THE EXTERIOR NONSTATIONARY PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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I. Introduction

Although the exterior *stationary* problem for the Navier–Stokes equations has been proved by Leray [1] to possess a solution under very general circumstances, it is unknown even in the case of small data whether Leray's solution of the problem is unique or whether it may be formed as the limit of a nonstationary solution as $t\to\infty$. In this paper we prove that for a particular class of prescribed boundary values there is exactly one stationary solution attainable as the limit, starting from rest, of a physically reasonable nonstationary solution. Our method is based on a global existence theorem for the initial boundary value problem which we prove under hypotheses that allow time dependent boundary values and a time dependent velocity at infinity. This theorem assures the unique solvability of the initial boundary value problem whenever there is an approximate solution which is sufficiently good and satisfies a stability condition. This existence theorem has also enabled us to state simple conditions sufficient to ensure the stability of nonstationary solutions of the Navier–Stokes equations defined in arbitrary three-dimensional regions.

The Navier–Stokes equations govern fluid motion in the theory of viscous incompressible flow. The exterior stationary problem for the Navier–Stokes equations consists of finding, in the region exterior to a closed bounded surface, time independent velocity and pressure functions which together solve the equations and are such that the velocity function assumes given values on the surface and tends to a prescribed limit at infinity. Of course, stationary flow occurs in nature only as the limit of nonstationary flow. Presumably solutions of the exterior stationary problem model fluid flows which may be obtained by performing the following idealized experiment with the right choice of prescribed data. An object is immersed in a fluid which occupies all three-dimensional space exterior to it. Initially both the object and the fluid are at rest and subsequently the object undergoes a smooth acceleration to some constant velocity which is then maintained indefinitely. During the period of acceleration, time dependent boundary values may be prescribed for the fluid velocity at the surface of the object, but these are stabilized as the object attains constant velocity. Afterwards, with conditions held fixed, the fluid is expected to approach steady motion as seen by an observer moving with the body. In this paper we will call a solution of the exterior stationary problem attainable if it occurs as the limit as $t \to \infty$ of a nonstationary solution of the Navier–Stokes equations which models such an experiment. The precise definition of *attainable solution*, given in Section 6, is made without reference to any pre-existing class of stationary solutions and in effect serves to introduce a new class of solutions.

Finn [2, 3, 4, 5] has studied the exterior stationary problem within the class of solutions, termed by him *physically reasonable*, which tend to a limit at infinity like $|x|^{-\frac{1}{2}-\varepsilon}$ for some $\varepsilon > 0$. For small data he proved both existence and uniqueness within this class. Further, he showed that flows described by physically reasonable stationary solutions exert drag forces and exhibit paraboloidal wake regions behind objects. Finn has conjectured [6] that for sufficiently small data these solutions are attainable in the sense described above and he proposed this problem to the author. In § 6 of this paper we prove Finn's conjecture when the difference between the physically reasonable solution and its limit at infinity is square summable, a condition equivalent to there being no net force exerted by the fluid on the object; see Finn [7]. Furthermore, when the physically reasonable solution is sufficiently small and satisfies this summability condition we prove that no other solution of the exterior stationary problem is attainable. In this case at least, Leray's solution of the exterior stationary problem is either identical to Finn's or else is unattainable and therefore of doubtful physical significance.

Section 7 contains two global existence theorems and two stability theorems. The first existence theorem is applicable to arbitrary spacial domains while the second is applicable only to interior domains — domains, either bounded or unbounded, for which the Poincaré inequality holds. A feature of the second theorem which greatly extends its potential usefulness for the study of flow in infinite pipes is that the total energy input made over infinite time is not assumed to be finite. The stability theorems we give follow readily from the existence theorems; they guarantee stability in the strict sense. That is, a solution $\mathbf{u}(x, t)$ is stable if, for any sufficiently small perturbation $\mathbf{u}_*(x)$ of the initial data, the initial boundary value problem has a solution which equals $\mathbf{u}(x, 0) + \mathbf{u}_*(x)$ at $t = 0$ and which converges to $\mathbf{u}(x, t)$ as $t \to \infty$. The boundary values prescribed for the perturbed solution are equal to those assumed by $\mathbf{u}$.

The results of this paper depend upon the investigation of the initial boundary value problem for the Navier–Stokes equations in cases where the prescribed data are nonhomogeneous and time dependent. In general we pose the problem in a noninertial coordinate frame, such as one attached to an accelerated body, thereby introducing a time dependent velocity at infinity. In order to define a generalized solution for this problem and in order to reduce the problem to a homogeneous one, it is necessary to introduce a class of admissible extensions of the prescribed initial and boundary data into the space-time region where a solution is sought. This is a somewhat delicate matter because the spacial domain is generally unbounded. It is important for much of our work that the class of admissible extensions contain certain known solutions and approximate solutions. Also, the class of generalized solutions defined in terms of these admissible extensions should include all classical solutions which are sufficiently well behaved at infinity. On the other hand, certain restrictions must be placed on the class of admissible extensions so that integrals appearing in the definition of generalized solution will make sense and so that the uniqueness of generalized solutions may be proved. In order to obtain uniqueness, we require that admissible extensions of the data represent only motions which remain unaccelerated at infinity relative to inertial coordinate frames. This type of condition is more natural in the definition of a generalized solution than a condition on the behavior of the pressure at infinity such as that used by Graffi [8] to prove uniqueness for classical solutions.

Section 2 is devoted to preliminaries. In Section 3 the initial boundary value problem is posed and its generalized solutions are shown to be unique and to satisfy an energy equality. Section 4 contains abstract conditions ensuring the convergence of Galerkin approximations to solutions and ensuring the convergence of nonstationary solutions to stationary solutions. Section 5 contains most of the a priori estimates on which the main results in sections 6 and 7 are based.

2. Preliminaries

The region occupied by the fluid is represented by an open subset Ω of R^3. The coordinates of position in Ω are denoted by $x=(x_1, x_2, x_3)$, and the time variable by t. The space-time domain $\Omega \times (0, T)$ is denoted by Q_T. We let $\mathbf{u}$ represent the flow velocity, p the pressure, and $\mathbf{f}$ the prescribed external force density; these are functions of x and t. The coefficient of kinematic viscosity is denoted by ν.

All functions in this paper are either R or R^3-valued; in the latter case they are denoted by bold faced letters. According to context, $L^p(\Omega)$ may denote either the space of pth power summable R-valued functions or of R^3-valued functions. Similarly $C_0^\infty(\Omega)$ denotes the space of smooth R (or R^3)-valued functions with compact support in Ω. There are corresponding spaces $L^p(Q_T)$ and $C_0^\infty(Q_T)$ of functions defined in Q_T. The Sobolev space consisting of functions in $L^2(\Omega)$ which have first and second derivatives in $L^2(\Omega)$ is denoted by $W_2^2(\Omega)$.

We call Ω an exterior domain if it is the exterior of a bounded closed surface $\partial\Omega$, and we call it an interior domain if the Poincaré inequality $\|\boldsymbol{\phi}\|_{L^2(\Omega)} \leq C_\Omega \|\nabla \boldsymbol{\phi}\|_{L^2(\Omega)}$ holds for all smooth functions $\boldsymbol{\phi}$ with compact support in Ω.

When necessary, the dependence of a constant on the fixed value of some variable is shown by writing the variable as a subscript, as for instance the constant C_Ω in Poincaré's inequality. Sometimes the same letter C will denote different constants within the same argument.

We employ the usual notation of vector analysis; in particular the ith components of $\mathbf{u} \cdot \nabla \mathbf{v}$ and $\Delta \mathbf{u}$ are $\sum_{j=1}^3 u_j \partial v_i / \partial x_j$ and $\sum_{j=1}^3 \partial^2 u_i / \partial x_j^2$ respectively. Some additional notation is needed:

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \left(\sum_{i=1}^3 u_i v_i \right) dx, \qquad \|\mathbf{u}\| = (\mathbf{u}, \mathbf{u})^{\frac{1}{2}}$$

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = \int_\Omega \left(\sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \right) dx, \qquad \|\nabla \mathbf{u}\| = (\nabla \mathbf{u}, \nabla \mathbf{u})^{\frac{1}{2}}$$

$$\int_\Omega \mathbf{u}\mathbf{w} : \nabla \mathbf{v}\, dx = (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = \int_\Omega \left(\sum_{i,j=1}^3 u_j \frac{\partial v_i}{\partial x_j} w_i \right) dx$$

$$\| \cdot \|_{Q_T}^2 = \int_0^T \| \cdot \|^2\, dt$$

$$D(\Omega) = \{\boldsymbol{\phi} : \boldsymbol{\phi} \in C_0^\infty(\Omega) \text{ and } \nabla \cdot \boldsymbol{\phi} = 0\}$$

$$D(Q_T) = \{\boldsymbol{\phi} : \boldsymbol{\phi} \in C_0^\infty(Q_T) \text{ and } \nabla \cdot \boldsymbol{\phi} = 0\}$$

$$J(\Omega) = \text{Completion of } D(\Omega) \text{ in the norm } \| \cdot \|$$

$$J(Q_T) = \text{Completion of } D(Q_T) \text{ in the norm } \| \cdot \|_{Q_T}$$

$$J_1(\Omega) = \text{Completion of } D(\Omega) \text{ in the norm } (\| \cdot \|^2 + \|\nabla \cdot \|^2)^{\frac{1}{2}}$$

$$J_1(Q_T) = \text{Completion of } D(Q_T) \text{ in the norm } (\| \cdot \|_{Q_T}^2 + \|\nabla \cdot \|_{Q_T}^2)^{\frac{1}{2}}.$$

The spaces $J(\Omega)$, $J(Q_T)$, $J_1(\Omega)$, and $J_1(Q_T)$ are Hilbert spaces. Elements of $J(Q_T)$ and $J_1 Q_T)$ need have no regularity with respect to t because ∇ denotes differentiation with respect to the x variables only.

The following lemmas are well known. The constant in Lemma 1 is due to Serrin [9]. Proofs of Lemmas 2 and 3 may be found in [10].

LEMMA 1. *For any domain $\Omega \subset R^3$, functions in $J_1(\Omega)$ satisfy the Sobolev inequality*

$$\int_\Omega |\boldsymbol{\phi}|^4 \, dx \leqslant 3^{-\frac{3}{2}} \|\boldsymbol{\phi}\| \cdot \|\nabla \boldsymbol{\phi}\|^3.$$

LEMMA 2. *For any domain Ω and point y in R^3, functions in $J_1(\Omega)$ satisfy*

$$\int_\Omega \frac{|\boldsymbol{\phi}(x)|^2}{|x-y|^2} \, dx \leqslant 4 \, \|\nabla \boldsymbol{\phi}\|^2.$$

LEMMA 3. *If Ω is contained in a strip of width C_Ω (many other conditions may be given), functions in $J_1(\Omega)$ satisfy Poincaré's inequality*

$$\|\boldsymbol{\phi}\| \leqslant C_\Omega \|\nabla \boldsymbol{\phi}\|.$$

3. The initial boundary value problem

We shall pose the initial boundary value problem for the Navier–Stokes equations in a form suitable for the study of flow exterior to a body which may undergo acceleration. In order that the region Ω occupied by the fluid may be time independent, we write the equations in a coordinate frame attached to the body. This frame will in general undergo translational acceleration relative to inertial frames, and the negative of this acceleration must be inserted into the equations of motion as a uniform force field applied throughout space to the fluid. When Ω is an exterior domain, say the exterior of a finite object with boundary $\partial\Omega$, we assume the existence of an inertial reference frame in which the fluid velocities tend to zero far from the object, and we denote by $-\mathbf{b}_\infty(t)$ the prescribed velocity with which the object moves relative to this inertial frame. Thus $(d/dt)\mathbf{b}_\infty(t)$, abbreviated $\mathbf{b}_{\infty t}(t)$, appears as a fictitious body force in the equations of motion when written in a coordinate frame attached to the object, and the condition at infinity, $\mathbf{u}_t(x, t) \to \mathbf{b}_{\infty t}(t)$ as $x \to \infty$, is imposed on the fluid's acceleration. Our formulation of the initial boundary value problem is also suitable for the study of flow in an interior domain, even for an unbounded one such as the interior of a pipe. However, if Ω is an interior domain we assume coordinate frames attached to $\partial\Omega$ are inertial and set $\mathbf{b}_{\infty t}(t) = 0$.

