THE STRUCTURE OF ALGEBRAS OF OPERATOR FIELDS

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Introduction

At the present time a great deal is known about the general theory of C^*-algebras. However, little has been done to clarify the precise structure of specific non-commutative C^*-algebras, for example, the group C^*-algebras of particular non-commutative non-compact groups. In this paper we present a number of results which together constitute program for determining the structure of many specific C^*-algebras; and apply them to describe completely the group C^*-algebra of $SL(2, C)$.

Our main tools will be algebras of operator fields defined on a locally compact Hausdorff space. Let T be a locally compact Hausdorff space, to each t in which there corresponds a C^*-algebra A_t. For different values of t the A_t are in general unrelated. By a full algebra of operator fields on T we mean a $*$-algebra A of functions x on T such that (i) $x(t) \in A_t$ for each t; (ii) $t \to \|x(t)\|$ is continuous on T and vanishes at infinity; (iii) for each t, $\{x(t) \mid x \in A\}$ is dense in A_t; (iv) A is complete in the norm $\|x\| = \sup_t \|x(t)\|$. Evidently A is itself a C^*-algebra; the A_t are called its component algebras.

Algebras of operator fields have been studied by various authors, for example in [8], [6], and [11] (a more complete bibliography will be found in [11]).

Our paper is divided into five chapters. The first chapter begins with the basic concept of a continuity structure; and then proceeds to the description of the dual space of a full algebra of operator fields in terms of the dual spaces of the component algebras. It ends with a description of all possible full subalgebras of a full algebra of operator fields—a special case of Glimm's generalization of the Stone–Weierstrass theorem (see [5]).

Chapter II takes up the problem of representing an arbitrary C^*-algebra A as a full algebra of operator fields. As usual we denote by $\hat{A}$ the dual space (i.e., the space of equivalence classes of irreducible $*$-representations) of A, equipped with the hull-kernel topology. (1) The transform of an element a of A is the function $T \to T_a$ on $\hat{A}$. Now A is isomorphic with the algebra of all transforms of elements of A; but the latter fails on two counts to be a full algebra of operator fields in the sense defined above. In the first place $\hat{A}$ need not be Hausdorff; in the second place the "norm-functions" $T \to \|T_a\|$ need not be continuous. However, in Chapter II we construct from $\hat{A}$ a compact Hausdorff space $\hat{A}^r$, called the regularized dual space of A; and modify the transform of each a in A so that it becomes an operator-valued function $\tilde{a}$ on $\hat{A}^r$ with continuous norm-function, called the regularized transform of a. The algebra $\tilde{A}$ of all regularized transforms is then a full algebra of operator fields on $\hat{A}^r$, called the regularized transform of A, and is isomorphic with A. The set of those t in $\hat{A}^r$ for which the component algebra A_t of $\tilde{A}$ is primitive is dense in $\hat{A}^r$, but will not in general coincide with $\hat{A}^r$. Also, $\tilde{A}$ need not be a maximal full algebra. If it is not maximal, then Glimm's theorem (Theorem 1.4) is precisely what we need in order to describe $\tilde{A}$ in terms of the maximal full algebra $\tilde{A}_{\max}$ containing $\tilde{A}$.

Thus, the determination of the structure of a C^*-algebra A is reduced to the study of $\tilde{A}_{\max}$, or, equivalently, the study of the continuity structure defined by $\tilde{A}$. In Chapter III it is shown that in certain cases (for example, if the irreducible representations of A are of uniformly bounded finite dimension), the possible continuity structures on $\hat{A}^r$ can be analyzed in terms of what we shall call fibre structures, which generalize the notion of fibre bundle. We will illustrate this in an important special case. Let T be a locally compact Hausdorff space, M_n the $n \times n$ total matrix algebra, G_n the group of all automorphisms $a \to u^{-1}au$ of M_n (where u is unitary), and $\mathcal{B}$ a fibre bundle with base space T, fibre space M_n, and group G_n. Then the family $C_0(\mathcal{B})$ of all continuous cross-sections of $\mathcal{B}$ which vanish at infinity forms a C^*-algebra whose irreducible representations are all n-dimensional, and whose dual space coincides with T. Conversely, it is shown in this chapter (Theorem 3.2) that any C^*-algebra A whose irreducible representations all have the same finite dimension n (such an A is called homogeneous of order n) is essentially equal to $C_0(\mathcal{B})$ for some fibre bundle $\mathcal{B}$ with base space $\hat{A}$, fibre space M_n, and group G_n. Using fibre structures, we can obtain a similar, though more complicated, description of many C^*-algebras which are not homogeneous.

Let T be a locally compact Hausdorff space, to each point of which a Hilbert space H_t is associated; and let a continuity structure F for vector fields (with values in the $\{H_t\}$) be given. In Chapter IV we construct from F a continuity structure for operator fields whose values are completely continuous operators on the H_t. The maximal full algebra of operator fields so obtained belongs to a special class of C^*-algebras which we call algebras with continuous trace. The main motivation for this construction lies in its usefulness

(1) For the definition and properties of the hull–kernel topology, we refer the reader to [1].

for Chapter V. As a matter of fact, the notion of an algebra with continuous trace seems to be a natural and important one, inasmuch as every *GCR* algebra has a composition series whose quotients are all algebras with continuous trace (Theorem 4.2). The question whether every algebra with continuous trace arises by the construction of this chapter from a continuity structure for *vector* fields amounts to a problem in the theory of fibre bundles; its answer is negative, even for homogeneous algebras.

Finally, in Chapter V, we apply the preceding results to find the detailed structure of the group C^*-algebra of $SL(2, C)$. The result (Theorems 5.3 and 5.4), as well as the steps by which it is obtained, is outlined in § 5.1. Observe that Theorem 5.4 can be interpreted in the light of Chapter III as saying that no "twists" occur in the fibre structure underlying this C^*-algebra.

I. Full algebras of operator fields

1.1. Continuity structures (1)

Let T be a locally compact Hausdorff space called the *base space*; and for each t in T, let A_t be a (complex) Banach space. A *vector field* (*with values in the* $\{A_t\}$) is a function x on T such that $x(t) \in A_t$ for each t in T. Obviously the vector fields form a complex linear space. If each A_t is a *-algebra, then the vector fields form a *-algebra under the pointwise operations; in that case the vector fields will usually be referred to as *operator fields*.

In this paper, either each A_t will be a Hilbert space or each A_t will be a C^*-algebrar

DEFINITION. *A continuity structure for T and the* $\{A_t\}$ is a linear space F of vecto. fields on T, with values in the $\{A_t\}$, satisfying:

(i) If $x \in F$, the real-valued function $t \to \|x(t)\|$ is continuous on T;

(ii) for each t in T, $\{x(t) \mid x \in F\}$ is dense in A_t.

If each A_t is a C^*-algebra, we require also that

(iii) F is closed under pointwise multiplication and involution.

If all A_t are equal to the same A, then the set of all constant functions on T to A forms a continuity structure, the so-called *product structure*.

Let us fix a continuity structure F for T, $\{A_t\}$.

DEFINITION. A vector field x is *continuous* (*with respect to* F) *at* t_0, if for each $\varepsilon > 0$, there is an element y of F and a neighborhood U of t_0 such that $\|x(t) - y(t)\| < \varepsilon$ for all t in U. We say that x is *continuous on* S if it is continuous at all points of S.

The following lemmas are easily verified:

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(1) References to previous work on this subject are given in the Introduction.

LEMMA 1.1. *If a vector field x is continuous with respect to F at t_0, then $t \to \|x(t)\|$ is continuous at t_0.*

LEMMA 1.2. *The vector fields continuous (with respect to F) at t_0 form a linear space, closed under multiplication by complex-valued functions on T which are continuous at t_0. If each A_t is a C^*-algebra, the vector fields continuous at t_0 are also closed under pointwise multiplication and involution.*

LEMMA 1.3. *A vector field x is continuous (with respect to F) at t_0 if and only if, for each y in F, the function $t \to \|x(t) - y(t)\|$ is continuous at t_0.*

LEMMA 1.4. *If a sequence of vector fields $\{x_n\}$ continuous (with respect to F) at t_0 converges uniformly on T to a vector field x, then x is continuous at t_0 (with respect to F).*

LEMMA 1.5. *For every t in T, and every α in A_t, there is a vector field x, continuous on T with respect to F, such that $x(t) = \alpha$.*

DEFINITION. If F' is another continuity structure for T and the $\{A_t\}$, then we shall say that F and F' are *strictly equivalent* ($F \sim F'$) if, for all t in T, a vector field is continuous at t with respect to F if and only if it is so with respect to F'.

LEMMA 1.6. *If F' is another continuity structure for T and the $\{A_t\}$, and if there exists a family G of vector fields such that*

(i) *each x in G is continuous on T with respect to both F and F', and*

(ii) *for each t in T, $\{x(t) \mid x \in G\}$ is dense in A_t,*

then $F \sim F'$.

Proof. Let F'' be the linear span of G. Then clearly F'' is a continuity structure contained in both F and F'. Combining the definition of continuity with Lemma 1.3, we see that $F \sim F''$; similarly $F' \sim F''$. Hence $F \sim F'$.

From here on, until Chapter IV, the A_t will always be C^*-algebras.

As in the Introduction, we make the following definition:

DEFINITION. A *full algebra of operator fields* is a family A of operator fields on T satisfying:

(i) A is a *-algebra, i.e., it is closed under all the pointwise algebraic operations;

(ii) for each x in A, the function $t \to \|x(t)\|$ is continuous on T and vanishes at infinity;

(iii) for each t, $\{x(t) \mid x \in A\}$ is dense in A_t;

(iv) A is complete in the norm $\|x\| = \sup_t \|x(t)\|$.

Clearly A is a C^*-algebra; hence (iii) could be strengthened to the statement that $\{x(t) \mid x \in A\} = A_t$. The algebra A_t will be called the *component of A at t*. We refer to T as the *base space*.

