ON THE GROWTH OF MEROMORPHIC SOLUTIONS OF THE DIFFERENTIAL EQUATION $(y')^m = R(z, y)$

BY

STEVEN B. BANK and ROBERT P. KAUFMAN(1)

University of Illinois, Urbana, Ill., U.S.A.

1. Introduction

The classical Yosida–Malmquist theorem [24] states that if $R(z, y)$ is a rational function of z and y, and if the differential equation $(y')^m = R(z, y)$, where m is a positive integer, possesses a transcendental meromorphic solution in the plane, then the equation must be of the form,

$$(y')^m = R_0(z) + R_1(z)y + \dots + R_n(z)y^n, \tag{1}$$

where $n \leqslant 2m$. The same conclusion holds (e.g. see [2]) if the equation possesses a meromorphic solution in a neighborhood of ∞ whose Nevanlinna characteristic is not $O(\log r)$ as $r \to \infty$. Similarly, the same conclusion holds if $R(z, y)$ is a rational function of y whose coefficients are analytic functions of z in a neighborhood of ∞ having no essential singularity at ∞. Other proofs and other generalizations of these theorems have been obtained by various authors including H. Wittich [20], [21], E. Hille [6], [7], Sh. Strelitz [17], I. Laine [12], [13], F. Gackstatter and I. Laine [3], and N. Steinmetz [16]. (Hille [8], [9] has also done extensive work on Briot-Bouquet equations $Q(w, w^{(k)}) = 0$, where Q is a polynomial.)

In the case when $m = 1$ in equation (1), it was proved by Wittich [23] that the order of growth of any solution $y_0(z)$ which is meromorphic in a neighborhood of ∞ and for which $T(r, y_0) \neq O(\log r)$ as $r \to \infty$, must be a positive integral multiple of $\frac{1}{2}$. However, this result does not extend to the case $m > 1$. It was shown several years ago by the authors [1, p. 298] that in the case $m = 2$, the equation (1) can possess transcendental meromorphic solutions whose order of growth is zero, although subsequent investigation revealed that in this case, the order of growth could not be strictly between zero and $\frac{1}{2}$.

---

(1) The research of both authors was supported in part by the National Science Foundation. For the first author, Grant No. MCS 78-02188, and for the second author, Grant No. MCS 76-07471.

In this paper, we consider the general case of equations (1), where m is an arbitrary positive integer, and the $R_j(z)$ are analytic functions in a neighborhood of ∞ having no essential singularity at ∞. It is shown (see § 2 below) that the order of growth of a meromorphic solution in a neighborhood of ∞, is either zero, a positive integral multiple of $\frac{1}{2}$, or a positive integral multiple of $\frac{1}{3}$. Conversely, we show that any such number is the order of growth of a transcendental meromorphic solution in the plane of an equation of the form (1). In addition, our methods permit us to determine the form of any meromorphic solution $y_0(z)$ in a neighborhood of ∞, whose order of growth is not a positive integral multiple of $\frac{1}{2}$, and for which $T(r, y_0) \neq O(\log r)$ as $r \to \infty$. We show (see § 5 below) that for some constants a, b, c, d with $ad-bc \neq 0$, the function $(ay_0(z)+b)/(cy_0(z)+d)$ has one of the four forms, (i) $\wp(g(z); \delta_1, \delta_2)$, (ii) $\wp'(g(z); \delta_1, \delta_2)$, (iii) $\wp^2(g(z); \delta_1 \delta_2)$, (iv) $\wp^3(g(z); \delta_1, \delta_2)$, where $\wp(z; \delta_1, \delta_2)$ is the Weierstrass $\wp$-function with certain primitive periods δ_1, δ_2, and where $g(z)$ is an analytic function in a slit region $D=\{z: |z|>K, \arg z \neq \pi\}$ for some $K>0$, with the property that the function $(g'(z))^q$ (where $q=2, 3, 4$, or 6 depending respectively on the forms (i), (ii), (iii), (iv)) can be extended to an analytic function in $|z|>K$ having no essential singularity at ∞. In our final result (§ 6), we show that for such a function $g(z)$, there always exist primitive periods δ_1, δ_2, such that the functions given by (i), (ii), (iii), (iv) (depending respectively on whether $q=2$, 3, 4, or 6) can be extended to be meromorphic functions in a neighborhood of ∞. In addition, for any elliptic function $w(z)$ and any analytic function $g(z)$ in the slit region D, which has the property that for some positive integer q the function $(g'(z))^q$ can be extended to be analytic in $|z|>K$ having no essential singularity at ∞, we derive a necessary condition (which is always satisfied if q is 2, 3, 4, or 6) for the function $w(g(z))$ to be extendable to a meromorphic function in a neighborhood of ∞.

2. The main result

We now state our main result. The proof will be completed in § 4.

THEOREM 1. *Let m be a positive integer, and let $Q(z, y)$ be a polynomial in y of degree at most $2m$, whose coefficients are analytic functions in a neighborhood of ∞ having no essential singularity at ∞. Let $y_0(z)$ be a meromorphic function in a neighborhood of ∞ which is a solution of the differential equation,*

$$(y')^m = Q(z, y), \tag{2}$$

and for which

$$T(r, y_0) \neq O(\log r) \quad \text{as} \quad r \to \infty. \tag{3}$$

Then the order of growth of $y_0(z)$ is either zero, a positive integral multiple of $\frac{1}{2}$, or a positive integral multiple of $\frac{1}{3}$. Conversely, any such number is the order of growth of a transcendental meromorphic solution in the plane of an equation of the form (2).

3. Preliminaries

If $f(z)$ is a meromorphic function in a neighborhood of ∞, say $|z| \geqslant K$, and if λ is a complex number or ∞, we will use the standard notation for the Nevanlinna functions $T(r, f)$, $m(r, \lambda, f)$, $n(r, \lambda, f)$ and $N(r, \lambda, f)$ (see [22, p. 49] or [2, p. 98]). (In the definitions of $n(r, \lambda, f)$ and $N(r, \lambda, f)$, only the λ-points lying in $K \leqslant |z| \leqslant r$ are considered.) The order of growth of f is $\limsup_{r\to\infty} \log T(r, f)/\log r$.

We will denote by $\mathcal{H}$, the field of all functions which are analytic in a neighborhood of ∞ and have no essential singularity at ∞. As usual, we identify two elements of $\mathcal{H}$ if they agree on a neighborhood of ∞, and we will call an element of $\mathcal{H}$ *nontrivial* if it is not identically zero.

We will require the following results concerning the Wiman–Valiron theory (see [19], [22], or [23].) If $w(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ is an analytic function in a neighborhood of ∞ such that $T(r, w) \neq O(\log r)$ as $r \to \infty$, let $M_1(r)$ denote $\max_{|z|=r} |w(z)|$ and let $k(r)$ denote the centralindex of $w(z)$. Then the following hold:

(a) For every $\alpha \geqslant 0$, $M_1(r)/r^{\alpha} \to +\infty$ as $r \to +\infty$.

(b) If q is a positive integer, there exists a set E in $(0, \infty)$ having finite logarithmic measure, such that if $r \notin E$ and z is a point on $|z| = r$ at which $|w(z)| = M_1(r)$, then for $j = 1, \ldots, q$,

$$w^{(j)}(z) = (k(r)/z)^j w(z)(1 + \delta_j(z)), \tag{4}$$

where $\delta_j(z) = o(1)$ as $r \to \infty$. In addition, for some $\alpha > 0$,

$$k(r) = O((\log M_1(r))^{\alpha}) \quad \text{as} \quad r \to \infty,\ r \notin E. \tag{5}$$

The order of $w(z)$ is also given by $\limsup_{r\to\infty} (\log k(r)/\log r)$.

(c) If $Q(z, w, w', \ldots, w^{(n)})$ is a nontrivial polynomial in $w, w', \ldots, w^{(n)}$, whose coefficients belong to $\mathcal{H}$, and if Q possesses only one nontrivial term of maximum total degree in $w, w', \ldots, w^{(n)}$, then the differential equation $Q(z, w, w', \ldots, w^{(n)}) = 0$ cannot possess a solution $w(z)$ which is analytic in a neighborhood of ∞ and for which $T(r, w) \neq O(\log r)$ as $r \to \infty$. (This follows easily, e.g. see [22, pp. 64–65], from Parts (a) and (b), since $k(r)$ is an unbounded increasing function for all sufficiently large r.)

4. Proof of Theorem 1

We will now prove a sequence of lemmas from which the theorem will immediately follow.

LEMMA 1. *Let* m, $Q(z, y)$ *and* $y_0(z)$ *be as in the statement of the theorem. Then:*

(i) *The degree of* $Q(z, y)$ *in* y *is at least* m.

(ii) *Let the factorization of* $Q(z, y)$ *into irreducible factors* (*e.g. see* [25, p. 31]) *be*

$$Q(z, y) = R(z)Q_1(z, y)^{m_1} \dots Q_q(z, y)^{m_q}, \tag{6}$$

where $R(z)$ *is a nontrivial element of* $\mathcal{H}$, $q \geq 1$ (*by Part* (i)), *the* m_j *are positive integers, and where the irreducible polynomials* $Q_j(z, y)$ *over* $\mathcal{H}$ *are monic and distinct. Let* j *denote any of the numbers* $1, \dots, q$. *Then the following are true:*

(a) *If the function* $Q_j(z, y_0(z))$ *has only finitely many zeros in a neighborhood of* ∞, *then* $Q_j(z, y)$ *is of the form* $y-a$, *where* a *is a constant.*

(b) *The function* $Q_j(z, y_0(z))$ *cannot be identically zero.*

(c) *If the meromorphic function* $Q_j(z, y_0(z))$ *has infinitely many zeros, say* $\{z_n\}$, *on* $|z| \geq K$ *for some* K, *and if* $F(z, y)$ *is any polynomial in* y, *with coefficients in* $\mathcal{H}$, *which is not the zero polynomial and which is relatively prime to* Q_j *as polynomials in* y *over* $\mathcal{H}$, *then for some* n_0, $F(z_n, y_0(z_n)) \neq 0$ *for all* $n \geq n_0$.

(d) *If* $m_j \notin \{m, 2m\}$, *then* $Q_j(z, y)$ *is of the form* $y-a$ *where* a *is a constant.*

Proof. Part (i): First, $Q(z, y)$ cannot be the zero polynomial in y, for otherwise $y_0(z)$ would be a constant function contradicting (3). Let d denote the degree of $Q(z, y)$ in y, and assume $d < m$. Then, in a neighborhood of ∞ where the coefficients of $Q(z, y)$ are analytic, and the leading coefficient is nowhere zero, the solution $y_0(z)$ can have no poles since the multiplicity α at such a pole would satisfy the relation $(\alpha+1)m = d\alpha$ contradicting $d < m$. Hence $y_0(z)$ would be analytic in a neighborhood of ∞. However, if $d < m$, then equation (2) has only one term of maximal total degree in y, y', and hence from § 3, Part (c), this equation cannot possess any analytic solutions in a neighborhood of ∞ satisfying (3). This contradiction proves that $d \geq m$ and hence Part (i) is proved.