We shall concern ourselves only with the global existence problem, that is with finding solutions defined in $\Omega \times (0, \infty)$, abbreviated Q_∞. We shall denote the fluid's initial velocity distribution by $\mathbf{b}_0(x, 0)$ and the generally time-dependent boundary values prescribed on $\partial\Omega \times (0, \infty)$ by $\mathbf{b}_0(x, t)$. Thus $\mathbf{b}_0$ is defined on ∂Q_∞.

Consider, then, the problem of finding a solution pair $\mathbf{u}, p$ of

$$\begin{aligned} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \nu \Delta \mathbf{u} + \mathbf{f} + \mathbf{b}_{\infty t} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{1}$$

in Q_∞ which takes given initial and boundary data, and for which $\mathbf{u}_t$ tends to a prescribed limit at infinity if Ω is an unbounded domain:

$$\begin{aligned} &\mathbf{u}(x,0)=\mathbf{b}_0(x,0) \quad \text{for } x\in\Omega \\ &\mathbf{u}(x,t)=\mathbf{b}_0(x,t) \quad \text{for } (x,t)\in\partial\Omega\times(0,\infty) \\ &\mathbf{u}_t(x,t)\to\mathbf{b}_{\infty t}(t) \quad \text{as } x\to\infty \text{ if } \Omega \text{ is an unbounded domain.} \end{aligned} \tag{2}$$

When Ω is an exterior domain we assume that $\mathbf{b}_0(x,0)\to\mathbf{b}_\infty(0)$ as $x\to\infty$. We always assume the prescribed data $\mathbf{f}(x,t)$, $\mathbf{b}_0(x,t)$, and $\mathbf{b}_{\infty t}(t)$ permit $\mathbf{b}_0$ to be extended continuously into Q_∞ as a solenoidal function $\mathbf{b}(x,t)$ which for all $T>0$ satisfies:

(A)
(i) $\mathbf{b}\in L^\infty(Q_T)$ and either $\nabla\mathbf{b}\in L^\infty(Q_T)$ or $\nabla\mathbf{b}\in L^2(Q_T)$,
(ii) $\mathbf{b}_t+\mathbf{b}\cdot\nabla\mathbf{b}-\nu\Delta\mathbf{b}-\mathbf{f}-\mathbf{b}_{\infty t}\in L^2(Q_T)$, and
(iii) $\|\mathbf{b}_t(x,t)-\mathbf{b}_{\infty t}(t)\|_{W_2^2(\Omega)}\leqslant C_T$ for all $t\in[0,T]$.

We call such extensions of the initial and boundary data *admissible*. Throughout this paper we set $\mathbf{g}=P(\mathbf{b}_t+\mathbf{b}\cdot\nabla\mathbf{b}-\nu\Delta\mathbf{b}-\mathbf{f}-\mathbf{b}_{\infty t})$, where P denotes the orthogonal projection of $L^2(\Omega)$ onto $J(\Omega)$.

We call $\mathbf{u}$ a *generalized solution* of (1), (2) in Q_∞ if $\mathbf{u}=\mathbf{v}+\mathbf{b}$ where $\mathbf{b}$ is an admissible extension of $\mathbf{b}_0$ into Q_∞ and $\mathbf{v}$ satisfies the following conditions for all $T>0$:

(B) $\mathbf{v}\in J_1(Q_T)$ and $\mathbf{v}_t\in J(Q_T)$,

(C) $\int_\Omega |(\mathbf{v}(x,t)|^4\,dx<C_T$ for $t\in(0,T)$,

(D) $\|\mathbf{v}(x,t)\|_{L^2(\Omega)}\to 0$ as $t\to 0$, and

(E) $\int_{Q_T}\{\mathbf{v}_t\cdot\boldsymbol{\phi}+(\nu\nabla\mathbf{v}-\mathbf{v}\mathbf{v}-\mathbf{b}\mathbf{v}-\mathbf{v}\mathbf{b}):\nabla\boldsymbol{\phi}+\mathbf{g}\cdot\boldsymbol{\phi}\}\,dx\,dt=0$

for all $\boldsymbol{\phi}\in J_1(Q_T)$.

When the initial data is in $W_2^2(\Omega)\cap J_1(\Omega)$ and both the velocity at infinity and the boundary values prescribed on $\partial\Omega\times(0,\infty)$ vanish, this definition of generalized solution is equivalent to that introduced by Kiselev and Ladyzhenskaya [11]. The solutions we actually obtain are easily seen to possess some additional regularity which makes possible a detailed investigation of their differentiability through application of results known for stationary solutions and for solutions of a linear nonstationary problem. To be precise, one may show that $\mathbf{v}$ has derivatives $\mathbf{v}_{xt}$ in $L^2(Q_T)$, and that $\mathbf{v}_t$ is weakly continuous with respect to t as an element of $L^2(\Omega)$. A proof that generalized solutions with this additional regularity are classical solutions if the data is sufficiently smooth has been given by Ladyzhenskaya [10]. The following lemma, needed in order to prove the uniqueness theorem, is central to the methods used in this paper.

LEMMA 4. *Suppose* $\mathbf{u}$ *is a generalized solution of* (1), (2) *in* Q_∞, *say* $\mathbf{u}=\mathbf{v}+\mathbf{b}$ *where* $\mathbf{v}$ *and* $\mathbf{b}$ *satisfy conditions* (A) *through* (E). *Let* $\bar{\mathbf{b}}$ *be an arbitrary admissible extension of* $\mathbf{b}_0$ *into* Q_∞ *and set* $\bar{\mathbf{v}}=\mathbf{u}-\bar{\mathbf{b}}$. *Then together* $\bar{\mathbf{v}}$ *and* $\bar{\mathbf{b}}$ *satisfy conditions* (A) *through* (E).

Proof. Notice that $\bar{\mathbf{v}}=\mathbf{v}+(\mathbf{b}-\bar{\mathbf{b}})$ satisfies conditions (B), (C), and (D) if $\mathbf{b}-\bar{\mathbf{b}}$ does. Now $\mathbf{b}-\bar{\mathbf{b}}$ vanishes at $t=0$ since initially both $\mathbf{b}$ and $\bar{\mathbf{b}}$ equal $\mathbf{b}_0$. In addition, (A) (iii) implies

$$\|\mathbf{b}_t(x,t)-\bar{\mathbf{b}}_t(x,t)\|_{W_2^2(\Omega)}\leqslant\|\mathbf{b}_t(x,t)-\mathbf{b}_{\infty t}(t)\|_{W_2^2(\Omega)}+\|\bar{\mathbf{b}}_t(x,t)-\mathbf{b}_{\infty t}(t)\|_{W_2^2(\Omega)}\leqslant C_T$$

for all $t\in[0,T]$. Consequently, if h represents $\mathbf{b}-\bar{\mathbf{b}}$ or any one of its first or second order x-partial derivatives, we have

$$\|h(t)\|^2=\int_\Omega\left(\int_0^t h_t(x,\tau)\,d\tau\right)^2dx=\int_\Omega\int_0^t\int_0^t h_t(x,\tau)\,h_t(x,\varrho)\,d\tau\,d\varrho\,dx$$

$$\leqslant\int_\Omega\int_0^t\int_0^t h_t^2(x,\tau)\,d\tau\,d\varrho\,dx=t\int_0^t\int_\Omega h_t^2(x,\tau)\,dx\,d\tau\leqslant t^2C_T^2$$

for all $t\in[0,T]$. Summing over all x-partial derivatives of order less than or equal to two, we get

$$\|\mathbf{b}(x,t)-\bar{\mathbf{b}}(x,t)\|_{W_2^2(\Omega)}\leqslant tC_T$$

for all $t\in[0,T]$. Thus $\bar{\mathbf{v}}$ satisfies condition (D). Condition (B) is satisfied because both $\mathbf{b}-\bar{\mathbf{b}}$ and $(\mathbf{b}-\bar{\mathbf{b}})_t$ are solenoidal, equal to zero on $\partial\Omega\times(0,\infty)$, and bounded in $W_2^2(\Omega)$ uniformly with respect to $t\in[0,T]$. Condition (C) follows from the uniformity with respect to t of the bound for $\mathbf{b}-\bar{\mathbf{b}}$ in $J_1(\Omega)$ and from the continuity of the imbedding of $J_1(\Omega)$ in $L^4(\Omega)$ given by Lemma 1.

In order to check that the integral

$$\int_{Q_T}\{\bar{\mathbf{v}}_t\cdot\boldsymbol{\phi}+(\nu\nabla\bar{\mathbf{v}}-\overline{\mathbf{v}\mathbf{v}}-\overline{\mathbf{b}\mathbf{v}}-\overline{\mathbf{v}\mathbf{b}}):\nabla\boldsymbol{\phi}+(\bar{\mathbf{b}}_t+\bar{\mathbf{b}}\cdot\nabla\bar{\mathbf{b}}-\nu\Delta\bar{\mathbf{b}}-\mathbf{f}-\mathbf{b}_{\infty t})\cdot\boldsymbol{\phi}\}\,dx\,dt$$

vanishes for all $\boldsymbol{\phi}\in J_1(Q_T)$ we substitute $\mathbf{v}+(\mathbf{b}-\bar{\mathbf{b}})$ for $\bar{\mathbf{v}}$ and reduce it to the corresponding integral for $\mathbf{v}$ and $\mathbf{b}$. This involves several integrations by parts. We have

$$\int_{Q_T}\nu\nabla(\mathbf{b}-\bar{\mathbf{b}}):\nabla\boldsymbol{\phi}\,dx\,dt=-\int_{Q_T}\nu\Delta(\mathbf{b}-\bar{\mathbf{b}})\cdot\boldsymbol{\phi}\,dx\,dt$$

because $\mathbf{b}-\bar{\mathbf{b}}\in W_2^2(\Omega)$ for all $t\in[0, T]$. In addition $\mathbf{b}$ and $\bar{\mathbf{b}}$ are solenoidal and satisfy conditions (A)(i), so the nonlinear terms may be integrated by parts as follows:

$$\int_{Q_T}[-(\mathbf{b}-\bar{\mathbf{b}})(\mathbf{b}-\bar{\mathbf{b}})-\bar{\mathbf{b}}(\mathbf{b}-\bar{\mathbf{b}})-(\mathbf{b}-\bar{\mathbf{b}})\bar{\mathbf{b}}]:\nabla\boldsymbol{\phi}\,dx\,dt$$
$$=\int_{Q_T}[-\mathbf{b}(\mathbf{b}-\bar{\mathbf{b}})-(\mathbf{b}-\bar{\mathbf{b}})\bar{\mathbf{b}}]:\nabla\boldsymbol{\phi}\,dx\,dt=\int_{Q_T}[\mathbf{b}\cdot\nabla(\mathbf{b}-\bar{\mathbf{b}})+(\mathbf{b}-\bar{\mathbf{b}})\cdot\nabla\bar{\mathbf{b}}]\cdot\boldsymbol{\phi}\,dx\,dt$$
$$=\int_{Q_T}[\mathbf{b}\cdot\nabla\mathbf{b}-\bar{\mathbf{b}}\cdot\nabla\bar{\mathbf{b}}]\cdot\boldsymbol{\phi}\,dx\,dt.$$

THEOREM 1. *The problem* (1), (2) *has at most one generalized solution.*

Proof. Suppose both $\mathbf{u}$ and $\bar{\mathbf{u}}$ are generalized solutions. Let $\mathbf{b}$ be any admissible extension of $\mathbf{b}_0$ into Q_∞. In view of Lemma 4 both $\mathbf{v}=\mathbf{u}-\mathbf{b}$ and $\bar{\mathbf{v}}=\bar{\mathbf{u}}-\mathbf{b}$ satisfy conditions (B) through (E) with $\mathbf{b}$. We will prove that $\mathbf{u}$ and $\bar{\mathbf{u}}$ coincide by showing that $\mathbf{w}=\mathbf{v}-\bar{\mathbf{v}}$ vanishes.