A full algebra of operator fields is evidently a continuity structure. If F is any continuity structure, let us define $C_0(F)$ to be the family of all vector fields x which are continuous on T with respect to F, and for which $t \to \|x(t)\|$ vanishes at infinity. In view of the preceding lemmas, $C_0(F)$ is a full algebra of operator fields—indeed, a maximal one. In fact, the following lemma is easily verified:

LEMMA 1.7. *For any full algebra A of operator fields on T, the following three conditions are equivalent:*

(i) *A is a maximal full algebra of operator fields;*
(ii) *$A = C_0(F)$ for some continuity structure F;*
(iii) *$A = C_0(A)$.*

Such a maximal full algebra A of operator fields may sometimes be called a *continuous direct sum* of the $\{A_t\}$. It is clearly *separating*, in the sense that, if $s, t \in T$, $s \neq t$, $\alpha \in A_s$, and $\beta \in A_t$, there is an x in A such that $x(s) = \alpha$, $x(t) = \beta$.

1.2. The dual spaces of algebras of operator fields

We recall that the *dual space* of a C^*-algebra A is the family $\hat{A}$ of all irreducible $*$-representations of A, topologized so that the closure of a subset W of $\hat{A}$ is the set of all those R in $\hat{A}$ such that $\bigcap_{S \in W}$ Kernel $(S) \subset$ Kernel (R). In this section we investigate the dual space of a full algebra of operator fields. Let T be a locally compact Hausdorff space; and let A be a full algebra of operator fields with base space T and component algebras $\{A_t\}_{t \in T}$. The following lemma is proved as Theorem 1, p. 301 of [11]: (1)

LEMMA 1.8. *If A is maximal, then any closed two-sided ideal I of A is of the form*

$$I = \{x \in A \mid x(t) \in I_t \text{ for all } t \text{ in } T\},$$

where, for each t, $I_t = \{x(t) \mid x \in I\}$.

THEOREM 1.1. *To each R in $\hat{A}$ there corresponds an element s of T and a Q in $(A_s)^\wedge$ such that*

$$R_x = Q_{x(s)} \quad (x \in A). \tag{1}$$

Proof. Assume first that A is maximal. Let I be the kernel of R, and define I_t as in Lemma 1.8. We consider the set Z of those t in T for which $I_t \neq A_t$. Assume now that Z contains two distinct elements t_1 and t_2. Let U_1 and U_2 be disjoint neighborhoods of t_1

(1) In the lemma as proved by Naimark in [11], T is assumed compact. This causes no trouble as we can adjoin the point at infinity to our T, and associate with it the 0-dimensional C*-algebra. Condition 2) of Naimark's version follows from the fact that every C*-algebra has an approximate identity.

and t_2 respectively; and denote by K_i the closed two-sided ideal of A consisting of all x which vanish outside U_i. Since $I_{t_i} \neq A_{t_i}$, it follows that $K_i \not\subset I$. On the other hand, $K_1 K_2 = \{0\}$. But this contradicts the fact that $R(A)$ has no ideal divisors of 0 (Lemma 2.5 of [8]); so Z has at most one element. If Z were void, Lemma 1.8 would give $I = A$. Thus Z has exactly one element s; and by Lemma 1.8 $I = \{x \in A \mid x(s) \in I_s\}$. Hence R induces an irreducible representation of A_s; and the theorem is proved for the case that A is maximal.

In the general case, let B be the maximal full algebra containing A. If $S \in \hat{A}$, there is an S' in $\hat{B}$ which acts in a space $H(S')$ containing $H(S)$, and such that $S' \mid A$ (1) coincides with S on $H(S)$. (2) Applying the preceding paragraph to S', we find a t in T and a Q in $(A_t)^{\hat{}}$ such that $S'_x = Q_{x(t)}$ $(x \in B)$. Since A is full, $S' \mid A$ is irreducible. So $H(S') = H(S)$, $S' \mid A = S$, and the theorem is proved.

Assume now that A is maximal. Then the uniqueness of the s and Q in Theorem 1.1 is evident. Thus there is a natural one-to-one correspondence between $\hat{A}$ and the set P of all pairs (t, Q), where $t \in T$ and $Q \in \hat{A}_t$. In the following two theorems we identify $\hat{A}$ with P (writing, for example, Kernel (s, Q) instead of Kernel (R), where R is given by (1)).

If J_t is a linear subspace of A_t for each t, let us define

$$\varliminf_{t \to s} J_t = \{x(s) \mid x \in A,\ x(t) \in J_t \text{ for all } t\}.$$

The topology of $\hat{A}$ is then given by the following theorem, the proof of which follows immediately from the definitions (and Theorem 1.1):

THEOREM 1.2. *Let A be maximal. If $W \subset \hat{A}$, we denote by W_t the set $\{Q \in \hat{A}_t \mid (t, Q) \in W\}$. An element $R = (t_0, Q^0)$ of $\hat{A}$ belongs to the closure of W if and only if*

$$\text{Kernel } (Q^0) \supset \varliminf_{t \to t_0} \{\bigcap_{S \in W_t} \text{Kernel } (S)\}.$$

(If W_t is void, $\bigcap_{S \in W_t}$ Kernel $(S) = A_t$).

COROLLARY. *If A is maximal and A_t is a simple dual C^*-algebra* (3) *for each t, then $\hat{A}$ is homeomorphic with T.*

Proof. It is well known (4) that each $\hat{A}_t$ contains only one element in this case. Now apply Theorems 1.1 and 1.2.

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(1) $S' \mid A$ denotes the restriction of S' to A.

(2) See, for example, Theorem 1, p. 274 of [11]. For C^*-algebras the hypothesis of a unit element, occurring in this reference, is inessential.

(3) A C^*-algebra is dual if it has a faithful representation by completely continuous operators. A simple dual C^*-algebra is one which is isomorphic with the algebra of all completely continuous operators on some Hilbert space.

(4) This is proved just as in the finite-dimensional case.

LEMMA 1.9. *If B is any C^*-algebra, W a family of closed two-sided ideals of B, and $J = \bigcap_{I \in W} I$, then*

$$\|x/J\| = \sup_{I \in W} \|x/I\| \quad (x \in B).$$

Proof. The natural homomorphism of B/J into the C^*-direct sum $\sum_{I \in W} B/I$ is an isomorphism, hence an isometry.

Lemma 1.9 could be rephrased as follows: If $\{I_\alpha\}$ is a decreasing net of closed two-sided ideals of B, and $J = \bigcap_\alpha I_\alpha$, then

$$\|x/J\| = \lim_\alpha \|x/I_\alpha\| \quad (x \in B).$$

The corresponding lemma for increasing nets is valid in a general Banach space; and its proof is extremely elementary.

THEOREM 1.3. *Suppose that all A_t are the same C^*-algebra B; and that A consists of all norm-continuous functions on T to B which vanish at infinity. Then the topology of $\hat{A}$ is that of $T \times \hat{B}$.*

Proof. By Theorem 1.1, $\hat{A}$ coincides as a set with $T \times \hat{B}$ (see the remark following Theorem 1.1).

I. Suppose that

$$(t_\alpha, Q^\alpha) \to (t_0, Q^0) \text{ in } \hat{A}. \tag{2}$$

If t_0 did not belong to the closure of $\{t_\alpha\}$, we could find an x in A with $x(t_\alpha) = 0$ for all α and $Q^0_{x(t_0)} \neq 0$; but this would contradict (2). Therefore $t_0 \in \overline{\{t_\alpha\}}$; and the same holds for any subnet of $\{t_\alpha\}$. Hence

$$t_\alpha \to t_0 \text{ in } T. \tag{3}$$

Let β be any element of $\bigcap_\alpha$ Kernel (Q^α); and choose an x in A whose value is β throughout some neighborhood of t_0. Then, by (3), $x \in$ Kernel (t_α, Q^α) for all large enough α; so that by (2) $x \in$ Kernel (t_0, Q^0), from which follows $\beta \in$ Kernel (Q^0). This shows that $\bigcap_\alpha$ Kernel $(Q^\alpha) \subset$ Kernel (Q^0), that is, Q^0 belongs to the closure of $\{Q^\alpha\}$ in $\hat{B}$. Since the same holds for any subnet, we have shown

$$Q^\alpha \to Q^0 \text{ in } \hat{B}. \tag{4}$$

Now (3) and (4) give

$$(t_\alpha, Q^\alpha) \to (t_0, Q^0) \text{ in } T \times \hat{B}. \tag{5}$$

II. Now assume (5), that is, (3) and (4). Let x be any element of $\bigcap_\alpha$ Kernel (t_α, Q^α); then

$$x(t_\alpha) \in \text{Kernel } (Q^\alpha) \text{ for each } \alpha. \tag{6}$$

By (3) and the norm-continuity of x, for each $\varepsilon > 0$ there is an α_0 such that

$$\|x(t_\alpha) - x(t_0)\| < \varepsilon \text{ for all } \alpha \succ \alpha_0.$$

This and (6) combine to give

$$\|Q^{\alpha}_{x(t_0)}\| < \varepsilon \text{ for } \alpha \succ \alpha_0.$$

Applying Lemma 1.9 to the last inequality, we have

$$\|x(t_0)/\bigcap_{\alpha \succ \alpha_0} \text{Kernel } (Q^{\alpha})\| < \varepsilon. \tag{7}$$

But, by (4) $\bigcap_{\alpha \succ \alpha_0}$ Kernel $(Q^{\alpha}) \subset$ Kernel (Q^0); so that from (7), $\|Q^0_{x(t_0)}\| < \varepsilon$. By the arbitrariness of ε, this gives $Q^0_{x(t_0)} = 0$, or $x \in$ Kernel (t_0, Q^0). We have proved that

$$\bigcap_{\alpha} \text{Kernel } (t_\alpha, Q^{\alpha}) \subset \text{Kernel } (t_0, Q^0),$$

hence that (t_0, Q^0) belongs to the closure of $\{(t_\alpha, Q^{\alpha})\}$ in $\hat{A}$. Since the same holds for any subnet of $\{(t_\alpha, Q^{\alpha})\}$, (2) must hold.

Now I. and II. show that (2) and (5) are equivalent. This proves the theorem.

