel measure on H^n. For each $\lambda \in \mathbf{R}-\{0\}$, π_λ is a unitary representation on the Hilbert space $L^2(\mathbf{R}^n)$, and if D is a homogeneous differential operator, then π_λ can be chosen so that

$$\pi_\lambda(D)=\begin{cases} |\lambda|^{d/2}\pi_1(D) & \text{if } \lambda>0 \\ |\lambda|^{d/2}\pi_{-1}(D) & \text{if } \lambda<0, \end{cases}$$

where d is the degree of homogeneity. Now the following may be derived immediately from Theorem 3. (It may also be obtained directly from Theorem 1.)

PROPOSITION 8. *Let L be a left invariant homogeneous differential operator on the Heisenberg group H^n. Suppose that $\pi_1(L^\tau)$ or $\pi_{-1}(L^\tau)$ has a nontrivial kernel in the space of C^∞ vectors of the representation space i.e. $\mathcal{S}(\mathbf{R}^n)$. Then L is not locally solvable.*

COROLLARY. *The Lewy operator*

$$L=\frac{\partial}{\partial x}-i\left(\frac{\partial}{\partial y}+x\frac{\partial}{\partial t}\right)$$

is unsolvable.

Proof. $\pi_1(L^\tau)=-(d/du+u)$. Let $\varphi(u)=e^{-u^2/2}$. Then $\pi_1(L^\tau)\varphi=0$, and clearly $\varphi\in\mathcal{S}(\mathbf{R})$.

We now use Theorem 3 to construct a family of unsolvable operators on the three dimensional Heisenberg group H with Lie algebra $\mathfrak{h}$ spanned by X, Y, T, and nonzero bracket $[X, Y]=T$. Then $\mathfrak{h}$ has dilations given by $\delta_r(X)=rX$, $\delta_r(Y)=rY$, $\delta_r(T)=r^2T$. We may realize π_λ on $L^2(\mathbf{R})$ so that

$$\pi_\lambda(X)=|\lambda|^{\frac{1}{2}}\frac{d}{dx},\quad \pi_\lambda(Y)=i\operatorname{sgn}\lambda|\lambda|^{\frac{1}{2}}x\cdot I,\quad \pi_\lambda(T)=i\lambda\cdot I, \tag{6.1}$$

where sgn denotes sign and I is the identity. The following result is then immediate.

LEMMA 3. *Let D be the ordinary differential operator*

$$D=\sum_{\substack{|\alpha|+|\beta|=d-2k\\ 0\leqslant k\leqslant[d/2]}} c_{\alpha,\beta}\left(\frac{d}{dx}\right)^{\alpha_1}x^{\beta_1}\left(\frac{d}{dx}\right)^{\alpha_2}x^{\beta_2}\cdots\left(\frac{d}{dx}\right)^{\alpha_j}x^{\beta_j}. \tag{6.2}$$

Then $D=\pi_1(L)$, where

$$L=\sum_{\substack{|\alpha|+|\beta|=d-2k\\ 0\leqslant k\leqslant[d/2]}} c_{\alpha,\beta}X^{\alpha_1}(-iY)^{\beta_1}X^{\alpha_2}(-iY)^{\beta_2}\cdots X^{\alpha_j}(-iY^{\beta_j})(-iT)^{[d-(|\alpha|+|\beta|)]/2}. \tag{6.3}$$

Furthermore, L is homogeneous of degree d.

Now suppose that $p(x)$, $q_1(x)$, $q_2(x)$ are monic complex valued polynomials where p consists only of odd degree terms while q_1 and q_2 consist only of even degree terms. Consider the ordinary differential operators

$$D_1=\frac{d}{dx}-q_1'(x)+q_1(x)\,p(x)\,p'(x), \tag{6.4}$$

and

$$D_2=\frac{d^2}{dx^2}-q_2''+q_1q_1'q_2'+(q_1q_1'q_2)'-q_1^2(q_1')^2q_2, \tag{6.5}$$

where $'$ denotes derivative. Now put $\varphi_1(x) = q_1(x)e^{-(p(x))^2/2}$ and $\varphi_2(x) = q_2(x)e^{-(q_1(x))^2/2}$. Then $\varphi_i \in \mathcal{S}(\mathbf{R})$ and $D_i\varphi_i = 0$, $i = 1, 2$. From Proposition 8 we therefore have