Let $\tau\in(0, T)$ be arbitrary, and let $\boldsymbol{\phi}\in J_1(Q_T)$ be equal to $\mathbf{w}$ for $t\leqslant\tau$ and vanish for $t>\tau$. Then subtraction of (E) for $\bar{\mathbf{v}}$ from (E) for $\mathbf{v}$ yields

$$\int_0^\tau\int_\Omega[\mathbf{w}_t\cdot\mathbf{w}+\nu\nabla\mathbf{w}:\nabla\mathbf{w}-(\mathbf{v}\mathbf{w}+\mathbf{w}\bar{\mathbf{v}}):\nabla\mathbf{w}-\mathbf{b}\mathbf{w}:\nabla\mathbf{w}-\mathbf{w}\mathbf{b}:\nabla\mathbf{w}]\,dx\,dt=0.$$

The terms $-\mathbf{v}\mathbf{w}:\nabla\mathbf{w}$ and $-\mathbf{b}\mathbf{w}:\nabla\mathbf{w}$ integrate by parts to zero. Hence, except possibly for values of τ in a set of measure zero,

$$\tfrac{1}{2}\|\mathbf{w}(\tau)\|^2+\nu\int_0^\tau\|\nabla\mathbf{w}\|^2dt-\int_0^\tau\int_\Omega[\mathbf{w}\bar{\mathbf{v}}:\nabla\mathbf{w}+\mathbf{w}\mathbf{b}:\nabla\mathbf{w}]\,dx\,dt=0. \tag{3}$$

Now $\bar{\mathbf{v}}$ is bounded in $L^4(\Omega)$, uniformly in $t\in(0, T)$, by some constant C_T. Thus, using the Schwarz inequality, Lemma 1, and Young's inequality, we obtain

$$|(\mathbf{w}\bar{\mathbf{v}},\nabla\mathbf{w})|\leqslant\|\nabla\mathbf{w}\|\left\{\int_\Omega\left(\sum_{i=1}^3 w_i^2\right)\left(\sum_{j=1}^3\bar{\mathrm{v}}_j^2\right)dx\right\}^{\frac{1}{2}}$$
$$\leqslant C_T\|\nabla\mathbf{w}\|\cdot\|\mathbf{w}\|_4\leqslant C_T\|\nabla\mathbf{w}\|\{\varepsilon\|\nabla\mathbf{w}\|+C_\varepsilon\|\mathbf{w}\|\}\leqslant\varepsilon C\|\nabla\mathbf{w}\|^2+C_{T\varepsilon}\|\mathbf{w}\|^2.$$

Here ε may be any positive number. Since $|\mathbf{b}|$ is pointwise bounded we have

$$|(\mathbf{w}\mathbf{b},\nabla\mathbf{w})|\leqslant\|\nabla\mathbf{w}\|\left\{\int_\Omega\left(\sum_{i=1}^3 w_i^2\right)\left(\sum_{j=1}^3 b_j^2\right)dx\right\}^{\frac{1}{2}}\leqslant C\|\nabla\mathbf{w}\|\cdot\|\mathbf{w}\|\leqslant\varepsilon C\|\nabla\mathbf{w}\|^2+\frac{1}{4\varepsilon}\|\mathbf{w}\|^2.$$

Thus, taking ε small enough, (3) yields $\frac{1}{2}\|\mathbf{w}(\tau)\|^2\leqslant C\int_0^\tau\|\mathbf{w}\|^2dt$. Setting $F(\tau)=\int_0^\tau\|\mathbf{w}\|^2dt$ we obtain

$$\frac{d}{d\tau}(e^{-2C\tau}F(\tau)) = e^{-2C\tau}(F'(\tau) - 2CF(\tau)) \leqslant 0.$$

Since $F(0)=0$, $F(\tau)$ must vanish identically and consequently $\|\mathbf{w}(t)\|=0$ for almost all $t>0$.

LEMMA 5 (Energy Equality). *Suppose* $\mathbf{u}$ *is a generalized solution of* (1), (2), *and let* $\mathbf{b}$ *be any admissible extension of* $\mathbf{b}_0$ *into* Q_∞. *Then, after redefinition on a set of measure zero if necessary, the difference* $\mathbf{v}=\mathbf{u}-\mathbf{b}$ *satisfies*

$$\tfrac{1}{2}\|\mathbf{v}(t)\|^2 + \nu\int_0^t \|\nabla \mathbf{v}\|^2 d\tau = -\int_0^t \{(\mathbf{v}\nabla\mathbf{b}, \mathbf{v}) + (\mathbf{g}, \mathbf{v})\}\, d\tau \tag{4}$$

for all $t>0$. *Further, for almost all* $t>0$,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 = -(\mathbf{v}\nabla\mathbf{b}, \mathbf{v}) - (\mathbf{g}, \mathbf{v}). \tag{5}$$

Proof. By Lemma 4, $\mathbf{v}$ and $\mathbf{b}$ together satisfy conditions (A) through (E). When we set

$$\boldsymbol{\phi}(x,t) = \begin{cases} \mathbf{v}(x,t), & 0 \leqslant t \leqslant \tau \\ 0 \quad , & \tau < t < T \end{cases}$$

in (E), the result is

$$\int_0^\tau \{(\mathbf{v}_t, \mathbf{v}) + \nu\|\nabla\mathbf{v}\|^2 - (\mathbf{v}\nabla\mathbf{v}, \mathbf{v}) - (\mathbf{b}\nabla\mathbf{v}, \mathbf{v}) - (\mathbf{v}\nabla\mathbf{v}, \mathbf{b}) + (\mathbf{g}, \mathbf{v})\}\, dt = 0.$$

The terms $(\mathbf{v}\nabla\mathbf{v}, \mathbf{v})$ and $(\mathbf{b}\nabla\mathbf{v}, \mathbf{v})$ integrate by parts to zero. Now (4) follows because $\mathbf{v}_t \cdot \mathbf{v}$ is the weak time derivative of $\frac{1}{2}\mathbf{v}^2$, and both are integrable over Q_T. Finally, $\|\mathbf{v}(t)\|^2$ is absolutely continuous and (5) follows by differentiation of (4).

4. Abstract lemmas concerning the choice of b

We shall employ Galerkin's method to prove the existence of generalized solutions. Let $\{\mathbf{a}^l(x)\}$ be a system of functions contained in $W_2^2(\Omega) \cap J_1(\Omega)$, complete in $J_1(\Omega)$, and orthonormal in $J(\Omega)$. Let $\mathbf{b}(x,t)$ be an admissible extension of the boundary data into Q_∞. Finally, let

$$\mathbf{v}^k(x,t) = \sum_{l=1}^{k} c_{kl}(t)\, \mathbf{a}^l(x) \quad (k=1,2,\ldots)$$

be the solution of the system $(l=1,2,\ldots,k)$ of ordinary differential equations

$$(\mathbf{v}_t^k, \mathbf{a}^l) + \nu(\nabla\mathbf{v}^k, \nabla\mathbf{a}^l) = -(\mathbf{v}^k\nabla\mathbf{v}^k, \mathbf{a}^l) - (\mathbf{b}\nabla\mathbf{v}^k, \mathbf{a}^l) - (\mathbf{v}^k\nabla\mathbf{b}, \mathbf{a}^l) - (\mathbf{g}, \mathbf{a}^l) \tag{6}$$

which satisfies the initial condition $\mathbf{v}^k(x, 0) = 0$. Equations (6) arise formally from (E) by specializing the choices of $\boldsymbol{\phi}$ and T. They can be rewritten in the form

$$\frac{d}{dt} c_{kl}(t) = \sum_{m=1}^{k} c_{km}(t)\,(\nu\Delta\mathbf{a}^m - \mathbf{b}\nabla\mathbf{a}^m - \mathbf{a}^m\nabla\mathbf{b}, \mathbf{a}^l) + \sum_{m,n=1}^{k} c_{km}(t)\,c_{kn}(t)\,(-\mathbf{a}^m\nabla\mathbf{a}^n, \mathbf{a}^l) - (\mathbf{g}, \mathbf{a}^l).$$

Standard theorems in the theory of ordinary differential equations ensure that this system has a unique solution $\mathbf{v}^k$ on some initial time interval which will be all $[0, \infty)$ if $\sum_{l=1}^{k} c_{kl}^2(t) = \|\mathbf{v}^k(t)\|^2$ remains finite as t increases. We shall see that this condition is met for any choice of admissible $\mathbf{b}$. Our existence proof for the initial boundary value problem (1), (2) will depend upon choosing $\mathbf{b}$ in such a way that the norms $\|\mathbf{v}_t^k(t)\|$ and $\|\nabla\mathbf{v}_t^k(t)\|$ as well as $\|\mathbf{v}^k(t)\|$ can be proved bounded, uniformly in k, on finite subintervals of $[0, \infty)$. Estimates for the growth of these norms will be based on identities (7) and (8) below. Equation (7), which is formally an expression of energy conservation, holds for as long as $\mathbf{v}^k$ continues to exist. It is obtained by multiplying each equation (6) by $c_{kl}(t)$, summing $\sum_{l=1}^{k}$, and noting that $(\mathbf{v}^k\cdot\nabla\mathbf{v}^k, \mathbf{v}^k)$ and $(\mathbf{b}\cdot\nabla\mathbf{v}^k, \mathbf{v}^k)$ integrate by parts to zero. Equation (8) also holds so long as $\mathbf{v}^k$ exists provided $\mathbf{g}_t \in L^2(Q_T)$ for all $T > 0$. Assuming this, (8) is obtained by differentiating each equation (6) with respect to t, multiplying by $(d/dt)c_{kl}(t)$, summing $\sum_{l=1}^{k}$, and noting that several terms integrate by parts to zero. Thus

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}^k\|^2 + \nu\|\nabla\mathbf{v}^k\|^2 = -(\mathbf{v}^k\nabla\mathbf{b}, \mathbf{v}^k) - (\mathbf{g}, \mathbf{v}^k) \tag{7}$$

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_t^k\|^2 + \nu\|\nabla\mathbf{v}_t^k\|^2 = -(\mathbf{v}_t^k\nabla\mathbf{b}, \mathbf{v}_t^k) - (\mathbf{v}_t^k\nabla\mathbf{v}^k, \mathbf{v}_t^k) - (\mathbf{b}_t\nabla\mathbf{v}^k, \mathbf{v}_t^k) - (\mathbf{v}^k\nabla\mathbf{b}_t, \mathbf{v}_t^k) - (\mathbf{g}_t, \mathbf{v}_t^k). \tag{8}$$

The initial values of $\|\mathbf{v}^k\|$ and $\|\mathbf{v}_t^k\|$ are both bounded uniformly in k. In fact we took $\|\mathbf{v}^k(0)\| = 0$ as an initial condition, and it follows from (6) that $\|\mathbf{v}_t^k(0)\| \leq \|\mathbf{g}(0)\|$.

If the admissible extension $\mathbf{b}$ is itself a solution of the Navier-Stokes equations then $\mathbf{g}$ is identically zero and consequently so are all the Galerkin approximations $\mathbf{v}^k$. By choosing an extension $\mathbf{b}$ which is "nearly a solution" one may hope for slow growth of the approximations $\mathbf{v}^k$ and hence through the following lemma for existence of a generalized solution.

LEMMA 6. *The problem* (1), (2) *has a generalized solution in* Q_∞ *if there is an admissible extension* $\mathbf{b}$ *of the boundary data into* Q_∞ *such that the corresponding Galerkin approximations satisfy for all* $T > 0$ *the estimates* $\|\mathbf{v}^k(t)\|$, $\|\mathbf{v}_t^k(t)\|$, $\|\nabla\mathbf{v}^k(t)\| \leq C_T$ *for all* $t \in [0, T]$.