Part ii(a): Let $Q_j(z, y) = \sum_{k=0}^{\lambda} a_k(z) y^k$, where $\lambda > 0$, the $a_k(z)$ belong to $\mathcal{H}$ and $a_\lambda(z) \equiv 1$. Assume that the function $f(z) = Q_j(z, y_0(z))$ has only finitely many zeros in a neighborhood of ∞, and set $w(z) = 1/f(z)$. Then $w(z)$ is analytic in a neighborhood of ∞, and since (e.g. [2, p. 100]), $T(r, w) = \lambda T(r, y_0) + O(\log r)$ as $r \to \infty$, clearly $T(r, w) \neq O(\log r)$ in view of

(3). Hence the Wiman–Valiron theory (§ 3) is applicable to $w(z)$. Set $M_1(r) = \max_{|z|=r} |w(z)|$ and for all sufficiently large r, let z_r denote a point on $|z| = r$ for which $|w(z_r)| = M_1(r)$. Then in view of § 3(a), it follows that,

$$r^{\alpha} |f(z_r)| \to 0 \quad \text{as} \quad r \to +\infty \quad \text{for all } \alpha > 0. \tag{7}$$

Since the coefficients $a_k(z)$ belong to $\mathcal{H}$, it is easy to see that if $a_k(z) \not\equiv 0$ then there are real constants α_k, K_k, L_k, and A_k, with K_k, L_k, and A_k positive, such that

$$K_k |z|^{\alpha_k} \leqslant |a_k(z)| \leqslant L_k |z|^{\alpha_k} \quad \text{for} \quad |z| \geqslant A_k. \tag{8}$$

(Of course, $\alpha_\lambda = 0$ and we may take $K_\lambda = L_\lambda = 1$.) Now $a_0(z) \not\equiv 0$ since $Q_j(z, y)$ is irreducible. In view of (7) and (8), it easily follows that for all sufficiently large r, say $r \geqslant r_0$, we have $|a_0(z_r)| \geqslant K_0 r^{\alpha_0} > |f(z_r)|$, and hence $y_0(z_r) \neq 0$. If we set $\Psi_k(z) = (a_k(z)/a_\lambda(z)) y_0(z)^{k-\lambda}$, then we have

$$f(z_r) = a_\lambda(z_r) y_0(z_r)^\lambda \left(1 + \sum_{k=0}^{\lambda-1} \Psi_k(z_r)\right). \tag{9}$$

Let I denote the set of all $k \in \{0, 1, ..., \lambda - 1\}$ for which $a_k(z) \not\equiv 0$, and let B denote the set of all $r \geqslant r_0$ for which $|y_0(z_r)|^{\lambda-k} > (\lambda+1) L_k K_\lambda^{-1} r^{\alpha_k - \alpha_\lambda}$, for all $k \in I$. Then clearly from (8), if $r \in B$, we have $|\Psi_k(z_r)| < (1/(\lambda+1))$ for all k, and hence from (9) (and the fact that $0 \in I$), we obtain $|f(z_r)| \geqslant L_0 r^{\alpha_0}$. This, of course, contradicts the definition of r_0. Hence B must be the empty set, and thus if $r \geqslant r_0$, there is an index $k \in I$, depending on r, for which $|y_0(z_r)|^{\lambda-k} \leqslant (\lambda+1) L_k K_\lambda^{-1} r^{\alpha_k - \alpha_\lambda}$. Hence if L denotes the maximum of the numbers $((\lambda+1) L_k K_\lambda^{-1})^{1/(\lambda-k)}$ for $k \in I$, and if σ denotes the maximum of the numbers $(\alpha_k - \alpha_\lambda)/(\lambda - k)$ for $k \in I$, then

$$|y_0(z_r)| \leqslant L r^{\sigma} \quad \text{for all } r \geqslant r_0. \tag{10}$$

Now let $Q_{j1}(z, y)$ denote $\partial Q_j(z, y)/\partial z$, and let $Q_{j2}(z, y)$ denote $\partial Q_j(z, y)/\partial y$. Then clearly,

$$w'(z) = -w(z)^2 (Q_{j1}(z, y_0(z)) + Q_{j2}(z, y_0(z)) y_0'(z)). \tag{11}$$

We distinguish three possibilities: (A) The polynomial $Q_{j1}(z, y)$ is the zero polynomial; (B) $Q_{j1}(z, y)$ is not the zero polynomial, but $Q_j(z, y)$ and $Q_{j1}(z, y)$ are not relatively prime as polynomials over $\mathcal{H}$; (C) $Q_{j1}(z, y)$ is not the zero polynomial, but $Q_j(z, y)$ and $Q_{j1}(z, y)$ are relatively prime as polynomials over $\mathcal{H}$.

In Case (A) clearly $Q_j(z, y)$ has constant coefficients, and since it is irreducible over $\mathcal{H}$, it must have the form $y - a$ which is the conclusion of Part ii(a). Case (B) is easily seen to be impossible because the irreducibility of $Q_j(z, y)$ would imply that $Q_j(z, y)$ divides $Q_{j1}(z, y)$ (as polynomials over $\mathcal{H}$), while $Q_{j1}(z, y)$ is clearly of smaller degree in y than $Q_j(z, y)$ since $Q_j(z, y)$ is monic. Thus to prove Part ii(a), it suffices to show Case (C) is impossible. If we assume Case (C) holds, then there exist polynomials $G_1(z, y)$ and $G_2(z, y)$ over $\mathcal{H}$ such that

$$G_1(z, y)Q_j(z, y) + G_2(z, y)Q_{j1}(z, y) = 1 \tag{12}$$

as polynomials in y over $\mathcal{H}$. We observe first that if $D(z, y)$ denotes any of the polynomials $Q_{j1}(z, y)$, $Q_{j2}(z, y)$, $G_1(z, y)$, $G_2(z, y)$, or $Q_k(z, y)$ for $k = 1, ..., q$, then since $D(z, y)$ has coefficients in $\mathcal{H}$, it easily follows from (10) that there are real constants $c > 0$ and σ_1 such that

$$|D(z_r, y_0(z_r))| \leqslant cr^{\sigma_1}, \quad \text{for all sufficiently large } r. \tag{13}$$

Since $f(z_r) = Q_j(z_r, y_0(z_r))$ satisfies (7), it now follows from (12), that

$$|G_2(z_r, y_0(z_r))\, Q_{j1}(z_r, y_0(z_r))| \geqslant \tfrac{1}{2} \quad \text{for all sufficiently large } r.$$

Applying (13) with $D = G_2$, we obtain,

$$|Q_{j1}(z_r, y_0(z_r))| \geqslant (1/2c)r^{-\sigma_1}, \tag{14}$$

for all sufficiently large r. Since each factor Q_k in Q satisfies (13), while the factor Q_j satisfies (7), we see that $r^{\alpha}|Q(z_r, y_0(z_r))| \to 0$ for each $\alpha > 0$ as $r \to +\infty$. From the differential equation (2), it then follows that $r^{\alpha}|y_0'(z_r)| \to 0$ for each $\alpha > 0$ as $r \to +\infty$. In view of (13) for $D = Q_{j2}$, we thus see that for all sufficiently large r, we have $|Q_{j2}(z_r, y_0(z_r))y_0'(z_r)| \leqslant (1/4c)r^{-\sigma_1}$. It now follows from (11) and (14) that

$$|w'(z_j)/w(z_r)| \geqslant M_1(r)(1/4c)r^{-\sigma_1}, \tag{15}$$

for all sufficiently large r. But by the Wiman–Valiron theory, relation (4) holds for all r outside of a set E of finite logarithmic measure, and hence together with (15), we obtain,

$$2k(r)/r \geqslant M_1(r)(1/4c)r^{-\sigma_1}, \tag{16}$$

for all sufficiently large r which lie outside E, where $k(r)$ is the centralindex of $w(z)$. Since $k(r)$ satisfies (5), and $M_1(r)$ grows faster than every power of r (by § 3(a)), clearly (16) is impossible for arbitrarily large r. Hence Case (C) is impossible and thus Part ii(a) is proved.

Part ii(b): If $f(z) = Q_j(z, y_0(z))$, then since (e.g. [2, p. 100]), $T(r, f) = \lambda T(r, y_0) + O(\log r)$, we cannot have $f(z) \equiv 0$ in view of assumption (3).

Part ii(c): In view of Part ii(b), the sequence $\{z_n\}$ of zeros of $Q_j(z, y_0(z))$ in $|z| \geqslant K$ must tend to ∞. Hence for all sufficiently large n, the point z_n cannot be a pole of $y_0(z)$ since the coefficients of $Q_j(z, y)$ are analytic on some neighborhood of ∞, and the leading coefficient is 1. If $F(z, y)$ is relatively prime to $Q_j(z, y)$,then as in (12), some linear combination of F and Q_j is 1, and it clearly follows that $F(z_n, y_0(z_n)) \neq 0$ for all sufficiently large n.

Part ii(d): Suppose now $m_j \notin \{m, 2m\}$ in (6), and let $f(z) = Q_j(z, y_0(z))$ be meromorphic on $|z| \geqslant K$ for some $K > 0$. If $f(z)$ has only finitely many zeros on $|z| \geqslant K$, the conclusion follows from Part ii(a). Hence we may assume that $f(z)$ has infinitely many zeros, say $\{z_n\}$, on $|z| \geqslant K$. Let α_n denote the multiplicity of the zero z_n for $f(z)$. In view of Part ii(c), no other function $Q_k(z, y_0(z))$ can vanish at z_n if n is sufficiently large (and, of course, in some neighborhood of ∞ the function $R(z)$ in (6) is analytic and nowhere zero), so $Q(z, y_0(z))$ has a zero of order $m_j\alpha_n$ at z_n. Thus from (2), y_0' has a zero at z_n, say of order λ_n, and $m\lambda_n = m_j\alpha_n$. But since the degree of Q is at most $2m$, we have $m_j < 2m$ by our assumption in this case. It follows that $\lambda_n < 2\alpha_n$ for all sufficiently large n. It is not possible for any α_n to be 1, since this would imply $\lambda_n = 1$, and thus $m = m_j$. Hence $\alpha_n > 1$ for all sufficiently large n, and hence $f'(z_n) = 0$. Since also $y_0'(z_n) = 0$, it follows that $Q_{j1}(z_n, y_0(z_n)) = 0$ for all sufficiently large n, where as in Part ii(a), $Q_{j1}(z, y)$ denotes $\partial Q_j(z, y)/\partial z$. If $Q_{j1}(z, y)$ were not the zero polynomial, it would follow from Part ii(c), that $Q_j(z, y)$ and $Q_{j1}(z, y)$ cannot be relatively prime. Since $Q_j(z, y)$ is irreducible over $\mathcal{H}$, it would follow that Q_j must divide Q_{j1} as polynomials over $\mathcal{H}$, and this would be impossible since the degree of Q_{j1} is smaller than that of Q_j because Q_j is monic. Hence $Q_{j1}(z, y)$ must be the zero polynomial over $\mathcal{H}$, and hence $Q_j(z, y)$ has constant coefficients. Since $Q_j(z, y)$ is irreducible over $\mathcal{H}$, it must have the form $y - a$ and this proves Part ii(d).

We will require the following form of Wittich's theorem [23]. We omit the proof since it is exactly the same as the proof given in [23] for the case when the Riccati equation has rational functions for coefficients. (We remark that if the Riccati equation is actually linear, the result follows immediately from the Wiman–Valiron theory (§ 3).)