THEOREM 6. *Let p, q_1, q_2 be polynomials as above and D_1, D_2 the ordinary differential operators defined by* (6.4) *and* (6.5). *Then D_1 and D_2 are of the form* (6.2). *Then the left invariant differential operators P_i, $i = 1, 2$ defined by $P_i = -L_i^\tau$, with L_i as in* (6.3), *are unsolvable. In particular, if $p(x) = x$ and $q_1(x) = 1$, then $P_1 = X - iY$ is the Lewy operator.*

7. Some counterexamples

We here exhibit two examples. The first is of a homogeneous left invariant operator L on the Heisenberg group for which there exists a smooth, $\mathcal{L}^2$ function f such that $L^\tau f = 0$, but which is locally solvable. This shows that the integer k of Theorem 1 cannot always be taken to be zero.

PROPOSITION 9. *Let X, Y, T be a basis for the three dimensional Heisenberg Lie algebra as defined in* Section 6, *and let $L = (Y^2 - iT)X$. Then L is locally solvable, but L^τ has a nontrivial kernel in $L^2(G)$, where G is the corresponding simply connected group.*

Proof. First, we claim that $Y^2 - iT$ and X are each locally solvable operators on G. For this note that since $[Y, T] = 0$, there is a system of coordinates (x_1, x_2, x_3) in which $Y = \partial/\partial x_1$ and $T = \partial/\partial x_2$. Hence $Y^2 - iT$ may be written as a constant coefficient differential operator, which is therefore locally solvable [15, Theorem 3.1.1]. Similarly X is a locally solvable operator. Thus L, being the composition of locally solvable operators, is again locally solvable.

To find $f \in \ker L^\tau$, we realize the representations of H^1 as in (6.1). Then for $\lambda > 0$,

$$\pi_\lambda(L^\tau) = \pi_\lambda(-X(Y^2 + iT)) = \lambda^{3/2}\frac{d}{dx}(x^2 + 1)\cdot I. \tag{7.1}$$

Therefore $a(x) = (x^2+1)^{-1} \in \mathcal{L}^2(\mathbf{R}) \cap \ker \pi_\lambda(L^\tau)$ for all $\lambda > 0$. Let $P\colon \mathcal{L}^2(\mathbf{R}) \to \mathcal{L}^2(\mathbf{R})$ be the orthogonal projection onto the subspace spanned by $a(x)$, and let α be a nonzero continuous function with compact support in $(0, \infty)$. Clearly P is a Hilbert–Schmidt operator; thus the Plancherel Theorem (see, e.g., Part II, Chapter III of [21]) implies that there is a unique function $f \in \mathcal{L}^2(G)$ determined by the condition

$$\mathfrak{F}(f)(\lambda) = \begin{cases} \alpha(\lambda)P, & \lambda > 0 \\ 0, & \lambda < 0, \end{cases}$$

where $\mathcal{F}$ is the extension to $L^2(G)$ of the operator defined by

$$\mathcal{F}(\varphi)(\lambda) = \int_G \varphi(g)\,\pi_\lambda(g)\,dg, \quad \varphi \in C_c^\infty(G).$$

We now claim that $L^\tau f = 0$ in the sense of distributions. For this, it suffices to prove that for any $\varphi \in C_0^\infty(G)$, for

$$\int f(g)\,L\varphi(g)\,dg = 0. \tag{7.2}$$

By the Plancherel formula, the left hand side of (7.2) is $\int \operatorname{tr}\,(\mathcal{F}(L\varphi)(\lambda)\,\mathcal{F}(f)(\lambda))\,|\lambda|\,d\lambda$. Since $\mathcal{F}(L\varphi)\,\mathcal{F}(f) = \mathcal{F}(\varphi)\pi_\lambda(L^\tau)\,\mathcal{F}_\lambda(f) = 0$, (7.2) is proved.