Now let A be a full algebra of operator fields on T with components $\{A_t\}$, and let B be the maximal full algebra of operator fields (with components $\{A_t\}$) which contains A. Theorem 1.2 gives us the topology of $\hat{B}$. The following lemma then gives that of $\hat{A}$, if we observe (Theorem 1.1) that each T in $\hat{B}$ is still irreducible when restricted to A.

LEMMA 1.10. *Let B be any C^*-algebra, and A any C^*-subalgebra of B such that $T|A$ is irreducible for each T in $\hat{B}$. Introduce into $\hat{B}$ the equivalence relation $\sim$ such that $T \sim S$ if and only if $T|A \cong S|A$. Then:*

(i) *Every R in $\hat{A}$ is of the form $T|A$ for some T in $\hat{B}$. Thus there is a natural identification of $\hat{A}$ with the set of equivalence classes $\hat{B}/\sim$.*

(ii) *With this identification, the topology of $\hat{A}$ coincides with the quotient topology of $\hat{B}/\sim$ derived from the topology of $\hat{B}$.*

Proof. To prove (i) we repeat the argument of the last paragraph of the proof of Theorem 1.1. The only mildly non-trivial part of the proof of (ii) consists in showing that if W is a closed subset of $\hat{B}$ and a union of $\sim$ classes, then $\tilde{W}$ (the set of equivalence classes contained in W) is closed in $\hat{A}$. Let $I = \{x \in B \mid T_x = 0 \text{ for all } T \text{ in } W\}$, $S \in \hat{B}$, and $S|A$ be an element of the closure of $\tilde{W}$. Then $A \cap I \subset$ Kernel (S); so that $S|A$ induces an irreducible representation S' of $A/A \cap I \cong A/I \subset B/I$, which extends to an irreducible representation T' of B/I acting in the same space as S'. If T is the element of $\hat{B}$ induced by T', we have

$T \in W$ (since W is closed), and $T|A = S|A$. So $S \sim T \in W$, whence $S \in W$, or $S|A \in \tilde{W}$. Thus $\tilde{W}$ is closed.

COROLLARY. *If A is a full algebra of operator fields (on a base space T) whose component algebras are all simple dual C^*-algebras,* (1) *then $\hat{A}$ is Hausdorff.*

Proof. If B is the maximal full algebra of operator fields containing A, $\hat{B} \cong T$ by the Corollary to Theorem 1.2. It is easy to show that, in the present case, the equivalence relation $\sim$ of Lemma 1.10 is a closed subset of $T \times T$, and that each equivalence class is compact. So $T/\sim$ is locally compact and Hausdorff. Now invoke Lemma 1.10.

1.3. Subalgebras of algebras of operator fields

We conclude this chapter with an interesting consequence of Glimm's generalization [5] of the Stone–Weierstrass Theorem.

If R is a relation, we write xRy to mean that the pair (x, y) belongs to R.

DEFINITION. Let A and B be C^*-algebras. An (A, B) *correlation* is a relation R contained in $A \times B$ such that, for some third C^*-algebra C and some *-homomorphisms f and g of A and B respectively *onto* C, we have

$$xRy \text{ if and only if } f(x) = g(y) \text{ (for all } x \text{ in } A \text{ and } y \text{ in } B).$$

An (A, B) correlation can also be described as a closed *-subalgebra R of the direct product algebra $A \times B$ such that $\{x \mid (x, y) \in R \text{ for some } y\} = A$ and $\{y \mid (x, y) \in R \text{ for some } x\} = B$.

Now let B be a maximal full algebra of operator fields on a base space T, with component algebras $\{A_t\}$. If r and s are distinct points of T and R is an (A_r, A_s) correlation, we define

$$B(r, s; R) = \{x \in B \mid x(r) R x(s)\}.$$

Clearly $B(r, s; R)$ is a full algebra of operator fields with the same component algebras A_t.

THEOREM 1.4. (Stone–Weierstrass–Glimm). *Let B be as in the preceding paragraph, and A any full algebra of operator fields contained in B, with the same components $\{A_t\}$. Then A is the intersection of those $B(r, s; R)$ (where $r \neq s$ and R is an (A_r, A_s) correlation) which contain A.*

Proof. Let A^0 be the intersection of all those $B(r, s; R)$ which contain A. Adjoin(2)

(1) See footnote (3) on p. 238.

(2) If A^0 already has a unit element, A_1^0 is the direct product of A^0 with the one-dimensional C^*-algebra.

to A^0 a unit element 1 not already in A^0, getting the C^*-algebra A_1^0. Let A_1 be the C^*-subalgebra of A_1^0 spanned by A and 1.

Let us denote by $P(A_1^0)$ the weak $*$-closure of the set of all pure states (i.e., indecomposable positive linear functionals f with $f(1)=1$) of A_1^0. Suppose now that f and g are distinct elements of $P(A_1^0)$ whose restrictions to A_1 coincide. We shall obtain a contradiction.

Suppose $f\,|\,A^0 \neq 0$. Then $f\,|\,A^0$ is a weak $*$-limit of pure states $\{h_\nu\}$ of A^0; and, for each ν, Theorem 1.1 enables us to write

$$h_\nu(x) = h'_\nu(x(t_\nu)), \tag{8}$$

where $t_\nu \in T$ and h'_ν is a pure state of A_{t_ν}. If $t_\nu \to \infty$ in T, then by (8) $h_\nu(x) \to 0$ for each x in A^0, whence $f \equiv 0$ on A^0, which was not the case. So $t_\nu \not\to \infty$, and we may pass to a subnet and assume that $t_\nu \to t$ in T. Then, whenever $x \in A^0$ and $x(t)=0$, we have by (8) $f(x) = \lim h_\nu(x) = 0$; so that f induces a continuous positive linear functional f' on A_t:

$$f(x) = f'(x(t)) \quad (x \in A^0). \tag{9}$$

If $f \equiv 0$ on A^0, (9) is trivially true (take $f'=0$). So we may always assume that f has the form (9). Similarly,

$$g(x) = g'(x(s)) \quad (x \in A^0), \tag{10}$$

where $s \in T$ and g' is a continuous positive linear functional on A_s.

Assume that $t=s$. Since $f \equiv g$ on A, and A is a full algebra of operator fields, (9) and (10) imply that $f \equiv g$ on A^0 and hence on A_1^0. This contradicts the distinctness of f and g.

Assume that $t \neq s$. Let U be the $*$-representation of A with cyclic vector ξ such that $f(x) = g(x) = (U_x\xi, \xi)$ for x in A. By (9) and (10) $U_x = 0$ if either $x(t)=0$ or $x(s)=0$; so U induces representations U' and U'' of A_t and A_s respectively. Clearly range (U) = range (U') = range (U''), and

$$U'_{x(t)} = U''_{x(s)} \text{ for } x \text{ in } A. \tag{11}$$

Thus U defines an (A_t, A_s) correlation R ($\alpha R \beta$ if and only if $U'_\alpha = U''_\beta$), and $A \subset B(t, s; R)$. It follows that $A^0 \subset B(t, s; R)$, so that (11) holds for all x in A^0. Thus, by (9) and (10), f and g coincide on A^0, and hence on A_1^0. This again contradicts the distinctness of f and g.

Thus we have reached a contradiction; and we conclude that A_1 separates the elements of $P(A_1^0)$. By Theorem 1 of [5], $A_1^0 = A_1$. Since $1 \notin A^0$, this implies that $A^0 = A$.

COROLLARY.(1) *A full separating algebra of operator fields is maximal.*

(1) For the definition of "separating" see the paragraph following Lemma 1.7. In case each component algebra is dual, this Corollary is essentially due to Kaplansky ([8], Theorem 3.3).

II. The representation of a C^*-algebra as an algebra of operator fields

2.1. The regularized transform of a C^*-algebra

For this chapter we fix an arbitrary C^*-algebra A.

Can A be represented as a full algebra of operator fields on some base space T, with components A_t? The answer is trivially "yes", unless the components A_t are restricted in some way. If the A_t are all required to be primitive,(1) the answer in general is "no".(2) But if we require only that the t for which A_t is primitive be *dense* in T, then the answer is always "yes". In this section we construct this representation of A.

As we mentioned before, the *dual space* $\hat{A}$ is the space of all unitary equivalence classes of irreducible $*$-representations of A, equipped with the hull-kernel topology.(3) Sometimes it is convenient to consider the space $\check{A}$ of all primitive ideals (i.e., kernels of elements of $\hat{A}$), also equipped with the hull-kernel topology; this we will call the *ideal dual space*. $\check{A}$ is obtained, both setwise and topologically, by identifying elements of $\hat{A}$ with the same kernel. By the *transform* of an element a of A we mean the function $T \to T_a$ on $\hat{A}$ (or the function $I \to a + I$ on $\check{A}$, according to context).

If R is any $*$-representation(4) of A, the function N_R on A defined by $N_R(x) = \|R_x\|$ is called the *norm-function* of R. The space of all norm-functions of $*$-representations of A, equipped with the topology of pointwise convergence on A, will be called $\mathfrak{N}$. The following lemma is easily verified:

LEMMA 2.1. *If $\{N^i\}$ is a net of norm-functions and $\lim_i N^i(x) = M(x)$ for all x in A, then M is the norm-function of some representation.*

COROLLARY. *$\mathfrak{N}$ is a compact Hausdorff space.*

Proof. This follows from Tychonoff's theorem, the preceding lemma, and the fact that $M(x) \leqslant \|x\|$ for x in A, M in $\mathfrak{N}$.

We now define $\hat{A}^r$ as the closure in $\mathfrak{N}$ of the set of all norm-functions N_R associated with elements R of $\hat{A}$. By the preceding corollary $\hat{A}^r$ is a compact Hausdorff space. To each N in $\hat{A}^r$ let $A_N = A/I_N$, where I_N is the closed two-sided ideal $\{x \mid N(x) = 0\}$; and to each x in A, associate the operator field $\tilde{x}$ on $\hat{A}^r$ defined by $\tilde{x}(N) = x/I_N \in A_N$. The family of all $\tilde{x}$ $(x \in A)$ will be called $\tilde{A}$. It is clearly a full algebra of operator fields on $\hat{A}^r$ (with

(1) A C^*-algebra is primitive if it has a faithful irreducible representation.