LEMMA 2. (Wittich [23]). *Given a Riccati equation,*

$$u' = R_0(z) + R_1(z)u + R_2(z)u^2, \tag{17}$$

where the $R_j(z)$ belong to $\mathcal{H}$. Let $u_0(z)$ be a solution of (17) *which is meromorphic in a neighborhood of ∞, and such that $T(r, u_0) \neq O(\log r)$ as $r \to \infty$. Then the order of growth of $u_0(z)$ is a positive integral multiple of $\frac{1}{2}$.*

Using this result, we can now prove:

LEMMA 3. *Let m, $Q(z, y)$, and $y_0(z)$ be as in the statement of Theorem 1. Then, at least one of the following holds:*

(a) *The order of growth of $y_0(z)$ is a positive integral multiple of $\frac{1}{2}$;*

(b) *The polynomial $Q(z, y)$ is of the form,*

$$Q(z, y) = R(z)(y-a_1)^{m_1} \dots (y-a_q)^{m_q}, \tag{18}$$

where $R(z)$ is a nontrivial element of $\mathcal{H}$; $a_1, \dots, a_q$ are distinct complex numbers, and $m_1, \dots, m_q$ are positive integers satisfying,

$$m \leqslant m_1 + \dots + m_q \leqslant 2m, \quad \textit{and} \quad m_j \notin \{m, 2m\} \quad \textit{for all } j. \tag{19}$$

Proof. If (b) fails to be true, then it follows easily from Lemma 1, Parts (i) and ii(d), that in the representation (6), we must have $m_j \in \{m, 2m\}$ for some $j \in \{1, \dots, q\}$. By renumbering if necessary, we may assume $m_1 \in \{m, 2m\}$. In this case we will show that the order of growth of y_0 is a positive integral multiple of $\frac{1}{2}$.

Suppose first that $m_1 = 2m$. Since the degree of $Q(z, y)$ is at most $2m$, clearly $Q_1(z, y)$ must be linear in y. Hence the equation (2) is of the form, $(y')^m = R(z)(y+B(z))^{2m}$, where $R(z)$ and $B(z)$ belong to $\mathcal{H}$. If we set $V = y_0'/(y_0+B)^2$, then $V(z)$ is meromorphic in a neighborhood of ∞. But since $y_0(z)$ satisfies (2), we have $V^m = R$, and hence V is actually analytic in a neighborhood of ∞, having no essential singularity at ∞. Since $y_0(z)$ satisfies the Riccati equation, $y' = V(y+B)^2$ whose coefficients belong to $\mathcal{H}$, it follows from Lemma 2, that (a) holds in this case.

Now assume that $m_1 = m$, and we consider the possibilities for q in the representation (6). If $q = 1$, then since the degree of $Q(z, y)$ cannot exceed $2m$, equation (2) must have one of the forms,

$$(y')^m = R(z)(y+B(z))^m \quad \text{or} \quad (y')^m = R(z)(y^2+B(z)y+A(z))^m, \tag{20}$$

where A, B and R belong to $\mathcal{H}$. As above, it again follows from Lemma 2 that the order of y_0 is a positive integral multiple of $\frac{1}{2}$, by setting $V = y_0'/(y_0+B)$ in the first case, and $V = y_0'/(y_0^2+By_0+A)$ in the second case.

Hence we are left with the case $m_1 = m$ and $q \geqslant 2$. Of course $Q_1(z, y)$ must be linear in y, or the degree of $Q(z, y)$ would exceed $2m$. We distinguish two subscases. Suppose first that for some $j \geqslant 2$, we have $m_j \in \{m, 2m\}$. Then we must have $q = 2$, $m_2 = m$, and $Q_2(z, y)$ is linear in y. Hence equation (2) is of the form $(y')^m = R(y+B)^m(y+A)^m$, where A, B, and R

belong to $\mathcal{H}$. As before, by setting $V = y_0'/(y_0+B)(y_0+A)$, it follows from Lemma 2 that conclusion (a) holds.

In the only case remaining, we have $m_1 = m$, $q \geqslant 2$, and $m_j \notin \{m, 2m\}$ for all $j \geqslant 2$. By Lemma 1, Part ii(d), it follows that for $j \geqslant 2$, each $Q_j(z, y)$ is of the form $y - a_j$, where a_j is a constant. Hence $y_0(z)$ satisfies the equation,

$$(y')^m = R(z)(y + B(z))^m (y - a_2)^{m_2} \dots (y - a_q)^{m_q}, \tag{21}$$

where B and R belong to $\mathcal{H}$, and $a_2, \dots, a_q$ are distinct constants. Since the degree in y of the right side of equation (21) is at most $2m$, and since $m_j \notin \{m, 2m\}$ for $j \geqslant 2$, we obviously have,

$$m_2 + \dots + m_q \leqslant m, \quad \text{and} \quad m_j < m \quad \text{for } j = 2, \dots, q. \tag{22}$$

We now assert that for each $j \in \{2, \dots, q\}$, the function $y_0(z) - a_j$ must have infinitely many zeros in every neighborhood of ∞. If we assume the contrary for some j, say for $j = 2$, then $v_0(z) = 1/(y_0(z) - a_2)$ is analytic in a neighborhood of ∞. Since y_0 satisfies (21), clearly $v_0(z)$ satisfies the equation,

$$(-1)^m (v')^m = R((a_2 + B)v + 1)^m (1 + b_3 v)^{m_3} \dots (1 + b_q v)^{m_q} v^{\sigma}, \tag{23}$$

where $b_j = a_2 - a_j$, and $\sigma = m - (m_2 + \dots + m_q)$. We observe that each $b_j \neq 0$, and $a_2 + B \not\equiv 0$, by the distinctness of the factors $Q_k(z, y)$ in (6). Hence, as a polynomial in v over $\mathcal{H}$, the degree of the right side of (23) is $2m - m_2$ which is greater than m by (22). Hence equation (23) possesses only one term of maximal total degree in v, v', and by the Wiman–Valiron theory (§ 3(c)), it must follow that for the analytic function $v_0(z)$, we have $T(r, v_0) = O(\log r)$ as $r \to \infty$. Of course, this leads to an immediate contradiction of our assumption (3) for y_0, and thus proves the assertion.

Returning to equation (21), let $j \geqslant 2$, and let z_0 be a zero of $y_0(z) - a_j$ of order d_j. If $|z_0|$ is sufficiently large, then by Lemma 1, Part ii(c), the right side of equation (21), when $y = y_0(z)$, has a zero at z_0 of multiplicity $m_j d_j$. From equation (21), y_0' also vanishes at z_0 with multiplicity $d_j - 1$, so clearly

$$d_j > 1, \quad (m - m_j) d_j = m, \quad \text{and} \quad m_j \geqslant m/2, \tag{24}$$

for $j = 2, \dots, q$. In view of (22), it now easily follows that $q \leqslant 3$, so either $q = 2$ or $q = 3$. In either case, equation (21) has only one term of maximal total degree in y, y', so by the Wiman–Valiron theory (§ 3(c)), $y_0(z)$ must have infinitely many poles in every neighborhood of ∞.

We now distinguish the two cases $q = 2$ and $q = 3$. If $q = 2$ and z_1 is a pole of $y_0(z)$ of order δ, then if $|z_1|$ is sufficiently large, it follows from equation (21) that $(\delta+1)m=\delta(m+m_2)$. Thus $m=\delta m_2$. Since $m_2<m$ by (22), we have $\delta>1$. But since $m_2 \geqslant m/2$ by (24), we must then have $\delta=2$, so $m=2m_2$. Hence equation (21) has the form

$$(y')^{2m_2} = R(z)(y+B(z))^{2m_2}(y-a_2)^{m_2}, \tag{25}$$

where B and R belong to $\mathcal{H}$. Setting $V=(y_0')^2/(y_0+B)^2(y_0-a_2)$, and noting that $V^{m_2}=R$, it follows as before that V belongs to $\mathcal{H}$. Now if we set $u_0=y_0'/(y_0+B)$. then $y_0=a_2+(u_0^2/V)$. Computing y_0' and substituting into the definition of V, we see that the meromorphic function u_0 satisfies the relation,

$$((2u_0'V-u_0V')/V^2)^2=(a_2+B+(u_0^2/V))^2. \tag{26}$$

Hence u_0 must satisfy one of the two Riccati equations defined by (26). Since both of these Riccati equations have coefficients belonging to the field $\mathcal{H}$, and since $T(r, y_0)=2T(r, u_0)+O(\log r)$ as $r\to\infty$, it follows from Lemma 2 that the order of growth of u_0, and hence of y_0, is a positive integral multiple of $\frac{1}{2}$.

Finally, we consider the case $q=3$. In this case if z_1 is a pole of $y_0(z)$ of order δ, and if $|z_1|$ is sufficiently large, then it follows from equation (21), that $(\delta+1)m=\delta(m+m_2+m_3)$, so $m=\delta(m_2+m_3)$. Since $m_j \geqslant m/2$ by (24), it easily follows that $\delta=1$, and $m_2=m_3=m/2$. Thus equation (21) is of the form,

$$(y')^{2m_2} = R(z)(y+B(z))^{2m_2}(y-a_2)^{m_2}(y-a_3)^{m_2}. \tag{27}$$

Since $y_0(z)$ satisfies (27), it easily follows that $y_1(z)=1/(y_0(z)-a_2)$ satisfies the equation,

$$(y')^{2m_2} = R_1(z)(y+B_1(z))^{2m_2}(y-b_1)^{m_2}, \tag{28}$$

where $R_1=R(a_2+B)^{2m_2}(a_2-a_3)^{m_2}$, $B_1=1/(a_2+B)$, and $b_1=1/(a_3-a_2)$. Since R_1 and B_1 obviously belong to $\mathcal{H}$, clearly (28) is an equation of the form (25), and we saw that any solution of (25) whose Nevanlinna characteristic is not $O(\log r)$ as $r\to\infty$, must have order of growth equal to a positive integral multiple of $\frac{1}{2}$. Since $T(r, y_1)=T(r, y_0)+O(\log r)$ as $r\to\infty$, it follows that the order of growth of y_1, and hence of y_0, is a positive integral multiple of $\frac{1}{2}$. This concludes the proof of Lemma 3.

LEMMA 4. *Let m, $Q(z, y)$, and $y_0(z)$ be as in the statement of Theorem 1. Then, at least one of the following holds:*

(a) *The order of growth of $y_0(z)$ is a positive integral multiple of $\frac{1}{2}$;*

(b) *There exist constants a, b, c, d, with $ad-bc\neq 0$, such that if $y_1(z)=(ay_0(z)+b)/(cy_0(z)+d)$, then $y_1(z)$ satisfies a differential equation of the form,*

$$(y')^m = R_1(z)(y-b_1)^{r_1} \dots (y-b_t)^{r_t}, \tag{29}$$

where R_1 is a nontrivial element of $\mathcal{H}$; $t \leqslant 4$; $b_1, \dots, b_t$ are distinct complex numbers, and where $r_1, \dots, r_t$ are positive integers satisfying the conditions,

$$r_1 + \dots + r_t = 2m, \quad 1 \leqslant r_j < m, \quad \text{and} \quad m = \lambda_j(m - r_j), \tag{30}$$

for $1 \leqslant j \leqslant t$, where λ_j is an integer greater than 1.