COROLLARY. *There is a smooth function $f' \in L^2(G)$ such that $L^\tau f' = 0$.*

Proof. Let $\varphi \in C_0^\infty(G)$, and put $f' = \varphi * f$. Then f' is smooth. Indeed, if D is any right-invariant differential operator, $D(\varphi * f) = D\varphi * f \in L^2$. Hence by Sobolev's lemma, $\varphi * f$ is smooth. Finally,

$$L^\tau(\varphi * f) = \varphi * L^\tau f = 0.$$

Our second example is of a nonsolvable operator L on a nilpotent group G with the property that $\pi_l(L^\tau)$ has trivial kernel for all $\pi_l \in \Gamma$. The group G has a Lie algebra $\mathfrak{g}$ spanned by X, Y, T, Z, with $[X, Y] = T$, $[X, T] = Z$, $[Y, T] = 0$, and $[\mathfrak{g}, Z] = (0)$. The dilations on $\mathfrak{g}$ are given by $\delta_r(X) = rX$, $\delta_r(Y) = rY$, $\delta_r(T) = r^2T$, and $\delta_r(Z) = r^3T$. G can be realized as a matrix group:

$$G = \left\{ \begin{pmatrix} 1 & x & \tfrac{1}{2}x^2 & z \\ 0 & 1 & x & t \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, y, t, z \in \mathbf{R} \right\}.$$

The representations in Γ act on $\mathcal{L}^2(\mathbf{R})$, and are given by

$$\pi_{a,c}(X) = a^{\frac{1}{3}}\frac{d}{dx}, \quad \pi_{a,c}(Y) = ia^{\frac{1}{3}}\left(c + \frac{x^2}{2}\right), \quad \pi_{a,c}(T) = ia^{\frac{2}{3}}x, \quad \pi_{a,c}(Z) = iaI \quad (c \in \mathbf{R}, a \in \mathbf{R} - \{0\}).$$

(The other irreducible unitary representations of G annihilate Z and reduce to representations of H^1.)

PROPOSITION 10. *$L = X + iY$ is not locally solvable on G, but $\pi_{a,c}(L^\tau)$ has trivial kernel for all $\pi_{a,c} \in \Gamma$.*

Proof. L is not locally solvable by Theorem 6.1.1 of [15]. On the other hand, $\pi_{a,c}(L^\tau)f = a^{\frac{1}{3}}(df/dx - cf - \frac{1}{2}x^2 f)$. Thus

$$\pi_{a,c}(L^\tau)f = 0 \quad \Rightarrow \quad f(x) = Ae^{-(cx+x^3/6)}, \quad A \in \mathbf{C}; \quad \text{if } A \neq 0,\ f \notin L^2(\mathbf{R}).$$

This proves the proposition.

Note. The operators $\pi_{a,c}(L^\tau)$ actually have bounded right inverses in $\mathcal{L}^2(\mathbf{R})$, given by

$$A_{a,c}f(x) = -a^{-\frac{1}{3}}e^{x^3/6+cx}\int_x^\infty f(t)\,e^{-t^3/6-ct}dt.$$

(See Section 4 of [2] for a proof.) In fact, the $A_{a,c}$ actually map $\mathcal{S}$ continuously into $\mathcal{S}$. To prove this, note that it suffices to prove that $f \in \mathcal{S} \Rightarrow A_{a,c}f \in \mathcal{S}$, by the closed graph theorem. Moreover, if f is differentiable, then

$$(A_{a,c}f)'(x) = \left(\frac{x^2}{2} + c\right)(A_{a,c}f)(x) + f(x);$$

by an easy induction it now suffices to prove that $f \in \mathcal{S} \Rightarrow (A_{a,c}f)$ is rapidly decreasing. It suffices (by homogeneity) to assume that $a = 1$. One checks easily that

$$(A_{1,c}f)(x) = \int_0^\infty e^{-cu}\,e^{-(x^2u/2)-(xu^2/2)-u^3/6}f(x+u)\,du.$$

Now it is easy to see that $A_{1,c}f$ decreases rapidly at $+\infty$ if f is rapidly decreasing. As for the behavior at $-\infty$, set $x = -|c| - 1 - y$, $y > 0$. Then

$$|(A_{1,c}f)(x)| \leq \int_0^\infty |g(y-u)| \exp\left[-\frac{u}{6}\left(u - \frac{3y}{2}\right)^2\right] \exp\left(-\frac{uy^2}{8}\right)du,$$

where $g(y) = f(-|c| - 1 - y)$. Split this integral into one from 0 to $u^{-3/2}$ and one from $u^{-3/2}$ to ∞ to see that $A_{1,c}f$ decreases rapidly at $-\infty$.