(2) Indeed, by the Corollary to Lemma 1.10, a *CCR* algebra whose dual space is not Hausdorff cannot be isomorphic to a full algebra of operator fields all of whose component algebras are primitive.

(3) See [1].

(4) The zero representation is admitted as a $*$-representation of A.

component algebras $\{A_N\}$), and is isomorphic with A. By the definition of $\hat{A}^r$, $\{N \mid A_N$ is primitive$\}$ is dense in $\hat{A}^r$.

Definition. The compact Hausdorff space $\hat{A}^r$ will be called the *regularized dual space* of A. The operator field $\tilde{x}$ is the *regularized transform* of x; and $\tilde{A}$ is the *regularized transform* of A.

2.2. The Hausdorff compactification of $\hat{A}$.

For the applications of our theory it is very useful to observe that $\hat{A}^r$ can be obtained by another construction, which uses the topology of $\hat{A}$ and nothing more. For this construction, let us note that $\hat{A}$ is always locally compact in the following sense:

Definition. A (not necessarily Hausdorff) topological space X is *locally compact* if, to each x in X and each neighborhood U of x, there is a compact neighborhood of x contained in U.

Theorem 2.1. (1) *For every C^*-algebra A, $\hat{A}$ is locally compact.*

Proof. Let U be an open neighborhood of an element T of $\hat{A}$. By the definition of the hull-kernel topology there is an element a of A such that $\|T_a\| = 1$ and $S_a = 0$ for all S in $\hat{A} - U$. Let $V = \{S \in \hat{A} \mid \|S_a\| \geqslant \frac{1}{2}\}$, $W = \{S \in \hat{A} \mid \|S_a\| > \frac{1}{2}\}$. By Lemma 4.3 of [8], V is compact. Since $S \to \|S_a\|$ is lower semi-continuous (Lemma 2.2 of [1]), W is open. Since $T \in W \subset V$, V is a compact neighborhood of T contained in U.

Now in a separate note [2], we have given a general construction for passing from a locally compact (not necessarily Hausdorff) space X to a compact *Hausdorff* space $H(X)$. Let us review that construction. Starting with a locally compact space X, we define $\mathcal{C}(X)$ as the family of all closed subsets of X. For each compact subset C of X and each finite family $\mathfrak{F}$ of non-void open subsets of X, let $U(C, \mathfrak{F})$ be the set of all Y in $\mathcal{C}(X)$ such that (i) $Y \cap C = \Lambda$, and (ii) $Y \cap B \neq \Lambda$ for each B in $\mathfrak{F}$. The set of all such $U(C, \mathfrak{F})$ forms a basis for the open sets of a topology for $\mathcal{C}(X)$; and $\mathcal{C}(X)$ with this topology is a compact Hausdorff space. Now $H(X)$ is defined as the closure in $\mathcal{C}(X)$ of the family of all closures $\{x\}^-$ of one-element subsets of X. Being a closed subset of $\mathcal{C}(X)$, $H(X)$ is compact and Hausdorff. As in [2], this $H(X)$ will be called the *Hausdorff compactification* of X.

A net $\{x_\nu\}$ of elements of X is *primitive* if $x_\nu \to y$ whenever there is a subnet $\{x'_\mu\}$ of $\{x_\nu\}$ such that $x'_\mu \to y$. By the *limit set* of a net $\{x_\nu\}$ we mean the set of all y such that $x_\nu \to y$.

(1) See p. 235 of [8] for the case that $\hat{A}$ is Hausdorff. It is stated on the same page of [8] that $\hat{A}$ need not be locally compact in the general case. Professor Kaplansky has informed the author that that statement was an error.

It is shown in [2] that $H(X)$ coincides with the family of all those subsets of X which are the limit sets of some primitive net of elements of X.

THEOREM 2.2. *The space $\mathcal{N}$ of all norm-functions on A (with the topology of pointwise convergence) is homeomorphic with the space $\mathcal{C}(\hat{A})$ of all closed subsets of $\hat{A}$ (or, equivalently, with the space $\mathcal{C}(\check{A})$ of all closed subsets of $\check{A}$). The homeomorphism is implemented by the mapping M which sends an element Y of $\mathcal{C}(\hat{A})$ into the norm-function M_Y given by*

$$M_Y(a) = \sup_{S \in Y} \|S_a\| \, (a \in A). \tag{1}$$

The image of $H(\hat{A})$ under M is precisely the regularized dual space $\hat{A}^r$.

Proof. By Lemma 1.9, the right side of (1) is $\|a/I\|$, where $I = \bigcap_{S \in Y} \text{Kernel}(S)$. Hence M_Y is a norm-function. Conversely, since every norm-function is of the form $a \to \|a/I\|$ (for some closed two-sided ideal I), and hence equal to M_Y where $Y = \{S \in \hat{A} \mid \text{Kernel}(S) \supset I\}$, the range of M is all of $\mathcal{N}$. Let Y and Z be distinct elements of $\mathcal{C}(\hat{A})$; in fact, let $T \in Y - Z$. Since Z is closed there is an a in A for which $T_a \neq 0$ and $S_a = 0$ for all S in Z. But then $M_Z(a) = 0$, and $M_Y(a) \neq 0$. So M is one-to-one.

By the compactness of $\mathcal{N}$ (Corollary of Lemma 2.1), M will be a homeomorphism if it is continuous, i.e., if $Z \to M_Z(a)$ is continuous on $\mathcal{C}(\hat{A})$ for each a in A. Fix a; and let Y be in $\mathcal{C}(\hat{A})$ and $\varepsilon > 0$. By Lemma 4.3 of [8], $C = \{S \in \hat{A} \mid \|S_a\| \geqslant M_Y(a) + \varepsilon\}$ is compact in $\hat{A}$; so that $W = \{Z \in \mathcal{C}(\hat{A}) \mid Z \cap C = \Lambda\}$ is a neighborhood of Y in $\mathcal{C}(\hat{A})$ on which $M_Z(a) \leqslant M_Y(a) + \varepsilon$. It follows that $Z \to M_Z(a)$ is upper semi-continuous. Now let T be an element of Y such that $\|T_a\| > M_Y(a) - \frac{1}{2}\varepsilon$. By the lower semi-continuity of $S \to \|S_a\|$ (Lemma 2.2 of [1]) there is a neighborhood U of T on which $\|S_a\| > M_Y(a) - \varepsilon$. Thus, if W' is the neighborhood of Y in $\mathcal{C}(\hat{A})$ consisting of all Z in $\mathcal{C}(\hat{A})$ which intersect U, we have $M_Z(a) > M_Y(a) - \varepsilon$ for all Z in W'. It follows that $Z \to M_Z(a)$ is lower semi-continuous. Being both lower and upper semi-continuous, $Z \to M_Z(a)$ must be continuous on $\mathcal{C}(\hat{A})$.

It remains only to show that the image of $H(\hat{A})$ under M is $\hat{A}^r$. This follows immediately from the fact that, if $T \in \hat{A}$ and $Y = \{T\}^-$, then M_Y is the norm-function of T. Thus the proof is complete.

In view of Theorem 2.2, we may sometimes identify the regularized dual space $\hat{A}^r$ with the Hausdorff compactification $H(\hat{A})$ of $\hat{A}$.

It is of some interest to observe that the compact Hausdorff topology of $\mathcal{C}(\hat{A})$ can be transferred to a compact Hausdorff topology for the space of all closed two-sided ideals of A, if we use the natural one-to-one correspondence between the latter space and $\mathcal{C}(\hat{A})$.

THEOREM 2.3. *The regularized transform $\tilde{A}$ of a C^*-algebra A is a maximal full algebra of operator fields if and only if the ideal dual space $\check{A}$ is Hausdorff.*

Proof. Let $\check{A}$ be Hausdorff. Then $\hat{A}^r$ is just the one-point compactification of $\check{A}$ (see Theorem 2.2); and $\tilde{A}$ is the algebra of all transforms of elements of A. To show that $\tilde{A}$ is maximal, it suffices by Theorem 1.4 to show that there are no correlations between the values of the $\tilde{x}$ in $\tilde{A}$ at different points. Such a correlation would imply that there were two distinct points I and J of $\check{A}$, and an irreducible $*$-representation T of A whose kernel contained both I and J. But then $K = \text{Kernel }(T)$ would belong to the closures of both $\{I\}$ and $\{J\}$; and $\check{A}$ would not be Hausdorff. Hence there are no correlations, and $\tilde{A}$ is maximal.

Now assume that $\check{A}$ is not Hausdorff. Then there is a primitive net $\{I_\nu\}$ of elements of $\check{A}$ whose limit set Y contains two distinct ideals J and K. Since closed sets separate points in $\check{A}$, one of $\{J\}$ and $\{K\}$ does not contain the other in its closure; say $K \notin \{J\}^- = Z$. Thus Z and Y are two elements of $\hat{A}^r$ (1) with $Z \subset Y$, $Z \neq Y$. If M is the mapping of Theorem 2.2, the fact that $Z \subset Y$ implies that $M_Z \leqslant M_Y$. So there is a $*$-homomorphism F of A_Y onto A_Z such that $\tilde{a}(Z) = F(\tilde{a}(Y))$ for all a in A. This, however, is a correlation between the values of the $\tilde{a}$ at the distinct points Z and Y of $\hat{A}^r$. So $\tilde{A}$ is not maximal.

COROLLARY. *If $\check{A}$ is Hausdorff, the algebra of all transforms of A is closed under multiplication by bounded continuous complex functions on $\check{A}$.*

III. C^*-algebras and fibre bundles

3.1. Extension of matrix units

We begin with two lemmas leading to a theorem which enables us to extend a finite system of "matrix units" throughout a neighborhood when they are given at a point.

Throughout this section we fix a full algebra A of operator fields on a locally compact Hausdorff base space T, with component algebras $\{A_t\}$.