Proof. We assume that (a) fails to hold. Then from Lemma 3, we know that $y_0(z)$ satisfies the differential equation,

$$(y')^m = R(z)(y-a_1)^{m_1} \dots (y-a_q)^{m_q}, \tag{31}$$

where $R(z)$ is a nontrivial element of $\mathcal{H}$, $a_1, \dots, a_q$ are distinct complex numbers, and the positive integers m_j satisfy (19).

We first show that for each $j = 1, \dots, q$, the function $y_0(z) - a_j$ must have infinitely many zeros in every neighborhood of ∞. This is very easy to prove, since under the change of variable $v = 1/(y - a_j)$, equation (31) becomes,

$$(-1)^m (v')^m = R(z)\, v^{2m-(m_1+\dots+m_q)} \prod_{k \neq j} (1 + (a_j - a_k)\, v)^{m_k} \tag{32}$$

The degree in v of the right side of (32) is $2m - m_j$ which by (19) cannot equal m. Hence equation (32) has only one nontrivial term of maximal total degree in v, v', and thus by the Wiman–Valiron theory (§ 3(c)), the Nevanlinna characteristic of any analytic solution of (32) in a neighborhood of ∞ must be $O(\log r)$ as $r \to \infty$, which proves the assertion in view of (3).

If $j \in \{1, \dots, q\}$, and z_1 is a zero of $y_0(z) - a_j$ of order λ_j whose modulus is sufficiently large, then from (31), y_0' vanishes at z_j so $\lambda_j > 1$, and $m(\lambda_j - 1) = m_j \lambda_j$. Hence, for each $j = 1, \dots, q$,

$$(m - m_j)\lambda_j = m, \quad 1 \leqslant m_j < m, \quad \text{and} \quad m_j \geqslant m/2. \tag{33}$$

We now distinguish two cases. Suppose first that $m_1 + \dots + m_q = m$. In this case, it follows from (33), that $q \leqslant 2$. Clearly $q = 2$, or otherwise $m_1 = m$ contradicting (19). Since $m_j \geqslant m/2$, it follows that $m_1 = m_2 = m/2$, and hence (31) has the form,

$$(y')^{2m_2} = R(z)(y-a_1)^{m_2}(y-a_2)^{m_2}. \tag{34}$$

If we set $v_0(z) = 1/(y_0(z) - a_1)$, then v_0 would satisfy the differential equation,

$$(v')^{2m_2} = R(z)(a_1 - a_2)^{m_2} v^{2m_2} (v - (1/(a_2 - a_1)))^{m_2}. \tag{35}$$

Of course, this is an equation of the form (25), and for such equations we proved that the order of growth of any solution $v(z)$ for which $T(r, v) \not\equiv O(\log r)$ as $r \to \infty$, must be a positive integral multiple of $\frac{1}{2}$. In view of our assumption (3), it would follow that the order of growth of $v_0(z)$, and hence of $y_0(z)$, would be a positive integral multiple of $\frac{1}{2}$, contradicting our assumption that conclusion (a) fails to hold.

Hence $m_1 + \ldots + m_q \neq m$, and so by (19) we must have,

$$m < m_1 + \ldots + m_q \leqslant 2m. \tag{36}$$

Thus equation (31) possesses only one term of maximal total degree in y, y', so in view of assumption (3) and the Wiman–Valiron theory (§ 3(c)), $y_0(z)$ cannot be analytic in some neighborhood of ∞, so $y_0(z)$ must have infinitely many poles in every neighborhood of ∞. If z_2 is a pole of $y_0(z)$ of order s whose modulus is sufficiently large, then from (31),

$$(s+1)m = (m_1 + \ldots + m_q)s. \tag{37}$$

Now from the last relation in (33) and the second inequality in (36), it follows that $q \leqslant 4$ (with equality holding only if $m_1 + \ldots + m_q = 2m$). Hence, if $m_1 + \ldots + m_q = 2m$, then in view of (33), equation (31) is already in the desired form (29), and we may take $t = q$, $b_j = a_j$, $r_j = m_j$, $R_1 = R$, and $y_1 = y_0$.

Thus we need only consider the case $m_1 + \ldots + m_q < 2m$ (in view of (36)). In this case, set $t = q+1$ (so $t \leqslant 4$), choose a complex number $a_t \notin \{a_1, \ldots, a_q\}$, and set $y_1 = 1/(y_0 - a_t)$. It is easily verified that y_1 is a solution of an equation of the form (29), where,

$$R_1 = (-1)^{-m} R(a_t - a_1)^{m_1} \ldots (a_t - a_q)^{m_q}, \tag{38}$$

$b_j = 1/(a_j - a_t)$ for $1 \leqslant j \leqslant q$, while $b_t = 0$, and where,

$$r_j = m_j \quad \text{for} \quad 1 \leqslant j \leqslant q, \quad \text{while} \quad r_t = 2m - (m_1 + \ldots + m_q). \tag{39}$$

Since we are assuming $m_1 + \ldots + m_q < 2m$, it follows (using (36) and (37)) that $1 \leqslant r_t < m$, $m = s(m - r_t)$, and $s > 1$. In view of (33), it now follows that the conditions (30) are all satisfied proving Lemma 4.

Before proceeding to solve equation (29), we require a simple result concerning elliptic functions. We recall that the *order* of an elliptic function $w(z)$ (which we will call the *elliptic order* of $w(z)$ to distinguish it from the order of growth of $w(z)$) is the number of poles (counting multiplicity) of $w(z)$ lying in the fundamental parallelogram. (Of course, we use the convention that if δ_1, δ_2 are primitive periods for $w(z)$, then the fundamental parallelogram consists of the interior of the parallelogram with vertices at 0, δ_1, δ_2, $\delta_1 + \delta_2$, together with the vertex 0, and the two sides intersecting at 0, but without the endpoints δ_1 and δ_2.) It is well-known (e.g. [15, p. 366]) that if $w(z)$ is of elliptic order q, then $w(z)$ assumes every complex value exactly q times in the fundamental parallelogram.

LEMMA 5. *Let $G(w)$ be a polynomial having constant coefficients, and let $w(z)$ be a nonconstant elliptic function of elliptic order q, which is a solution of the differential equation $(w')^q = G(w)$. Then:*

(a) *If c_0 and c_1 are complex numbers satisfying $c_1^q = G(c_0)$, then there exists a complex number ζ such that $w(\zeta) = c_0$ and $w'(\zeta) = c_1$.*

(b) *Any solution of the differential equation $(w')^q = G(w)$ which is meromorphic and nonconstant in a region of the plane must be of the form $w(z+K)$ where K is a constant.*

Proof. Part (a): If c_0 is a root of $G(w)$, and if ζ is a point for which $w(\zeta) = c_0$, then clearly $w'(\zeta) = 0 = c_1$. Hence we may assume that $G(c_0) \neq 0$. Then, from the differential equation it follows that all roots of the equation, $w(z) = c_0$ are simple, and hence there are q distinct roots $z_1, \ldots, z_q$ of $w(z) = c_0$ in the fundamental parallelogram. Since $c_1 \neq 0$, the equation, $y^q - c_1^q = 0$, has q distinct nonzero solutions for y, say $c_1, \ldots, c_q$. Assume that $w'(z_j) \neq c_1$ for $j \in \{1, \ldots, q\}$. From the differential equation it follows that for each j, $(w'(z_j))^q = c_1^q$, and hence from our assumption, the value of $w'(z_j)$ is one of the $q-1$ numbers $c_2, \ldots, c_q$. Thus for at least two distinct values of j (say $j = r$ and $j = n$), we have $w'(z_j) = c_k$ for some $k \in \{2, \ldots, q\}$, so that

$$w(z_r) = c_0 = w(z_n) \quad \text{and} \quad w'(z_r) = c_k = w'(z_n). \tag{40}$$

Then if we set,

$$w_1(z) = w(z + z_n - z_r), \tag{41}$$

it easily follows from (40) that $w(z)$ and $w_1(z)$ are both solutions of the initial-value problem,

$$w'' = (w')^2 G'(w)/qG(w), \quad w(z_r) = c_0, \quad w'(z_r) = c_k, \tag{42}$$

and hence must coincide by the standard uniqueness theorem for ordinary differential equations (e.g. [2, p. 19]) since the right side of the differential equation in (42) is analytic as a function of (w, w') around (c_0, c_k). Hence $z_n - z_r$ is a period of $w(z)$ which obviously contradicts the fact that z_n and z_r are distinct numbers both lying in the fundamental parallelogram. This contradiction proves that $w'(z_j) = c_1$ for some $j \in \{1, \ldots, q\}$ and we may take $\zeta = z_j$ proving Part (a).

Part (b): If $w_0(z)$ is another solution of the differential equation $(w')^q = G(w)$, which is meromorphic and nonconstant in a region D, then obviously there exists a point $z_r \in D$

such that $c_0 = w_0(z_r)$ is not ∞ or a root of $G(w)$. Setting $c_k = w_0'(z_r)$, we have $c_k^q = G(c_0)$ so by Part (a), there is a complex number z_n such that $w(z_n) = c_0$ and $w'(z_n) = c_k$. Hence if we define $w_1(z)$ by (41), then clearly $w_0(z)$ and $w_1(z)$ are both solutions of the analytic initial-value problem (42) and thus must coincide as in the proof of Part (a). This proves Part (b).

LEMMA 6. *Let m, $Q(z, y)$, and $y_0(z)$ be as in the statement of Theorem 1. Then, at least one of the following holds:*

(a) *The order of growth of $y_0(z)$ is a positive integral multiple of $\frac{1}{2}$,*

(b) *There exist constants a_1, b_1, c_1, d_1, with $a_1 d_1 - b_1 c_1 \neq 0$, such that if $y_2 = (a_1 y_0 + b_1)/(c_1 y_0 + d_1)$, then $y_2(z)$ satisfies a differential equation having one of the following forms:*

$$(y')^2 = R_2(z)(y - e_1)(y - e_2)(y - e_3), \tag{43}$$

$$(y')^3 = R_2(z)(y - \beta)^2 (y + \beta)^2, \tag{44}$$

$$(y')^4 = R_2(z)(y - \beta)^2 y^3, \tag{45}$$

$$(y')^6 = R_2(z)(y - \beta)^3 y^4. \tag{46}$$

Here, R_2 is a nontrivial element of $\mathcal{H}$, the e_j are distinct constants whose sum is zero, and β is a nonzero constant.

Furthermore, in Case (b), *there exist primitive periods δ_1, δ_2, for the Weierstrass $\wp$-function $\wp(z; \delta_1, \delta_2)$, and a function $g(z)$ which is analytic in a slit region, $D = \{z: |z| > K, \arg z \neq \pi\}$ for some $K > 0$, such that the following hold:*

(A) *If $y_2(z)$ satisfies* (43), *then $(g'(z))^2 = R_2(z)/4$ and $y_2(z) = \wp(g(z); \delta_1, \delta_2)$.*

(B) *If $y_2(z)$ satisfies* (44), *then $(g'(z))^3 = 2R_2(z)/27$ and $y_2(z) = \wp'(g(z); \delta_1, \delta_2)$.*

(C) *If $y_2(z)$ satisfies* (45), *then $(g'(z))^4 = R_2(z)/4^4$ and $y_2(z) = \wp^2(g(z); \delta_1, \delta_2)$.*

(D) *If $y_2(z)$ satisfies* (46), *then $(g'(z))^6 = R_2(z)/6^6$ and $y_2(z) = \wp^3(g(z); \delta_1, \delta_2)$.*

Proof. We assume that the order of growth of $y_0(z)$ is not a positive integral multiple of $\frac{1}{2}$. Then from Lemma 4, we know that some linear fractional transform y_1 of y_0 satisfies an equation of the form (29) where (30) is satisfied. Since $r_j \geq m/2$, it follows easily that t is either 3 or 4, and we distinguish these possibilities.