8. Hypoellipticity and local solvability

It is well known that if D is any differential operator which is hypoelliptic, then D^τ is locally solvable. Furthermore, there is an example, due to Kannai [16] of a hypoelliptic differential operator which is not locally solvable. However, in the context of homogeneous, left invariant differential operators, there are no known examples of unsolvable hypoelliptic operators.

PROPOSITION 11. *Let D be a hypoelliptic, left invariant differential operator on a Lie group. Then D is locally solvable if and only if D^τ is again hypoelliptic.*

Proof. By the above comments it suffices to show that if D is hypoelliptic and locally solvable, then D^τ is hypoelliptic. By [25, Theorem 15.4], D locally solvable implies it has a local fundamental solution σ in a neighborhood U of 0. However, if D is also hypoelliptic, then $D\sigma(x)=0$ for $x\in U-\{0\}$ implies σ is smooth in $U-\{0\}$. Now the operator k: $f\mapsto f*\sigma$, $f\in C_0^\infty(G)$ is a local right inverse for D, and hence its transpose k^τ given by

$$k^\tau f(y)=f*\check{\sigma},$$

where $\check{\sigma}$ is the distribution defined by $\check{\sigma}(h)=\sigma(\check{h})$ with $\check{h}(x)=h(x^{-1})$. Since $\check{\sigma}$ is again smooth in $U-\{0\}$, k^τ is pseudo local in U, i.e. f smooth in an open set $V\subset U$ implies $k^\tau f$ is again smooth in V. This proves that D^τ is hypoelliptic.

9. Open questions

We collect here some unanswered questions suggested by either our results or our methods.

(1) Proposition 6 in Section 4 asserts that if $l\mapsto P_l$ is a smoothly varying family on G and if α is a C^∞ function in l with compact support, then there is a function $f\in \mathcal{S}(G)$ with $\pi_l(f)=\alpha(l)P_l$, provided that G meets a further stringent condition (that there be an ideal $\mathfrak{m}$ of $\mathfrak{g}$ which is polarizing for all functionals l in general position). Can one prove this proposition for a more general class of nilpotent Lie groups? More generally, can one find general necessary and sufficient conditions on a set of operators $\{A_l\}$ so that there exists a function $f\in\mathcal{S}(G)$ with $\pi_l(f)=A_l$, $\forall\pi_l\in\Gamma$, the generic representations in $\hat{G}$?

(2) Let L be a homogeneous left-invariant operator on G; suppose that $\pi_l(L^\tau)$, regarded as an operator from $\mathcal{S}(G)$ to $\mathcal{S}(G)$, has a continuous left inverse A_l for all l in general position. There exists an integer s_l such that A_l is continuous from the $\|\ \|_{s_l}$ Sobolev seminorm on $\mathbf{R}^k$ to the $\|\ \|_0$ seminorm. If the A_l vary continuously with l, *and* if the s_l are uniformly bounded, *and* if one has an appropriate bound on the norms of the operators A_l: $\|\ \|_{s_l}\to\|\ \|_0$, then a procedure like that of [24] or [2] (compare also [19]) should prove that L is locally solvable. Can one weaken these hypotheses? In particular, can one settle this matter for two-step nilpotent Lie groups? Note that the existence of the A_l, even for $s_l=0$, is insufficient in general (see Section 7).

(3) Conversely, what are the representation-theoretic implications of local solvability of L? In particular, must there be an open set $\Gamma'\subset\hat{G}$ of full Plancherel measure such that

$\pi(L^\tau)$ is invertible in some sense for all $\pi \in \Gamma'$? Also, can the necessary conditions be strengthened in order to give a representation-theoretic proof of the unsolvability of the example in Section 7? Finally, is it possible to find a global fundamental solution σ to an equation of the form $L\sigma = Z\delta$, for some homogeneous, locally solvable operator Z?

References

[1] Birkhoff, G., Representability of Lie algebras and Lie groups by matrices. *Ann. of Math.*, 38 (1937), 526–532.

