SUBSPACES AND QUOTIENTS OF $l_p \oplus l_2$ AND X_p(1)

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0. Introduction

Much progress has been made in recent years in describing the structure of $L_p = L_p[0, 1]$, and, in particular, the $\mathcal{L}_p$ spaces (complemented subspaces of L_p which are not Hilbert space) have been studied extensively. The obvious or natural $\mathcal{L}_p$ spaces are l_p, $l_p \oplus l_2$, $(l_2 \oplus l_2 \oplus \dots)_p$ and L_p itself. These were the only known examples until H. P. Rosenthal [18] discovered the space X_p (see below). This space perhaps seemed pathological when first introduced; however, it now appears that X_p plays a fundamental role in the study of L_p and $\mathcal{L}_p$ spaces.

The discovery of X_p permitted the list of separable $\mathcal{L}_p$ spaces to be increased to 9 in number [18]. Then G. Schechtman [20], again using X_p, showed that there are an infinite number of mutually non-isomorphic separable $\mathcal{L}_p$ spaces, and recently Bourgain, Rosenthal and Schechtman [2] succeeded in constructing uncountably many such spaces. It now appears improbable that a complete classification of the separable $\mathcal{L}_p$ spaces will be obtained. However, it might be possible to classify the "smaller" $\mathcal{L}_p$ spaces. For example it was proved in [11] that the only $\mathcal{L}_p$ subspace of l_p $(1 < p < \infty)$ is l_p. Also all complemented subspaces of $l_p \oplus l_2$ and $(l_2 \oplus l_2 \oplus \dots)_p$ are known (see [4], [21] and [17]). (X_p is, for $p > 2$, a $\mathcal{L}_p$ space which embeds into $l_p \oplus l_2$ and thus into $(l_2 \oplus l_2 \oplus \dots)_p$, but does not embed into these spaces as a complemented subspace.)

One question with which we are concerned in this paper is "What are the $\mathcal{L}_p$ subspaces X of $l_p \oplus l_2$ $(1 < p \neq 2 < \infty)$?" We answer this in Section 2 for those X with an unconditional basis (although every separable $\mathcal{L}_p$ space is known to have a basis [10], it is a major un-

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solved problem as to whether each one has an unconditional basis). More precisely, we prove in Theorem 2.1 that if $1<p<2$ then X is isomorphic to either l_p or $l_p \oplus l_2$. In proving this result we obtain a representation of unconditional basic sequences in $l_p \oplus l_2$ which might prove useful elsewhere (Lemma 2.3).

In Theorem 2.12 we show if $2<p<\infty$ and X is a $\mathcal{L}_p$ subspace of $l_p \oplus l_2$ with an unconditional basis, then X is isomorphic to l_p, $l_p \oplus l_2$ or X_p. The fact that X_p enters into the $p>2$ case necessitates our proving several preliminary results which are of interest in their own right. In Proposition 2.5 we show if X is a subspace of $l_p \oplus l_2$ $(2<p<\infty)$ and $T: L_p \to X$ is a bounded linear operator, then T factors through X_p. A consequence of this, Corollary 2.6, is that the class of $\mathcal{L}_p$ subspaces of $l_p \oplus l_2$ $(2<p<\infty)$ is the same as the class of complemented subspaces of X_p. In Theorem 2.9 we prove that if X is isomorphic to a complemented subspace of X_p and X_p is isomorphic to a complemented subspace of X, then X is isomorphic to X_p. Theorem 2.10 shows that X_p is primary. This means if X_p is isomorphic to $Y \oplus Z$ then either Y or Z is isomorphic to X_p.

Finally, in Section 3 we are concerned with a specific case of the following general question: if Y is a given $\mathcal{L}_p$ space, give necessary and sufficient conditions to insure that if X is a subspace of L_p which satisfies these conditions, then X is isomorphic to a subspace of Y (i.e. X embeds into Y). For example it was shown in [9] (respectively, [5]) that a subspace X of L_p, $2<p<\infty$ (respectively, $1<p<2$) embeds into l_p if and only if X does not contain an isomorph of l_2 (respectively, there exists $\lambda<\infty$ so that every normalized basic sequence in X has a subsequence which is λ-equivalent to the unit vector basis for l_p).

In Theorem 3.1 we give a sufficient condition (which is trivially necessary) for the space $l_p \oplus l_2$ $(2<p<\infty)$. Namely, if X is a subspace of L_p which is isomorphic to a quotient of a subspace of $l_p \oplus l_2$, then X embeds into $l_p \oplus l_2$. Theorem 3.1 of course implies that if X is a $\mathcal{L}_q$ subspace of $l_q \oplus l_2$ $(1<q<2;\ 1/p+1/q=1)$ then X^* is a $\mathcal{L}_p$ subspace of $l_p \oplus l_2$, so that Theorem 2.1 can be derived from Theorem 2.12. However, Theorem 2.1 is simpler to prove than Theorem 2.12 and the proof of Theorem 3.1 is terribly complicated, so we prefer to give a direct proof for Theorem 2.1. Moreover, this presentation allows Sections 2 and 3 to be read independently of each other.

1. Preliminary material

In this section we present some background material and also set certain notation. Our terminology is standard Banach space terminology—any terms not defined below may be found in the books of Lindenstrauss and Tzafriri ([14] and [15]).

A *subspace* of a Banach space shall be understood to be closed and infinite dimensional unless otherwise noted. If S is a subset of a Banach space, then $[S]$ is the closed linear span of S. We write $X \sim Y$ if X and Y are isomorphic. All *operators* are bounded and linear. If (X_n) is a sequence of Banach spaces, $(\sum X_n)_p$ is the space $\{(x_n): x_n \in X_n$ for all n and $\|(x_n)\| = (\sum \|x_n\|^p)^{1/p} < \infty\}$. B_X is the closed unit ball of the Banach space X. If basic sequences (x_i) and (y_i) are equivalent we write $(x_i) \sim (y_i)$.

We denote the norm in L_p by $\|\cdot\|_p$.

The Haar system is an unconditional basis for L_p $(1<p<\infty)$ and we let its unconditional basis constant be λ_p. If (x_i) is an unconditional basic sequence with unconditional constant K in L_p $(1<p<\infty)$ then $\|\sum a_i x_i\|_p$ may be calculated by means of the "square function". Thus

$$\|\sum a_i x_i\|_p \overset{KK_p}{\sim} \left(\int (\sum a_i^2 |x_i(s)|^2)^{p/2} ds\right)^{1/p} \tag{1.1}$$

where K_p is a constant arising from the Khinchine inequality, (a_i) are scalars and "$\overset{M}{\sim}$" means that each side is no greater than M times the other side. Thus $A \overset{M}{\sim} B$ means $A \leqslant MB$ and $B \leqslant MA$. Note by (1.1) if (y_i) is an unconditional basic sequence in L_p and $|y_i(s)| = |x_i(s)|$ for all $s \in [0, 1]$, then (y_i) is equivalent to (x_i). This observation was used in a clever way by Schechtman [19] and we employ it in the sequel. We shall also require the following well known inequalities.

Let (x_i) be a normalized unconditional basic sequence in L_p with unconditional constant K. Then

$$(KK_p)^{-1} (\sum |a_i|^p)^{1/p} \leqslant \|\sum a_i x_i\|_p \leqslant KK_p (\sum a_i^2)^{1/2} \quad \text{if} \quad 2<p<\infty \quad \text{and} \quad (a_i) \quad \text{are scalars} \tag{1.2}$$

and

$$(KK_p)^{-1} (\sum a_i^2)^{1/2} \leqslant \|\sum a_i x_i\|_p \leqslant KK_p (\sum |a_i|^p)^{1/p} \quad \text{if} \quad 1<p<2. \tag{1.3}$$

We use the basic results of Kadec and Pelczynski [13] which we now recall. Let

$$M_p(\varepsilon) = \{f \in L_p(m): m\{t: |f(t)| \geqslant \varepsilon \|f\|_p\} \geqslant \varepsilon\}$$

where m is a finite measure. If (x_n) is a normalized unconditional basic sequence in L_p $(2<p<\infty)$ with $x_i \in M_p(\varepsilon)$ for all i and some $\varepsilon>0$, then (x_i) is equivalent to the unit vector basis of l_2. If $(x_i) \nsubseteq M_p(\varepsilon)$ for any $\varepsilon>0$ then for every $\delta>0$, some subsequence of (x_i) is $(1+\delta)$-equivalent to the unit vector basis of l_p. Of course $(x_i) \subseteq M_p(\varepsilon)$ implies $\|x_i\|_2 \geqslant \varepsilon^{3/2}$ for all i and $(x_i) \nsubseteq M_p(\varepsilon)$ for any $\varepsilon>0$ means $\inf_i \|x_i\|_2 = 0$.

Much of our interest centers around $l_p \oplus l_2$ and X_p. We shall write $|x|_p$ for the l_p-part of the norm of a vector $x \in l_p \oplus l_2$ and similarly $|x|_2$ for the l_2-part.

Let $w = (w_i)$ (a *weight sequence*) be a sequence of positive scalars. $X_{p,w}$ is defined to be the completion of the space of all sequences of scalars (a_n) with only finitely many $a_n \neq 0$ under the norm

$$\|(a_n)\|_{p,w} = \max\left(\left(\sum |a_n|^p\right)^{1/p}, \left(\sum |a_n w_n|^2\right)^{1/2}\right).$$

Rosenthal [18] showed that for all weight sequences w, $X_{p,w}$ is complemented in L_p and if $w_i \downarrow 0$ with $\sum w_i^{2p/(p-2)} = \infty$ then $X_{p,w}$ is not isomorphic to a complemented subspace of $l_p \oplus l_2$. He also showed, if the weight sequence $v = (v_n)$ also satisfies for all $\varepsilon > 0$, $\sum_{\{n: v_n < \varepsilon\}} v_n^{2p/(p-2)} = \infty$, then $X_{p,w}$ and $X_{p,v}$ are isomorphic. This is the space we call X_p. For any weight sequence w, $X_{p,w}$ is isomorphic to one of the spaces l_p, l_2, $l_p \oplus l_2$ or X_p.

$(e_i)_{i=1}^{\infty}$ will often be used to denote the natural basis for some $X_{p,w}$ space which is isomorphic to X_p, and we write for $x = \sum_{n=1}^{\infty} a_n e_n \in X_{p,w}$,

$$|x|_p = \left(\sum |a_n|^p\right)^{1/p} \quad \text{and} \quad |x|_2 = |x|_{2,w} = \left(\sum |a_n w_n|^2\right)^{1/2}.$$

The most important tool we need for this paper is the "blocking technique" introduced in [11] in its simplest form and then developed in later papers (e.g. see [12], [6], [5]). Briefly, if (E_n) is a shrinking finite dimensional decomposition (shrinking f.d.d.) for X and T is an operator from X into Y where Y has an f.d.d. (F_n), then there exist blockings (E_n') ($E_n' = [E_i]_{i=k(n)+1}^{k(n+1)}$ for certain integers $k(1) < k(2) < \ldots$) of (E_n) and (F_n') of (F_n) so that TE_n' is essentially contained in $F_n' + F_{n+1}'$ for each n. The overlap between TE_n' and TE_{n+1}' in F_{n+1}' causes some problems which can sometimes be overcome (e.g. see [5]). We use these tricks below where we describe them in more detail. The technical difficulties are particularly troublesome in Section 3, in part because the operator T is defined only on a subspace of X.

2. Subspaces of $l_p \oplus l_2$ and X_p

The first part of this section is devoted to a proof of

THEOREM 2.1. *Let X be a subspace of L_p $(2 < p < \infty)$ which has an unconditional basis and which is isomorphic to a quotient of $l_p \oplus l_2$. Then there is a subspace U of l_p (possibly $U = \{0\}$) so that X is isomorphic to U or $U \oplus l_2$.*

COROLLARY 2.2. *If X is a $\mathcal{L}_q$ subspace of $l_q \oplus l_2$ $(1 < q < 2)$ with an unconditional basis, then X is isomorphic to either l_q or $l_q \oplus l_2$.*

Proof of Theorem 2.1. Let (x_i) be a normalized unconditional basis for X and let Q be a quotient mapping of $l_p \oplus l_2$ onto X. There are two plausible cases.