LEMMA 3.1. *Let s be an element of T, and let $\pi_1, \ldots, \pi_n$ be a finite number of pairwise orthogonal non-zero projections in A_s. Then there exist a neighborhood U of s, and n elements $p_1, \ldots, p_n$ of A, such that*

(i) $p_i(s) = \pi_i \quad (i = 1, 2, \ldots, n)$;

(ii) *for each t in U, the $p_1(t), \ldots, p_n(t)$ are pairwise orthogonal non-zero projections in A_t.*

The proof of this lemma is essentially contained in Part A of the proof of Lemma 2.5 of [1].

(1) $Y \in \hat{A}^r$ in virtue of the remark preceding Theorem 2.2.

LEMMA 3.2. *Suppose that $s \in T$, π_1 and π_2 are projections in A_s, and α is an element of A_s such that $\alpha^* \alpha = \pi_1$, $\alpha \alpha^* = \pi_2$. Suppose further that p_1 and p_2 are elements of A such that*

(i) *$p_i(s) = \pi_i (i = 1, 2)$; and*

(ii) *there is a neighborhood U of s such that $p_1(t)$ and $p_2(t)$ are projections for all t in U. Then there is an element q in A, and a neighborhood V of s, such that $q(s) = \alpha$ and*

$$(q^* q)(t) = p_1(t), \quad (qq^*)(t) = p_2(t) \textit{ for } t \textit{ in } V.$$

Proof. Choosing an element h' in A such that $h'(s) = \alpha$, and setting $h = p_2 h' p_1$, we have

$$h(s) = \pi_2 \alpha \pi_1 = \alpha, \tag{1}$$

and, for t in U,

$$(p_2 h)(t) = (h p_1)(t) = h(t). \tag{2}$$

Now consider the positive square root $g = (h^* h)^{\frac{1}{2}}$. We have by (1)

$$g(s) = (h(s))^* h(s))^{\frac{1}{2}} = (\alpha^* \alpha)^{\frac{1}{2}} = \pi_1 = p_1(s). \tag{3}$$

For t in U, by (2)

$$(h^* h)(t) = p_1(t)(h^* h)(t) p_1(t);$$

hence, since $p_1(t)$ is a projection,

$$g(t) = p_1(t) g(t) p_1(t) \quad (t \in U). \tag{4}$$

Now by (3) $(g - p_1)(s) = 0$. Hence, narrowing U if necessary we may assume that

$$\|(g - p_1)(t)\| < \tfrac{1}{8} \quad (t \in U). \tag{5}$$

Let φ be a continuous non-negative-valued function on the reals such that $\varphi(0) = 0$ and

$$\varphi(r) = \frac{1}{r} \text{ if } |r - 1| \leq \tfrac{1}{8}.$$

Forming the element $q = h \cdot \varphi(g)$, we have for t in U

$$(q(t))^* q(t) = \varphi(g(t))\, h(t)^* h(t)\, \varphi(g(t)) = (\varphi(g(t))\, g(t))^2 = (\psi(g(t)))^2, \tag{6}$$

where

$$\psi(r) = r\, \varphi(r) = \begin{cases} 0 & \text{if } r = 0, \\ 1 & \text{if } |r - 1| \leq \tfrac{1}{8}. \end{cases} \tag{7}$$

Combining (4), (5), and (7), we find that $\psi(g(t)) = p_1(t)$ for t in U; so, by (6),

$$(q(t))^* q(t) = p_1(t) \quad (t \in U). \tag{8}$$

Now
$$q(s) = h(s)\varphi(g(s)) = \alpha\pi_1 = \alpha; \tag{9}$$

so $(qq^*)(s) = \alpha\alpha^* = \pi_2 = p_2(s)$, that is,

$$(qq^* - p_2)(s) = 0. \tag{10}$$

On the other hand, for t in U, by (2)

$$p_2(t)(qq^*)(t) = p_2(t)h(t)\varphi(g(t))q^*(t) = h(t)\varphi(g(t))q^*(t) = (qq^*)(t). \tag{11}$$

In view of (8), $q(t)$ is a partial isometry for t in U; thus, $(qq^*)(t)$ is a projection, which by (11) is contained in $p_2(t)$. If $(qq^*)(t) \neq p_2(t)$ for some t in U, then $\|(p_2 - qq^*)(t)\| = 1$. By (10) and the continuity of $\|(p_2 - qq^*)(t)\|$, the neighborhood U can be narrowed so that for t in U this cannot happen. Then, for all t in U,

$$(q(t))^* q(t) = p_1(t), \quad q(t)(q(t))^* = p_2(t);$$

and this with (9) completes the proof.

THEOREM 3.1. *Suppose that $s \in T$, and that B is a finite dimensional *-subalgebra of A_s. Then there is a neighborhood U of s, and a mapping $\beta \to x_\beta$ of B into A, such that*

(i) $x_\beta(s) = \beta$ *for all* β in B;

(ii) *for each t in U, $\beta \to x_\beta(t)$ is a *-isomorphism of B onto a finite-dimensional *-subalgebra of A_t.*

Proof. Let $B^1, \ldots, B^r$ be the minimal two-sided ideals of B; and let $\{\beta^i_{jk}\}_{j,k=1,\ldots,n_i}$ form a basis of B^i, where

$$\beta^i_{jk}\beta^i_{pq} = \delta_{kp}\beta^i_{jq}, \quad (\beta^i_{jk})^* = \beta^i_{kj}.$$

Using Lemmas 3.1 and 3.2, we choose a neighborhood U of s, and elements p^i_j $(i = 1, \ldots, r;\ j = 1, \ldots, n_i)$ and q^i_{j1} $(i = 1, \ldots, r;\ j = 2, \ldots, n_i)$ of A such that

(i) $p^i_j(s) = \beta^i_{jj}$, $q^i_{j1}(s) = \beta^i_{j1}$;

(ii) for each t in U, the $p^i_j(t)$ are $\sum_{i=1}^{r} n_i$ orthogonal non-zero projections;

(iii) for each t in U, $i = 1, \ldots, r$, and $j = 2, \ldots, n_i$

$$(q^i_{j1}(t))^* q^i_{j1}(t) = p^i_1(t),$$
$$q^i_{j1}(t)(q_{j1}(t))^* = p^i_j(t).$$

Now define $q^i_{jk} = q^i_{j1}(q^i_{k1})^*$. We easily verify that the linear map of B into A which carries β^i_{jk} into q^i_{jk} has the required properties.

3.2. Homogeneous C^*-algebras

DEFINITION. A C^*-algebra A is *homogeneous of order* n if every irreducible *-representation of A is of the same finite dimension n.

Here is a way of constructing homogeneous C^*-algebras from fibre bundles. Let M_n be the C^*-algebra of all $n \times n$ matrices (with complex entries), and G_n the group of all automorphisms of M_n of the form $a \to u^{-1}au$, where u is a unitary matrix in M_n. Let T be a locally compact Hausdorff space, and $\mathcal{B}$ a fibre bundle ([12], p. 9) with bundle space B, base space T, fibre space M_n, and group G_n. If p is the canonical projection of B onto T, $A_t = p^{-1}(t)$ $(t \in T)$, $\{V_j\}$ is a covering of T by coordinate neighborhoods, and $\{\varphi_j\}$ the corresponding coordinate functions ([12], p. 7), we can transfer to each fibre A_t the algebraic operations and the C^*-algebraic norm of M_n via the mapping $\alpha \to \varphi_j(t, \alpha)$ (where j is so chosen that $t \in V_j$); this makes each A_t into a C^*-algebra isomorphic with M_n. The operations in A_t thus defined are clearly independent of the choice of j.

Now let $C_0(\mathcal{B})$ denote the family of all continuous cross-sections x of $\mathcal{B}$ which vanish at infinity (that is, x is a continuous function on T to B such that $p(x(t)) = t (t \in T)$ and $\lim_{t \to \infty} \|x(t)\| = 0$). Clearly $C_0(\mathcal{B})$ is a C^*-algebra under the pointwise operations and the supremum norm. In fact it is a maximal full algebra of operator fields (1) with component algebras $\{A_t\}$. Thus, by the Corollary of Theorem 1.2, $C_0(\mathcal{B})$ is homogeneous of order n, and $C_0(\mathcal{B})^\wedge$ coincides with T (both setwise and topologically).

Now the converse of this is also true:

THEOREM 3.2. *Every homogeneous C^*-algebra A of order n is isomorphic with some $C_0(\mathcal{B})$, where $\mathcal{B}$ is a fibre bundle with base space $\hat{A}$, fibre space M_n, and group G_n.*

Proof. In the first place, $\hat{A}$ is locally compact and Hausdorff by [8], Theorem 4.2. If we identify A with the algebra of its transforms, A becomes a maximal full algebra of operator fields on $\hat{A}$ ([8], Theorem 4.1 and Lemma 4.3; also the Corollary of Theorem 1.4 of this paper). Let B denote the set of all pairs (T, α), where $T \in \hat{A}$ and $\alpha \in T(A)$.

We shall construct a fibre bundle with bundle space B. For this we choose (i) a covering of $\hat{A}$ by open sets $\{U_i\}$, and (ii) for each i a map f_i of M_n into A, such that, whenever

(1) See § 1.1.

$T \in U_i$, the mapping $a \to T_{fi(a)}$ is a *-isomorphism of M_n onto $T(A)$. These choices are possible by Theorem 3.1. Now let $\mathcal{B}$ be the fibre bundle with bundle space B, base space $\hat{A}$, projection sending (T, α) into T, fibre space M_n, group G_n, coordinate neighborhoods $\{U_i\}$, and coordinate functions $\{\varphi_i\}$ sending (T, a) into $(T, T_{fi(a)})$. As a matter of fact, for $\mathcal{B}$ to be a fibre bundle the coordinate transformations ([12], p. 8) must be continuous on the intersections $U_i \cap U_j$. This amounts to saying that, for each a in M_n, $a \to g_T(a)$ is continuous on $U_i \cap U_j$, where $g_T(a) = b$ is the element of M_n defined by the condition that $T_{fi(a)} = T_{fj(b)}$. But this follows easily from the continuity of the norm-functions $T \to \|T_x\|$ $(x \in A)$.