[2] Corwin, L., A representation-theoretic criterion for local solvability of left-invariant differential operators on nilpotent Lie groups. *Trans. Amer. Math. Soc.*, 264 (1981), 113–120.

[3] Corwin, L. & Greenleaf, F. P., Character formulas and spectra of compact nilmanifolds. *J. Functional Analysis*, 21 (1976), 123–154.

[4] —— Fourier transforms of smooth functions on certain nilpotent Lie groups. *J. Functional Analysis*, 37 (1980), 203–217.

[5] —— Rationally varying polarizing subspaces in nilpotent Lie algebras. *Proc. Amer. Math. Soc.*, 81 (1981), 27–32.

[6] Corwin, L., Greenleaf, F. P. & Penney, R., A general character formula for the distribution kernels of primary projections in L^2 of a nilmanifold. *Math. Ann.*, 225 (1977), 21–37.

[7] Duflo, M. & Wigner, D., Convexité pour des operateurs differentiels invariants sur les groupes de Lie. *Math. Z.*, 167 (1979), 61–80.

[8] Folland, G. & Stein, E. M., Estimates for the ∂_b complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27 (1974), 429–522.

[9] Geller, D., Fourier analysis on the Heisenberg group II: Local solvability and homogeneous distributions. *Comm. Partial Diff. Eq.*, 5 (1980), 475–560.

[10] Greiner, P., Kohn, J. J. & Stein, E. M., Necessary and sufficient conditions for solvability of the Lewy equation. *Proc. Nat. Acad. Sci. U.S.A.*, 72 (1975), 3287–3289.

[11] Grušin, V. V., On a class of hypoelliptic operators. *Mat. Sb.*, 83 (125) (1970), 456–473 (= *Math. USSR-Sb.*, 12 (1970), 458–476).

[12] —— On a class of elliptic pseudodifferential operators degenerate on a submanifold. *Mat. Sb.*, 84 (126): 2 (1971), 163–195. (= *Math. USSR-Sb.*, 13: 2 (1971), 155–185).

[13] Helffer, B. & Nourrigat, J., Théorèmes d'indice pour les operateurs à coefficients polynomiaux. Preprint.

[14] —— Characterisation des operateurs hypoelliptiques homogènes invariants a gauche sur un groupe de Lie nilpotent gradué. *Comm. Partial Differential Equations*, 4 (8) (1979), 899–958.

[15] Hörmander, L., *Linear Partial Differential Operators*. Springer-Verlag, Berlin, 1963.

[16] Kannai, Y., An unsolvable hypoelliptic differential operator. *Israel J. Math.*, 9 (3) (1971), 306–315.

[17] Kato, T., *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin, 1966.

[18] Kirillov, A. A., Unitary representations of nilpotent Lie groups. *Uspehi Mat. Nauk.*, 17 (1962), 57–110.

[19] Knapp, A. & Stein, E. M., Intertwining operators for semisimple Lie groups. *Ann. of Math.*, 93 (1971), 489–578.

[20] Lions, G., Hypoellipticité et resolubilité d'operateurs differentiels sur les groupes nilpotents de rang 2. *C.R. Acad. Sci. Paris*, 290 (1980), 271–274.

[21] PUKANSZKY, L., *Leçons sur les representations des groupes*. Dunod, Paris, 1967.
[22] —— On the characters and the Plancherel formula of nilpotent groups. *J. Functional Analysis*, 1 (1967), 255–280.
[23] ROCKLAND, C., Hypoellipticity on the Heisenberg group–representation-theoretic criteria. *Trans. Amer. Math. Soc.*, 240 (1978), 1–52.
[24] ROTHSCHILD, L. P., Local solvability of left-invariant differential operators on the Heisenberg group. *Proc. Amer. Math. Soc.*, 74 (1979), 383–388.
[25] —— Local solvability of second order differential operators on nilpotent Lie groups. *Ark. Math.* To appear.
[26] ROTHSCHILD, L. P. & STEIN, E. M., Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137 (1976), 247–320.
[27] VERGNE, M., Construction de sous-algèbres subordonées à un element du dual d'une algèbre de Lie résoluble. *C. R. Acad. Sci. Paris, Sér. A*, 270 (1970), 173–175, 704–707.

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