Assume first that $t = 4$. It follows from (30), that $\lambda_1^{-1} + \ldots + \lambda_4^{-4} = 2$, and since the λ_j are integers exceeding 1, we must have $\lambda_j = 2$ for $j = 1, 2, 3, 4$. Hence $r_j = m/2$ for each j so m is even, and if we set $r = m/2$, then y_1 satisfies the differential equation,

$$(y')^{2r} = R_1(z)(y - b_1)^r (y - b_2)^r (y - b_3)^r (y - b_4)^r. \tag{47}$$

If we set $R_3 = (y_1')^2/(y_1 - b_1) \dots (y_1 - b_4)$, then $R_3(z)$ is meromorphic in a neighborhood of ∞, and $R_3^r = R_1$. Hence R_3 is a nontrivial element of $\mathcal{H}$, and y_1 satisfies the differential equation,

$$(y')^2 = R_3(z)(y - b_1)(y - b_2)(y - b_3)(y - b_4). \tag{48}$$

If we set $y_2 = (y_1 - b_4)^{-1} - \sum_{j=1}^{3} (b_j - b_4)^{-1}/3$, then it is easily verified that y_2 satisfies a differential equation of the form (43), where R_2 is a nontrivial element of $\mathcal{H}$ and where the e_j are distinct constants whose sum is zero. In view of this latter condition it is well-known (e.g. [15, pp. 403–404]) that there exist a pair of primitive periods δ_1, δ_2 such that the Weierstrass $\wp$-function $\wp(z) = \wp(z; \delta_1, \delta_2)$ satisfies the equation,

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \tag{49}$$

Now let $K > 0$ be so large that $y_2(z)$ is meromorphic on $|z| > K$, and $R_2(z)$ is analytic and nowhere zero on $|z| > K$. With the region D as in the statement of the lemma, there exists an analytic branch of $(R_2(z)/4)^{\frac{1}{2}}$ on D. Let $g_1(z)$ denote a primitive of this branch on D, so that $(g_1'(z))^2 = R_2(z)/4$. Choose a point $z_0 \in D$ so that $b_0 = y_2(z_0)$ does not belong to the set $\{e_1, e_2, e_3, \infty\}$, and set $b_1 = y_2'(z_0)/g_1'(z_0)$. Then from (43), we have $b_1^2 = 4(b_0 - e_1)(b_0 - e_2)(b_0 - e_3)$. In view of (49) and the fact that $\wp(z)$ is of elliptic order 2, it follows from Lemma 5, that there is a point z_1 such that $\wp(z_1) = b_0$ and $\wp'(z_1) = b_1$. Now for $z \in D$, set

$$g(z) = g_1(z) + z_1 - g_1(z_0), \tag{50}$$

and $y_3(z) = \wp(g(z))$. Then from (49), it easily follows that $y_3(z)$ also satisfies the differential equation (43) on D, and clearly,

$$y_3(z_0) = b_0 = y_2(z_0) \quad \text{and} \quad y_3'(z_0) = y_2'(z_0). \tag{51}$$

By our choice of b_0 and b_1, there exists an analytic branch $F(u)$ of $(4(u - e_1)(u - e_2)(u - e_3))^{\frac{1}{2}}$ in a neighborhood of $u = b_0$, such that $F(b_0) = b_1$. From (43) and (51), it now easily follows that $y_2(z)$ and $y_3(z)$ are both solutions of the analytic initial value problem,

$$y' = g'(z)F(y), \quad y(z_0) = b_0, \tag{52}$$

and hence must coincide by the uniqueness theorem for ordinary differential equations. This proves the representation described in Part (A).

We now consider the case where $t = 3$ in (29) and (30). Then $\lambda_1^{-1} + \lambda_2^{-1} + \lambda_3^{-1} = 1$, and by renumbering if necessary, we may assume $\lambda_1 \leq \lambda_2 \leq \lambda_3$. It is clearly not possible for λ_1 to exceed 3, so λ_1 is either 2 or 3. If $\lambda_1 = 2$, then $\lambda_2^{-1} + \lambda_3^{-1} = \frac{1}{2}$. Clearly then $\lambda_2 > 2$. If $\lambda_2 = 3$, then $\lambda_3 = 6$, while if $\lambda_2 = 4$, then $\lambda_3 = 4$. It is clearly not possible for λ_2 to exceed 4 if $\lambda_1 = 2$. Secondly, if $\lambda_1 = 3$, then $\lambda_2^{-1} + \lambda_3^{-1} = \frac{2}{3}$. If $\lambda_2 = 3$ then $\lambda_3 = 3$. It is clearly not possible for λ_2 to exceed 3 if $\lambda_1=3$. Hence the possibilities for $(\lambda_1, \lambda_2, \lambda_3)$ in (30) are (3, 3, 3), (2, 4, 4), and (2, 3, 6), and we consider each case separately.

Suppose first that $(\lambda_1, \lambda_2, \lambda_3)=(3, 3, 3)$. Then each r_j in (29) is $2m/3$ so that m is a multiple of 3. Hence if we set $r=m/3$, then $y_1(z)$ satisfies the equation,

$$(y')^{3r} = R_1(z)(y-b_1)^{2r}(y-b_2)^{2r}(y-b_3)^{2r}. \tag{53}$$

Setting $R_3=(y_1')^3/(y_1-b_1)^2(y_1-b_2)^2(y_1-b_3)^2$, it follows that $R_3(z)$ is meromorphic in a neighborhood of ∞, and $R_3^r=R_1$. Hence R_3 is a nontrivial element of $\mathcal{H}$, and y_1 satisfies the differential equation,

$$(y')^3 = R_3(z)(y-b_1)^2(y-b_2)^2(y-b_3)^2. \tag{54}$$

If we set $y_2=(y_1-b_3)^{-1}-\sum_{j=1}^{2}(b_j-b_3)^{-1}/2$, then it is easily verified that y_2 satisfies an equation of the form (44) where R_2 is a nontrivial element of $\mathcal{H}$, and β is a nonzero constant. In view of this later condition, it follows from well-known results (e.g. [15, p. 403]) that there exist primitive periods δ_1, δ_2 such that $\wp(z)=\wp(z;\delta_1\,\delta_2)$ satisfies the differential equation, $(\wp')^2=4\wp^3+\beta^2$. Hence $\wp'(z)$ satisfies the equation,

$$(\wp''(z))^3 = (27/2)(\wp'(z)-\beta)^2(\wp'(z)+\beta)^2. \tag{55}$$

Choosing K sufficiently large as before, there exists an analytic function $g_1(z)$ on D such that $(g_1')^3=2R_2/27$. Choose a point $z_0\in D$ such that $b_0=y_2(z_0)$ does not belong to the set $\{\beta, -\beta, \infty\}$, and again set $b_1=y_2'(z_0)/g_1'(z_0)$. Then in view of (44), (53), and the fact that $\wp'$ is of elliptic order 3, it follows from Lemma 5 that there is a point z_1 such that $\wp'(z_1)=b_0$ and $\wp''(z_1)=b_1$. Setting $y_3(z)=\wp'(g(z))$, where $g(z)$ is defined by (50), it easily follows that $y_2(z)$ and $y_3(z)$ are both solutions of equation (44) and that (51) holds. Hence if $F(u)$ denotes the analytic branch of $(27(u-\beta)^2(u+\beta)^2/2)^{1/3}$ around $u=b_0$, satisfying $F(b_0)=b_1$, then it is easily verified that $y_2(z)$ and $y_3(z)$ are both solutions of the initial-value problem (52) and thus coincide. This proves the representation described in Part (B).

Now assume $(\lambda_1, \lambda_2, \lambda_3)=(2, 4, 4)$ in (30). Then it easily follows from (29) and (30) that m is a multiple of 4, and that y_1 satisfies a differential equation,

$$(y')^4 = R_3(z)(y-b_1)^2(y-b_2)^3(y-b_3)^3, \tag{56}$$

where R_3 is a nontrivial element of $\mathcal{H}$, and the b_j are distinct constants. Then if we set, $y_2=(y_1-b_3)^{-1}-(b_2-b_3)^{-1}$, it is easy to verify that y_2 satisfies a differential equation of the form (45), where R_2 is a nontrivial element of $\mathcal{H}$, and β is a nonzero constant. From this latter condition, it follows as before that there exist primitive periods δ_1, δ_2, such that $\wp(z) = \wp(z; \delta_1, \delta_2)$ satisfies the equation, $(\wp')^2 = 4\wp^3 - 4\beta\wp$. Hence $G = \wp^2$ satisfies,

$$(G')^4 = 4^4(G-\beta)^2 G^3. \tag{57}$$

For $K>0$ sufficiently large, there exists an analytic function $g_1(z)$ on D such that $(g_1')^4 = R_2/4^4$. Choose $z_0 \in D$ such that $b_0 = y_2(z_0)$ does not belong to the set $\{0, \beta, \infty\}$, and set $b_1 = y_2'(z_0)/g_1'(z_0)$. Since G is of elliptic order 4, it follows from (45), (57) and Lemma 5 that for some point z_1, we have $G(z_1) = b_0$ and $G'(z_1) = b_1$. Setting $y_3(z) = G(g(z))$, where $g(z)$ is defined by (50), it follows that $y_3(z)$ also satisfies (45), and that (51) holds. Then if $F(u)$ is the analytic branch of $(4^4(u-\beta)^2u^3)^{1/4}$ around $u = b_0$ satisfying $F(b_0) = b_1$, it is easy to see that $y_2(z)$ and $y_3(z)$ are both solutions of the initial-value problem (52) and thus coincide. This proves the representation described in Part (C).

The only remaining possibility in (30) is that $(\lambda_1, \lambda_2, \lambda_3) = (2, 3, 6)$. It easily follows that m is a multiple of 6, and that $y_1(z)$ satisfies an equation,

$$(y')^6 = R_3(z)(y-b_1)^3(y-b_2)^4(y-b_3)^5, \tag{58}$$

where R_3 is a nontrivial element of $\mathcal{H}$, and the b_j are distinct constants. If we set $y_2 = (y_1 - b_3)^{-1} - (b_2 - b_3)^{-1}$, then it is easily verified that y_2 satisfies an equation of the form (46), where R_2 is a nontrivial element of $\mathcal{H}$ and the constant β is nonzero. As before, there exist primitive periods δ_1, δ_2 such that $\wp(z) = \wp(z; \delta_1, \delta_2)$ satisfies the equation, $(\wp')^2 = 4\wp^3 - 4\beta$. It easily follows that $G_1 = \wp^3$ astisfies the differential equation $(G_1')^6 = 6^6(G_1 - \beta)^3 G_1^4$. If $K>0$ is sufficiently large, let $g_1(z)$ be an analytic function on D such that $(g_1')^6 = R_2/6^6$. Choosing a point $z_0 \in D$ such that $b_0 = y_2(z_0)$ does not belong to the set $\{0, \beta, \infty\}$, and setting $b_1 = y_2'(z_0)/g_1'(z_0)$, it follows from Lemma 5 that for some point z_1, we have $G_1(z_1) = b_0$ and $G_1'(z_1) = b_1$. Setting $y_3(z) = G_1(g(z))$ where $g(z)$ is defined by (50), it is easy to see that $y_3(z)$ satisfies equation (46) and the conditions (51). Then if $F(u)$ is the analytic branch of $(6^6(u-\beta)^3u^4)^{1/6}$ around $u = b_0$ satisfying $F(b_0) = b_1$, it now follows easily that $y_2(z)$ and $y_3(z)$ both satisfy the initial-value problem (52) and hence must coincide. This proves the representation described in Part (D), and concludes the proof of Lemma 6.