Case 1. There exist $\varepsilon_n \downarrow 0$ and a sequence (N_n) of disjoint infinite subsets of $\mathbf{N}$ such that

$$x_i \in M_p(\varepsilon_n) \setminus M_p(\varepsilon_{n-1}) \quad \text{for} \quad i \in N_n. \tag{2.1}$$

Case 2. There exists $\varepsilon > 0$ such that for all $0 < \delta < \varepsilon$,

$$\{x_i \colon x_i \in M_p(\delta) \setminus M_p(\varepsilon)\} \quad \text{is finite.} \tag{2.2}$$

Our first objective is to show that Case 1 is impossible. Let (f_i) be the unconditional basis for X^* which is biorthogonal to (x_i) and assume Case 1 holds. Then for each n, $(x_i)_{i \in N_n}$ is an unconditional basic sequence in X which is equivalent to the unit vector basis of l_2. Thus $(f_i)_{i \in N_n}$ is also equivalent to the unit vector basis of l_2. Since Q is a quotient map, Q^* is an embedding of X^* into $l_q \oplus l_2$ $(1/q + 1/p = 1)$ and thus since $1 < q < 2$ we have (see e.g. [18])

$$\lim_{\substack{i \in N_n \\ i \to \infty}} |Q^* f_i|_q = 0.$$

In particular there exist integers $m_n \in N_n$ so that (f_{m_n}) is equivalent to the unit vector basis of l_2. However, by (2.1) a subsequence of (x_{m_n}) is equivalent to the unit vector basis of l_p [13], and this is impossible.

Our discussion of Case 2 requires the following lemma, the proof of which uses an idea due to Schechtman [19].

LEMMA 2.3. *Let (z_i) be an unconditional basic sequence in $l_p \oplus l_2$ $(1 < p < \infty)$. Then there is a monotonely unconditional basic sequence (x_i) in l_p and an orthogonal sequence (y_i) in l_2 such that if $w_i = x_i \oplus y_i \in l_p \oplus l_2$, then (w_i) is equivalent to (z_i).*

Proof. Let (e_n) be the unit vector basis for l_p and let (δ_n) be the unit vector basis for l_2. By a standard perturbation argument we can assume that for each n only finitely many of the z_i's have non-zero nth coordinates with respect to the basis $\{(e_n \oplus 0), (0 \oplus \delta_n)\}_{n=1}^{\infty}$ for $l_p \oplus l_2$. Embed $l_p \oplus l_2$ into $L_p[-1, 1]$ in such a way that $(e_n \oplus 0)_{n=1}^{\infty}$ is a sequence of L_p-normalized indicator functions of disjoint subsets of $[-1, 0)$ and $(0 \oplus \delta_n)_{n=1}^{\infty}$ are the Rademacher functions on $[0, 1]$. Let $z_i = x_i + y_i$ where $x_i \in [(e_n \oplus 0)_{n=1}^{\infty}]$ and $y_i \in [(0 \oplus \delta_n)_{n=1}^{\infty}]$.

The sequence (z_i) is then equivalent to $(r_i \otimes x_i + r_i \otimes y_i)$ in $L_p([0, 1] \times [-1, 1])$, where (r_i) are the Rademacher functions on $[0, 1]$. Now the terms of the monotonely unconditional sequence $(r_i \otimes x_i)$ are measurable with respect to a purely atomic sub-sigma field of $[0, 1] \times [-1, 0]$ so that $[(r_i \otimes x_i)]$ embeds isometrically into l_p. Furthermore $(r_i \otimes y_i)$ is equivalent to an orthogonal sequence in l_2. Q.E.D.

Let us return to the proof of Theorem 2.1. Assume Case 2 holds and let $\varepsilon>0$ be as in (2.2). Since $[\{x_i: x_i \in M_p(\varepsilon)\}]$ is either finite dimensional or isomorphic to l_2 ([13]) we may assume that for all $\delta>0$

$$\{x_i: x_i \in M_p(\delta)\} \quad \text{is finite.} \tag{2.3}$$

As before let (f_i) be the basis for X^* which is biorthogonal to (x_i). We shall show $[(f_i)]$ embeds into l_q, which by [11] yields that $[(f_i)]$ is isomorphic to $(\sum_{j=1}^{\infty} [f_i]_{i=n(j)}^{n(j+1)-1})_q$ for some $1=n(1)<n(2)<\dots$, and thus X is isomorphic to $(\sum_{i=1}^{\infty} [x_i]_{i=n(j)}^{n(j+1)-1})_p$, whence X embeds into l_p. By Lemma 2.3 we may assume $f_i=g_i \oplus h_i$ where (g_i) is a K-unconditional basic sequence in l_q and $h_i=|h_i|_2\delta_i$ ((δ_i) is the unit vector basis of l_2).

By (2.3), no subsequence of (x_i) is equivalent to (δ_i) and so the same is true of (f_i). Thus there exists $\delta>0$ and an integer n such that $|g_i|_q \geqslant \delta$ for $i \geqslant n$. Define T: $[(f_i)_{i=n}^{\infty}] \to l_q$ to be the natural projection;

$$T\left(\sum_{i=n}^{\infty} a_i(g_i \oplus h_i)\right) = \sum_{i=n}^{\infty} a_i g_i.$$

Then T is an isomorphism, for if $w=\sum_{i=n}^{\infty} a_i(g_i \oplus h_i)$ then by (1.3),

$$\left(\sum_{i=n}^{\infty} a_i^2\right)^{1/2} \leqslant (KK_p\delta^{-1})\left|\sum_{i=n}^{\infty} a_i g_i\right|_q$$

and so $\|Tw\| \leqslant \|w\| \leqslant KK_p\delta^{-1}\|Tw\|$. Q.E.D.

Proof of Corollary 2.2. By Theorem 2.1, $X^* \sim U$ or $X^* \sim U \oplus l_2$ for some infinite dimensional subspace U of l_p. Since X^* is complemented in L_p, U is also complemented, and hence by [11], $U \sim l_p$. Q.E.D.

We turn now to the case $2<p<\infty$. Our first result (Proposition 2.5) says that every operator from L_p into a subspace of $l_p \oplus l_2$ factors through X_p. We begin with a simple blocking lemma.

LEMMA 2.4. *Let X be a Banach space with a shrinking* f.d.d. *(E_n), let Y have* f.d.d. *(F_n) and let $1 \leqslant p<\infty$. If $T: X \to Y$ is a bounded linear operator, then there exist integers $0=k(1)<k(2)<\dots$ so that if $E_n'=[E_i]_{i=k(n)+1}^{k(n+1)}$ and $F_n'=[F_i]_{i=k(n)+1}^{k(n+1)}$ then $T: (\sum E_n')_p \to (\sum F_n')_p$ is bounded.*

Proof. Let P_k be the natural projection of Y onto $[F_i]_{i=1}^{k}$, $P^k=I-P_k$ and for $k<l$, $P_k^l=P_l-P_k$. The conclusion of the lemma means there exists $C<\infty$ so that if $x_n \in E_n'$ and $x=\sum x_n$ then

$$\left(\sum \|P_{k(n)}^{k(n+1)}Tx\|^p\right)^{1/p} \leqslant C\left(\sum \|x_n\|^p\right)^{1/p}.$$

We may assume both (E_n) and (F_n) are bimonotone f.d.d.'s. By the blocking technique there exist $0 = k(1) < k(2) < \dots$ such that

(a) $x \in [E_j]_{j=k(i)+1}^{k(i+1)} \equiv E'_i$ and $i < n$ implies $\|P^{k(n+1)}Tx\| \leq 2^{-n-i}\|x\|$, and
(b) $x \in E'_i$ for $i > n$ implies $\|P_{k(n)}Tx\| \leq 2^{-n-i}\|x\|$.

Let $x_n \in E'_n$ so that $\sum_{n=1}^{\infty} \|x_n\|^p = 1$. Then

$$\left(\sum_{n=1}^{\infty}\left\|P_{k(n)}^{k(n+1)}T\left(\sum_{i=1}^{\infty}x_i\right)\right\|^p\right)^{1/p} \leq \left(\sum_{n=1}^{\infty}\left\|\sum_{i=1}^{n-2}P_{k(n)}^{k(n+1)}Tx_i\right\|^p\right)^{1/p}$$

$$+\left(\sum_{n=1}^{\infty}\left\|P_{k(n)}^{k(n+1)}T(x_{n-1}+x_n+x_{n+1})\right\|^p\right)^{1/p}+\left(\sum_{n=1}^{\infty}\left\|\sum_{i=n+2}^{\infty}P_{k(n)}^{k(n+1)}Tx_i\right\|^p\right)^{1/p}$$

$$\leq \left(\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n-2}2^{-n+1-i}\|x_i\|\right)^p\right)^{1/p}+\left(\sum_{n=1}^{\infty}\|T\|^p(\|x_{n-1}\|+\|x_n\|+\|x_{n+1}\|)^p\right)^{1/p}$$

$$+\left(\sum_{n=1}^{\infty}\left(\sum_{i=n+2}^{\infty}2^{-n-1-i}\|x_i\|\right)^p\right)^{1/p} \qquad \text{(by (a) and (b))}$$

$$\leq \left(\sum_{n=1}^{\infty}(2^{-n+1})^p\right)^{1/p}+3\|T\|\left(\sum_{n=1}^{\infty}\|x_n\|^p\right)^{1/p}+\left(\sum_{n=1}^{\infty}(2^{-n-1})^p\right)^{1/p} \leq 3\|T\|+3. \quad \text{Q.E.D.}$$

PROPOSITION 2.5. *Let X be a subspace of $l_p \oplus l_2$ $(2 < p < \infty)$ and let $T: L_p \to X$ be a bounded linear operator. Then T factors through X_p.*

Proof. We wish to find operators $R: L_p \to X_p$ and $S: X_p \to X$ so that $T = SR$. For $x \in X$, $\|x\| = \max(|x|_p, |x|_2)$. By a theorem of Maurey [16] we may assume T is $\|\cdot\|_2 - |\cdot|_2$ bounded; i.e. there exists $K < \infty$ so that $|Tx|_2 \leq K\|x\|_2$ [indeed by Maurey's theorem there exists a change of density φ making the operator induced by T on $L_2(\varphi\, d\mu)$ bounded].

By Lemma 2.4 there exists a blocking (E_n) of the Haar basis for L_p so that $T: (\sum (E_n, \|\cdot\|_p))_p \to (X, |\cdot|_p)$ is bounded. To see this embed $(X, |\cdot|_p)$ into l_p and block the unit vector basis there. Thus if we define for $x = \sum x_n$, $x_n \in E_n$,

$$|||x||| = \max((\textstyle\sum \|x_n\|_p^p)^{1/p}, \quad (\sum \|x_n\|_2^2)^{1/2})$$

we have $T: (\sum E_n, |||\cdot|||) \to (X, \|\cdot\|)$ is bounded. Since $p > 2$ by (1.2) the natural injection $i: L_p \to (\sum E_n, |||\cdot|||)$ is bounded. Thus we will be done once we check that the completion of $(\sum E_n, |||\cdot|||)$ is complemented in $X_{p,w}$ for some w.

To see this let $H_n = [h_i]_1^{2^{k(n)}}$ where (h_i) are the Haar functions in L_p, and $k(n)$ is chosen so that $H_n \supseteq E_n$. Then $(\sum H_n, |||\cdot|||)$ is isomorphic to $X_{p,w}$ for some w, where as above

$$|||\textstyle\sum x_n||| = \max((\sum \|x_n\|_p^p)^{1/p}, \quad (\sum \|x_n\|_2^2)^{1/2}).$$

Indeed $(f_i^n)_{i=1}^{2^{k(n)}}$ is a basis for H_n where

$$f_i^n = \chi_{[(i-1)2^{-k(n)}, i2^{-k(n)}]}.$$

Suppose

$$x_n = \sum_{i=1}^{2^{k(n)}} \alpha_i^n f_i^n / |||f_i^n|||.$$

Note $|||f_i^n||| = \|f_i^n\|_p$. Then

$$(\sum_n \|x_n\|_p^p)^{1/p} = (\sum_n \sum_i |\alpha_i^n|^p)^{1/p},$$

while

$$(\sum \|x_n\|_2^2)^{1/2} = (\sum_n \sum_i |\alpha_i^n w_i^n|^2)^{1/2}$$

where $w_i^n = \|f_i^n\|_2$.

Clearly $(\sum E_n, |||\cdot|||)$ is norm 1 complemented in $(\sum H_n, |||\cdot|||)$ by means of the orthogonal projection. This proves the proposition. Q.E.D.