Proof. The given estimates insure that a subsequence $\{\mathbf{v}^{k_n}\}$ can be selected from $\{\mathbf{v}^k\}$ such that $\{\mathbf{v}^{k_n}\}$ and $\{\mathbf{v}_t^{k_n}\}$ have weak limits $\mathbf{v} \in J_1(Q_T)$ and $\mathbf{v}_t \in J(Q_T)$ respectively, for all

$T>0$. We assume these weak limits are, if necessary, redefined on a set of measure zero so as to satisfy for all $t>0$ the estimates given for each $\mathbf{v}^k$. According to the Rellich theorem [12], the convergence $\mathbf{v}^{k_n}\to\mathbf{v}$ is strong in $L^2(Q')$ for compact subsets Q' of Q_∞.

In view of Lemma 1 the given estimates imply

$$\int_\Omega (v_i^k)^4\,dx \leqslant C_T \quad \text{for all } t\in[0,T] \quad \text{and } k=1,2,\ldots.$$

Thus the limit $\mathbf{v}$ must satisfy condition (C). Integrating the preceding inequality over time and applying the Schwarz inequality yields $\int_{Q_T}(v_i^k v_j^k)^2\,dx\,dt\leqslant C_T$. Hence it suffices to test the weak $L^2(Q_T)$ convergence $v_i^{k_n}v_j^k \to v_i v_j$ by test functions $\psi\in C_0^\infty(Q_T)$. Now

$$\int_{Q_T}(v_i^{k_n}v_j^{k_n}-v_iv_j)\,\psi\,dx=\int_{Q_T}v_i^{k_n}(v_j^{k_n}-v_j)\,\psi\,dx+\int_{Q_T}(v_i^{k_n}-v_i)\,v_j\,\psi\,dx\to 0$$

as $n\to\infty$ for all $\psi\in C_0^\infty(Q_T)$ since $(v_j^{k_n}-v_j)\,\psi\to 0$ strongly in $L^2(Q_T)$.

It is easily shown that (E) holds for all $\boldsymbol{\phi}\in J_1(Q_T)$ if it holds for all $\boldsymbol{\phi}$ of the form $\boldsymbol{\phi}=\sum_{l=1}^m c_l(t)\,\mathbf{a}^l(x)$ with arbitrary coefficients $c_l(t)$. For such $\boldsymbol{\phi}$, (E) becomes

$$\int_0^T\sum_{l=1}^m c^l(t)\int_\Omega[\mathbf{v}_t\cdot\mathbf{a}^l+(\nu\nabla\mathbf{v}-\mathbf{v}\mathbf{v}-\mathbf{b}\mathbf{v}-\mathbf{v}\mathbf{b}):\nabla\mathbf{a}^l+\mathbf{g}\cdot\mathbf{a}^l]\,dx\,dt=0$$

which is certainly valid in view of (6) and the weak $L^2(Q_T)$ convergence of $\mathbf{v}^{k_n}$, $\mathbf{v}_t^{k_n}$, $\nabla\mathbf{v}^{k_n}$, and $v_i^{k_n}v_j^{k_n}$.

In order to see that $\mathbf{v}$ assumes the initial data note that $\mathbf{v}(t)\to 0$ strongly in $L^2(\Omega)$ as $t\to 0$ if the approximations do, uniformly in k. Now

$$\|\mathbf{v}^k(t)\|^2=\int_0^t\frac{d}{d\tau}\|\mathbf{v}^k(\tau)\|^2\,d\tau=\int_0^t\int_\Omega 2\,\mathbf{v}^k(\tau)\cdot\mathbf{v}_t^k(\tau)\,dx\,d\tau\leqslant 2\,\|\mathbf{v}^k\|_{L^2(Q_t)}\,\|\mathbf{v}_t^k\|_{L^2(Q_t)}\to 0$$

as $t\to 0$, uniformly in k, since $\|\mathbf{v}^k(t)\|$, $\|\mathbf{v}_t^k(t)\|\leqslant C_T$.

By making an appropriate choice of $\mathbf{b}$, the following lemma enables us to study the behavior of a solution as $t\to\infty$.

LEMMA 7. *Let $\mathbf{u}$ be a generalized solution of* (1), (2) *in Q_∞. Suppose that $\mathbf{b}$ is an admissible extension of the given data into Q_∞ for which the difference $\mathbf{v}=\mathbf{u}-\mathbf{b}$ satisfies $\|\nabla\mathbf{v}\|_{L^2(Q_\infty)}$, $\|\nabla\mathbf{v}_t\|_{L^2(Q_\infty)}<\infty$. Then $\mathbf{u}\to\mathbf{b}$ as $t\to\infty$ in the sense that $\|\nabla\mathbf{v}(t)\|_{L^2(\Omega)}$ and $\|\mathbf{v}(t)\|_{L^2(\Omega')}$ tend to zero as $t\to\infty$, where Ω' is any finite subset of Ω.*

Proof. According to the Schwarz inequality

$$\left|\frac{d}{dt}\|\nabla\mathbf{v}\|^2\right| \leqslant 2\|\nabla\mathbf{v}\|\cdot\|\nabla\mathbf{v}_t\|.$$

Integration with respect to time yields

$$\int_0^\infty \left|\frac{d}{dt}\|\nabla\mathbf{v}\|^2\right| dt \leqslant 2\|\nabla\mathbf{v}\|_{L^2(Q_\infty)}\|\nabla\mathbf{v}_t\|_{L^2(Q_\infty)} < \infty.$$

Because the integrals $\int_0^\infty |(d/dt)\|\nabla\mathbf{v}\|^2|\,dt$ and $\int_0^\infty \|\nabla\mathbf{v}\|^2 dt$ are both finite, $\|\nabla\mathbf{v}\|$ must converge to zero as $t\to\infty$. According to Lemma 2 then, $\|\mathbf{v}\|_{L^2(\Omega')}$ must also converge to zero as $t\to\infty$ for any bounded subdomain Ω' of Ω.

5. A priori estimates

Throughout this section Galerkin approximations $\mathbf{v}^k$ will be denoted simply by $\mathbf{v}$ without the superscript. Furthermore, $\mathbf{b}$ will always denote an admissible extension of the data into Q_∞ and T will always be an arbitrary positive number. The first three lemmas below contain energy-type estimates for the Galerkin approximations derived under successively stronger assumptions regarding $\mathbf{b}$ and Ω. If a generalized solution $\mathbf{u}$ of (1), (2) exists, these estimates also apply to $\mathbf{v}=\mathbf{u}-\mathbf{b}$ because of the energy equality, or rather its derivative (5). One consequence of Lemma 8 is that the Galerkin approximations exist on the whole interval $[0,\infty)$. Furthermore, using only estimate (9) below one can prove that the problem (1), (2) always possesses a weak solution of the type introduced by Hopf [13].

LEMMA 8. *Let* $B=\dfrac{1}{2\nu}\|\mathbf{b}\|^2_{L^\infty(Q_T)}$. *Then*

$$\|\mathbf{v}(T)\|^2+\nu\int_0^T\|\nabla\mathbf{v}\|^2\,dt \leqslant E^2(T) \tag{9}$$

and

$$\tfrac{1}{2}\nu\|\nabla\mathbf{v}(T)\|^2 \leqslant E(T)\{\|\mathbf{v}_t(T)\|+BE(T)+\|\mathbf{g}(T)\|\}, \tag{10}$$

where

$$E(T)=e^{BT}\int_0^T\|\mathbf{g}(t)\|\,e^{-Bt}\,dt.$$

Proof. Integrating by parts and applying the Schwarz inequality gives

$$|(\mathbf{v}\nabla\mathbf{b},\mathbf{v})| \leqslant \tfrac{1}{2}\nu\|\nabla\mathbf{v}\|^2+B\|\mathbf{v}\|^2.$$

Thus from either (5) or (7) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \tfrac{1}{2}\nu\|\nabla\mathbf{v}\|^2 \leqslant B\|\mathbf{v}\|^2 + \|\mathbf{g}\|\cdot\|\mathbf{v}\|,$$

and hence $(d/dt)\|\mathbf{v}\| \leqslant B\|\mathbf{v}\| + \|\mathbf{g}\|$ and $\|\mathbf{v}(t)\| \leqslant E(t)$. Since $(d/dt)\,E(t) = BE(t) + \|\mathbf{g}(t)\|$, we have

$$\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 \leqslant \frac{d}{dt}E^2(t),$$

from which (9) follows. By the Schwarz inequality $|\tfrac{1}{2}(d/dt)\|\mathbf{v}\|^2| \leqslant \|\mathbf{v}\|\cdot\|\mathbf{v}_t\|$. Thus (10) is obtained as follows:

$$\tfrac{1}{2}\nu\|\nabla\mathbf{v}\|^2 \leqslant \left|\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2\right| + B\|\mathbf{v}\|^2 + \|\mathbf{g}\|\cdot\|\mathbf{v}\| \leqslant E\{\|\mathbf{v}_t\| + BE + \|\mathbf{g}\|\}.$$

LEMMA 9. *Suppose* $|(\boldsymbol{\phi}\cdot\nabla\mathbf{b}, \boldsymbol{\phi})| \leqslant (\nu - \varrho)\|\nabla\boldsymbol{\phi}\|^2$ *holds for some* $\varrho > 0$, *all* $\boldsymbol{\phi} \in J_1(\Omega)$, *and all* $t \in [0, T]$. *Then*

$$\tfrac{1}{2}\|\mathbf{v}(T)\|^2 + \varrho\int_0^T \|\nabla\mathbf{v}\|^2 dt \leqslant \tfrac{1}{2}W^2(T) \tag{11}$$

and

$$\varrho\|\nabla\mathbf{v}(T)\|^2 \leqslant W(T)\{\|\mathbf{v}_t(T)\| + \|\mathbf{g}(T)\|\}, \tag{12}$$

where $W(T) = \int_0^T \|\mathbf{g}(t)\|\,dt$.

Proof. From either (5) or (7) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \nu\|\nabla\mathbf{v}\|^2 \leqslant (\nu - \varrho)\|\nabla\mathbf{v}\|^2 + \|\mathbf{g}\|\cdot\|\mathbf{v}\|,$$

hence $(d/dt)\|\mathbf{v}\| \leqslant \|\mathbf{g}\|$ and $\|\mathbf{v}(T)\| \leqslant W(T)$. Thus

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \varrho\|\nabla\mathbf{v}\|^2 \leqslant \|\mathbf{g}\|\,W(t) = \frac{1}{2}\frac{d}{dt}W^2(t),$$

from which (11) follows. We obtain (12) by using the Schwarz inequality:

$$\varrho\|\nabla\mathbf{v}\|^2 \leqslant \left|\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2\right| + \|\mathbf{g}\|\,W \leqslant W\{\|\mathbf{v}_t\| + \|\mathbf{g}\|\}.$$

Remark. If the hypothesis of Lemma 9 holds only for $t \in [T_0, T]$, with $T_0 > 0$, one still has (11) and (12) but with T_0 replacing 0 as the lower limit of the integral and with $W(T) = \int_{T_0}^T \|\mathbf{g}\|\,dt + W_{T_0}$, where W_{T_0} is an upper bound for the values of $\|\mathbf{v}^k(T_0)\|$.