In order to compute the order of growth of the function $y_2(z)$ in Lemma 6, we require the following result.

LEMMA 7. *Let D be a region of the form, $\{z: |z| > K, \arg z \neq \pi\}$ for some $K>0$. Let $g(z)$ be an analytic function in D such that as $z \to \infty$ in D,*

$$g'(z) = cz^{\alpha}(1+o(1)) \quad \text{and} \quad g''(z) = z^{\alpha-1}(c\alpha + o(1)), \tag{59}$$

for some constants α *and* c, *with* $\alpha > -1$ *and* $c \neq 0$. *Let* $w_0(z)$ *be a nonconstant elliptic function and assume that* $y(z) = w_0(g(z))$ *is meromorphic in a neighborhood of* ∞. *Then for some constants* $K_1 > 0$ *and* $K_2 > 0$, *the inequalities* $n(r, \infty, y) \geqslant K_1 r^{2+2\alpha}$ *and* $T(r, y) \geqslant K_2 r^{2+2\alpha}$ *hold for all sufficiently large* r.

Proof. Choose a constant $B > 0$ which is greater than the length of the longer diagonal of the fundamental parallelogram for $w_0(z)$, and set $A = (2+2B)/|c|$, where c is as in (59). For a point z_0 in the right half-plane, with $|z_0| = r$, let $D(z_0)$ denote the closed disk, $|z - z_0| \leqslant Ar^{-\alpha}$. For $\zeta \in D(z_0)$, clearly,

$$r - Ar^{-\alpha} \leqslant |\zeta| \leqslant r + Ar^{-\alpha}, \tag{60}$$

and since $\alpha > -1$, it easily follows from the first inequality in (60) that if r is sufficiently large, then $D(z_0)$ lies in the slit region D so that the estimates (59) are valid on $D(z_0)$. From (59) and (60), we see that

$$|g''(\zeta)| \leqslant (|\alpha|\ |c| + 1)(r \pm Ar^{-\alpha})^{\alpha-1} \quad \text{on } D(z_0), \tag{61}$$

if $r = |z_0|$ is sufficiently large (where the plus sign is used if $\alpha \geqslant 1$, while the minus sign is used if $-1 < \alpha < 1$). Since the radius of $D(z_0)$ is $Ar^{-\alpha}$, and since $\alpha > -1$, we see from (61) that if $r = |z_0|$ is sufficiently large, then

$$|g'(z) - g'(z_0)| \leqslant 2\mathrm{A}(|\alpha|\ |c| + 1)r^{-1} \quad \text{for } z \in D(z_0). \tag{62}$$

For fixed z_0, define the function $h(z)$ on $D(z_0)$ by

$$g(z) = g(z_0) + (z - z_0)g'(z_0) + h(z), \tag{63}$$

so that $h'(z) = g'(z) - g'(z_0)$ and $h(z_0) = 0$. In view of (62), we see that if $r = |z_0|$ is sufficiently large, then

$$|h(z)| \leqslant 2A^2(|\alpha|\,|c| + 1)r^{-1-\alpha} \quad \text{for } z \in D(z_0). \tag{64}$$

Let w be a point in the disk $|w - g(z_0)| \leqslant B$, and write

$$g(z) - w = f(z) + h(z), \tag{65}$$

where (from (63)), $f(z) = g(z_0) - w + (z - z_0)g'(z_0)$ on $D(z_0)$. In view of the first estimate in (59) and the definition of A, it easily follows that if $r = |z_0|$ is sufficiently large, then on the boundary of $D(z_0)$ we have $|f(z)| \geqslant 1$, and hence in view of (64), $|f(z)| > |h(z)|$ since $\alpha > -1$. Since it is easy to see that the linear function $f(z)$ has its zero inside $D(z_0)$, it follows from Rouché's theorem (and (65)) that $g(z) - w$ has a zero inside $D(z_0)$ if

$|w-g(z_0)| \leqslant B$. Thus we have shown that if z_0 is a point in the right half-plane with $r = |z_0|$ sufficiently large, then the image under $g(z)$ of the interior of $D(z_0)$ contains the disk $|w-g(z_0)| \leqslant B$. By definition of B, the latter disk must contain a pole of the elliptic function w_0, and hence $y(z) = w_0(g(z))$ has a pole on the interior of $D(z_0)$. Thus clearly, if $q_1(r)$ denotes the maximum number of disjoint open disks of the form $|z-z_0| < A|z_0|^{-\alpha}$ which lie in the set J_r defined by Re $(z) \geqslant 0$, $r/2 \leqslant |z| \leqslant r$, then for all sufficiently large r we have $n(r, \infty, y) \geqslant q_1(r)$. Since the radius $t(z_0)$ of each such disk clearly satisfies $t(z_0) \leqslant K_0 r^{-\alpha}$, where $K_0 = \max\{A, 2^\alpha A\}$, it suffices to compute the maximum number $q(r)$ of disjoint open disks of radius $s = K_0 r^{-\alpha}$ which lie in J_r, for then $q_1(r) \geqslant q(r)$. Let $D_1, \ldots, D_{q(r)}$ be disjoint open disks of radius s lying in J_r, and let z_j be the center of D_j. Then clearly, if I_r denotes the set defined by Re $(z) \geqslant s$, $(r/2)+s < |z| < r-s$, and if $z \in I_r$, then the open disk of radius s around z clearly lies in J_r, and hence by the definition of $q(r)$ must have a point in common with some D_j. It follows that $|z-z_j| < 2s$, and hence the disks of radius $2s$ around $z_1, \ldots, z_{q(r)}$ cover I_r. Hence the area of I_r must be at most $4\pi s^2 q(r)$. But since $\alpha > -1$, an elementary estimate on the area of I_r shows that this area exceeds $c_1 r^2$ for some fixed $c_1 > 0$ if r is sufficiently large, and hence $q(r)$ exceeds $(c_1/4\pi K_0^2) r^{2+2\alpha}$. Since $n(r, \infty, y) \geqslant q(r)$, the conclusions of the lemma now follow immediately.

LEMMA 8. *Let m, $Q(z, y)$, and $y_0(z)$ be as in the statement of Theorem 1. Assume that Case* (b) *in Lemma* 6 *holds, and let $y_2(z)$ and $R_2(z)$ be as in that case. Let the Laurent expansion of $R_2(z)$ around ∞ be,*

$$R_2(z) = c_0 z^d + c_1 z^{d-1} + \ldots, \quad \text{with } c_0 \neq 0. \tag{66}$$

Then the following are true:

(A) *If $y_2(z)$ satisfies equation* (43), *then $d \geqslant -2$, and both $y_0(z)$ and $y_2(z)$ have order of growth equal to $d+2$.*

(B) *If $y_2(z)$ satisfies equation* (44), *then $d \geqslant -3$, and both $y_0(z)$ and $y_2(z)$ have order of growth equal to* $(2d/3)+2$.

(C) *If $y_2(z)$ satisfies equation* (45), *then $d \geqslant -4$, and both $y_0(z)$ and $y_2(z)$ have order of growth equal to* $(d/2)+2$.

(D) *If $y_2(z)$ satisfies equation* (46), *then $d \geqslant -6$, and both $y_0(z)$ and $y_2(z)$ have order of growth equal to* $(d/3)+2$.

In all of the four cases (A), (B), (C), (D), *above, if λ denotes the order of growth of $y_0(z)$, then the following hold:*

(a) *If $\lambda = 0$, then $T(r, y_0) = O(\log^2 r)$ as $r \to \infty$.*

(b) *If $\lambda > 0$, then λ is either a positive integral multiple of $\frac{1}{2}$ or $\frac{1}{3}$, and in addition, there are positive constants K_1 and K_2 such that*

$$K_1 r^\lambda \leqslant T(r, y_0) \leqslant K_2 r^\lambda \quad \textit{for all sufficiently large } r. \tag{67}$$

Proof. Each of the equations (43)–(46) are of the form $(y')^q = R_2(z) G(y)$, where q is 2, 3, 4, or 6 respectively, and $G(y)$ has constant coefficients. It follows from [4, Th. 4] or [1, §§ 3, 4], that if $R_2(z)$ has the Laurent expansion (66) around ∞, then as $r \to \infty$, (i) $T(r, y_2) = O(\log r)$ if $d/q < -1$, (ii) $T(r, y_2) = O(\log^2 r)$ if $d/q = -1$, and (iii) $T(r, y_2) = O(r^{2(d/q)+2})$ if $d/q > -1$. Since $T(r, y_0) = T(r, y_2) + O(\log r)$ as $r \to \infty$, it follows from assumption (3) on y_0, that (i) cannot hold so $d/q \geqslant -1$. If $d/q = -1$, then by (ii), y_2 and y_0 have zero order of growth, and conclusion (a) holds. Assume now $d/q > -1$. From the representations (A)–(D) in Lemma 6, the function $y_2(z)$ is of the form $w_0(g(z))$, where w_0 is a nonconstant elliptic function, and where $g(z)$ is analytic in a slit region $\{z: |z| > K, \arg z \neq \pi\}$, and in view of (66), both g' and g'' possess expansions of the form (59) with $a = d/q$. Hence by Lemma 7, together with (iii), we conclude that y_2 and hence y_0 have order of growth precisely $2(d/q)+2$, and (67) holds with $\lambda = 2(d/q)+2$. Finally, since $q \in \{2, 3, 4, 6\}$, clearly λ is either an integral multiple of $\frac{1}{2}$ or $\frac{1}{3}$, which concludes the proof of Lemma 8.

Proof of Theorem 1. The first conclusion of Theorem 1 is contained in Lemmas 6 and 8.

For the second conclusion, we note first that it was shown in [1, § 5] that the function, $y_0(z) = \wp(\log((z + (z^2-4)^{\frac{1}{2}})/2;\ 1,\ 2\pi i)$, is a transcendental meromorphic function in the plane whose order of growth is zero, and satisfies the differential equation,

$$(y')^2 = (z^2-4)^{-1}(4y^3 - g_2 y - g_3), \tag{68}$$

where g_2 and g_3 are the invariants for $\wp(z; 1, 2\pi i)$.