COROLLARY 2.6. *Every $\mathcal{L}_p$ subspace X of $l_p \oplus l_2$ $(2 < p < \infty)$ is isomorphic to a complemented subspace of X_p.*

Proof. Let $T: L_p \to X$ be a projection. By Proposition 2.5 there exist $R: L_p \to X_p$ and $S: X_p \to X$ so that $T = SR$. Then RS is a projection of X_p onto RX which is isomorphic to X. Q.E.D.

COROLLARY 2.7. *A quotient of L_p which embeds into $l_p \oplus l_2$ $(2 < p < \infty)$ is isomorphic to a quotient of X_p.*

LEMMA 2.8. *There exists $M_p < \infty$ so that if T is a bounded linear operator on $X_{p,w}$ for some weight sequence $w = (w_n)$, then there exists a weight sequence $v = (v_n)$ so that $|T|_{2,v} \leq M_p \|T\|$ and $|||x||| = \max(|x|_p, |x|_{2,v})$ is M_p-equivalent to $\|x\|$.*

In other words we can renorm $X_{p,w}$ by $|||\cdot|||$, another X_p-norm, so that T is bounded with respect to the $|\cdot|_{2,v}$ part of the norm.

Proof. We shall use M_p below to denote constants depending solely on p. Let (e_n) be the natural basis for $X_{p,w}$ so that

$$\|\sum a_n e_n\| = \max((\sum |a_n|^p)^{1/p}, (\sum |a_n w_n|^2)^{1/2})$$

and define

$$\tilde{e}_n = w_n r_n + g_n \in L_p(0, 1)$$

where (r_n) are the Rademacher functions supported on $[0, \frac{1}{2})$ and (g_n) are disjointly supported functions on $[\frac{1}{2}, 1]$ with $\|g_n\|_p = 1$, and $\|g_n\|_2 \leqslant w_n$. Then $(e_n) \overset{M_p}{\sim} (\tilde{e}_n)$ and

$$|\sum a_n e_n|_{2,w} \overset{M_p}{\sim} \|\sum a_n \tilde{e}_n\|_2.$$

Let $\tilde{T}$ be the operator on $[(\tilde{e}_n)] \subseteq L_p$ induced by T. Then $\tilde{T}$ is bounded and so by [7], there exists a change of density φ, $\varphi > \frac{1}{2}$ on $[0, 1]$, with $\int_0^1 \varphi(t)\,dt = 1$ which makes $\tilde{T}$ L_2-bounded. By this we mean if $e_n' = \tilde{e}_n/\varphi^{1/p}$ and T' is the operator on $[(e_n')] \subseteq L_p(\varphi\, dm)$ induced by $\tilde{T}$, then $\|T'\|_{L_2(\varphi\, dm)} \leqslant M_p \|T\|$. We claim for all scalars (a_n);

$$\max\left((\sum |a_n|^p)^{1/p},\ \|\sum a_n e_n'\|_{L_2(\varphi\, dm)}\right) \overset{M_p}{\sim} \|\sum a_n e_n'\|_{L_p(\varphi\, dm)} = \|\sum a_n \tilde{e}_n\|_p.$$

Indeed "$\leqslant$" is clear since (e_n') are disjointly supported norm 1 vectors in $L_p(\varphi dm)$ and $2 < p$. To see "$\geqslant$" observe that

$$\|\sum a_n e_n'\|_{L_2(\varphi\, dm)} = \left(\int |\sum a_n \tilde{e}_n|^2 \phi^{(p-2)/p} dm\right)^{1/2} \geqslant (\tfrac{1}{2})^{(p-2)/2p} \|\sum a_n \tilde{e}_n\|_2.$$

Hence

$$\|\sum a_n \tilde{e}_n\|_p \overset{M_p}{\sim} \max\left((\sum |a_n|^p)^{1/p},\ \|\sum a_n \tilde{e}_n\|_2\right) \leqslant \max\left((\sum |a_n|^p)^{1/p},\ 2^{(p-2)/2p} \|\sum a_n e_n'\|_{L_2(\varphi\, dm)}\right)$$

which proves the claim.

Let

$$v_n^2 = w_n^2 + \|g_n \varphi^{-1/p}\|^2_{L_2(\varphi\, dm)}.$$

To finish the proof we need only check that

$$(\sum a_n^2 v_n^2)^{1/2} \overset{M_p}{\sim} \|\sum a_n e_n'\|_{L_2(\varphi\, dm)}.$$

But

$$\|\sum a_n e_n'\|^2_{L_2(\varphi\, dm)} = \|\sum a_n w_n r_n \varphi^{-1/p}\|^2_{L_2(\varphi\, dm)} + \|\sum a_n g_n \varphi^{-1/p}\|^2_{L_2(\varphi\, dm)} \overset{M_p}{\sim} \sum |a_n w_n|^2 + \sum a_n^2 \|g_n \varphi^{-1/p}\|^2_{L_2(\varphi\, dm)},$$

since the g_n's are disjointly supported, and

$$M_p(\sum a_n^2 w_n^2)^{1/2} \geqslant \|\sum a_n w_n r_n\|_p = \|\sum a_n w_n r_n \varphi^{-1/p}\|_{L_p(\varphi\, dm)} \geqslant \|\sum a_n w_n r_n \varphi^{-1/p}\|_{L_2(\varphi\, dm)}$$
$$= \left(\int |\sum a_n w_n r_n|^2 \varphi^{(p-2)/p} dm\right)^{1/2} \geqslant 2^{(2-p)/2p} (\sum |a_n w_n|^2)^{1/2} \qquad \text{Q.E.D.}$$

We are finally ready to prove

THEOREM 2.9. *If X is isomorphic to a complemented subspace of X_p $(1<p<\infty)$ and X contains a complemented subspace isomorphic to X_p, then X is isomorphic to X_p.*

Proof. By duality we may assume $2<p<\infty$. As above, let (e_n) be the natural basis of $X_p=X_{p,w}$;

$$\|\sum a_n e_n\| = \max\left((\sum |a_n|^p)^{1/p}, (\sum |a_n w_n|^2)^{1/2}\right).$$

By Lemma 2.8, we may assume the projection $P: X_p \to X$ satisfies

$$|P|_{2,w} = K < \infty.$$

By Lemma 2.4 there exists a blocking $E_n=[e_i]_{k(n)+1}^{k(n+1)}$ of (e_i) such that

$$P: (\sum (E_n, \|\cdot\|))_p \to (\sum (E_n, \|\cdot\|))_p$$

is bounded.

For $x=\sum x_n$, $x_n \in E_n$, define $|x|_p=(\sum \|x_n\|^p)^{1/p}$. Then we see $\|x\| \sim \max(|x|_p, |x|_{2,w})$. Define

$$\tilde{X}_p = (X_p \oplus X_p \oplus \ldots)_{p,2}.$$

By this we mean if $x_n \in X_p$ then

$$\|(x_n)\|_{\tilde{X}_p} = \max\left((\sum |x_n|_p^p)^{1/p}, (\sum |x_n|_{2,w}^2)^{1/2}\right).$$

Claim: $\tilde{X}_p$ is isomorphic to X_p.

Let us assume the claim and finish the proof. As usual we write $X \sim Y$ if X and Y are isomorphic. Since X_p is complemented in X, there exists W so that

$$X \sim X_p \oplus W \sim X_p \oplus X_p \oplus W \sim X_p \oplus X.$$

Thus we need only show $X_p \sim X_p \oplus X$. Let $X \oplus Z = X_p$ where $Z = \ker P$. Then since P is bounded both in $|\cdot|_p$ and $|\cdot|_{2,w}$ we have for $(y_n) \subseteq X$ and $(z_n) \subseteq Z$,

$$\max\left((\sum |y_n+z_n|_p^p)^{1/p}, (\sum |y_n+z_n|_{2,w}^2)^{1/2}\right) \sim \max\left((\sum |y_n|_p^p+|z_n|_p^p)^{1/p}, (\sum |y_n|_{2,w}^2+|z_n|_{2,w}^2)^{1/2}\right).$$

Thus

$$\begin{aligned} X_p \sim \tilde{X}_p &= ((X \oplus Z) \oplus (X \oplus Z) \oplus \ldots)_{p,2} \\ &\sim X \oplus (Z \oplus (X \oplus Z) \oplus (X \oplus Z) \oplus \ldots)_{p,2} \\ &\sim X \oplus (Z \oplus X \oplus Z \oplus X \oplus \ldots)_{p,2} \\ &\sim X \oplus ((Z \oplus X) \oplus (Z \oplus X) \oplus \ldots)_{p,2} \sim X \oplus \tilde{X}_p \sim X \oplus X_p. \end{aligned}$$

It remains only to prove the claim that $X_p \sim \tilde{X}_p$. Let e_i^n be the ith basis vector in the nth copy of X_p in $\tilde{X}_p$. It is enough to show

$$\left\| \left(\sum_{i=1}^{\infty} \alpha_i^n e_i^n \right)_{n=1}^{\infty} \right\|_{\tilde{X}_p} \sim \max \left[\left(\sum_{i,n} |\alpha_i^n|^p \right)^{1/p}, \left(\sum_{i,n} |\alpha_i^n w_i|^2 \right)^{1/2} \right], \tag{2.4}$$

since the expression on the right is an X_p-norm. Now

$$\left\| \left(\sum_{i=1}^{\infty} \alpha_i^n e_i^n \right)_{n=1}^{\infty} \right\|_{\tilde{X}_p} = \max \left[\left(\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \left\| \sum_{i=k(j)+1}^{k(j+1)} \alpha_i^n e_i^n \right\|^p \right)^{1/p}, \left(\sum_{i,n} |\alpha_i^n w_i|^2 \right)^{1/2} \right]$$

which dominates the right side of (2.4). On the other hand,

$$\left(\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \left\| \sum_{i=k(j)+1}^{k(j+1)} \alpha_i^n e_i^n \right\|^p \right)^{1/p} \leqslant 2 \max \left(\left(\sum_{i,n} |\alpha_i^n|^p \right)^{1/p}, \left(\sum_{n,j} \left(\sum_{i=k(j)+1}^{k(j+1)} |\alpha_i^n w_i|^2 \right)^{p/2} \right)^{1/p} \right)$$

$$\leqslant 2 \max \left(\left(\sum_{i,n} |\alpha_i^n|^p \right)^{1/p}, \left(\sum_{n,i} |\alpha_i^n w_i|^2 \right)^{1/2} \right)$$

since $p > 2$. This proves (2.4) and the theorem. Q.E.D.

THEOREM 2.10. *X_p $(1 < p < \infty)$ is primary.*

Proof. Let $X_p = X \oplus Z$. In [1] an argument of Casazza and Lin [3] was used to show that either Y or Z contains a complemented isomorph of X_p. By Theorem 2.9 this space is isomorphic to X_p. Q.E.D.

Recall that one of our objectives in this section is to characterize the $\mathcal{L}_p$ subspaces of $l_p \oplus l_2$ $(2 < p < \infty)$ with an unconditional basis. The main tools we shall need are Theorem 2.9, Lemma 2.3, Corollary 2.6 and the following proposition.

PROPOSITION 2.11. *Let X be a subspace of $l_p \oplus l_2$ $(2 < p < \infty)$ with a normalized basis $x_n = y_n \oplus z_n$ where (y_n) is a basic sequence in l_p and (z_n) is a basic sequence in l_2. Assume $|z_n|_2 \to 0$ as $n \to \infty$. Then either X embeds into l_p or X_p is isomorphic to a complemented subspace of X.*

Proof. If l_2 does not embed into X, then X embeds into l_p [9]. Thus we may assume X contains a copy of l_2.