LEMMA 10. *Suppose* $|(\boldsymbol{\phi}\cdot\nabla\mathbf{b}, \boldsymbol{\phi})| \leqslant (\nu - \varrho)\|\nabla\boldsymbol{\phi}\|^2$ *and* $\|\boldsymbol{\phi}\|^2 \leqslant \varrho/\omega\,\|\nabla\boldsymbol{\phi}\|^2$ *hold for some* ϱ, $\omega > 0$, *all* $\boldsymbol{\phi} \in J_1(\Omega)$, *and all* $t \in [0, T]$. *Then* (11) *holds as well as*

$$\|\mathbf{v}(T)\| \leqslant V(T) \tag{13}$$

and

$$\varrho\|\nabla \mathbf{v}(T)\|^2 \leqslant V(T)\{\|\mathbf{v}_t(T)\| + \|\mathbf{g}(T)\|\}, \tag{14}$$

where $V(T) = e^{-\omega T}\int_0^T \|\mathbf{g}(t)\| e^{\omega t} dt$ *remains bounded if* $\|\mathbf{g}(t)\|$ *does and tends to zero as* $t \to \infty$ *if* $\|\mathbf{g}(t)\|$ *does.*

Proof. From either (5) or (7) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \varrho\|\nabla \mathbf{v}\|^2 \leqslant \|\mathbf{g}\| \cdot \|\mathbf{v}\|,$$

hence

$$\frac{d}{dt}\|\mathbf{v}\| + \omega\|\mathbf{v}\| \leqslant \|\mathbf{g}\| \quad \text{and} \quad \|\mathbf{v}(T)\| \leqslant V(T).$$

Using the Schwarz inequality again yields (14):

$$\varrho\|\nabla \mathbf{v}\|^2 \leqslant \left|\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2\right| + \|\mathbf{g}\| \cdot \|\mathbf{v}\| \leqslant V\{\|\mathbf{v}_t\| + \|\mathbf{g}\|\}.$$

The behavior claimed for $V(t)$ is obvious since if $0 \leqslant \tau \leqslant T$, then

$$V(T) \leqslant \frac{1}{\omega} e^{\omega(\tau - T)} \sup_{[0,\tau]} \|\mathbf{g}(t)\| + \frac{1}{\omega} \sup_{[\tau, T]} \|\mathbf{g}(t)\|.$$

The following lemma does not apply directly to generalized solutions because its proof is based in part on identity (8) for the Galerkin approximations. However the estimates (15) together with Lemmas 6 and 9 ensure convergence of the approximations $\mathbf{v}^k$ to a function $\mathbf{v}$, which also satisfies (15), such that $\mathbf{u} = \mathbf{v} + \mathbf{b}$ is a generalized solution. By uniqueness then, Lemma 11 may be viewed as applying to generalized solutions.

LEMMA 11. *Suppose, in addition to the hypothesis of Lemma 9, that* $\gamma \equiv \sup_{Q_T}|b_t(x,t) - \mathbf{b}_{\infty t}(t)| \cdot |x| < \infty$, *and that* Φ *defined by*

$$\Phi = \tan^{-1}\|\mathbf{g}(0)\|^2 + \{\tfrac{1}{4}\varrho^{-5}(1 + \sup_{t \in [0,T]}\|\mathbf{g}(t)\|)W(T) + 8\gamma^2\varrho^{-2}\}W^2(T)$$
$$+ \int_0^T \{4\varrho^{-1}|\mathbf{b}_{\infty t}(t)|^2 W^2(t) + 2\|g_t(t)\|\}\,dt$$

satisfies $\Phi < \pi/2$. *Then*

$$\|\mathbf{v}_t(T)\|^2 \leqslant \tan\Phi \quad \textit{and} \quad \varrho\int_0^T \|\nabla \mathbf{v}_t\|^2\,dt \leqslant 4\Phi(1 + \tan^2\Phi). \tag{15}$$

Proof. We begin by estimating each term on the right side of (8). By hypothesis $|(\mathbf{v}_t \cdot \nabla \mathbf{b}, \mathbf{v}_t)| \leqslant (\nu - \varrho)\|\nabla \mathbf{v}_t\|^2$ for some $\varrho > 0$. Using the Schwarz inequality, Lemma 1, and Young's inequality to estimate the second term we obtain

$$
\begin{aligned}
|(\mathbf{v}_t \nabla \mathbf{v}, \mathbf{v}_t)| &\leqslant \left\{ \int_\Omega \sum_{i,j=1}^{3} v_{ix_j}^2 \, dx \right\}^{\frac{1}{2}} \left\{ \int_\Omega \sum_{i,j=1}^{3} v_{it}^2 v_{jt}^2 \, dx \right\}^{\frac{1}{2}} \\
&= \|\nabla \mathbf{v}\| \left\{ \int_\Omega |\mathbf{v}_t|^4 dx \right\}^{\frac{1}{2}} \\
&\leqslant 3^{-\frac{3}{4}} \|\nabla \mathbf{v}\| \, \|\mathbf{v}_t\|^{\frac{1}{2}} \, \|\nabla \mathbf{v}_t\|^{\frac{3}{2}} \leqslant 3^{-\frac{3}{4}} \|\nabla \mathbf{v}\| \left\{ \tfrac{1}{4} \sigma^3 \|\mathbf{v}_t\|^2 + \tfrac{3}{4} \sigma^{-1} \|\nabla \mathbf{v}_t\|^2 \right\}
\end{aligned}
$$

where $\sigma > 0$ is arbitrary. If $\|\nabla \mathbf{v}\|$ is positive we can set $\sigma = 3^{\frac{1}{4}} \varrho^{-1} \|\nabla \mathbf{v}\|$ and thereby obtain

$$
|(\mathbf{v}_t \cdot \nabla \mathbf{v}, \mathbf{v}_t)| \leqslant \tfrac{1}{4} \varrho^{-3} \|\nabla \mathbf{v}\|^4 \|\mathbf{v}_t\|^2 + \tfrac{1}{4} \varrho \|\nabla \mathbf{v}_t\|^2
$$

which clearly holds regardless of whether $\|\nabla \mathbf{v}\|$ is positive or not. Finally, applying inequality (12) of Lemma 9 to the term in $\|\nabla \mathbf{v}\|^4$ on the right we get

$$
|(\mathbf{v}_t \cdot \nabla \mathbf{v}, \mathbf{v}_t)| \leqslant \tfrac{1}{4} \varrho^{-4} \{ \|\mathbf{v}_t\| + \|\mathbf{g}\| \} \, W(t) \, \|\mathbf{v}_t\|^2 \cdot \|\nabla \mathbf{v}\|^2 + \tfrac{1}{4} \varrho \|\nabla \mathbf{v}_t\|^2.
$$

The term $(\mathbf{v} \cdot \nabla \mathbf{b}_t, \mathbf{v}_t)$ is handled through integration by parts and an application of Lemma 2:

$$
\begin{aligned}
|(\mathbf{v} \nabla \mathbf{b}_t, \mathbf{v}_t)| = |(\mathbf{v} \nabla \mathbf{v}_t, \mathbf{b}_t - \mathbf{b}_{\infty t})| &\leqslant \|\nabla \mathbf{v}_t\| \left\{ \gamma^2 \int_\Omega \frac{|\mathbf{v}|^2}{|x|^2} \, dx \right\}^{\frac{1}{2}} \\
&\leqslant 2\gamma \|\nabla \mathbf{v}_t\| \cdot \|\nabla \mathbf{v}\| \leqslant \tfrac{1}{4} \varrho \|\nabla \mathbf{v}_t\|^2 + 4 \gamma^2 \varrho^{-1} \|\nabla \mathbf{v}\|^2.
\end{aligned}
$$

Lemma 2 is used again to estimate the term $(\mathbf{b}_t \nabla \mathbf{v}, \mathbf{v}_t)$ as follows:

$$
\begin{aligned}
|(\mathbf{b}_t \nabla \mathbf{v}, \mathbf{v}_t)| &\leqslant |((\mathbf{b}_t - \mathbf{b}_{\infty t}) \nabla \mathbf{v}_t, \mathbf{v})| + |(\mathbf{b}_{\infty t} \nabla \mathbf{v}_t, \mathbf{v})| \leqslant \|\nabla \mathbf{v}_t\| \left\{ \int_\Omega |\mathbf{b}_t - \mathbf{b}_{\infty t}|^2 |\mathbf{v}|^2 \, dx \right\}^{\frac{1}{2}} \\
&\quad + \|\nabla \mathbf{v}_t\| \left\{ \int_\Omega |\mathbf{b}_{\infty t}|^2 |\mathbf{v}|^2 \, dx \right\}^{\frac{1}{2}} \leqslant \|\nabla \mathbf{v}_t\| \{ 2\gamma \|\nabla \mathbf{v}\| + |\mathbf{b}_{\infty t}| \, \|\mathbf{v}\| \} \\
&\leqslant \tfrac{3}{8} \varrho \|\nabla \mathbf{v}_t\|^2 + 4 \gamma^2 \varrho^{-1} \|\nabla \mathbf{v}\|^2 + 2 \varrho^{-1} |\mathbf{b}_{\infty t}(t)|^2 \|\mathbf{v}\|^2.
\end{aligned}
$$

Having estimated each term on the right side of (8), we get

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\mathbf{v}_t\|^2 + \tfrac{1}{8} \varrho \|\nabla \mathbf{v}_t\|^2 \leqslant \{ \tfrac{1}{4} \varrho^{-4} (\|\mathbf{v}_t\| + \|\mathbf{g}\|) \|\mathbf{v}_t\|^2 \, W(t) + 8 \gamma^2 \varrho^{-1} \} \|\nabla \mathbf{v}\|^2 \\
+ 2 \varrho^{-1} |\mathbf{b}_{\infty t}(t)|^2 \|\mathbf{v}\|^2 + \|\mathbf{g}_t\| \, \|\mathbf{v}_t\|. \qquad (16)
\end{aligned}
$$

Dropping the second term on the left of (16) and multiplying by $2(1 + \|\mathbf{v}_t\|^4)^{-1}$ gives

$$
\frac{d}{dt} \tan^{-1} \|\mathbf{v}_t\|^2 \leqslant 2 \{ \tfrac{1}{4} \varrho^{-4} (1 + \|\mathbf{g}\|) \, W(t) + 8 \gamma^2 \varrho^{-1} \} \|\nabla \mathbf{v}\|^2 + 4 \varrho^{-1} |\mathbf{b}_{\infty t}(t)|^2 \|\mathbf{v}\|^2 + 2 \|\mathbf{g}_t\|.
$$

Because of (11), this can be integrated from 0 to T and the result is $\|\mathbf{v}_t(T)\|^2 \leqslant \tan \Phi$. Now multiplying each term of (16) except $\varrho/8\|\nabla \mathbf{v}_t\|^2$ by $2(1+\|\mathbf{v}_t\|^4)^{-1}$ and multiplying the term $\varrho/8\|\nabla \mathbf{v}_t\|^2$ by $2(1+\tan^2\Phi)^{-1}$ the inequality is preserved and we may integrate to obtain

$$\varrho\int_0^T \|\nabla \mathbf{v}_t\|^2\,dt \leqslant 4\Phi(1+\tan^2\Phi).$$

6. On the attainability of stationary solutions

In this section we shall think of Ω as an exterior domain although the proofs of our theorems do not depend upon this assumption. Our main interest, however, is in the implications that these theorems have for the exterior stationary problem. This problem consists of finding a solution pair $\mathbf{w}$, p of

$$\begin{aligned} \mathbf{w}\cdot\nabla\mathbf{w} &= -\nabla p + \nu\Delta\mathbf{w} + \boldsymbol{\zeta} \\ \nabla\cdot\mathbf{w} &= 0 \end{aligned} \tag{17}$$

in Ω which assumes prescribed data $\mathbf{w}_0(x)$ on $\partial\Omega$ and tends to a prescribed limit $\mathbf{w}_\infty$ as $x\to\infty$. Leray [1] proved that *the exterior stationary problem has at least one solution if the boundary* $\partial\Omega$ *and boundary data* $\mathbf{w}_0$ *are sufficiently smooth, if* $\mathbf{w}_0$ *also satisfies* $\int_{\partial\Omega}\mathbf{w}_0\cdot\mathbf{n}\,ds = 0$, *and if* $|x|\boldsymbol{\zeta}(x)\in L^2(\Omega)$. Although Leray only proved that $\mathbf{w}(x)\to\mathbf{w}_\infty$ as $x\to\infty$ in a generalized sense, Finn [2, 6] later proved that Leray's solution actually tends continuously to $\mathbf{w}_\infty$ as $x\to\infty$. Leray's solutions are as smooth as the data allow and satisfy $\|\nabla\mathbf{w}\|<\infty$, but little more is known about them. Whether or not they provide physically acceptable models of fluid flow is uncertain as, in particular, it is unknown whether they are unique or stable or whether they occur as limits of time dependent motions. The following definitions make precise the notion of *attainable solution* which we described in Section 1.