Next, we verify without difficulty that, if an operator field X on $\hat{A}$ is continuous with respect to the continuity structure A (in the sense of § 1.1), then $T \to (T, X(T))$ is a continuous cross-section of $\mathcal{B}$, and conversely. Thus the family $C_0(\mathcal{B})$ of all continuous cross-sections of $\mathcal{B}$ vanishing at infinity coincides(1) with the maximal full algebra of operator fields containing A; and this is A, since A is maximal. The proof is now complete.

Fix an integer n and a locally compact Hausdorff space T. Two fibre bundles $\mathcal{B}$ and $\mathcal{B}'$ with base space T, fibre space M_n, and group G_n will be said to be *weakly equivalent* if there exists a third such fibre bundle $\mathcal{B}''$ such that (i) $\mathcal{B}$ and $\mathcal{B}''$ are equivalent in the sense of [12], p. 11, and (ii) $\mathcal{B}''$ is induced from $\mathcal{B}'$ by a homeomorphism of T onto itself. It is easy to see that, if $\mathcal{B}$ and $\mathcal{B}'$ are two such fibre bundles, $C_0(\mathcal{B})$ and $C_0(\mathcal{B}')$ are *-isomorphic if and only if $\mathcal{B}$ and $\mathcal{B}'$ are weakly equivalent. Thus the problem of classifying all homogeneous C^*-algebras of given order n and with given dual space T is reduced by Theorem 3.2 to that of classifying to within weak equivalence all fibre bundles with base space T, fibre space M_n, and group G_n, or, equivalently (see [12], p. 36), of classifying all principal fibre bundles with base space T and group G_n. Generally speaking, for given n and T, there will exist many inequivalent such bundles; so that a homogeneous C^*-algebra is not fully determined by its order and its dual space.

3.3. Fibre structures

The notion of fibre structure, which we shall now introduce, lies in between the general notion of a continuity structure and the special continuity structures arising (as in § 3.2) from fibre bundles. It permits the "fibre" to vary essentially from point to point of the base space.

This section is confined to definitions and elementary facts. All proofs are of a routine nature, and are omitted in the interest of brevity. It seems probable that fibre structures

(1) Here we are identifying the operator field X with the cross-section $T \to (T, X(T))$.

will eventually prove to be of some interest; but we have so far not obtained any results about them substantial enough to justify more than passing mention.

Let T be a fixed locally compact Hausdorff base space; and suppose that a C^*-algebra A_t is given for each t in T. An operator field is a function x on T such that $x(t) \in A_t$ for each t.

DEFINITION. A *fibre element* is a triple $\mathfrak{J} = (C, W, I)$, where

(i) $C = C(\mathfrak{J})$ is a C^*-algebra, called the *fibre* of $\mathfrak{J}$,

(ii) $W = W(\mathfrak{J})$ is an open subset of T, called the *domain* of $\mathfrak{J}$,

(iii) for each t in W, I_t is a $*$-isomorphism of C into (but not necessarily onto) A_t.

DEFINITION. A *fibre structure* (for T, $\{A_t\}$) is a family $\mathfrak{B}$ of fibre elements such that:

(i) For each t in T, each $\varepsilon > 0$, and each pair of elements α_1 and α_2 of A_t, there is a fibre element (C, W, I) in $\mathfrak{B}$, and elements β_1, β_2 of C, such that $t \in W$ and $\|I_t(\beta_i) - \alpha_i\| < \varepsilon$ $(i = 1, 2)$;

(ii) If $\mathfrak{J} = (C, W, I)$ and $\mathfrak{J}' = (C', W', I')$ are in $\mathfrak{B}$, $t \in W \cap W'$, $\alpha \in C$, $\alpha' \in C'$, $\varepsilon > 0$, and $\|I_t\alpha - I'_t\alpha'\| < \varepsilon$, then $\|I_s\alpha - I'_s\alpha'\| < \varepsilon$ for all s in some neighborhood of t.

For the time being we fix a fibre structure $\mathfrak{B}$ (for T, $\{A_t\}$).

DEFINITION. An operator field x *is continuous* at a point s of T (with respect to $\mathfrak{B}$) if and only if, for each $\varepsilon > 0$, there exists a fibre element (C, W, I) in $\mathfrak{B}$ such that $s \in W$, and an α in C such that $\|I_t\alpha - x(t)\| < \varepsilon$ for all t in some neighborhood of s.

PROPOSITION 3.1. *Let W be an open subset of T. The family $\mathfrak{F}$ of operator fields which are continuous everywhere on W (with respect to $\mathfrak{B}$) is closed under addition, multiplication, involution, multiplication by continuous complex functions on W, and under the operation of passing to uniform limits.* If $x \in \mathfrak{F}$, $t \to \|x(t)\|$ *is continuous on* W.

Let $C_0(\mathfrak{B})$ denote the family of all operator fields x which are continuous everywhere on T with respect to $\mathfrak{B}$ and for which $\lim_{t\to\infty} \|x(t)\| = 0$. From Proposition 3.1 we see that $C_0(\mathfrak{B})$ is a maximal full algebra of operator fields (with the A_t as component algebras). Any continuity structure $\mathfrak{F}$ which is strictly equivalent to $C_0(\mathfrak{B})$ (i.e., such that $C_0(\mathfrak{B}) = C_0(\mathfrak{F})$; see § 1.1) will be said to be *derived from* $\mathfrak{B}$.

If each A_t is isomorphic with M_n (for some n independent of t), then any fibre structure (for T, $\{A_t\}$) is strictly equivalent to some fibre bundle with base space T, fibre space M_n, and group G_n (see § 3.2), in a sense which the reader can easily make precise.

The question now arises: Are all continuity structures derived from fibre structures? In general the answer to this is "no". For example, it is not hard to construct a continuity structure in which one of the component algebras, say A_s, is the algebra of all complex continuous functions on [0, 1], while all the other component algebras are finite-dimensional. Since A_s has no non-trivial finite-dimensional *-subalgebras, it is clear that in this case there exist no fibre structures at all. On the other hand, if each A_s has "enough" finite-dimensional *-subalgebras, the answer is "yes".

THEOREM 3.3. *Suppose that each A_t has the following property: To each α, β in A_t and each $\varepsilon > 0$, there are a finite-dimensional *-subalgebra C of A_t, and elements α', β' of C, such that $\|\alpha' - \alpha\| < \varepsilon$ and $\|\beta' - \beta\| < \varepsilon$.*

Then every continuity structure (for T, $\{A_t\}$) is derived from a fibre structure.

The proof of this theorem falls out of Theorem 3.1 almost immediately.

If each A_t is dual, the hypothesis of Theorem 3.3 is obviously satisfied.

IV. Algebras with continuous trace

4.1. Definition and elementary properties

If S is a representation of an algebra A, $H(S)$ will denote the space of S. If Q is a linear operator, dim (Q) means the dimension of the closure of the range of Q. $\mathrm{Tr}(a)$ is the trace of the operator a.

DEFINITION. *A C^*-algebra A will be said to have a continuous trace* if it is a *CCR* algebra (1) whose dual space $\hat{A}$ is Hausdorff, and if, for each T in $\hat{A}$, there is an a in A and a neighborhood U of T such that, for all S in U, S_a is a one-dimensional projection in $H(S)$.

The phrase "continuous trace" will be justified in Theorem 4.1.

LEMMA 4.1. *Let A be a C^*-algebra with continuous trace, U an open subset of $\hat{A}$, and a an element of A such that S_a is a projection for all S in U. Then $S \to \dim (S_a)$ is continuous on U.*

Proof. Let T be in U. Choose an element b of A so that S_b is a one-dimensional projection for all S in some neighborhood of T. Further let $\pi_1, \ldots, \pi_n$ be orthogonal one-dimensional projections in $H(T)$ whose sum is T_a; and, for $i = 1, 2, \ldots, n$, let α_i be a partial isometry in $H(T)$ such that $\alpha_i^* \alpha_i = \pi_i$, $\alpha_i \alpha_i^* = T_b$. According to Lemmas 3.1 and 3.2 there are elements p_i, q_i of A $(i = 1, 2, \ldots, n)$ such that (i) $T_{p_i} = \pi_i$, $T_{q_i} = \alpha_i$, and (ii) for all S in some neighborhood of T, the S_{p_i} are pairwise orthogonal projections and $S_{q_i{}^*q_i} = S_{p_i}$, $S_{q_i q_i{}^*} = S_b$. Let $p = \sum_{i=1}^n p_i$. Now since S_b is one-dimensional for S near to T, it follows from (ii) that the same holds for S_{p_i}. Thus S_p is an n-dimensional projection for each S sufficiently

(1) A *CCR* algebra is a C^*-algebra A such that T_a is completely continuous for all T in $\hat{A}$ and a in A. For the basic facts about *CCR* algebras, see [8].

near to T; and $T_p = T_a$. From this, the continuity of the mapping $S \to \|S_p - S_a\|$, and the fact that S_a is a projection, we conclude that S_a is n-dimensional for all S sufficiently close to T.

An element a of A is said to be *boundedly represented*(1) if there is an integer n such that $\dim(T_a) \leqslant n$ for all T in $\hat{A}$.

THEOREM 4.1. *If A is a C^*-algebra with continuous trace, the map $S \to \mathrm{Tr}(S_a)$ is continuous on $\hat{A}$ for all boundedly represented elements a of A.*

Proof. Since the boundedly represented elements form a *-subalgebra of A, we may as well assume that a is Hermitian.