Since $|z_n|_2 \to 0$, we can assume without loss of generality that $|z_n|_2 < 1$ for each n. For a subspace Y of X, let $\delta(Y) = \sup \{|y|_2 : \|y\| = 1\}$. Note that since X contains a copy of l_2, if $\dim X/Y < \infty$, then $\delta(Y) = 1$. By the blocking technique [11] there exists $0 = k(1) < k(2) < \dots$ such that if $E_n = [(y_i)_{k(n)+1}^{k(n+1)}]$ and $F_n = [(z_i)_{k(n)+1}^{k(n+1)}]$, then (E_n) is an l_p-f.d.d. for $[(y_n)]$ and (F_n) is an l_2-f.d.d. for $[(z_n)]$. Thus if $u_n \in E_n$, then $|\sum u_n|_p \sim (\sum |u_n|_p^p)^{1/p}$ and a similar statement holds for (F_n). Also by our above remark we can insure that $\delta([x_i]_{k(n)+1}^{k(n+1)}) \geqslant \frac{1}{2}$ for each n. Since $|z_n|_2 \to 0$, we can find $k(n) < q(n) < k(n+1)$ such that if $H_n = [(x_i)]_{k(n)+1}^{q(n)}$ then

$$1 > \delta(H_n) > 0 \quad \text{for each } n,$$

$$\sum_{n=1}^{\infty} \delta(H_n)^{2p/(p-2)} = \infty \quad \text{and} \quad \lim_{n\to\infty} \delta(H_n) = 0.$$

Let $e_n \in H_n$ so that $\|e_n\| = 1$ and $|e_n|_2 = \delta(H_n)$. Clearly $[(e_n)]$ is isomorphic to X_p. We must show it is also complemented in X. Thus we wish to find $\tilde{f}_n \in X^*$ so that $(\tilde{f}_n)$ is biorthogonal to (e_n) and $P(x) = \sum \tilde{f}_n(x) e_n$ is a bounded operator, and hence a projection onto $[(e_n)]$.

Let f_n be the functional on H_n defined by $f_n(h) = \langle h, e_n |e_n|_2^{-2}\rangle$. Then

$$|f_n|_p = \max_{\substack{|h|_p=1 \\ h\in H_n}} \langle h, e_n |e_n|_2^{-2}\rangle \leqslant \max_{\substack{|h|_p=1 \\ h\in H_n}} |h|_2 |e_n|_2^{-1} = 1,$$

since $|e_n|_2 = \delta(H_n)$ and $\|\cdot\| = |\cdot|_p$ on H_n. Thus f_n is a norm 1 functional on H_n in the l_p norm. Extend f_n to a functional $\tilde{f}_n$ on X by letting $\tilde{f}_n(x_i) = 0$ if $i < k(n)$ or $i > q(n)$. Since (y_i) and (z_i) are basic, we have

$$|\tilde{f}_n|_p \leqslant K \quad \text{and} \quad |\tilde{f}_n|_2 \leqslant K |f_n|_2 = K |e_n|_2^{-1}$$

where K is twice the larger basis constant of (y_i) and (z_i). Moreover, since (E_n) and (F_n) are p- and 2-f.d.d.'s respectively, and $|e_n|_p \leqslant 1$, we see that $P(x) = \sum \tilde{f}_n(x) e_n$ is bounded.

Q.E.D.

THEOREM 2.12. *If X is a $\mathcal{L}_p$ subspace of $l_p \oplus l_2$ $(2 < p < \infty)$ with an unconditional basis, then X is isomorphic to l_p, $l_p \oplus l_2$ or X_p.*

Proof. By Corollary 2.6, X is isomorphic to a complemented subspace of X_p. By Lemma 2.3 we may assume X is embedded into $l_p \oplus l_2$ in such a way that it has a normalized unconditional basis (x_i), $x_i = y_i \oplus z_i$, where (y_i) is an unconditional basic sequence in l_p and (z_i) is an unconditional basic sequence in l_2. There are two possibilities:

(1) there exists $\varepsilon > 0$ so that if $M = \{i: |z_i|_2 < \varepsilon\}$ then

$$\lim_{\substack{i\to\infty \\ i\in M}} |z_i|_2 = 0,$$

(2) there exists $\varepsilon_n \downarrow 0$ so that for all n, $M_n = \{i: \varepsilon_{n-1} > |z_i|_2 \geqslant \varepsilon_n\}$ is infinite.

Suppose (1) holds. If l_2 does not embed into $[(x_i)]_{i\in M}$, then by [9] X is isomorphic to l_p or $l_p \oplus l_2$ depending upon whether $\mathbf{N} \setminus M$ is finite or infinite. If l_2 embeds into $[(x_i)]_{i\in M}$ then by Proposition 2.11 and Theorem 2.10 $[(x_i)]_{i\in M}$ and hence X is isomorphic to X_p.

If (2) holds then by a diagonal argument we can find infinite $M'_n \subset M_n$ so that $(x_i)_{i\in M_n', n\in \mathbf{N}}$ is a small perturbation of a block basis of the natural basis for X_p. It follows that $X' = [x_i]_{i\in M_n', n\in\mathbf{N}}$ is isomorphic to X_p and of course X' is complemented in X, so again by Theorem 2.10, X is isomorphic to X_p. Q.E.D.

We do not know how to extend the above results to an arbitrary $\mathcal{L}_p$ subspace, X, of $l_p \oplus l_2$. Of course one approach would be to show every $\mathcal{L}_p$ space has an unconditional basis, or perhaps just an unconditional f.d.d. Unfortunately we do not even know how to handle the latter case. We illustrate the difficulties encountered in trying to show X has an unconditional f.d.d. with the following.

Example 2.13. *There exists an* f.d.d. *for* $l_p \oplus l_2$ *which cannot be blocked to be an unconditional* f.d.d. (*This is false in* l_p [11].)

Indeed let (δ_i) be the unit vector basis of l_2 and (e_i) the unit vector basis of l_p. Let $E_1 = [0 \oplus \delta_1]$ and for $n \geq 2$, $E_n = [e_{n-1} \oplus \delta_{n-1}, 0 \oplus \delta_n]$. It is easily checked that E_n is an f.d.d. for $l_p \oplus l_2$. Also if $F_n = [E_i]_{i=k(n)+1}^{k(n+1)}$ is any blocking of (E_i), let

$$f_1 = 0 \oplus \delta_{k(2)},$$

$$f_n = e_{k(n)} \oplus (\delta_{k(n)} + \delta_{k(n+1)}) \quad \text{for} \quad n > 1.$$

Then $f_n \in F_n$ for all n and

$$\left\| \sum_{n=1}^{m} f_n \right\| \sim m^{1/2}$$

while

$$\left\| \sum_{n=1}^{m} (-1)^n f_n \right\| \sim m^{1/p}. \qquad \text{Q.E.D.}$$

3. Quotients of subspaces of $l_p \oplus l_2$ $(2 < p < \infty)$

In this section we prove

THEOREM 3.1. *Let X be a subspace of L_p $(2 < p < \infty)$ which is isomorphic to a quotient of a subspace Y of $l_p \oplus l_2$. Then X embeds into $l_p \oplus l_2$.*

COROLLARY 3.2. *Let Z be a $\mathcal{L}_q$ subspace of $l_q \oplus l_2$ $(1 < q < 2)$. Then Z^* is isomorphic to a $\mathcal{L}_p$ subspace of $l_p \oplus l_2$ $(1/p + 1/q = 1)$ and hence to a complemented subspace of X_p.*

COROLLARY 3.3. *Let X be a subspace of L_p $(2 < p < \infty)$. Then X is isomorphic to a quotient of X_p if and only if X is isomorphic both to a quotient of L_p and to a subspace of $l_p \oplus l_2$.*

Proofs of the corollaries. The first corollary follows directly from Theorem 3.1 and Corollary 2.6 while the second follows from Theorem 3.1 and Proposition 2.5. Q.E.D.

The remainder of this section is devoted to the proof of Theorem 3.1. Since $l_p \oplus l_2$ embeds into X_p, we can regard Y as a subspace of X_p and let (e_n) be the natural basis for X_p. So for $y = \sum a_n e_n \in X_p$,

$$\|y\| = \max(|y|_p, |y|_2)$$

where

$$|y|_p = (\sum |a_n|^p)^{1/p} \quad \text{and} \quad |y|_2 = (\sum |a_n w_n|^2)^{1/2}$$

for a suitable sequence $1 > w_n \downarrow 0$. Let Q be a mapping from Y onto X so that $\|Q\| = 1$ and

$$KQB_Y \supseteq B_X$$

for a certain constant K.

Notice that to prove Theorem 3.1 it is sufficient to define a blocking (H_n) of the Haar system (h_n) for L_p so that for some $\beta > 0$ and every $x \in X$ with $x = \sum x_n$ $(x_n \in H_n)$, we have:

$$\max(\|x\|_2, (\sum \|x_n\|_p^p)^{1/p}) \geqslant \beta \|x\|_p. \tag{3.1}$$

Indeed, if $x = \sum x_n$ $(x_n \in H_n)$, then by (1.2) we have

$$(\sum \|x_n\|_p^p)^{1/p} \leqslant \lambda_p K_p \|x\|_p$$

so (3.1) implies that the operator

$$i: X \to (\sum (H_n, \|\cdot\|_p)_p) \oplus L_2$$

defined by

$$ix = ((x_n), x)$$

where $x = \sum x_n$ $(x_n \in H_n)$, is an isomorphism from X into a space which is isometric to a subspace of $l_p \oplus l_2$.

We would like to construct the blocking (H_n) of the Haar system (h_n) so that if $x = \sum x_n \in X$ $(x_n \in H_n)$, then we can find $y_n \in Y$ so that $Qy_n = x_n$, $|y|_2 \leqslant K\|x_n\|_2$, $\|y_n\| \leqslant K\|x_n\|_p$, and the terms of (y_n) have pairwise disjoint supports relative to the basis (e_n) of X_p. Set $y = \sum y_n$; since $Qy = x$, we have if $\|y\| = |y|_2$ that

$$\|x\|_p \leqslant \|y\| = (\sum |y_n|_2^2)^{1/2} \leqslant K(\sum \|x_n\|_2^2)^{1/2}$$

while if $\|y\| = |y|_p$, then

$$\|x\|_p \leqslant \|y\| = (\sum |y_n|_p^p)^{1/p} \leqslant (\sum \|y_n\|^p)^{1/p} \leqslant K (\sum \|x_n\|_p^p)^{1/p}.$$

Consequently, (3.1) would be satisfied.

Of course, we cannot do all of this, but we carry out the spirit of this approach. The main technical problem is that we need to check that Q is essentially a quotient mapping from $(Y, |\cdot|_2)$ onto $(X, \|\cdot\|_2)$; this is the content of Lemma 3.4. A second problem is that for any blocking (H_n) of (h_n), there may be vectors $x \in X$ with $x = \sum x_n$ $(x_n \in H_n)$ so that some of the x_n's are not in X. A third difficulty is that Q is not defined on all of X_p, so it is technically troublesome to do blocking arguments relative to the basis (e_n) of X_p.

In order to state Lemma 3.4, we need a definition. For $K \leqslant L$ and $x \in X$, set

$$W_L(x) = \inf \{|y|_2 \colon y \in Y, \|y\| \leqslant L\|x\|_p, Qy = x\}.$$

It is easy to check that the inf in the definition is really a minimum.

Let P_n denote the natural norm one projection from L_p onto $[h_i]_{i=1}^n$. Of course, P_n is the restriction to L_p of the orthogonal projection from L_2 onto $[h_i]_{i=1}^n$.

LEMMA 3.4. *There are $M \geqslant K$ and $\lambda < \infty$ so that for every $\varepsilon > 0$ there exists an $n \in \mathbf{N}$ so that if $x \in X$ and $P_n x = 0$ then*

$$W_M(x) \leqslant \max (\varepsilon\|x\|_p, \lambda\|x\|_2).$$

The proof of Lemma 3.4 will be postponed for a while. To fix the main ideas in the derivation of Theorem 3.1 from Lemma 3.4, we first sketch the proof in a special case which avoids the second and third technical difficulties mentioned above. We assume that X has a basis (w_n) which is a block basis of the Haar system, say

$$w_n \in [h_i]_{i=s(n)}^{s(n+1)-1} \quad (1 = s(1) < s(2) < \ldots).$$

Letting $P'_n = P_{s(n+1)-1}$, we have that $P'_n X \subseteq X$ for all n. The P'_n's are the partial sum operators associated with the blocking $H'_n = [h_i]_{i=s(n)}^{s(n+1)-1}$ of the Haar basis.

We will also assume that Q can be extended to an operator (also denoted by Q) from X_p into L_p, and that the extended operator also has norm one.