Definition. We shall say that data $\mathbf{b}_0$, $\mathbf{b}_\infty$, $\mathbf{f}$ for the initial boundary value problem (1), (2) satisfies condition (F) with respect to data $\mathbf{w}_0$, $\mathbf{w}_\infty$, $\boldsymbol{\zeta}$ for the exterior stationary problem if and only if

(F)
- (i) $\mathbf{b}_0$ has an admissible extension into Q_∞,
- (ii) the initial data $\mathbf{b}_0(x, 0)$ vanishes, and
- (iii) for all sufficiently large t there holds $\mathbf{b}_0(x,t)=\mathbf{w}_0(x)$ for all $x\in\partial\Omega$, $\mathbf{b}_\infty(t)=\mathbf{w}_\infty$, and $\mathbf{f}(x,t)=\boldsymbol{\zeta}(x)$ for $x\in\Omega$.

Definition. For prescribed data $\mathbf{w}_0$, $\mathbf{w}_\infty$, $\boldsymbol{\zeta}$, we shall say that a locally square summable function $\mathbf{w}(x)$ is an *attainable solution* of the exterior stationary problem if and only if, for some choice of data $\mathfrak{b}_0$, $\mathfrak{b}_\infty$, $\mathfrak{f}$ which satisfies condition (F) with respect to $\mathbf{w}_0$, $\mathbf{w}_\infty$, $\boldsymbol{\zeta}$, the initial boundary value problem (1), (2) possesses a generalized solution $\mathbf{u}(x, t)$ which converges to $\mathbf{w}(x)$ as $t \to \infty$ in the sense that $\|\mathbf{u}(x, t) - \mathbf{w}(x)\|_{L^2(\Omega')} \to 0$ for every bounded subdomain Ω' of Ω.

The main results of this section are contained in the following two theorems.

THEOREM 2. *Suppose that for prescribed data* $\mathbf{w}_0$, $\mathbf{w}_\infty$, $\boldsymbol{\zeta}$ *the exterior stationary problem possesses a classical solution* $\mathbf{w}(x)$ *which satisfies* $\sup_{x\in\Omega} |\mathbf{w}(x) - \mathbf{w}_\infty| \cdot |x| < \nu/2$ *and* $\|\mathbf{w}(x) - \mathbf{w}_\infty\| < \infty$. *Then, for this prescribed data,* $\mathbf{w}(x)$ *is the only possible attainable solution.*

THEOREM 3. *For prescribed data* $\mathbf{w}_0$, $\mathbf{w}_\infty$, $\boldsymbol{\zeta}$, *a classical solution* $\mathbf{w}(x)$ *of the exterior stationary problem is an attainable solution provided* $\sup_{x\in\Omega} |\mathbf{w}(x) - \mathbf{w}_\infty| \cdot |x|$, $\|\mathbf{w}(x) - \mathbf{w}_\infty\|$, $\|\nabla \mathbf{w}\|$, *and* $\sup_{x\in\Omega} |\mathbf{w}(x)|$ *are sufficiently small.*

Before proving these theorems we shall discuss their hypotheses. Finn [3] has shown that a stationary solution's behavior at infinity is controlled by that of the fundamental solution tensor associated with Oseen's linearized equations [14] if for some $\varepsilon > 0$ the solution tends like $|x|^{-\frac{1}{2}-\varepsilon}$ to its limit at infinity. Such solutions, which we follow Finn in calling *physically reasonable*, have a number of important properties. In particular they satisfy the sharper estimate

$$\sup_{x\in\Omega} |\mathbf{w}(x) - \mathbf{w}_\infty| \cdot |x| < C, \tag{18}$$

which appears as a hypothesis in our Theorems 2 and 3. Finn [5] proved the following theorem concerning the existence and uniqueness of physically reasonable solutions. *The exterior stationary problem has a solution* $\mathbf{w}$ *satisfying* (18) *if the boundary* $\partial\Omega$ *is sufficiently smooth, if the boundary data* $\mathbf{w}_0(x)$ *specified on* $\partial\Omega$ *is sufficiently close to* $\mathbf{w}_\infty$, *and if the external force density* $\boldsymbol{\zeta}(x)$ *is sufficiently small. Provided* (18) *holds with* $C = \nu/2$, $\mathbf{w}$ *is unique among all physically reasonable solutions taking the same data.* Although any physically reasonable solution has finite Dirichlet integral $\|\nabla \mathbf{w}\|$, the hypothesis $\|\mathbf{w}(x) - \mathbf{w}_\infty\| < \infty$ appearing in our theorems is significant. Its meaning is clarified by another theorem due to Finn [7]. *A physically reasonable solution* $\mathbf{w}$ *of the exterior stationary problem with* $\boldsymbol{\zeta} \equiv 0$ *satisfies* $\|\mathbf{w}(x) - \mathbf{w}_\infty\| < \infty$ *if and only if the net force due to the flux of momentum across* $\partial\Omega$ *and to the stress exerted by the fluid on* $\partial\Omega$ *vanishes.* Of course, the net force due to the surrounding fluid on a body in steady motion is equal and opposite to the net external force applied to the body. So the condition $\|\mathbf{w}(x) - \mathbf{w}_\infty\| < \infty$ simply means that there is no net external force applied to the body. Several examples of such motion are (i) steady flow about a body which propels itself by maintaining a momentum flux across portions of its boundary, (ii) steady flow about a body which propels itself by moving tangentially portions of its boundary, as by belts, (iii) steady flow about a body which, perhaps due to an external torque, rotates about an axis of symmetry. For small data, then, there is a broad and physically important class of steady solutions which satisfy the hypotheses of our theorems.

Proof of Theorem 2. Suppose, for the data $\mathbf{w}_0$, $\mathbf{w}_\infty$, $\boldsymbol{\zeta}$, that $\overline{\mathbf{w}}(x)$ is an attainable solution of the exterior stationary problem. Then for some choice of data $\mathbf{b}_0$, $\mathbf{b}_\infty$, $\mathbf{f}$ satisfying (F) the initial boundary value problem (1), (2) has a generalized solution $\mathbf{u}(x, t)$ such that $\|\mathbf{u}(x, t)-\overline{\mathbf{w}}(x)\|_{L^2(\Omega')}\to 0$ as $t\to\infty$ for every bounded $\Omega'\subset\Omega$.

Let T be a number sufficiently large that the conditions of (F) (iii) hold for all $t\geqslant T$. Let $\alpha(t)$ be a continuously differentiable real valued function defined for all $t\geqslant 0$ which vanishes for $t\leqslant T$ and equals 1 for $t\geqslant T+1$. Now, if we let $\overline{\mathbf{b}}$ be any admissible extension of $\mathbf{b}_0$ into Q_∞, the function $\mathbf{b}$ defined by

$$\mathbf{b}(x, t) = (1-\alpha(t))\,\overline{\mathbf{b}}(x, t)+\alpha(t)\,\mathbf{w}(x)$$

is an admissible extension of $\mathbf{b}_0$ into Q_∞ which equals $\mathbf{w}(x)$ for all $t\geqslant T+1$. In order to check this one needs to known $\|\mathbf{w}(x)-\mathbf{w}_\infty\|_{W_2^2(\Omega)}<\infty$. We have assumed $\|\mathbf{w}(x)-\mathbf{w}_\infty\|<\infty$; Finn [5] has shown that the derivatives of $\mathbf{w}$ are square summable.

Let $\mathbf{v}=\mathbf{u}-\mathbf{b}$. According to Lemmas 5 and 8, $\|\mathbf{v}(T+1)\|$ is finite. By hypothesis, the number ϱ defined by $(\nu-\varrho)/2=\sup_\Omega|\mathbf{w}(x)-\mathbf{w}_\infty|\cdot|x|$ is positive. Using Lemma 2 we find

$$\begin{aligned}|(\boldsymbol{\phi}\nabla\mathbf{w},\boldsymbol{\phi})| = |(\boldsymbol{\phi}\nabla\boldsymbol{\phi},\mathbf{w}-\mathbf{w}_\infty)| &\leqslant \left\{\int_\Omega\sum_{i,j=1}^{3}\phi_{ix_j}^2\,dx\right\}^{\frac12}\left\{\int_\Omega\sum_{i,j=1}^{3}\phi_i^2(w-w_\infty)_j^2\,dx\right\}^{\frac12}\\ &= \|\nabla\boldsymbol{\phi}\|\left\{\int_\Omega\left(\sum_{i=1}^{3}\phi_i^2\right)\left(\sum_{j=1}^{3}(w-w_\infty)_j^2\right)dx\right\}^{\frac12}\\ &\leqslant \|\nabla\boldsymbol{\phi}\|\left\{\frac{(\nu-\varrho)^2}{4}\int_\Omega\frac{|\boldsymbol{\phi}|^2}{|x|^2}\,dx\right\}^{\frac12}\leqslant(\nu-\varrho)\|\nabla\boldsymbol{\phi}\|^2. \end{aligned}\tag{19}$$

holds for all $\boldsymbol{\phi}\in J_1(\Omega)$. Thus for all $t\geqslant T+1$ the hypothesis of Lemma 9 is satisfied, and according to the remark following Lemma 9 we have

$$\tfrac12\|\mathbf{v}(t)\|^2+\varrho\int_{T+1}^{t}\|\nabla\mathbf{v}\|^2\,d\tau\leqslant\tfrac12\left\{\int_{T+1}^{t}\|\mathbf{g}\|\,d\tau+\|\mathbf{v}(T+1)\|\right\}^2=\tfrac12\|\mathbf{v}(T+1)\|^2. \tag{20}$$

Now let Ω' be an arbitrary bounded subdomain of Ω. We will show that $\overline{\mathbf{w}}=\mathbf{w}$ in Ω'. Since $\mathbf{b}(x, t)=\mathbf{w}(x)$ for all $t\geqslant T+1$ we have

$$\begin{aligned}\lim_{t\to\infty}\|\mathbf{v}(x,t)-(\overline{\mathbf{w}}(x)-\mathbf{w}(x))\|_{L^2(\Omega')} &= \lim_{t\to\infty}\|\mathbf{v}(x,t)-(\overline{\mathbf{w}}(x)-\mathbf{b}(x,t))\|_{L^2(\Omega')}\\ &= \lim_{t\to\infty}\|\mathbf{u}(x,t)-\overline{\mathbf{w}}(x)\|_{L^2(\Omega')}=0.\end{aligned}$$

If $\mathbf{w} \neq \overline{\mathbf{w}}$ in $L^2(\Omega')$, then $\lim_{t\to\infty}\|\mathbf{v}(x,t)\|_{L^2(\Omega')}$ is positive. It follows from Lemma 2 that $\|\nabla\mathbf{v}(x,t)\| > C$ for some $C>0$ and all sufficiently large t. Therefore $\int_{T+1}^{t}\|\nabla\mathbf{v}\|^2 d\tau \to \infty$ as $t\to\infty$, contradicting (20).