Now let n be a positive integer. As in the last case of Lemma 6, there exist primitive periods δ_1, δ_2 such that $G_1(z) = \wp^3(z; \delta_1, \delta_2)$ satisfies the equation, $(G_1')^6 = 6^6(G_1 + (\frac{1}{4}))^3 G_1^4$. Set $G_2(z) = G_1(z/3)$ so that $(G_2')^6 = (4G_2+1)^3 G_2^4$. But if $G_3(z) = G_2(e^{i\pi/3} z)$, then also $(G_3')^6 = (4G_3+1)^3 G_3^4$. Since G_2 has elliptic order 6, it follows from Lemma 5, Part (b), that for some constant K, $G_3(z) \equiv G_2(z+K)$. Evaluating at $z=0$, we see that $K/3$ is a pole of $\wp(z; \delta_1, \delta_2)$, and hence K is a period of $G_2(z)$. Thus $G_2(e^{i\pi/3} z) \equiv G_2(z)$, and it easily follows that $G_4(\zeta) = G_2(\zeta^{1/6})$ is single-valued on $|\zeta| < \infty$. Clearly $G_4(\zeta)$ is meromorphic on $0 < |\zeta| < \infty$ since there exists an analytic branch of $\zeta^{1/6}$ in a neighborhood of any point $\zeta_0 \neq 0$, but in addition, it follows easily from the definition of $G_2(z)$, that $\zeta = 0$ is actually an isolated singularity of $G_4(\zeta)$, and is, in fact, a pole.

Hence $G_4(\zeta)$ is meromorphic in the plane, and $y_0(z) = G_4(z^n)$ satisfies the equation

$$(y')^6 = (n/3)^6 z^{n-6}(y + (\tfrac{1}{4}))^3 y^4, \tag{69}$$

which is an equation of the form (46). Since $y_0(z)$ clearly satisfies assumption (3), it follows from Lemma 8, Part (D), that y_0 is a transcendental meromorphic solution in the plane of equation (69), whose order of growth is precisely $n/3$, where n is any preassigned positive integer.

For transcendental meromorphic solutions of order $n/2$, we give three diverse examples of such solutions.

First, $y(z) = z^{-n/2} \tan z^{n/2}$ is a transcendental meromorphic solution of order $n/2$ of the Riccati equation,

$$y' = (n/2z) - (n/2z)y + (n/2)z^{n-1}y^2. \tag{70}$$

Secondly, $y_1(z) = \cos z^{n/2}$ is an entire transcendental solution of order $n/2$ of the equation,

$$(y')^2 = (n^2/4)z^{n-2}(1 - y^2). \tag{71}$$

Thirdly, we know that there exist primitive periods δ_1, δ_2, such that $G(z) = \wp^2(z; \delta_1, \delta_2)$ satisfies equation (57) with $\beta = \frac{1}{4}$. By an argument very similar to that used earlier for $\wp^3$, it is easy to see that the function $G_1(z) = G(z/4)$ satisfies the condition $G_1(z) \equiv G_1(iz)$. From this it follows that $y_0(z) = G_1(z^{n/4})$ is meromorphic on the plane, satisfies the differential equation,

$$(y')^4 = (n/4)^4 z^{n-4}(y - (\tfrac{1}{4}))^2 y^3, \tag{72}$$

and is of order of growth $n/2$ by Lemma 8, Part (C).

This concludes the proof of Theorem 1.

5. Remarks

The results in Lemmas 6 and 8 permit us to obtain a representation of those solutions of equation (2) which satisfy condition (3) and whose order of growth is not a positive integral multiple of $\frac{1}{2}$. We summarize these results now.

THEOREM 2. *Let m be a positive integer, and let $Q(z, y)$ be a polynomial in y whose coefficients belong to the field $\mathcal{H}$ described in § 3. Let $y_0(z)$ be a meromorphic function defined in a neighborhood of ∞ which satisfies the differential equation, $(y')^m = Q(z, y)$, and which has the property that $T(r, y_0) \neq O(\log r)$ as $r \to \infty$. Let λ denote the order of growth of $y_0(z)$, and assume that λ is not a positive integral multiple of $\frac{1}{2}$. Then there exist constants a_1, b_1, c_1, d_1, with $a_1 d_1 - b_1 c_1 \neq 0$, such that if $y_2 = (a_1 y_0 + b_1)/(c_1 y_0 + d_1)$, then the following are true:*

(a) *If* $\lambda=0$, *then* $y_2(z)$ *must have one of the forms described in* (A), (B), (C), (D), *of the statement of Lemma* 6, *where in the expansion* (66) *of* $R_2(z)$, *we have* $d=-2$, -3, -4, *or* -6, *depending respectively on the form* (A), (B), (C), *or* (D).

(b) *If* $\lambda>0$, *then* $y_2(z)$ *must have one of the forms described in* (B), (D), *of the statement of Lemma* 6, *where in the expansion* (66) *of* $R_2(z)$, *the integer* d *is not a multiple of* 3, *and* $d>-3$ *for form* (B), *while* $d>-6$ *for form* (D).

2. This remark concerns those solutions of equation (2) which are meromorphic in a neighborhood of ∞, which satisfy condition (3), and which have zero order of growth. In Lemma 8, it was shown that any such solution $y_0(z)$ satisfies the condition $T(r, y_0)=O(\log^2 r)$ as $r\to\infty$. We remark here that for any such solution, $T(r, y_0)\neq o(\log^2 r)$ as $r\to\infty$, which is in accord with the conjecture (still unproven) of the authors [1, p. 290] that arbitrary equations of the form $F(z, y, y')=0$, where F is a polynomial in all its arguments, cannot possess transcendental meromorphic solutions whose Nevanlinna characteristic is $o(\log^2 r)$ as $r\to\infty$. Although we will not give a detailed proof of this fact, we will outline the argument. As stated in Part (a) of Theorem 2, if $y_0(z)$ is a solution whose order of growth is zero, then some linear fractional transform y_2 of y_0 is of the form $w_0(g(z))$, where $w_0(z)$ is a nonconstant elliptic function, and $g(z)$ has the properties (59) where $\alpha=-1$. In this case, one can modify the proof of Lemma 7 to show that for all sufficiently large r, $T(r, y_2)\geqslant K_1\log^2 r$ where $K_1>0$ is fixed, and hence $T(r, y_0)\geqslant K_2\log^2 r$ where $K_2>0$ is fixed. To see this, we let A be a constant satisfying $0<A<\frac{1}{3}$, and $1-6A+A^2>0$, and we choose $B>0$ satisfying the condition, $B<(1-A)^{-2}(A-6A^2+A^3)|c|/2$ where c is as in (59). As in Lemma 7, we denote by $D(z_0)$, the disk $|z-z_0|\leqslant Ar$ where $|z_0|=r$ and z_0 belongs to the right half plane. Using the estimates (59) where $\alpha=-1$, and defining $h(z)$ by (63), we find that $|h(z)|\leqslant 2|c|A^2/(1-A)^2$ on $D(z_0)$. Decomposing $g(z)-w$ as in (65), it follows exactly as in the proof of Lemma 7 (by using Rouché's theorem and our choice of B), that if $r=|z_0|$ is sufficiently large, then the image under $g(z)$ of the interior of $D(z_0)$ contains the disk $|w-g(z_0)|\leqslant B$. Now subdivide the fundamental parallelogram for $w_0(z)$ by drawing lines parallel to its edges, into congruent parallelograms $\Omega_1, \ldots, \Omega_s$, whose longer diagonal has length less than B, and set $\varphi(\zeta)=\prod_{j=1}^{s}(\zeta-w_0(\zeta_j))^{-1}$, where ζ_j is the center of Ω_j. Clearly any disk of radius B contains a point of the form $\zeta_j+n_1\delta_1+n_2\delta_2$, where n_1, n_2 are integers and δ_1, δ_2 are primitive periods for $w_0(z)$. Hence from the mapping property of $g(z)$ proved above, it follows that if $r=|z_0|$ is sufficiently large, then $\varphi(y_2(z))$ has a pole on $D(z_0)$. Choosing z_0 to be of the form 2^k where k is a sufficiently large integer, it follows that for some fixed integer k_0, we have $n(2^{k+1}, \infty, \varphi(y_2(z)))\geqslant k-k_0$ if $k>k_0$: thus, $n(r, \infty, \varphi(y_2(z)))\geqslant K_3\log r$ for all sufficiently large r, where $K_3>0$ is fixed, and

hence $T(r, \varphi(y_2(z))) \geq K_4 \log^2 r$ where $K_4 > 0$ is fixed. Since $T(r, \varphi(y_2(z))) = sT(r, y_2) + O(\log r)$ as $r \to \infty$, our assertion easily follows.

6. An additional result

In this section, we consider the functions described in Parts (A)–(D) of Lemma 6, and we show that there always exist primitive periods δ_1, δ_2 such that these functions are actually meromorphic in a neighborhood of ∞.

THEOREM 3. *Let q be a positive integer, and let $R_2(z)$ be an analytic function in a neighborhood of ∞, which is not identically zero, and which has no essential singularity at ∞. Let the Laurent expansion of R_2 around ∞ be*

$$R_2(z) = b_0 z^d + b_1 z^{d-1} + \dots, \quad \text{for } |z| > K, \text{ where } b_0 \neq 0. \tag{73}$$

Let D be the region $\{z\colon |z| > K, \arg z \neq \pi\}$, and let $g(z)$ be an analytic function on D such that $(g'(z))^q = \alpha R_2(z)$ where α is a nonzero constant. Then:

(a) *If $w_0(z)$ is a nonconstant elliptic function with the property that the function $w_0(g(z))$ can be extended to be meromorphic in a neighborhood of ∞, then $e^{2\pi i d/q}$ must be either a fourth root of 1 or a sixth root of 1.*

(b) *If $q=2$, there always exist primitive periods δ_1, δ_2, such that each of the functions $\wp(g(z); \delta_1, \delta_2)$, $\wp^2(g(z); \delta_1, \delta_2)$, and $\wp^3(g(z); \delta_1, \delta_2)$ can be extended to be meromorphic in a neighborhood of ∞.*

(c) *If $q=3$, there always exist primitive periods δ_1, δ_2, such that both of the functions $\wp'(g(z); \delta_1, \delta_2)$ and $\wp^3(g(z); \delta_1, \delta_2)$ can be extended to be meromorphic in a neighborhood of ∞.*

(d) *If $q=4$, there always exist primitive periods δ_1, δ_2 such that $\wp^2(g(z); \delta_1, \delta_2)$ can be extended to be meromorphic in a neighborhood of ∞.*

(e) *If $q=6$, there always exist primitive periods δ_1, δ_2 such that $\wp^3(g(z); \delta_1, \delta_2)$ can be extended to be meromorphic in a neighborhood of ∞.*

Proof. Clearly $g'(z)$ possesses a convergent expansion,

$$g'(z) = z^{d/q} \sum_{j=0}^{\infty} c_j z^{-j} \text{ in } D, \text{ where } c_0 \neq 0, \tag{74}$$

and where $z^{d/q}$ denotes the principal branch of the power function in D. If d/q is not an integer, it follows that for some constant K_1, $g(z)$ possesses the convergent expansion in D,

$$g(z) = z^{d/q} \sum_{j=0}^{\infty} (c_j z^{-j+1}/(-j+1+(d/q))) + K_1, \tag{75}$$

since the infinite series in (75) converges for $|z| > K$.