We can get a blocking (E'_n) of the natural basis (e_n) for X_p and a blocking of (H'_n) (which we continue to denote by (H'_n)) so that QE'_n is essentially contained in $H'_n + H'_{n+1}$ for $n = 1, 2, \ldots$; let us assume that QE'_n is actually a subset of $H'_n + H'_{n+1}$. Therefore, for any $L \geqslant K$,

$$\begin{gathered} \text{if} \quad x \in X \cap [H'_i]_{i=n+1}^m \quad \text{then there is} \quad y \in [E'_i]_{i=n}^m \quad \text{so that} \\ (P'_m - P'_n)Qy = x, \quad \|y\| \leqslant L\|x\|_p, \quad |y|_2 = W_L(x) \end{gathered} \tag{3.2}$$

(since if $z=\sum z_i$ $(z_i \in E_i')$ and $Qz=x$, then setting $y=\sum_{i=n}^m z_i$ we have $(P_m'-P_n')Qy=x$, $\|y\|\leqslant\|z\|$ and $|y|_2\leqslant|z|_2$).

Let $\varepsilon_n\downarrow 0$ so that $\varepsilon_1=K$, $\sum_{n=2}^\infty \varepsilon_n<1$ and use Lemma 3.4 to get constants $M\geqslant K$ and λ so that we can choose $0=k(1)<k(2)<\dots$ to satisfy

$$W_M(x)\leqslant \max(\varepsilon_n\|x\|_p, \lambda\|x\|_2) \quad \text{if} \quad x\in X \quad \text{and} \quad P'_{k(n)-1}x=0. \tag{3.3}$$

We claim that the blocking

$$H_n=H'_{k(n)+1}+\dots+H'_{k(n+1)}$$

of (h_n) satisfies (3.1). Indeed, let $x=\sum x_n\in X$ with $x_n\in H_n$. Since each x_n is also in X, we can by (3.2) and (3.3) choose

$$y_n\in E'_{k(n)}+\dots+E'_{k(n+1)}$$

so that

$$(P'_{k(n+1)}-P'_{k(n)})Qy_n=x_n, \quad \|y_n\|\leqslant M\|x_n\|_p \quad \text{and} \quad |y_n|_2\leqslant \max(\varepsilon_n\|x_n\|_p, \lambda\|x_n\|_2).$$

Now (y_{2n}) and (y_{2n-1}) are both disjointly supported relative to the basis (e_n) for X_p, so if we assume, for definiteness, that $\frac{1}{2}\|x\|_p\leqslant\|\sum_{n=1}^\infty x_{2n-1}\|_p$ we get by Tong's diagonal principle (cf. Proposition 1.c.8 in [14]) that the linear extension, S, of the operator which for $n=1,2,3,\dots$ takes $y\in E'_{k(2n-1)}+\dots+E'_{k(2n)}$ to $(P'_{k(2n)}-P'_{k(2n-1)})Qy$ and vanishes on $[(E_i': i\notin\bigcup_{n=1}^\infty\{k(2n-1), k(2n-1)+1, \dots, k(2n)\})]$ has norm at most $\|Q\|$ times the unconditional constant of (H_n). Consequently, we have

$$\begin{aligned}\tfrac{1}{2}\|x\|_p &\leqslant \left\|S\left(\sum_{n=1}^\infty y_{2n-1}\right)\right\|_p\\ &\leqslant \lambda_p\left\|\sum_{n=1}^\infty y_{2n-1}\right\|\leqslant \lambda_p \max\left[\left(\sum|y_n|_p^p\right)^{1/p}, \left(\sum|y_n|_2^2\right)^{1/2}\right]\\ &\leqslant \lambda_p\max\left[(M+1)\left(\sum\|x_n\|_p^p\right)^{1/p}, \lambda\left(\sum\|x_n\|_2^2\right)^{1/2}\right];\end{aligned}$$

that is, (3.1) is satisfied for $\beta=(2\lambda_p)^{-1}\min((M+1)^{-1},\lambda^{-1})$.

Remark 3.5. Schechtman observed in [19] that every unconditional basic sequence in L_p is equivalent to a block basis of the Haar system, which puts one of the simplifying assumptions above in perspective. The other simplifying assumption can be replaced by the assumption that the operator Q, considered as an operator from Y into L_p, factors through X_p. It may be that every operator from a subspace of $l_p\oplus l_2$ into L_p factors through X_p; if so, the derivation of Theorem 3.1 from Lemma 3.4 given below can be simplified somewhat.

In deriving Theorem 3.1 from Lemma 3.4 in the general case, we use several lemmas. Given $A \subseteq V^*$, we use the symbol $A^\top$ to denote the annihilator of A in V.

LEMMA 3.6. *Suppose T is an operator from the reflexive space $(Z, \|\cdot\|)$ onto V, $KTB_Z \supseteq B_V$, S is a finite rank operator from Z, and $(v_n^*)_{n=1}^\infty \subseteq V^*$ with $[(v_n^*)] = V^*$. Suppose that $|\cdot| \leqslant \|\cdot\|$ is another norm on Z. For $M \geqslant K$ and $x \in V$, set*

$$W_M(x) = \inf \{|z| : z \in Z,\ \|z\| \leqslant M\|x\|,\ Tz = x\}.$$

Then given any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ so that if $x \in [(v_n^*)_{n=1}^m]^\top$, then there is $z \in Z$ so that

$$\|z\| \leqslant 2M\|x\|, \quad |z| \leqslant (2+\varepsilon) \max(\varepsilon\|x\|, W_M(x)), \quad \|Sz\| \leqslant \varepsilon\|x\| \quad \text{and} \quad Tz = x.$$

Proof. Suppose the lemma is false for a given $M \geqslant K$ and a given $\varepsilon > 0$. Then we can find for $n = 1, 2, \ldots$ unit vectors x_n in $[(v_i^*)_{i=1}^n]^\top$ so that if for some n there is $z \in Z$ so that $\|z\| \leqslant 2M$, $|z| \leqslant (2+\varepsilon) \max(\varepsilon, W_M(x_n))$ and $Tz = x_n$, then $\|Sz\| > \varepsilon$.

For each $n \in \mathbf{N}$, pick $z_n \in Z$ with $\|z_n\| \leqslant M$, $|z_n| = W_M(x_n)$, and $Tz_n = x_n$. This can be done since the "inf" in the definition of $W_M(\cdot)$ is easily seen to be a minimum. Since S has finite rank, there exist integers $n(1) < n(2) < \ldots$ so that $\|Sz_{n(i)} - Sz_{n(j)}\| < \varepsilon$ for all i and j. By passing to a subsequence of $(n(j))_{j=1}^\infty$, we can also assume that

$$\sup_j W_M(x_{n(j)}) \leqslant \max(\varepsilon, (1+\varepsilon) W_M(x_{n(1)})).$$

Now $x_n \to 0$ weakly, so we can find for all $N = 1, 2, \ldots$ a vector

$$y_N = \sum_{i=N}^\infty a_i^N x_{n(i)}$$

with

$$\sum_{i=N}^\infty |a_i^N| = \sum_{i=N}^\infty a_i^N = 1$$

and $\|y_N\| \to 0$. Letting

$$w_N = \sum_{i=N}^\infty a_i^N z_{n(i)},$$

we have

$$\|z_{n(1)} - w_N\| \leqslant 2M, \quad \|S(z_{n(1)} - w_N)\| < \varepsilon, \quad |z_{n(1)} - w_N| \leqslant (2+\varepsilon) \max(\varepsilon, W_M(x_{n(1)}))$$

and

$$\|T(z_{n(1)} - w_N) - x_{n(1)}\| \to 0 \quad \text{as} \quad N \to \infty.$$

Thus if we define the convex set C by

$$C = \{z \in Z: \|z\| \leqslant 2M, \quad \|Sz\| \leqslant \varepsilon, \quad |z| \leqslant (2+\varepsilon) \max (\varepsilon, W_M(x_{n(1)}))\}$$

then $x_{n(1)}$ is in the closure of TC. But C is closed, since $|\cdot|$ is continuous, and hence TC is closed, because Z is reflextive, whence $x_{n(1)} \in TC$. Q.E.D.

Remark 3.7. The proof shows that the reflexivity assumption in Lemma 3.6 can be dropped if we replace the "$Tz=x$" conclusion by "$\|Tz-x\|<\varepsilon$". In fact, an open mapping argument shows that the reflexivity assumption can be dropped if we merely replace the "$\|z\| \leqslant 2M\|x\|$" conclusion by "$\|z\| \leqslant (2+\varepsilon) M\|x\|$".

If A is a subset of the normed space Z, and $z \in Z$, $d(z, A)$ denotes the distance from z to A, and $A^\perp$ is the annihilator of A in Z^*.

LEMMA 3.8. *Suppose that V is a subspace of Z, V_1 is a finite codimensional subspace of V, and $F_1 \subseteq F_2 \subseteq \ldots$ are finite dimensional subspaces of Z^* with $\bigcup_{j=1}^\infty F_j$ dense in Z^*. Then for all $\varepsilon > 0$ there is $m \in \mathbf{N}$ so that if $z \in F_m^\top$ then*

$$d(z, V_1) \leqslant (2+\varepsilon) d(z, V).$$

Proof. Let $T: Z^* \to Z^*/V^\perp$ be the quotient mapping; of course, under the usual identification of V^* with $Z^*/V^\perp$, Tz^* is just the restriction of z^* to V. Since $\dim V_1^\perp/V^\perp = \dim V/V_1 < \infty$ and $\bigcup_{j=1}^\infty F_j$ is dense in Z^*, given $\varepsilon>0$ we can pick $m \in \mathbf{N}$ to satisfy

$$(1+\varepsilon) TB_{F_m} \supseteq TB_{V_1^\perp}.$$

Let $z \in F_m^\top$ and pick $f \in B_{V_1^\perp}$ so that $d(z, V_1) = f(z)$. Select $g \in (1+\varepsilon) B_{F_m}$ so that $Tg = Tf$. Then $f - g \in (2+\varepsilon) B_{V^\perp}$ and hence

$$d(z, V_1) = f(z) = (f-g)(z) \leqslant (2+\varepsilon) d(z, V). \qquad \text{Q.E.D.}$$

LEMMA 3.9. *Suppose V is a subspace of Z, F is a finite dimensional subspace of Z so that*

$$F \cap V \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq V$$

where $\dim F_j < \infty$ *and $\bigcup_{j=1}^\infty F_j$ is dense in V. Then for each $\varepsilon > 0$ there is $m \in \mathbf{N}$ so that for each $z \in Z$,*

$$d(z, F_m) \leqslant (1+\varepsilon) d(z, V) + (2+\varepsilon) d(z, F).$$

Proof. We need to show that there is $m \in \mathbf{N}$ so that for every $z \in F$,

$$d(z, F_m) \leqslant (1+\varepsilon) d(z, V). \tag{3.4}$$

This is sufficient, because if $z \in Z$, we can pick $x \in F$ so that $d(z, F) = \|z-x\|$. Then (3.4) yields

$$d(z, F_m) \leqslant \|z-x\| + d(x, F_m) \leqslant \|z-x\| + (1+\varepsilon) d(x, V)$$
$$\leqslant (2+\varepsilon)\|z-x\| + (1+\varepsilon) d(z, V) = (2+\varepsilon) d(z, F) + (1+\varepsilon) d(z, V).$$

The elegant proof of (3.4) which follows is due to T. Figiel. First assume $F \cap V = \{0\}$ and for $n = 1, 2, \ldots$ define real functions f_n on the unit sphere $S_F = \{z \in F: \|z\| = 1\}$ of F by

$$f_n(z) = d(z, F_n)/d(z, V).$$

The f_n's are continuous functions which decrease pointwise to the constantly one function, hence the convergence is uniform on the compact set S_F by Dini's Theorem. Now just choose m so that $f_m(z) \leqslant 1+\varepsilon$ for all $z \in S_F$.

In the general case, let $T: Z \to Z/(F \cap V)$ be the quotient mapping. Now for any $z \in Z$, $d(z, V) = d(Tz, TV)$ and $d(z, F_n) = d(Tz, TF_n)$ $(n = 1, 2, \ldots)$ since V and all the F_n's contain $F \cap V$. Consequently, the general case follows from the special case by passing to the quotient space $Z/(F \cap V)$. Q.E.D.

LEMMA 3.10. *Suppose Z is reflexive, V is a subspace of Z, (G_n) is an* f.d.d. *for Z, and $R_n: Z \to G_1 + \ldots + G_n$ are the natural projections. Given $\varepsilon > 0$ and $n \in \mathbf{N}$, there exists $m \in \mathbf{N}$ so that for each $x \in V$,*

$$d(R_n x, V) \leqslant \max\,(2\|(R_m - R_n)x\|, \varepsilon\|x\|).$$

Proof. This is Lemma 3.7 in [5] with the second parenthesis placed correctly. Q.E.D.