Fix an element of T of $\hat{A}$. Since $\dim(T_a)$ is finite, $\mathrm{Sp}(T_a)$ (the spectrum of T_a) consists of finitely many distinct non-zero real numbers $r_1, \ldots, r_m$, together (possibly) with 0. By Lemma 3.1 there are m elements $e_1, \ldots, e_m$ of A such that (i) for all S near enough to T, the S_{e_i} $(i = 1, \ldots, m)$ are m orthogonal projections, and (ii) $T_b = T_a$, where $b = \sum_{i=1}^m r_i e_i$. Now by Lemma 4.1 $\mathrm{Tr}(S_b) = \sum_{i=1}^m r_i \dim S_{e_i}$ has a constant value, namely $\mathrm{Tr}(T_a)$, on some neighborhood of T. On the other hand, by continuity of the norm, $T_b = T_a$ implies $\lim_{S \to T} \|S_{b-a}\| = 0$. Since dim (S_{b-a}) is uniformly bounded on some neighborhood of T, the latter statement implies that $\lim_{S \to T} \mathrm{Tr}\,(S_{b-a}) = 0$; whence $\lim_{S \to T} \mathrm{Tr}(S_a) = \mathrm{Tr}(T_a)$.

THEOREM 4.2. *Every GCR algebra*(2) *has a composition series all of whose quotients are C^*-algebras with continuous trace.*

Proof. In view of the structure theorem for *CCR* algebras ([8], Theorem 6.2), it is sufficient to assume that A is *CCR* with a Hausdorff dual space, and to show that A has a non-zero closed two-sided ideal I with continuous trace.

By Lemma 3 of [9] there is a non-zero positive element a of A such that aAa is a commutative set. Let I be the smallest closed two-sided ideal containing a. Since A is *CCR* and $\hat{A}$ is Hausdorff, the same is true for I. The commutativity of aAa implies that $\dim(T_a) \leqslant 1$ for each T in $\hat{A}$. Further, if $T \in \hat{I}$, $a \notin \mathrm{Kernel}\,(T)$. It follows that, for each T in $\hat{I}$, T_a is a positive multiple of a one-dimensional projection. Thus, if $S \in \hat{I}$, we may apply to the element a some suitable real continuous function f so that $T_{f(a)}$ is a one-dimensional projection throughout a neighborhood of S. Hence I has continuous trace.

THEOREM 4.3. *Every homogeneous C^*-algebra has continuous trace.*

This follows easily from Theorem 3.2.

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(1) For this notion, see [1].

(2) For the definition of a *GCR* algebra, and of a composition series, see [8].

4.2. Spatially constructed algebras with continuous trace

All the vector fields (see § 1.1) so far discussed in this paper have been operator fields, that is, their values have been elements of C^*-algebras. In this section we deal with vector fields whose values are vectors in Hilbert spaces.

For this section we fix a locally compact Hausdorff space T to each point t of which there corresponds a complex Hilbert space H_t. A *vector field* will be a function x on T such that $x(t) \in H_t (t \in T)$; an *operator field* will be a function a on T such that, for each t, $a(t)$ is a bounded linear operator on H_t. We also fix at the outset a continuity structure F for vector fields (see § 1.1). Continuity of vector fields will always be with respect to F. If x and y are continuous vector fields, (¹) the polarization identity assures us that $t \to (x(t), y(t))$ is continuous.

The Gram-Schmidt orthogonalization process yields the following lemma:

LEMMA 4.2. *If $x_1, \ldots, x_n$ are continuous vector fields such that, at some point s, the $x_1(s), \ldots, x_n(s)$ are linearly independent, then $x_1(t), \ldots, x_n(t)$ are linearly independent for all t near enough to s. In fact there are continuous vector fields $y_1, \ldots, y_n$ such that, for all t near enough to s, the $y_1(t), \ldots, y_n(t)$ form an orthonormal set in H_t spanning the same space as $x_1(t), \ldots, x_n(t)$.*

The following easy technical lemmas will be useful in what follows. Their verification is left to the reader.

LEMMA 4.3. *If Q is a bounded operator on a Hilbert space H, and P and P' are projections with $P \leqslant P'$, then*

$$\|P'QP' - PQP\| \leqslant \|Q - PQP\|.$$

In particular,

$$\|P'QP' - Q\| \leqslant 2\|Q - PQP\|.$$

LEMMA 4.4. *Let $u_1, \ldots, u_n$ and $u_1', \ldots, u_n'$ be two orthonormal sets of vectors in a Hilbert space, such that $\|u_i' - u_i\| < \varepsilon \quad (i = 1, 2, \ldots, n)$. Then*

$$\|P' - P\| < 2n\varepsilon,$$

where P and P' are the projections onto the spaces spanned by the u_i and the u_i' respectively.

DEFINITION. An operator field a will be called *almost finite-dimensional* (a.f.d.) *around* a point s of T if, for each $\varepsilon > 0$, there exist (i) a neighborhood U of s, (ii) a positive number k, and (iii) a finite set $x_1, \ldots, x_n$ of continuous vector fields which are linearly independent at each point on U, such that:

(¹) A vector field is continuous if it is continuous at all points of T (with respect to F).

(a) $\|a(t)\| \leqslant k$ for t in U,

(b) $\|P(t)a(t)P(t) - a(t)\| < \varepsilon$ for t in U,

where $P(t)$ is the projection onto the space spanned by $x_1(t), \ldots, x_n(t)$.

LEMMA 4.5. *If a and b are operator fields which are a.f.d. around s, then $a+b$, λa, ab, and a^* are also a.f.d. around s (where λ is a complex constant).*

Proof. Let $\varepsilon > 0$. Choose a positive k, a neighborhood U of s, and two finite sets $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_m$ of continuous vector fields linearly independent at each point of U, such that for all t in U:

$$\|a(t)\| \leqslant k, \ \|b(t)\| \leqslant k; \tag{1}$$

$$\|P(t)a(t)P(t) - a(t)\| < \varepsilon/6; \tag{2}$$

$$\|P'(t)b(t)P'(t) - b(t)\| < \varepsilon/6. \tag{3}$$

(Here $P(t)$ and $P'(t)$ are the projections onto the spaces spanned by $x_1(t), \ldots, x_n(t)$ and $x'_1(t), \ldots, x'_m(t)$ respectively.)

Now the $x_1(s), \ldots, x'_m(s)$ need not be linearly independent. Assume that

$$x_1(s), \ldots, x_n(s), x'_1(s), \ldots, x'_r(s) \tag{4}$$

are linearly independent, while, for $i = r+1, \ldots, m$, x''_i is such a linear combination of the $x_1, \ldots, x_n, x'_1, \ldots, x'_r$ that

$$x''_i(s) = x'_i(s). \tag{5}$$

Narrow the neighborhood U, if necessary, so $x_1, \ldots, x_n, x'_1, \ldots, x'_r$ are linearly independent everywhere in U. Define $P_0(t)$ as the projection onto the space spanned by $x_1(t), \ldots, x_n(t)$, $x'_1(t), \ldots, x'_r(t)$; and $P''(t)$ as the projection onto the space spanned by the $x'_1(t), \ldots, x'_r(t)$, $x''_{r+1}(t), \ldots, x''_n(t)$. Evidently

$$P(t) \leqslant P_0(t), \ P''(t) \leqslant P_0(t) \ (t \in U). \tag{6}$$

From (5) and Lemma 4.4, it follows that

$$\lim_{t \to s} \|P''(t) - P'(t)\| = 0. \tag{7}$$

Now, for $t \in U$,

$$\begin{aligned} \|P''(t)b(t)P''(t) - P'(t)b(t)P'(t)\| &\leqslant \|(P''(t) - P'(t))b(t)P''(t)\| + \|P'(t)b(t)(P''(t) - P'(t))\| \\ &\leqslant 2k\|P''(t) - P'(t)\|. \end{aligned} \tag{8}$$

It follows from (3), (7), and (8) that U may be further narrowed so that for all t in U

$$\|P''(t)b(t)P''(t) - b(t)\| < \varepsilon/3. \tag{9}$$

Now combining (6), (9), and Lemma 4.3, we obtain

$$\|P_0(t)b(t)P_0(t) - b(t)\| < 2\varepsilon/3. \tag{10}$$

Again by (2), (6), and Lemma 4.3,

$$\|P_0(t)a(t)P_0(t) - a(t)\| < \varepsilon/3. \tag{11}$$

Adding (10) and (11), we get for t in U

$$\|P_0(t)(a(t) + b(t))P_0(t) - (a(t) + b(t))\| < \varepsilon.$$

Therefore $a + b$ is a.f.d. around s.

Observe from (10) and (11) that, for t in U,

$$\|b(t)P_0(t) - b(t)\| < 4\varepsilon/3,$$
$$\|P_0(t)a(t) - a(t)\| < 2\varepsilon/3.$$

But then

$$\|P_0(t)a(t)b(t)P_0(t) - a(t)b(t)\| \leqslant \|(P_0(t)a(t) - a(t))b(t)P_0(t)\| + \|a(t)(b(t)P_0(t) - b(t))\|$$
$$< 2k\varepsilon$$

Therefore ab is a.f.d. around s.

Next, it follows from (2) that

$$\|P(t)a^*(t)P(t) - a^*(t)\| < \varepsilon/6 \quad (t \in U).$$

Hence a^* is a.f.d. around s. Now it is trivial that the *a.f.d.* property is preserved on multiplication by a scalar. This completes the proof.

DEFINITION. An operator field a is *weakly continuous* at s if, for all continuous vector fields x and y (or, equivalently, for all x and y in F), the numerical function $t \to (a(t)x(t), y(t))$ is continuous at s.

We shall say simply that a is *weakly continuous* if it is weakly continuous everywhere on T. The following lemma is easily verified.

LEMMA 4.6. *If $\{a_n\}$ is a sequence of operator fields each of which is a.f.d. around s and weakly continuous at s, and if $a_n(t) \to a(t)$ (in norm) uniformly on a neighborhood of s, then a is a.f.d. around s and weakly continuous at s.*

LEMMA 4.7. *Suppose that a is an operator field which is a.f.d. around s and weakly continuous at s. Then $t \to \|a(t)\|$ is continuous at s.*

Proof. Fix $\varepsilon > 0$; and choose a positive k, a neighborhood U of s, and continuous vector fields $x_1, \ldots, x_n$ which are linearly independent everywhere in U, such that, for all t in U,

$$\|a(t)\| \leqslant k,$$
$$\|P(t)a(t)P(t) - a(t)\| < \varepsilon/2, \tag{12}$$

where $P(t)$ is the projection onto the space spanned by the $x_i(t)$. By Lemma 4.2, we may assume that the $x_i(t)$ form an orthonormal set for each t in U. Then the continuity at s of the n^2 matrix elements $(a(t)x_i(t), x_j(t))$ assures us that $t \to \|P(t)a(t)P(t)\|$ is continuous at s. Combining this with (12), we see that U can be further narrowed so that, for all t in U, $|\,\|a(t)\| - \|a(s)\|\,| < \varepsilon$. This completes the proof.