To prove Part (a), set $\sigma = e^{2\pi i/q}$. If σ^d is real, then $\sigma^d = \pm 1$ and the conclusion holds. Hence we may assume σ^d is not real, and, in particular, d/q is not an integer, so (75) holds. Since the analytic continuation of $z^{d/q}$ once around the origin results in $\sigma^d z^{d/q}$, it follows from (75) that if $w_0(g(z))$ can be extended to be meromorphic in a neighborhood of ∞, then we must have $w_0(\zeta) \equiv w_0(\sigma^d \zeta + K_1(1-\sigma^d))$ on an open set in the plane, and hence everywhere. Thus if δ_1, δ_2 are primitive periods for $w_0(z)$, then $\sigma^d\delta_1$, $\sigma^d\delta_2$, $\sigma^{-d}\delta_1$, and $\sigma^{-d}\delta_2$ are also periods for $w_0(z)$. Hence there exist integers m_1, m_2, n_1, n_2, such that

$$\sigma^d\delta_1 = m_1\delta_1 + m_2\delta_2 \quad \text{and} \quad \sigma^d\delta_2 = n_1\delta_1 + n_2\delta_2, \tag{76}$$

and there exist integers M_1, M_2, N_1, N_2 such that,

$$\sigma^{-d}\delta_1 = M_1\delta_1 + M_2\delta_2 \quad \text{and} \quad \sigma^{-d}\delta_2 = N_1\delta_1 + N_2\delta_2. \tag{77}$$

It follows that

$$\sigma^{2d} - (m_1 + n_2)\sigma^d + m_1 n_2 - m_2 n_1 = 0 \tag{78}$$

and

$$\sigma^{-2d} - (M_1 + N_2)\sigma^{-d} + M_1 N_2 - M_2 N_1 = 0. \tag{79}$$

Since σ^d is assumed non-real, it easily follows from (78) and (79) that $(m_1 n_2 - m_2 n_1)(M_1 N_2 - M_2 N_1) = 1$ and $(m_1 + n_2)^2 < 4(m_1 n_2 - m_2 n_1)$. From these relations, we see that

$$m_1 n_2 - m_2 n_1 = 1, \tag{80}$$

and $(m_1 + n_2)^2 < 4$. Thus, if we set $k = m_1 + n_2$, then $k = 0$, -1, or 1, and $\sigma^d = (k \pm (k^2 - 4)^{1/2})/2$. If $k = 0$, then $\sigma^d = \pm i$ which are fourth roots of 1. If $k = -1$, then $\sigma^d = e^{\pm 2\pi i/3}$ which are cube roots of 1, while if $k = 1$, then $\sigma^d = e^{\pm i\pi/3}$ which are sixth roots of 1. This proves Part (a).

We next observe that if d/q is an integer, then from (74), we have $g(z) = h(z) + c\,(\operatorname{Log} z)$ on D, where $h(z)$ is analytic on $|z| > K$, c is a constant, and where $\operatorname{Log} z$ denotes the principal branch of the logarithm. It is clear that if primitive periods δ_1, δ_2 are chosen so that $2\pi i c$ is of the form $r_1\delta_1 + r_2\delta_2$, where r_1, r_2 are integers, then for any elliptic function $w_0(z)$ with these primitive periods, the function $w_0(g(z))$ is actually meromorphic in a neighborhood of ∞. Hence for the remainder of the proof, we can assume,

$$d/q \text{ is not an integer, so (75) holds.} \tag{81}$$

Now assume $q = 2$. In view of (81), $d/q = n + \frac{1}{2}$ for some integer n. Let δ_1, δ_2 be nonzero complex numbers with a nonreal ratio, such that

$$K_1 = r_1\delta_1 + r_2\delta_2, \quad \text{for some integers } r_1,\ r_2. \tag{82}$$

Then from (75) and the fact that $\wp(z; \delta_1, \delta_2)$ is an even function, the conclusion of Part (b) now follows.

Next suppose $q=3$. In view of (81), we have $d/q = n \pm \frac{1}{3}$, and thus $\sigma^d = e^{2\pi i d/q} = e^{\pm 2\pi i/3}$. We now assert that if δ_1, δ_2 are chosen to be nonzero complex numbers with a nonreal ratio, which satisfy condition (82) and which satisfy equations (76) for some integers m_1, m_2, n_1, n_2, then $\wp'(g(z); \delta_1, \delta_2)$ and $\wp^3(g(z); \delta_1, \delta_2)$ are both meromorphic in a neighborhood of ∞. (We observe that in view of the identity $\sigma^{2d} = -1 - \sigma^d$, such a pair (δ_1, δ_2) always exists, since we may take $(\delta_1, \delta_2) = (K_1, K_1\sigma^d)$ if $K_1 \neq 0$, and $(\delta_1, \delta_2) = (1, \sigma^d)$ if $K_1 = 0$.) To prove our assertion, we note that since $\sigma^{-d} = \sigma^{2d}$, it follows that if (76) holds, then so does (77) for some integers M_1, M_2, N_1, N_2. Hence (78) and (79) both hold, and thus (80) holds. It is well-known (e.g. [11, p. 125]) that this implies that $(\sigma^d\delta_1, \sigma^d\delta_2)$ is also a pair of primitive periods for $\wp(z; \delta_1, \delta_2)$ so that $\wp(z; \delta_1, \delta_2)$ coincides with $\wp(z; \sigma^d\delta_1, \sigma^d\delta_2)$ as functions of z. From the well-known fact (e.g. [15, p. 374]) that the $\wp$-function is homogeneous of degree -2 as a function of $(z; \delta_1, \delta_2)$ it then follows that

$$\wp(z; \delta_1, \delta_2) \equiv \sigma^{-2d}\wp(\sigma^{-d}z; \delta_1, \delta_2) \quad \text{as functions of } z. \tag{83}$$

Since $\sigma^{-3d} = 1$, we see that $\wp'(z; \delta_1, \delta_2)$ coincides with $\wp'(\sigma^{-d}z; \delta_1, \delta_2)$ as functions of z, and $\wp^3(z; \delta_1, \delta_2)$ coincides with $\wp^3(\sigma^{-d}z; \delta_1, \delta_2)$ as functions of z. Since the analytic continuation of the functions $z^{\pm 1/3}$ once around the origin, multiplies these functions by either σ^d or σ^{-d}, our assertion now follows easily from (75), (82), and the fact that $d/q = n \pm \frac{1}{3}$ in this case. This proves Part (c).

Assume now $q=4$, so in view of (81), either $d/q = n + \frac{1}{2}$ or $d/q = n \pm \frac{1}{4}$ for some integer n. In the first case, the conclusion of Part (d) follows from the proof of Part (b). In the second case, $\sigma^d = \pm i$, and as in the proof of Part (c), it follows using (83) that if (δ_1, δ_2) satisfies both (82) and equations (76) for some integers m_1, m_2, n_1, n_2, then $\wp^2(z; \delta_1, \delta_2)$ coincides with $\wp^2(\sigma^d z; \delta_1, \delta_2)$. (As in the proof of Part (c), such a pair (δ_1, δ_2) always exists.) Since $d/q = n \pm \frac{1}{4}$, it now easily follows from the representation (75) that $\wp^2(g(z); \delta_1, \delta_2)$ is meromorphic in a neighborhood of ∞, proving Part (d).

Finally, if $q=6$, then the cases $d/q = n + \frac{1}{2}$, and $d/q = n \pm \frac{1}{3}$ were covered in the proofs of Part (b) and Part (c) respectively. In view of (81), it suffices to consider only the cases $d/q = n \pm \frac{1}{6}$. Since $\sigma^d = e^{\pm \pi i/3}$, it follows exactly as in the proof of Part (c) that if (δ_1, δ_2) is again chosen to satisfy (82) and equations (76) for some choice of m_1, m_2, n_1, n_2, then using (83), the function $\wp^3(g(z); \delta_1, \delta_2)$ is meromorphic in a neighborhood of ∞. (As in Part (c), (δ_1, δ_2) can be taken to be $(K_1, K_1\sigma^d)$ if $K_1 \neq 0$, or $(1, \sigma^d)$ if $K_1 = 0$, since $\sigma^{2d} = \sigma^d - 1$ in this case.) This concludes the proof of Theorem 3.

Bibliography

[1] BANK, S. & KAUFMAN, R., On meromorphic solutions of first-order differential equations. *Comment. Math. Helv.*, 51 (1976), 289–299.

[2] BIEBERBACH, L., *Theorie der gewöhnlichen Differentialgleichungen*, 2nd ed., Band 66. Springer-Verlag, Berlin, 1965.

[3] GACKSTATTER, F. & LAINE, I., Zur theorie der gewöhnlichen Differentialgleichungen im Komplexen. To appear.

[4] GOL'DBERG, A., On single-valued solutions of first-order differential equations, (Russian). *Ukrain. Mat. Z.*, 8 (1956), 254–261.

[5] HAYMAN, W., *Meromorphic Functions*. Oxford Math. Monographs, Clarendon Press, Oxford, 1964.

[6] HILLE, E., On some generalizations of the Malmquist theorem. *Math. Scand.*, 39 (1976), 59–79.

[7] —— Finiteness of the order of meromorphic solutions of some nonlinear ordinary differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 72: 29 (1973/1974), 331–336.

[8] —— Remarks on Briot-Bouquet differential equations, I. *Comment. Math. Univ. Carolinæ*, XXI (1978), 113–132.

[9] —— Some remarks on Briot-Bouquet differential equations, II. *J. Math. Anal. Appl.*, 65 (1978), 572–585.

[10] —— *Ordinary Differential Equations in the Complex Domain*. Wiley-Interscience, New York, 1976.

[11] —— *Analytic Function Theory, Vol. II*. Ginn, Boston, Mass., 1962.

[12] LAINE, I., On the behavior of the solutions of some first-order differential equations. *Ann. Acad. Sci. Fenn., Ser. A I*, 497 (1971), 26 pp.

[13] —— Admissible solutions of some generalized algebraic differential equations. *Publ. Univ. Joensuu, Ser. B*, 10 (1974), 6 pp.

[14] NEVANLINNA, R., *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*. Gauthier-Villars, Paris, 1929.

[15] SAKS, S. & ZYGMUND, A., *Analytic Functions*. Monografie Mat., Tom 28 (Engl. transl.), Warsaw, 1952.

[16] STEINMETZ, N., Bermerkung zu einem Satz von Yosida. To appear.

[17] STRELITZ, SH., A remark on meromorphic solutions of differential equations. *Proc. Amer. Math. Soc.*, 65 (1977), 255–261.

[18] —— *Asymptotic Properties of Analytic Solutions of Differential Equations* (Russian). Vilnius, 1972.

[19] VALIRON, G., *Lectures on the General Theory of Integral Functions*. Edouard Privat, Toulouse, 1923.

[20] WITTICH, H., Zur Theorie der Riccatischen Differentialgleichung. *Math. Ann.*, 127 (1954), 433–450.

[21] —— Einige Eigenschaften der Lösungen von $w' = a(z) + b(z)w + c(z)w^2$. *Arch. Math.* (*Basel*), 5 (1954), 226–232.

[22] —— *Neuere Untersuchungen über eindeutige analytische Funktionen*. Ergebnisse der Math., Heft 8, Springer-Verlag, Berlin, 1955.

[23] —— Eindeutige Lösungen der Differentialgleichung $w' = R(z, w)$. *Math. Z.*, 74 (1960), 278–288.

[24] YOSIDA, K., A generalization of Malmquist's theorem. *Japan. J. Math.*, 9 (1933), 253–256.

[25] ZARISKI, O. & SAMUEL, P., *Commutative Algebra, Vol. I*. Van Nostrand, Princeton, N. J., 1958.

Received March 2, 1979