We turn to the derivation of Theorem 3.1 from Lemma 3.4. By perturbing the space X in L_p slightly, we can assume without loss of generality that $\bigcup_{n=1}^{\infty} [h_i]_{i=1}^{n} \cap X$ is dense in X. A formal consequence of this is that for all $N = 1, 2, \ldots$, $\bigcup_{n=N}^{\infty} [h_i]_{i=N}^{n} \cap X$ is dense in $[h_i]_{i=N}^{\infty} \cap X$. Let $M \geqslant K$ and λ be constants which satisfy the conditions of Lemma 3.4, and recall that Q denotes a norm one operator from the subspace Y of X_p onto X which satisfies $KQB_Y \supseteq B_X$. Eventually we will verify that (3.1) holds for $\beta = 16^{-1} \min\,[(12M)^{-1}, (32\lambda)^{-1}]$. Let $\varepsilon_n \downarrow 0$ so that $\varepsilon_1 < \min\,(8^{-p}\beta^2, 2^{-7})$ and $2\varepsilon_{n+1} < \varepsilon_n$ for $n = 1, 2, \ldots$.

We define a blocking (H'_n) of the Haar system and a blocking (E_n) of the natural basis for X_p to satisfy conditions (3.5)–(3.10), where P'_n denotes the natural projection from L_p onto $H'_1 + \ldots + H'_n$ and R_n denotes the natural projection from X_p onto $E_1 + \ldots + E_n$; $P'_0 \equiv 0$ and $R_0 \equiv 0$.

(3.5) *If $x \in X$ and $P'_k x = 0$, then*

$$W_M(x) \leqslant \max\,(\varepsilon_k \|x\|_p, \lambda\|x\|_2).$$

(3.6) *If $x \in X$ and $P'_k x = 0$, then there is $y \in Y$ which satisfies*

$$\|R_{k-1} y\| \leqslant \varepsilon_k \|x\|_p, \quad \|y\| \leqslant 2M\|x\|_p, \quad |y|_2 \leqslant 3 \max\,(\varepsilon_k\|x\|_p, W_M(x)), \quad \textit{and} \quad Qy = x.$$

(3.7) *If* $x \in X$, $1 \leqslant i < k$, *and* $P'_i x = 0 = (I - P'_k)x$, *then there is* $y \in Y$ *which satisfies*

$$\|R_{i-1}y\| \leqslant 2\varepsilon_i \|x\|_p, \quad \|(I-R_k)y\| \leqslant \varepsilon_k \|x\|_p, \quad \|y\| \leqslant 3M\|x\|,$$

$$|y|_2 \leqslant 4 \max(\varepsilon_i \|x\|_p, W_M(x)) \quad \textit{and} \quad Qy = x.$$

(3.8) *If* $x \in X$, *then*

$$d(P'_{k-1}x, X) \leqslant \max(\varepsilon_{k+1}\|x\|_p, 2\|(P'_k - P'_{k-1})x\|_p).$$

(3.9) *If* $z \in L_p$ *and* $P'_k z = 0$, *then*

$$d(z, X \cap (I - P'_{k-1})L_p) \leqslant 3d(z, X).$$

(3.10) *If* $1 \leqslant i < k$ *and* $z \in L_p$ *with* $P'_{i-1}z = 0$, *then*

$$d(z, X \cap (P'_k - P'_{i-1})L_p) \leqslant 2d(z, X \cap (I - P'_{i-1})L_p) + 3d(z, (P'_{k-1} - P'_{i-1})L_p).$$

Suppose that $H'_1 + \ldots + H'_{k-1} = [h_i]_{i=1}^n$ and $E_1 + \ldots + E_{k-1} = [e_i]_{i=1}^s$ have been defined. Now if $m > n$ is large enough and we set

$$H'_k = [h_i]_{i=n+1}^m$$

then (3.5), (3.6) and (3.8) will be satisfied by, respectively, Lemma 3.4, Lemma 3.6 and Lemma 3.10. That (3.9) will be true for large m follows from Lemma 3.8. To see this, set $Z = L_p$, $V = X$, $V_1 = X \cap (I - P'_{k-1})L_p$, $\varepsilon = 1$, $F_j = [h_i]_{i=1}^j \subseteq L_q = L_p^*$ $(1/p + 1/q = 1)$, and apply Lemma 3.8. Similarly, (3.10) is satisfied if m is large enough by Lemma 3.9. To see this, for each fixed $1 \leqslant i < k$ apply Lemma 3.9 with $Z = (I - P'_{i-1})L_p$, $V = X \cap (I - P'_{i-1})L_p$, $F = (P'_{k-1} - P'_{i-1})L_p$, $F_j = [h_r]_{r=p(i)+1}^j \cap X$, (where $H'_1 + \ldots + H'_{i-1} = [h_r]_{r=1}^{p(i)}$) and $\varepsilon = 1$.

Now fix $m > n$ so that (3.5), (3.6) and (3.8)–(3.10) sare satisfied. We need to get $t > s$ so that (3.7) will be true if we set

$$E_k = [e_j]_{j=s+1}^t.$$

Call statement (3.6) with "i" substituted for "k" $(3.6)_i$. For $1 \leqslant i < k$ and a small $\delta > 0$ we can apply $(3.6)_i$ to a finite δ-net (say, A_i) of the unit sphere of $X \cap (H'_i + \ldots + H'_k)$ to get a finite set (say, B_i) in Y so that for all $x \in A_i$ there is $y \in B_i$ which satisfies the conditions in $(3.6)_i$ with ε_k replaced by δ. Now we choose $t > s$ so that, setting $E_k = [e_j]_{j=s+1}^t$, we have for $y \in \bigcup_{i=1}^{k-1} B_i$, $\|(I - R_k)y\| < 2^{-1}\varepsilon_k$. It is easy to check that if $\delta > 0$ is small enough relative to the strictly positive numbers ε_k and $\inf [W_M(x) \colon x \in H'_1 + \ldots + H'_k, \|x\| = 1]$ then (3.7) is satisfied.

Now we choose $0 = n(1) < n(2) < \dots$ with $n(j) - n(j-1) \geq 4$ so that if

$$x = \sum_{i=n(k)+1}^{n(k+1)} x_i \quad \text{with} \quad x_i \in H_i'$$

then

$$\min_{n(k)+2<j<n(k+1)-2} \|x_{j-1}\|_p + \|x_j\|_p + \|x_{j+1}\|_p \leq \tfrac{1}{2}\, \varepsilon_{k+1} \|x\|_p. \tag{3.11}$$

This is possible by (1.2). Finally, we define the blocking which satisfies (3.1): set

$$H_k = H'_{n(k)+1} + \dots + H'_{n(k+1)}.$$

Suppose that $x \in X$, $\|x\|_p = 1$, $x = \sum x_i$ $(x_i \in H_i')$. By (3.11) we can select for $k = 1, 2, \dots,$ $n(k) + 2 < j(k) < n(k+1) - 2$ so that

$$\|x_{j(k)-1}\|_p + \|x_{j(k)}\|_p + \|x_{j(k)+1}\|_p \leq 2^{-1}\varepsilon_{k+1}, \tag{3.12}$$

and set, for notational convenience, $j(0) + 2 = j(0) + 1 = j(0) = 1$; $j(0) - 1 = 0$. Since (ε_k) is decreasing and $k + 1 \leq j(k) - 1$, we have from (3.12) and (3.8) that

$$d\left(\sum_{i=1}^{j(k)-2} x_i, X\right) \leq \varepsilon_{k+1}$$

hence

$$d\left(\sum_{i=j(k-1)+2}^{j(k)-2} x_i, X\right) \leq \tfrac{3}{2}\varepsilon_k + \varepsilon_{k+1} \leq 2\varepsilon_k$$

whence by applying (3.10) and (3.9) to the vector

$$z = \sum_{i=j(k-1)+2}^{j(k)-2} x_i$$

we can find

$$z_k \in X \cap H'_{j(k-1)+1} + \dots + H'_{j(k)-1} \tag{3.13}$$

so that

$$\left\| \sum_{i=j(k-1)+2}^{j(k)-2} x_i - z_k \right\|_p \leq 12\varepsilon_k. \tag{3.14}$$

Therefore,

$$\left\| x - \sum_{k=1}^{\infty} z_k \right\|_p \leq 13 \sum_{k=1}^{\infty} \varepsilon_k < \tfrac{1}{2}. \tag{3.15}$$

By (3.13), (3.7) and (3.5), (and the fact that $j(k-1) > k$ for $k > 1$) we can get $y_k \in Y$ so that

$$\left.\begin{aligned}&\|R_{j(k-1)-1}\,y_k\| \leqslant 2\varepsilon_k \|z_k\|_p,\\&\|(I-R_{j(k)-1})\,y_k\| \leqslant \varepsilon_k\|z_k\|_p,\\&\|y_k\| \leqslant 3M\|z_k\|_p,\ Qy_k = z_k,\\&|y_k|_2 \leqslant 4\max(\varepsilon_k\|z_k\|_p,\ \lambda\|z_k\|_2).\end{aligned}\right\} \tag{3.16}$$

In particular, y_k is, essentially, in $E_{j(k-1)}+\dots+E_{j(k)-1}$, so that the terms of the sequence (y_k) are, essentially, disjointly supported relative to the basis (e_n) of X_p.

Set

$$y = \sum_{k=1}^{\infty} y_k.$$

Since $Qy = \sum_{k=1}^{\infty} z_k$, we have from (3.15) that

$$\|y\| \geqslant \left\|\sum_{k=1}^{\infty} z_k\right\| \geqslant \tfrac{1}{2}. \tag{3.17}$$

Now

$$\left|\sum_{k=1}^{\infty}(R_{j(k)-1}-R_{j(k-1)-1})\,y_k\right|_p = \left(\sum_{k=1}^{\infty}|(R_{j(k)-1}-R_{j(k-1)-1})\,y_k|_p^p\right)^{1/p}$$

$$\leqslant \left(\sum_{k=1}^{\infty}|y_k|_p^p\right)^{1/p},$$

and by (3.16)

$$\left\|y-\sum_{k=1}^{\infty}(R_{j(k)-1}-R_{j(k-1)-1})\,y_k\right\| \leqslant 3\sum_{k=1}^{\infty}\varepsilon_k\|z_k\|_p < \tfrac{1}{4},$$

so if $\|y\| = |y|_p$, we have by (3.17) and (3.16) that

$$\tfrac{1}{4} \leqslant \left(\sum_{k=1}^{\infty}|y_k|_p^p\right)^{1/p} \leqslant \left(\sum_{k=1}^{\infty}\|y_k\|^p\right)^{1/p} \leqslant 3M\left(\sum_{k=1}^{\infty}\|z_k\|_p^p\right)^{1/p}. \tag{3.18}$$

Similarly, since $\sum_{k=1}^{\infty}\varepsilon_k < 2^{-6}$ we get that if $\|y\| = |y|_2$, then

$$\tfrac{1}{4} \leqslant \left(\sum_{k=1}^{\infty}|y_k|_2^2\right)^{1/2} \leqslant 8\lambda\left(\sum_{k=1}^{\infty}\|z_k\|_2^2\right)^{1/2}. \tag{3.19}$$

Recalling that

$$\beta = 16^{-1}\min\,[(12M)^{-1},\,(32\lambda)^{-1}],$$

we have from (3.18) and (3.19) that

$$\max\left[\left(\sum_{k=1}^{\infty}\|z_k\|_p^p\right)^{1/p},\ \left(\sum_{k=1}^{\infty}\|z_k\|_2^2\right)^{1/2}\right] \geqslant 16\beta. \tag{3.20}$$

Using the fact that the Haar system is a monotone basis for L_p and for L_2, we have if $r \in \{2, p\}$ that

$$\sum_{k=1}^{\infty} \left\| \sum_{i=n(k)+1}^{n(k+1)} x_i \right\|_r^r \geqslant 4^{-r} \sum_{k=1}^{\infty} \left(\left\| \sum_{i=n(k)+1}^{j(k)-2} x_i \right\|_r^r + \left\| \sum_{i=j(k)+2}^{n(k+1)} x_i \right\|_r^r \right)$$

$$\geqslant 8^{-r} \sum_{k=1}^{\infty} \left\| \sum_{i=j(k-1)+2}^{j(k)-2} x_i \right\|_r^r$$

$$\geqslant 8^{-r} \left(\sum_{k=1}^{\infty} \|z_k\|_r^r - 12^r \sum_{k=1}^{\infty} \varepsilon_k \right) \quad \text{(by 3.14)}$$

$$\geqslant 8^{-r} \sum_{k=1}^{\infty} \|z_k\|_r^r - \beta^r \quad \left(\text{since } 4^p \sum_{k=1}^{\infty} \varepsilon_k < \varepsilon^2 \right).$$

Thus from (3.20) it follows that

$$\max \left[\left(\sum_{k=1}^{\infty} \left\| \sum_{i=n(k)+1}^{n(k+1)} x_i \right\|_p^p \right)^{1/p}, \left(\sum_{k=1}^{\infty} \left\| \sum_{i=n(k)+1}^{n(k+1)} x_i \right\|_2^2 \right)^{1/2} \right] \geqslant \beta,$$

which is (3.1). Q.E.D.