Proof of Theorem 3. For the initial boundary value problem we prescribe data $\mathbf{b}_0(x,t) = \psi(t)\mathbf{w}_0(x)$, $\mathbf{b}_\infty(t) = \psi(t)\mathbf{w}_\infty$, and $\mathbf{f}(x,t) = \psi(t)\boldsymbol{\zeta}(x)$ where

$$\psi(t) = \begin{cases} \frac{1}{2}t^2 & 0 \leqslant t < 1 \\ 1 - \frac{1}{2}(t-2)^2 & 1 \leqslant t < 2 \\ 1 & 2 \leqslant t. \end{cases}$$

Clearly $\mathbf{b}(x,t) = \psi(t)\mathbf{w}(x)$ is an admissible extension of $\mathbf{b}_0$ into Q_∞, and the data satisfies (F). Note that

$$\sup_{[0,\infty)}|\psi(t)| = 1, \quad \sup_{[0,\infty)}|\psi'(t)| = 1, \quad \sup_{[0,\infty)}|\psi^2(t)-\psi(t)| = \tfrac{1}{4},$$

$$\int_0^\infty |\psi'(t)|\,dt = 1, \quad \int_0^\infty |\psi''(t)|\,dt = 2, \quad \int_0^\infty |\psi'(t)|^2\,dt = \tfrac{2}{3},$$

$$\int_0^\infty |\psi^2(t)-\psi(t)|\,dt = \tfrac{7}{30}, \quad \text{and} \quad \int_0^\infty |\psi'(t)\,(2\,\psi(t)-1)|\,dt = \tfrac{1}{2}.$$

We now show that $\mathbf{b}$ satisfies the hypotheses of Lemmas 9 and 11. Assume $\sup_\Omega|\mathbf{w}(x) - \mathbf{w}_\infty|\cdot|x| < \nu/2$. It follows that $\sup_{Q_\infty}|\mathbf{b}(x,t)-\mathbf{b}_\infty(t)|\cdot|x| < \nu/2$, and we see as in (19) that $|(\boldsymbol{\phi}\cdot\nabla\mathbf{b},\boldsymbol{\phi})| \leqslant (\nu-\varrho)\|\nabla\boldsymbol{\phi}\|^2$ for some $\varrho>0$ and all $\boldsymbol{\phi}\in J_1(\Omega)$. We also have $\gamma<\nu/2$.

Because $P(\psi\mathbf{w}\cdot\nabla\mathbf{w} - \nu\psi\Delta\mathbf{w} - \psi\boldsymbol{\zeta}) = 0$, we have $\mathbf{g}(t) = P(\psi'(t)[\mathbf{w}-\mathbf{w}_\infty] + [\psi^2(t)-\psi(t)]\mathbf{w}\cdot\nabla\mathbf{w})$, hence $\|\mathbf{g}(t)\| \leqslant \psi'(t)\|\mathbf{w}-\mathbf{w}_\infty\| + |\psi^2(t)-\psi(t)|\cdot\|\mathbf{w}\cdot\nabla\mathbf{w}\|$, and $\|\mathbf{g}_t(t)\| \leqslant |\psi''(t)|\cdot\|\mathbf{w} - \mathbf{w}_\infty\| + |\psi'(t)(2\psi(t)-1)|\cdot\|\mathbf{w}\cdot\nabla\mathbf{w}\|$. Thus it may be seen that $\|\mathbf{g}(0)\| = 0$,

$$\sup_{[0,\infty)}\|\mathbf{g}(t)\| \leqslant \|\mathbf{w}-\mathbf{w}_\infty\| + \tfrac{1}{4}\sup_\Omega|\mathbf{w}(x)|\cdot\|\nabla\mathbf{w}\|,$$

$$W(t) \leqslant \int_0^\infty \|\mathbf{g}(t)\|\,dt \leqslant \|\mathbf{w}-\mathbf{w}_\infty\| + \tfrac{7}{30}\sup_\Omega|\mathbf{w}(x)|\cdot\|\nabla\mathbf{w}\|,$$

and

$$\int_0^\infty \|\mathbf{g}_t(t)\|\,dt \leqslant 2\,\|\mathbf{w}-\mathbf{w}_\infty\| + \tfrac{1}{2}\sup_\Omega|\mathbf{w}(x)|\cdot\|\nabla\mathbf{w}\|.$$

Furthermore

$$\int_0^\infty |\mathbf{b}_{\infty t}(t)|^2 W^2(t)\,dt \leqslant \tfrac{2}{3}|\mathbf{w}_\infty|^2(2\,\|\mathbf{w}-\mathbf{w}_\infty\|^2 + \tfrac{1}{9}\sup_\Omega|\mathbf{w}(x)|^2\cdot\|\nabla\mathbf{w}\|^2).$$

One easily sees from these estimates that $\mathbf{b}$ satisfies the hypotheses of Lemmas 9 and 11 if $\mathbf{w}$ satisfies the hypotheses of Theorem 3. So the Galerkin approximations satisfy inequalities (11), (12), and (15). According to Lemma 6, they converge to a limit $\mathbf{v}$ such that $\mathbf{u}=\mathbf{v}+\mathbf{b}$ is a generalized solution of (1), (2). Because $\mathbf{v}$ also satisfies (11) and (15), Lemma 7 ensures that $\mathbf{u}(t)\to\mathbf{w}$ as $t\to\infty$.

7. Global existence and stability

To prove the unique solvability of the initial boundary value problem for $\Omega\subset R^3$ and all $t\geqslant 0$ we require essentially two hypotheses regarding an extension $\mathbf{b}$ of the prescribed initial and boundary data $\mathbf{b}_0$ into Q_∞. One, namely that $(\nu-\varrho)\|\nabla\boldsymbol{\phi}\|^2\geqslant-(\boldsymbol{\phi}\nabla\mathbf{b},\boldsymbol{\phi})$ for all $\boldsymbol{\phi}\in J_1(\Omega)$ and some $\varrho>0$, serves as a stability condition. It insures that a solution's growth through the nonlinear term is dominated by viscous damping, and in effect occurs only in response to a forcing term $\mathbf{g}$ which represents the extent of $\mathbf{b}$'s failure to be itself a solution. The second hypothesis is that $\mathbf{b}$ be "nearly a solution" so that the forcing term is small. In the following two theorems Ω may be any open subset of R^3.

THEOREM 4. *The initial boundary value problem* (1), (2) *has a generalized solution in* Q_∞ *if only there is an admissible extension* $\mathbf{b}$ *of* $\mathbf{b}_0$ *into* Q_∞ *such that* $\sup_{Q_\infty}|\mathbf{b}(x,t)-\mathbf{b}_\infty(t)|\cdot|x|<\nu/2$, $\sup_{Q_\infty}|\mathbf{b}_t(x,t)-\mathbf{b}_{\infty t}(t)|\cdot|x|<\infty$, $\int_0^\infty\mathbf{b}_{\infty t}^2(t)\,dt<\infty$, $\sup_{[0,\infty)}\|\mathbf{g}(t)\|<\infty$, *and such that* $\int_0^\infty\|\mathbf{g}(t)\|\,dt$ *and* $\int_0^\infty\|\mathbf{g}_t(t)\|\,dt$ *are sufficiently small.*

Proof. The stability condition, needed to apply Lemma 9, is proved for $\mathbf{b}$ in the same way it was for $\mathbf{w}$ in (19) by using the assumption $\sup|\mathbf{b}(x,t)-\mathbf{b}_\infty(t)|\cdot|x|<\nu/2$. The existence of a solution then readily follows from Lemmas 6, 9, and 11.

THEOREM 5. *A solution* $\mathbf{u}$ *of the initial boundary value problem* (1), (2) *is stable if* (i) $\mathbf{u}$ *is itself an admissible extension of* $\mathbf{b}_0$ *into* Q_∞, (ii) $\sup_{Q_\infty}|\mathbf{u}(x,t)-\mathbf{b}_\infty(t)|\cdot|x|<\nu/2$, (iii) $\sup_{Q_\infty}|\mathbf{u}_t(x,t)-\mathbf{b}_{\infty t}(t)|\cdot|x|<\infty$, $\sup_{[0,\infty)}|\mathbf{b}_{\infty t}(t)|<\infty$, *and* $\int_0^\infty\mathbf{b}_{\infty t}^2(t)\,dt<\infty$.

Proof. Consider the initial boundary value problem for a solution $\bar{\mathbf{u}}$ subject to the same external force, assuming the same boundary values on $\partial\Omega\times(0,\infty)$ and tending to the same limit at infinity as $\mathbf{u}$, but taking different initial data, say $\bar{\mathbf{u}}(x,0)=\mathbf{b}_0(x,0)+\mathbf{u}_*(x)$. We will show it has a solution $\bar{\mathbf{u}}$ in Q_∞ and that $\|\nabla(\bar{\mathbf{u}}-\mathbf{u})\|\to 0$ as $t\to\infty$ if the perturbation $\mathbf{u}_*$ of the initial data is in $J_1(\Omega)\cap W_2^2(\Omega)$, and if $\|\mathbf{u}_*\|_{W_2^2(\Omega)}$ and $\sup_\Omega|\mathbf{u}_*(x)|\cdot|x|$ are small.

Let ψ be a real valued function defined on $[0,\infty)$ such that $\psi(0)=1$, and $\psi(t)=0$ for $t\geqslant 1$. Assume ψ is smooth so that $|\psi|$, $|\psi'|$, and $|\psi''|$ are bounded. Clearly $\bar{\mathbf{b}}(x,t)=\mathbf{u}(x,t)+\psi(t)\mathbf{u}_*(x)$ is an admissible extension of the initial and boundary data into Q_∞. Furthermore, $\sup_{Q_\infty}|\bar{\mathbf{b}}(x,t)-\mathbf{b}_\infty(t)|\cdot|x|<\nu/2$ and $\sup_{Q_\infty}|\bar{\mathbf{b}}_t(x,t)-\mathbf{b}_{\infty t}(t)|\cdot|x|<\infty$ if $\sup_\Omega|\mathbf{u}_*(x)|\cdot|x|$ is sufficiently small.

The forcing term $\bar{\mathbf{g}}(t)=P(\bar{\mathbf{b}}_t+\bar{\mathbf{b}}\cdot\nabla\bar{\mathbf{b}}-\nu\Delta\bar{\mathbf{b}}-\mathbf{f}-\mathbf{b}_{\infty t})$ vanishes for $t\geqslant 1$, since there $\bar{\mathbf{b}}$ is a solution of (1). To complete the proof through application of Lemmas 6, 7, 9, and 11, we need only show $\|\bar{\mathbf{g}}(t)\|\leqslant C\|\mathbf{u}_*\|_{W_2^2(\Omega)}$ and $\|\bar{\mathbf{g}}_t(t)\|\leqslant C\|\mathbf{u}_*\|_{W_2^2(\Omega)}$, so that the integrals $\int_0^1\|\bar{\mathbf{g}}\|\,dt$ and $\int_0^1\|\bar{\mathbf{g}}_t\|\,dt$ will be small if $\|\mathbf{u}_*\|_{W_2^2(\Omega)}$ is.

Assuming Ω satisfies the ordinary cone condition, we have [12] Sobolev's inequality $\sup_\Omega|\mathbf{u}_*(x)|\leqslant C\|\mathbf{u}_*\|_{W_2^2(\Omega)}$. Since $\mathbf{u}$ satisfies (A) we see that $\sup_{Q_1}|\mathbf{u}|$, $\sup_{Q_1}|\mathbf{u}_t|$, $\sup_{[0,1]}\|\nabla\mathbf{u}_t\|$, and either $\sup_{Q_1}|\nabla\mathbf{u}|$ or $\sup_{[0,1]}\|\nabla\mathbf{u}(t)\|$ are all finite. Thus estimating term by term and assuming $\|\mathbf{u}_*\|_{W_2^2(\Omega)}<1$ one readily finds

$$\|\bar{\mathbf{g}}(t)\|=\|P(\psi'\mathbf{u}_*+\psi\mathbf{u}\nabla\mathbf{u}_*+\psi\mathbf{u}_*\nabla\mathbf{u}+\psi^2\mathbf{u}_*\nabla\mathbf{u}_*-\nu\psi\Delta\mathbf{u}_*)\|\leqslant C\|\mathbf{u}_*\|_{W_2^2(\Omega)}$$

and similarly

$$\|\bar{\mathbf{g}}_t(t)\|=\|P(\psi''\mathbf{u}_*+\psi'\mathbf{u}\nabla\mathbf{u}_*+\psi\mathbf{u}_t\nabla\mathbf{u}_*+\psi'\mathbf{u}_*\nabla\mathbf{u}+\psi\mathbf{u}_*\nabla\mathbf{u}_t+2\psi\psi'\mathbf{u}_*\nabla\mathbf{u}_*-\nu\psi'\Delta\mathbf{u}_*)\|$$
$$\leqslant C\|\mathbf{u}_*\|_{W_2^2(\Omega)}.$$

for some constant C.