In order to prove Lemma 3.4, we need several lemmas which may not be as routine as Lemmas 3.6, 3.8, 3.9 and 3.10. The first lemma restates the notation set up at the beginning of this section, except that X is not required to embed into L_p and it is convenient to regard Y as a subspace of $l_p \oplus l_2$.

LEMMA 3.11. *Let Y be a subspace of $l_p \oplus l_2$, $2 < p < \infty$, Q a norm one operator from Y onto X, $KQB_Y \supseteq B_X$, and V a subspace of X which is isomorphic to l_2. Set for $x \in X$,*

$$W_K(x) = \inf \{ |y|_2 : y \in Y, \|y\| \leqslant K\|x\|, Qy = x \}$$

where for $y = y_1 \oplus y_2 \in l_p \oplus l_2$, $|y|_2 \equiv \|y_1\|$. Then there exists $\delta = \delta(p, K) > 0$ and a finite co-dimensional subspace V_1 of V so that for all $x \in V_1$,

$$W_K(x) \geqslant \delta d(V, l_2)^{-1} \|x\|.$$

Proof. Since X is $2K$-isomorphic to a quotient of a subspace of L_p, X has type 2 with constant $\leqslant 2K_p K$, so by Maurey's extension theorem [16] there is a projection P from X onto V so that

$$\|P\| \leqslant \gamma_2(P) \leqslant 2K_p K d(V, l_2).$$

Again by Maurey's theorem, there is an operator

$$S: l_p \oplus l_2 \to V$$

so that

$$Sy = PQy \quad (y \in Y), \quad \|S\| \leqslant 4K_p^2 K d(V, l_2).$$

Since the restriction of S to l_p is compact (as is any operator from l_p into l_2; cf. Proposition 2.c.3 in [14]), given $\varepsilon > 0$, there is $N = N(\varepsilon)$ so that

$$\|Sz\| \leqslant \varepsilon K^{-1} \|z\|$$

if $z \in l_p$ and the first N coordinates of z are zero.

Now let V_1 be any finite codimensional subspace of V such that for all $x \in V_1$,

$$d(x, S[(e_i)_{i=1}^N]) \geqslant (1+\varepsilon)^{-1} \|x\|$$

where (e_i) is the unit vector basis for l_p. (For example, if F is a finite dimensional subspace of X^* which is $1+\varepsilon$-norming over $S[(e_i)_{i=1}^N]$, we can let $V_1 = V \cap F^{\top}$.)

Suppose that $x \in V_1$, $\|x\| = 1$, and choose $y \in Y$ with $\|y\| \leqslant K$, $Qy = x$, and $|y|_2 = W_K(x)$. Write

$$y = y_1 + y_2 + y_3, \quad y_1 \in [(e_i)_{i=1}^N],$$

$$y_2 \in [(e_i)_{i=N+1}^{\infty}], \quad y_3 \in l_2.$$

Then $x = Sy_1 + Sy_2 + Sy_3$, but

$$\|Sy_2\| \leqslant \varepsilon, \quad \|x - Sy_1\| \geqslant (1+\varepsilon)^{-1}$$

so that

$$\begin{aligned}(1+\varepsilon)^{-1} - \varepsilon \leqslant \|x - S(y_1 + y_2)\| &= \|Sy_3\| \\ &\leqslant 4K_p^2 K d(V, l_2) \|y_3\| = 4K_p^2 K d(V, l_2) |y|_2 \\ &= 4K_p^2 K d(V, l_2) W_K(x).\end{aligned}$$

This gives the desired conclusion for any

$$\delta < (4K_p^2 K)^{-1}. \qquad \text{Q.E.D.}$$

Remark 3.12. Notice that in Lemma 3.11, if $(v_i^*) \subseteq V^*$ and $[(v_i^*)] = V^*$, then V_1 can be taken to be of the form $[(v_i^*)_{i=1}^n]^{\top}$ for some n.

Remark 3.13. The definition of $W_K(\cdot)$ and $|\cdot|_2$ given in Lemma 3.11 is the same as that given in the beginning of this section if we regard Y as being contained in $X_{p,(w_n)}$ and $X_{p,(w_n)} \subseteq l_p \oplus l_2$ in the natural way; i.e., the nth basis vector for $X_{p,w}$ is $e_n \oplus w_n \delta_n \in l_p \oplus l_2$.

LEMMA 3.14. *Suppose that Z is reflexive and has an* f.d.d. (E_n), *W is a subspace of Z such that $\bigcup_{n=1}^{\infty} W \cap [(E_i)_{i=1}^n]$ is dense in W, and T is a norm one operator from W into some* *space V. Given any $L<\infty$, $\varepsilon_k \downarrow 0$, and a weakly null normalized sequence (x_n) in V, there is a subsequence (y_n) of (x_n) so that if $y=\sum a_n y_n$, $\|y\|=1$, and if $z \in W$ with $\|z\| \leqslant L$, $Tz=y$ with $z=\sum z_i$ $(z_i \in E_i)$, then there are $1 \leqslant m(1)<m(2)<\ldots$ and $w_k \in W \cap [(E_i)_{i=m(k)}^{m(k+1)-1}]$ so that*

$$\left\| \sum_{i=m(k)}^{m(k+1)-1} z_i - w_k \right\| < \varepsilon_k, \; \|Tw_k - a_k y_k\| < \varepsilon_k.$$

Proof. We can consider V to be embedded in $C[0, 1]$ in such a way that the operator T has an extension to a norm one operator from Z into $C[0, 1]$. By passing to a subsequence of (x_n), we can also assume that (x_n) is a block basis of some basis for $C[0, 1]$. Therefore Lemma 3.14 is a simple consequence of the following blocking lemma:

LEMMA 3.15. *Suppose that Z is reflexive and has an* f.d.d. *(E_n), W is a subspace of Z such that $\bigcup_{n=1}^{\infty} W \cap [(E_i)_{i=1}^{n}]$ is dense in W, T is a norm one operator from Z into V, and V has an* f.d.d. *(F_n'). Given any $L<\infty$ and $\varepsilon_k \downarrow 0$, there is a blocking (F_n) of (F_n') so that if $1 \leqslant n(1)<n(2)<\ldots$ and $x \in V$,*

$$x = \sum x_k \quad \text{with} \quad x_k \in F_{n(k)+1} + \ldots + F_{n(k+1)-1}, \quad \|x\| = 1$$

and if $z \in W$ with $\|z\| \leqslant L$, $Tz=x$, where $z=\sum z_i$ $(z_i \in E_i)$, then there are $1 \leqslant j(1)<j(2)<\ldots$ so that for every $k=1, 2, 3, \ldots$

$$d\left(\sum_{i=j(k)}^{j(k+1)-1} z_i, \; W \cap [(E_i)_{i=j(k)}^{j(k+1)-1}]\right) < \varepsilon_{n(k)}$$

and

$$\left\| x_k - T \sum_{i=j(k)}^{j(k+1)-1} z_i \right\| < \varepsilon_{n(k)}.$$

Proof. Since the concluding condition on (E_n) becomes more restrictive as we pass to blockings of (E_n), we can assume by passing to blockings of (E_n) and (F_n') that TE_n is essentially contained in $F_n' + F_{n+1}'$ for all $n=1, 2, \ldots$. The technical condition we use is:

$$\|(R_m - R_n) Ty\| < \delta_n \|y\| \quad \text{for} \quad y \in [(E_j)_{j=1}^{n-1} \cup (E_j)_{j=m+1}^{\infty}] \tag{3.21}$$

where R_n is the natural projection from V onto $[F_i']_{i=1}^{n}$ and where $\delta_n \downarrow 0$ at a rate which will be specified in (3.27a) and (3.27b). Next, by passing to a further blocking of (E_n) (and the corresponding blocking of (F_n'), to preserve (3.21)) we can by Lemma 3.10 assume that if $y \in W$, $y=\sum y_n$ with $y_n \in E_n$, then

$$d\left(\sum_{i=1}^{k-1} y_i, W\right) \leqslant \max\left(\delta_k \|y\|, 2\|y_k\|\right) \quad \text{for } k=1, 2, \ldots. \tag{3.22}$$

Moreover, as in the verification of (3.10), we have from Lemma 3.9 that we can assume, by passing to a further blocking of (E_n), that for $y \in [E_i]_{i=n}^{\infty}$, $1 \leqslant n \leqslant m < \infty$,

$$d(y, W \cap [(E_i)_{i=n}^{m+1}]) \leqslant 2d(y, W \cap [(E_i)_{i=n}^{\infty}]) + 3d(y, [(E_i)_{i=n}^{m}]). \tag{3.23}$$

Also, by Lemma 3.8 we can guarantee that if $y \in [E_i]_{i=n+1}^{\infty}$ for some $n = 1, 2, \ldots$, then

$$d(y, W \cap [(E_i)_{i=n}^{\infty}]) \leqslant 3d(y, W). \tag{3.24}$$

Putting together (3.23) and (3.24), we have that if $y \in [E_i]_{i=n+1}^{\infty}$ for some $n = 1, 2, \ldots$, and $n \leqslant m$, then

$$d(y, W \cap [(E_i)_{i=n}^{m+1}]) \leqslant 6d(y, W) + 3d(y, [(E_i)_{i=n}^{m}]). \tag{3.25}$$

Finally, by Sublemma 3.16 (see below), we define $1 = m(1) < m(2) < \ldots$ so that if $y = \sum y_i \in W$, $(y_i \in E_i)$, then for each $k = 1, 2 \ldots$

$$\min_{m(k)+1<j<m(k+1)-1} \|y_{j-1}\| + \|y_j\| + \|y_{j+1}\| < \delta_k \|y\|. \tag{3.26}$$

Set for $k = 1, 2, \ldots$

$$F_k = [(F_i')_{i=m(k)}^{m(k+1)-1}].$$

Suppose $1 \leqslant n(1) < n(2) < \ldots$ and

$$x_k \in [(F_j)_{j=n(k)+1}^{n(k+1)-1}] = [(F_i')_{i=m(n(k)+1)}^{m(n(k+1))-1}]$$

with $\|\sum x_k\| = 1$ and $z \in W$ with $\|z\| \leqslant L$, $Tz = x$. Write

$$z = \sum z_i \quad (z_i \in E_i)$$

and, using (3.26), choose $j(k)$ for $k = 1, 2, \ldots$ so that $m(n(k)) + 1 < j(k) < m(n(k)+1) - 1$ and $\|z_{j(k)-1}\| + \|z_{j(k)}\| + \|z_{j(k)+1}\| \leqslant \delta_{n(k)} \|z\|$. Then by (3.25) and (3.22) we have for $k = 1, 2, \ldots$

$$\begin{aligned}
& d\left(\sum_{i=j(k)}^{j(k+1)-1} z_i, W \cap [(E_i)_{i=j(k)}^{j(k+1)-1}]\right. \\
&\leqslant \|z_{j(k)}\| + \|z_{j(k+1)-1}\| + d\left(\sum_{i=j(k)+1}^{j(k+1)-2} z_i,\ W \cap [(E_i)_{i=j(k)}^{j(k+1)-1}]\right) \\
&\leqslant \delta_{n(k)}\|z\| + \delta_{n(k+1)}\|z\| + 6d\left(\sum_{i=j(k)+1}^{j(k+1)-2} z_i, W\right) \\
&\leqslant 2\delta_{n(k)}\|z\| + 6\left[d\left(\sum_{i=1}^{j(k+1)-2} z_i, W\right) + d\left(\sum_{i=1}^{j(k)} z_i, W\right)\right] \\
&\leqslant 2\delta_{n(k)}\|z\| + 6[\max(\delta_{j(k+1)-1}\|z\|, 2\|z_{j(k+1)-1}\|) \\
&\qquad + \max(\delta_{j(k)+1}\|z\|, 2\|z_{j(k)+1}\|)] \\
&\leqslant 26\delta_{n(k)}\|z\| \leqslant 26\delta_{n(k)} L.
\end{aligned}$$

This gives the first conclusion as long as

$$\delta_i < (26L)^{-1}\varepsilon_i \quad \text{for} \quad i = 1, 2, \ldots. \tag{3.27a}$$

Lastly, since for $k = 1, 2, \ldots,$

$$x_k = (R_{m(n(k+1))-1} - R_{m(n(k)+1)-1})\, Tz$$

and, by (3.21) (which applies because $j(k) < m(n(k)+1)$ and $m(n(k+1)) - 1 < j(k+1)$),

$$\left\| (R_{m(n(k+1))-1} - R_{m(n(k)+1)-1})\, T\left(\sum_{i=1}^{j(k)-1} z_i + \sum_{i=j(k+1)}^{\infty} z_i\right)\right\| \leqslant \delta_{m(n(k)+1)-1} 3K\|z\| \leqslant \delta_{n(k)} 3KL,$$

where K is the basis constant for (E_n). Consequently,

$$\left\| x_k - T\left(\sum_{i=j(k)}^{j(k+1)-1} z_i\right)\right\| \leqslant \delta_{n(k)} 3KL$$

so the second conclusion follows as long as

$$\delta_i < (3KL)^{-1}\varepsilon_i \quad \text{for} \quad i = 1, 2, \ldots. \tag{3.27b}$$

Q.E.D.

In the proof of Lemma 3.15 we used the following simple sublemma:

SUBLEMMA 3.16. *Suppose that (E_n) is a boundedly complete* f.d.d. *for a space Z. Given any n and $\varepsilon > 0$, there is $m > n$ so that if $z \in Z$, $z = \sum_{i=1}^{\infty} z_i$ $(z_i \in E_i)$, then*

$$\min_{n<i<m} \|z_{i-1}\| + \|z_i\| + \|z_{i+1}\| < \varepsilon \|z\|.$$

Proof. If the sublemma is false for a certain n and $\varepsilon > 0$, then we can find $z^k \in Z$ for $k = 1, 2, \ldots$ so that $\|z^k\| = 1$,

$$z^k = \sum_{i=1}^{\infty} z_i^k, \;\; (z_i^k \in E_i), \quad \text{and} \quad \min_{1<j<k} \|z_{n+j}^k\| + \|z_{n+j+1}^k\| + \|z_{n+j+2}^k\| > \varepsilon.$$

By passing to a subsequence of (z^k), we can assume that for each $i = 1, 2, 3, \ldots$, there is $z_i \in E_i$ so that

$$\|z_i^k - z_i\| \to 0 \quad \text{as} \quad k \to \infty.$$

Then

$$\inf_{1<j<\infty} \|z_{n+j}\| + \|z_{n+j+1}\| + \|z_{n+j+2}\| \geqslant \varepsilon$$

and $(\|\sum_{i=1}^{k} z_i\|)_{k=1}^{\infty}$ is bounded by the basis constant for (E_n), which contradicts the boundedly completeness of (E_n). Q.E.D.

A sequence (x_n) in a Banach space is said to be a *symmetric X_p sequence with weight* $w \geq 0$ provided

$$\|\sum a_n x_n\| = \max\left((\sum |a_n|^p)^{1/p}, w(\sum |a_n|^2)^{1/2}\right)$$

for all sequences of scalars (a_n).

LEMMA 3.17. *Suppose that Y is a subspace of X_p $(p>2)$, T is a norm one operator from Y onto X, and $KTB_Y \supseteq B_X$. There is a constant A so that if (x_n) is a normalized symmetric X_p sequence in X with weight $w \geq 0$, then*

$$\limsup_{n\to\infty} W_{AK}(x_n) \leq AKw,$$

where for $x \in X$ and $L \geq K$,

$$W_L(x) = \inf\{|y|_2 : y \in Y, \|y\| \leq L, Ty = x\}.$$

Proof. For $w=0$ (i.e., if (x_n) is isometrically equivalent to the unit vector basis of l_p), Lemma 3.17 is a special case of Lemma III.4 in [5], because X_p can be embedded into L_p in such a way that $\|\cdot\|_2$ is equivalent to $|\cdot|_2$ on L_p. (The p-Banach–Saks assumption in [5] is satisfied only by the space $[(x_n)]$ and not necessarily by X, but Lemma III.4 can be applied to the restriction of T to $T^{-1}[(x_n)]$.) So we assume $w>0$. However, we should mention that the proof below—which is much simpler than the proof of Lemma III.4 in [5]—can be easily modified to take care of the case $w=0$.

We can also assume that $\bigcup_{n=1}^{\infty}(Y \cap [e_i]_{i=1}^n)$ is dense in Y, where (e_n) is the natural basis for X_p.

Choose m so that

$$m^{1/p} = wm^{1/2} \tag{3.28}$$

and assume (by perturbing the norm on X and increasing K by a constant factor at most) that m is an integer.

Let $0<\varepsilon<1$. If the conclusion is false for the constant $A=5$, we can assume, by passing to a subsequence of (x_n), that

$$5Kw < W_{5K}(x_n) \quad (n=1,2,\dots) \tag{3.29}$$

and, by Lemma 3.14, that if

$$x = \sum_{n=1}^{m} x_n, \quad y \in Y, \quad \|y\| \leq K\|x\| = Km^{1/p},$$

and $Ty=x$, then there are $(y_i)_{i=1}^m$ in Y which are disjointly supported relative to the basis (e_n) for X_p so that

$$\left\| y - \sum_{i=1}^{m} y_i \right\| < \varepsilon \quad \text{and} \quad \max_{1 \leqslant i \leqslant m} \|x_i - Ty_i\| < \varepsilon. \tag{3.30}$$

If such a y is chosen so that also $|y|_2 = W_K(x)$, then

$$\left(\sum_{i=1}^{m} |y_i|_2^2 \right)^{1/2} = \left| \sum_{i=1}^{m} y_i \right|_2 \leqslant |y|_2 + \varepsilon = W_K(x) + \varepsilon. \tag{3.31}$$

Moreover, since

$$\left(\sum_{i=1}^{m} \|y_i\|^p \right)^{1/p} \leqslant 2 \left\| \sum_{i=1}^{m} y_i \right\| \leqslant 2\|y\| + 2\varepsilon \leqslant 2Km^{1/p} + 2\varepsilon,$$

we have, if $\varepsilon > 0$ is small enough, that

$$\|y_i\| < 4K \quad \text{for at least } m/2 \text{ values of} \quad i,\ 1 \leqslant i \leqslant m, \tag{3.32}$$

which we assume, for definiteness, to be $1 \leqslant i \leqslant m/2$.

Note that by (3.30) and (3.32) we have

$$W_{5K}(x_i) \leqslant |y_i|_2 + K\varepsilon \quad \text{for} \quad 1 \leqslant i \leqslant m/2. \tag{3.33}$$

Putting everything together, we get

$$\begin{aligned}
5Km^{1/p} &= 5Kwm^{1/2} \quad \text{(by 3.28)} \\
&\leqslant \sqrt{2} \left[\left(\sum_{i=1}^{m/2} W_{5K}(x_i)^2 \right)^{1/2} \right] \quad \text{(by 3.29)} \\
&\leqslant \sqrt{2} \left[\left(\sum_{i=1}^{m/2} |y_i|_2^2 \right)^{1/2} + K\varepsilon (m/2)^{1/2} \right. \quad \text{(by 3.33)} \\
&\leqslant \sqrt{2}\, [W_K(x) + \varepsilon(1 + Km^{1/2})] \quad \text{(by 3.31)} \\
&\leqslant \sqrt{2}\, [K\|x\| + \varepsilon(1 + Km^{1/2})] \\
&= \sqrt{2}\, [Km^{1/p} + \varepsilon(1 + Km^{1/2})] \quad \text{(by 3.28)}
\end{aligned}$$

which is a contradiction if $\varepsilon > 0$ is sufficiently small. Q.E.D.

We now turn to the proof of Lemma 3.4. We can assume, without loss of generality, that $\bigcup_{n=1}^{\infty} (Y \cap [(e_i)_{i=1}^n])$ is dense in Y and $\bigcup_{n=1}^{\infty} (X \cap [(h_i)_{i=1}^n])$ is dense in X, where (e_i) is the usual basis for X_p and (h_i) is the Haar basis for L_p.

Suppose that the conclusion is false for a value of M which will be specified momentarily. Then for each fixed $k = 1, 2, \ldots$, we can find a sequence $(x_n^k)_{n=1}^{\infty}$ of unit vectors in X which is a block basis of the Haar system so that

$$W_M(x_n^k) > k\|x_n^k\|_2 \quad (n = 1, 2, \ldots) \tag{3.34}$$

and

$$\inf_n W_M(x_n^k) > 0. \tag{3.35}$$

By passing to a subsequence of each $(x_n^k)_{n=1}^\infty$, we can in view of Theorem 1.14 of [8] assume that each $(x_n^k)_{n=1}^\infty$ sequence is M_p-equivalent to a symmetric X_p sequence with weight w_k. In view of (3.35), we have from Lemma 3.17 (or Lemma III.4 in [5]) that $w_k > 0$ for all $k = 1, 2, \ldots$, as long as M is sufficiently large.

Now for each $k = 1, 2, \ldots$, define m_k by

$$m_k^{1/p} = w_k m_k^{1/2} \tag{3.36}$$

and assume (by adjusting M_p, if necessary) that each m_k is an integer. As was already alluded to, if M is large enough we have from Lemma 3.17 that if (x_n) is any sequence in X which is M_p-equivalent to a symmetric X_p sequence with, say, weight $w > 0$, then $\limsup_n W_M(x_n) < Mw$. (This specifies our choice of M, as was promised above.) Consequently, we can assume that for all n and k

$$W_M(x_n^k) \leqslant Mw_k. \tag{3.37}$$

Notice that for each $k = 1, 2, \ldots$, the sequence $(y_n^k)_{n=1}^\infty$ defined by

$$y_n^k = m_k^{-1/p} \sum_{j=nm_k}^{(n+1)m_k - 1} x_j^k$$

is M_p-equivalent to the unit vector basis for l_2, so if $n = n(k)$ is sufficiently large, we have from Lemma 3.11 that $W_M(y_n^k) \geqslant \delta$, where $\delta = \delta(p, M_p, K) > 0$ does not depend on k. Assume without loss of generality that $n(k) = 1$ for all k; i.e.,

$$W_M\left(\sum_{j=1}^{m_k} x_j^k\right) \geqslant \delta m_k^{1/p} \quad (k = 1, 2, \ldots). \tag{3.38}$$

Recalling that $(x_n^k)_{n=1}^\infty$ is a block basis of (h_n) and thus orthogonal, we have for $k = 1, 2, \ldots$:

$$\begin{aligned}
\left\|\sum_{j=1}^{m_k} x_j^k\right\|_2 &= \left(\sum_{j=1}^{m_k} \|x_j^k\|_2^2\right)^{1/2} \\
&\leqslant k^{-1}\left(\sum_{j=1}^{m_k} W_M(x_j^k)^2\right)^{1/2} \quad \text{(by 3.34)} \\
&\leqslant k^{-1} m_k^{1/2} M w_k \quad \text{(by 3.37)} \\
&\leqslant k^{-1} M M_p \left\|\sum_{j=1}^{m_k} x_j^k\right\|_n \quad \text{(by 3.36)}.
\end{aligned}$$

That is, if we set

$$z_k = \left\| \sum_{j=1}^{m_k} x_j^k \right\|_p^{-1} \sum_{j=1}^{m_k} x_j^k,$$

then there is a constant B so that for $k=1, 2, \ldots,$

$$\|z_k\|_2 < k^{-1}B$$

and hence by [13], (z_k) has a subsequence which is equivalent to the unit vector basis of l_p. However, by (3.38) and (3.36),

$$w_M(z_k) \geqslant \delta M_p^{-1},$$

and this is a contradiction. Q.E.D.

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