Although the hypotheses of Theorems 4 and 5 are natural for the exterior problem, the assumptions concerning a solution's behavior at infinity are unnatural and restrictive in the case of an interior domain Ω. For instance, a solution describing flow through an infinite pipe would not tend to a constant at infinity. Theorems 6 and 7 apply only to interior domains and the constant C_Ω appearing in their statements is that of the Poincaré inequality, Lemma 3.

THEOREM 6. *Let Ω be an interior domain. The initial boundary value problem* (1), (2) *has a generalized solution* $\mathbf{u}$ *for all* $t\geqslant 0$ *if only there is an admissible extension* $\mathbf{b}$ *of the data* $\mathbf{b}_0$ *into* Q_∞ *such that* $\sup_{Q_\infty}|\nabla\mathbf{b}|<\nu C_\Omega^{-2}$, $\sup_{Q_\infty}|\mathbf{b}_t|<\infty$, *and such that* $\sup_{[0,\infty)}\|\mathbf{g}(t)\|$ *and* $\sup_{[0,\infty)}\|\mathbf{g}_t(t)\|$ *are sufficiently small. Moreover,* $\|\mathbf{u}(t)-\mathbf{b}(t)\|$ *converges to zero as* $t\to\infty$ *if* $\|\mathbf{g}(t)\|$ *does, and converges exponentially to zero as* $t\to\infty$ *if* $\|\mathbf{g}(t)\|$ *does.*

Proof. Let $\varrho=\nu-C_\Omega^2\sup_{Q_\infty}|\nabla\mathbf{b}|$. Then for $\boldsymbol{\phi}\in J_1(\Omega)$

$$|(\boldsymbol{\phi}\nabla\mathbf{b},\boldsymbol{\phi})|\leqslant\int\left\{\sum_{i,j=1}^{3}\left(\frac{\partial b_j}{\partial x_i}\right)^2\right\}^{\frac12}\left\{\sum_{i,j=1}^{3}\phi_i^2\phi_j^2\right\}^{\frac12}dx=\int|\nabla\mathbf{b}|\cdot|\boldsymbol{\phi}|^2dx\leqslant(\nu-\varrho)\|\nabla\boldsymbol{\phi}\|^2.$$

Thus Lemma 10 applies and may be used to estimate terms on the right side of (8). Using estimate (14), Lemma 1, and the Poincaré inequality, we find that each Galerkin approximation satisfies

$$|(\mathbf{v}_t\nabla\mathbf{v},\mathbf{v}_t)|\leqslant\|\nabla\mathbf{v}\|\cdot\|\mathbf{v}_t\|^2_{L^4(\Omega)}\leqslant 3^{-\frac34}\varrho^{-\frac12}C_\Omega^{\frac12}V^{\frac12}(t)(\|\mathbf{v}_t\|^{\frac12}+\|\mathbf{g}\|^{\frac12})\|\nabla\mathbf{v}_t\|^2.$$

Integrating by parts, applying the Schwarz and Poincaré inequalities, and using (13), gives

$$|(\mathbf{b}_t\nabla\mathbf{v},\mathbf{v}_t)+(\mathbf{v}\nabla\mathbf{b}_t,\mathbf{v}_t)+(\mathbf{g}_t,\mathbf{v}_t)|\leqslant\{2V(t)\sup_{Q_\infty}|\mathbf{b}_t|+C_\Omega\|\mathbf{g}_t\|\}\|\nabla\mathbf{v}_t\|$$

$$\leqslant\mu\|\nabla\mathbf{v}_t\|^2+\mu^{-1}\{V(t)\sup_{Q_\infty}|\mathbf{b}_t|+\tfrac{1}{2}C_\Omega\|\mathbf{g}_t\|\}^2$$

for any $\mu>0$. Thus (8) yields

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_t\|^2\leqslant-\{\varrho-\mu-3^{-\frac{3}{4}}\varrho^{-\frac{1}{4}}C_\Omega^{\frac{1}{2}}V^{\frac{1}{4}}(t)(\|\mathbf{g}\|^{\frac{1}{2}}+\|\mathbf{v}_t\|^{\frac{1}{2}})\}\|\nabla\mathbf{v}_t\|^2$$

$$+\mu^{-1}\{V(t)\sup_{Q_\infty}|\mathbf{b}_t|+\tfrac{1}{2}C_\Omega\|\mathbf{g}_t\|\}^2.$$

We recall $\sup_{[0,\infty)}V(t)\leqslant\varrho^{-1}C_\Omega^2\sup_{[0,\infty)}\|\mathbf{g}(t)\|$ in Lemma 10. Now assume $\sup\|\mathbf{g}\|$ is sufficiently small, and take μ small enough so that

$$\varrho-\mu-3^{-\frac{3}{4}}\varrho^{-\frac{1}{4}}C_\Omega^{\frac{1}{2}}V^{\frac{1}{4}}(t)\|\mathbf{g}\|^{\frac{1}{2}}\geqslant A>0$$

for some constant A and all $t\geqslant0$. Then

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_t\|^2\leqslant-\{A-C_1\|\mathbf{v}_t\|^{\frac{1}{2}}\}\|\nabla\mathbf{v}_t\|^2+C_2$$

where C_1 and C_2 are constants which may be taken as small as we please by assuming $\sup\|\mathbf{g}\|$ and $\sup\|\mathbf{g}_t\|$ are sufficiently small. We may, and shall, assume $C_2<(A/4C_\Omega^2)(3A/4C_1)^2$. Finally, we assume $\sup\|\mathbf{g}\|<(3A/4C_1)^2$, so that for each Galerkin approximation $\|\mathbf{v}_t(0)\|<(3A/4C_1)^2$. We claim $\|\mathbf{v}_t(t)\|<(3A/4C_1)^2$ for all $t\geqslant0$. Otherwise there must be a least value of t, say $t=t_1$, such that $\|\mathbf{v}_t\|=(3A/4C_1)^2$. Since $\|\mathbf{v}_t(t)\|<(3A/4C_1)^2$ for all $t\in[0,t_1)$, $(d/dt)\|\mathbf{v}_t(t_1)\|\geqslant0$. On the other hand $\{A-C_1\|\mathbf{v}_t\|^{\frac{1}{2}}\}>0$ at t_1, and consequently at t_1

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_t\|^2\leqslant-\{A-C_1\|\mathbf{v}_t\|^{\frac{1}{2}}\}C_\Omega^{-2}\|\mathbf{v}_t\|^2+C_2.$$

Substituting $(3A/4C_1)^2$ for $\|\mathbf{v}_t(t_1)\|$ gives a contradiction:

$$\frac{d}{dt}\|\mathbf{v}_t(t_1)\|<-\left\{A-C_1\left(\frac{3A}{4C_1}\right)\right\}C_\Omega^{-2}\left(\frac{3A}{4C_1}\right)^2+\frac{A}{4}C_\Omega^{-2}\left(\frac{3A}{4C_1}\right)^2=0.$$

We've shown that $\|\mathbf{v}_t^k(t)\|$ is bounded, uniformly in $t\geqslant0$ and in k. By (14) then, so is $\|\nabla\mathbf{v}^k(t)\|$. Hence existence of a generalized solution in Q_∞ follows from Lemma 6. The convergence of $\|\mathbf{u}(t)-\mathbf{b}(t)\|$ to zero as $t\to\infty$ follows from Lemma 10 if $\|\mathbf{g}(t)\|\to0$ as $t\to\infty$.

THEOREM 7. *Let $\Omega\subset R^3$ be an interior domain. A solution* $\mathbf{u}$ *of the initial boundary value problem* (1), (2) *is stable if* (i) $\mathbf{u}$ *is itself an admissible extension of b_0 into Q_∞,* (ii) $\sup_{Q_\infty}|\nabla\mathbf{u}|<\nu C_\Omega^{-2}$, *and* (iii) $\sup_{Q_\infty}|\mathbf{u}_t|<\infty$.

Proof. Consider the initial boundary value problem for a solution $\bar{\mathbf{u}}$ subject to the same external force and assuming the same boundary values on $\partial\Omega\times(0,\infty)$ as $\mathbf{u}$, but taking different initial data, say $\bar{\mathbf{u}}(x,0)=\mathbf{b}_0(x,0)+\mathbf{u}_*(x)$. We will show it has a solution $\bar{\mathbf{u}}$ in Q_∞ and that $\|\bar{\mathbf{u}}(t)-\mathbf{u}(t)\|\to 0$ exponentially as $t\to\infty$ if the perturbation $\mathbf{u}_*$ of the initial data is in $J_1(\Omega)$ and if $\|\mathbf{u}_*\|_{W_2^2(\Omega)}$ and $\sup_\Omega|\nabla\mathbf{u}_*|$ are sufficiently small.

Let $\psi(t)$ be defined as in the proof of Theorem 5 and let $\bar{\mathbf{b}}(x,t)=\mathbf{u}(x,t)+\psi(t)\mathbf{u}_*(x)$. Assuming Ω satisfies the cone condition, $\bar{\mathbf{b}}$ is an admissible extension of the given data into Q_∞ and $\sup_{Q_\infty}|\bar{\mathbf{b}}_t|<\infty$. Also, $\sup_{Q_\infty}|\nabla\bar{\mathbf{b}}|<\nu C_\Omega^{-2}$ if $\sup_\Omega|\nabla\mathbf{u}_*|$ is sufficiently small.

For $t\geqslant 1$ we know $\bar{\mathbf{b}}$ is a solution (1) and $\bar{\mathbf{g}}(t)=P(\bar{\mathbf{b}}_t+\bar{\mathbf{b}}\cdot\nabla\bar{\mathbf{b}}-\nu\Delta\bar{\mathbf{b}}-\mathbf{f})$ is identically zero. Exactly as was done in the proof of Theorem 5, we may show $\|\bar{\mathbf{g}}(t)\|\leqslant C\|\mathbf{u}_*\|_{W_2^2(\Omega)}$ and $\|\bar{\mathbf{g}}_t(t)\|\leqslant C\|\mathbf{u}_*\|_{W_2^2(\Omega)}$ for all $t\geqslant 0$. Theorem 7 now follows by an application of Theorem 6.

Remarks. In an earlier paper [15] we proved stability of stationary solutions with respect to perturbations $\mathbf{u}_*(x)\in J_1(\Omega)\cap W_2^2(\Omega)$ without a restriction on $\sup_\Omega|\mathbf{u}_*(x)|\cdot|x|$ as made in the proof of Theorem 5, or a restriction on $\sup_\Omega|\nabla\mathbf{u}_*(x)|$ as made in the proof of Theorem 7. These extra restrictions are unnecessary here also, but without them we need some additional, rather tedious, local estimates of the type used to prove local existence for large data.

Although available methods of proving global existence for the three-dimensional problem seem closely tied up with the matter of stability, it is otherwise in two dimensions. For $\Omega\subset R^2$ the Sobolev inequality $\|\boldsymbol{\phi}\|_4^2\leqslant\|\nabla\boldsymbol{\phi}\|\cdot\|\boldsymbol{\phi}\|$, for $\boldsymbol{\phi}\in J_1(\Omega)$, is stronger than the corresponding inequality in three dimensions, and this enables one to prove the following theorem which is well known for the homogeneous problem [10]. *For $\Omega\subset R^2$, the initial boundary value problem* (1), (2) *has a global generalized solution for any data b_0 which can be admissibly extended into Q_∞.* Despite this, the stability problem seems more difficult in two dimensions than in three. For an interior two-dimensional domain Ω, Theorem 7 holds and is proved almost exactly as it is for $\Omega\subset R^3$. But if Ω is a two-dimensional exterior domain it seems necessary to assume $|\mathbf{u}(x,t)-\mathbf{u}_\infty(t)|\leqslant C(|x|\log|x|)^{-1}$ in order to prove stability. This condition is evidently not physically reasonable, at least not for the study of flow past a cylinder. Smith [16] has shown that within a class of "physically reasonable" stationary solutions, any solution $\mathbf{u}(x)$ which is constant on the surface of a cylinder and which tends to a constant $\mathbf{u}_\infty$ at infinity faster than $|x|^{-\frac{1}{2}}$ is identically equal to $\mathbf{u}_\infty$ throughout Ω.

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