LEMMA 4.8. *Let a and b be two operator fields which are both a.f.d. around s and weakly continuous at s. Then $a+b$, λa (λ complex), ab, and a^* are all a.f.d. around s and weakly continuous at s.*

Proof. In view of Lemma 4.5, the only non-trivial step is to show that ab is weakly continuous at s.

Choose $k > 0$, a neighborhood U of s, and projections $P_0(t)\,(t \in U)$ as in the proof of Lemma 4.5, so that (1), (10), and (11) hold. If we define

$$q(t) = (P_0(t)a(t)P_0(t))(P_0(t)b(t)P_0(t)),$$

then by the continuity at s of the matrix elements of $P_0(t)a(t)P_0(t)$ and $P_0(t)b(t)P_0(t)$, we conclude that q is weakly continuous at t_0. But by (1), (10), and (11),

$$\|q(t) - a(t)b(t)\| < k\varepsilon \quad (t \in U).$$

From this and the weak continuity of q at s, we deduce that of ab at s.

DEFINITION. We denote by A the family of all operator fields a on T which are a.f.d. and weakly continuous everywhere on T, and which vanish at infinity (that is, $\lim_{t\to\infty} \|a(t)\| = 0$).

A is a *-algebra in virtue of Lemma 4.8. By Lemma 4.7 we may introduce into A the sup norm

$$\|a\| = \sup_{t \in T} \|a(t)\|;$$

then by Lemma 4.6 A is complete. In fact, A is a C^*-algebra.

LEMMA 4.9. *For each t, $A_t = \{b(t) \mid b \in A\}$ consists of all completely continuous operators on H_t.*

The proof is easy and is omitted.

Using Lemmas 4.9 and 4.7, we verify:

THEOREM 4.4. *A is a maximal full algebra of operator fields, whose component algebra A_t at t is the algebra of all completely continuous operators on H_t.*

THEOREM 4.5. *A is a C^*-algebra with continuous trace.*

Proof. By Theorem 4.4 and the Corollary of Theorem 1.2, $\hat{A}$ can be identified with T. Thus A is CCR and $\hat{A}$ is Hausdorff. If $s \in T$, and x is a continuous vector field not vanishing at s, there clearly exists an element of A coinciding on a neighborhood of s with projection onto the one-dimensional space spanned by $x(t)$. So A has continuous trace.

Thus, to every continuity structure F for vector fields on T there corresponds an algebra A with continuous trace, constructed as above, and having T as its dual space. This A will be said to be *derived* from F.

At this point it is natural to ask whether every algebra A with continuous trace is derived from some continuity structure F for vector fields on $\hat{A}$. Also, if A is derived from some F, is that F in any sense unique? Both these questions can be answered in the negative by considering homogeneous algebras (see Theorem 4.3).

Indeed, let T be a locally compact Hausdorff space, n a positive integer, and for each t in T let an n-dimensional Hilbert space H_t be given. If F and F' are two continuity structures for vector fields on T (with values in the $\{H_t\}$), we shall say that F and F' are *equivalent* if for each t there is a unitary operator U_t on H_t such that a vector field x is continuous with respect to F if and only if $t \to U_t(x(t))$ is continuous with respect to F'. It is left to the reader to verify that there is a natural one-to-one correspondence between equivalence classes of continuity structures F (for vector fields) and equivalence classes of principal fibre bundles with base space T and group U_n (all $n \times n$ unitary matrices).

Now let F be a continuity structure for vector fields on T (with values in the $\{H_t\}$), and $\mathcal{B}_u$ a corresponding principal bundle with base space T and group U_n. Form the algebra A_F with continuous trace derived from F, and let $\mathcal{B}_g$ be a principal bundle, with base space T and group(1) $G_n = U_n/Z_n$ (see § 3.2), corresponding to A_F. On the other hand, the natural homomorphism of U_n onto G_n induces a natural mapping Φ which carries principal bundles with group U_n into principal bundles (with the same base space) with group G_n; and it is easy to verify that $\Phi(\mathcal{B}_u)$ is equivalent to $\mathcal{B}_g$. Thus the passage from a continuity

(1) Z_n denotes the center of U_n.

structure for vector fields to the derived algebra with continuous trace will be similar in structure to the mapping Φ from U_n-bundles to G_n-bundles. In particular, for fixed T and n, the question whether every homogeneous algebra of degree n with dual space T is derived from a continuity structure for vector fields amounts to asking whether Φ is *onto*, i.e., whether every principal bundle with base space T and group $G_n = U_n/Z_n$ can be obtained by the mapping Φ from a principal bundle with base space T and group U_n. The author is indebted to Professors Spanier and Steenrod for an example of a T for which the answer to this question is negative. Again, the question whether a continuity structure for vector fields on T is determined to within equivalence by the derived homogeneous algebra amounts to asking whether Φ is necessarily one-to-one. The answer here is again negative. The same homogeneous algebra can be derived from essentially different continuity structures for vector fields.

We conclude this chapter with a theorem which will be of importance in Chapter V. As before, let T be a locally compact Hausdorff space, H_t a Hilbert space for each t in T, and F a continuity structure for T, $\{H_t\}$.

THEOREM 4.6. *Let a be a positive operator field on T (i.e., each $a(t)$ is a positive operator on H_t); and let s be an element of T such that:*

(i) *a is weakly continuous at s (with respect to F);*

(ii) *$a(t)$ has a trace for all t in some neighborhood of s, and the map $t \to \operatorname{Tr}(a(t))$ is continuous at s.*

Then a is a.f.d. around s.

Proof. Fix $\varepsilon > 0$; and choose a projection π on H_s, of finite dimension r, such that $a(s) - \pi a(s)\pi$ is positive and

$$\operatorname{Tr}(a(s) - \pi a(s)\pi) < \varepsilon^2/9. \tag{13}$$

Choose a neighborhood U of s in which $\operatorname{Tr}(a(t))$ is bounded, and r continuous vector fields $x_1, \ldots, x_r$, orthonormal everywhere in U, such that $P(s) = \pi$ (where $P(t)$ is the projection onto the space spanned by $x_1(t), \ldots, x_r(t)$).

If b is a positive operator field and $t \in U$, let us set

$$\begin{aligned}
b^{11}(t) &= P(t)b(t)P(t),\\
b^{12}(t) &= P(t)b(t)(1-P(t)),\\
b^{21}(t) &= (b^{12}(t))^* = (1-P(t))b(t)P(t),\\
b^{22}(t) &= (1-P(t))b(t)(1-P(t)).
\end{aligned}$$

Now, for t in U,

$$\operatorname{Tr}(a(t)) = \operatorname{Tr}(a^{11}(t)) + \operatorname{Tr}(a^{22}(t)); \tag{14}$$

so by (13)

$$\operatorname{Tr}(a^{22}(s)) < \varepsilon^2/9. \tag{15}$$

Now the weak continuity of a at s implies that $\operatorname{Tr}(a^{11}(t))$ is continuous at s. Combining this with (14) and (15), and the continuity of $\operatorname{Tr}(a(t))$ at s, we can narrow U so that

$$\operatorname{Tr}(a^{22}(t)) < \varepsilon^2/9 \quad \text{for } t \text{ in } U.$$

Hence, since $a^{22}(t)$ is positive,

$$\|a^{22}(t)\| < \varepsilon^2/9 \quad \text{for } t \text{ in } U. \tag{16}$$

Let $b(t)$ be the positive square root of $a(t)$. Then (16) becomes

$$\|(b^{12}(t))^* b^{12}(t) + (b^{22}(t))^2\| < \varepsilon^2/9 \, (t \in U);$$

from which we obtain, for $t \in U$,

$$\|b^{22}(t)\| < \varepsilon/3, \quad \|b^{12}(t)\| = \|b^{21}(t)\| < \varepsilon/3. \tag{17}$$

Hence, for $t \in U$,

$$\|b(t) - P(t)b(t)P(t)\| = \|b^{12}(t) + b^{21}(t) + b^{22}(t)\| < \varepsilon.$$

It follows that b is *a.f.d.* around s. By Lemma 4.5, $b^2 = a$ is also a.f.d. around s.

COROLLARY. *If a is a positive operator field on T which is everywhere weakly continuous with respect to F, and if* $\operatorname{Tr}(a(t))$ *exists and is continuous everywhere and vanishes at* ∞ *(in T), then a belongs to the algebra with continuous trace which is derived from F.*

V. The group algebra of the 2×2 complex unimodular group

5.1. Introduction

In this last chapter, with the help of the preceding chapters, we deduce the precise structure of the group C^*-algebra of the 2×2 complex unimodular group G, that is, the group of all complex 2×2 matrices of determinant 1. We shall first remind the reader of some concepts and results which will be used.

The irreducible unitary representations of G have been known for some time (see [3] and [4]). They are intimately related to the fractional linear transformations of the complex plane C. We note here for later use the following fact. [1] If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, the corresponding transformation

[1] See p. 420 of [3].

$$z \to \frac{\alpha z + \gamma}{\beta z + \delta}$$

of C induces a transformation of Lebesgue measure described by the factor $|\beta z + \delta|^{-4}$. In fact, if f is summable over C with respect to Lebesgue measure,

$$\int_C f(z)\, dz = \int_C |\beta z + \delta|^{-4} f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) dz. \tag{1}$$

Now, apart from the trivial identity representation, which we will call I, the irreducible unitary representations of G are classified into two series, the *principal* and the *supplementary* series. The representations $T^{m,\varrho}$ of the principal series are indexed by an integer m and a real number ϱ. The space $H_{m,\varrho}$ of $T^{m,\varrho}$ is the Hilbert space $L_2(C)$ of complex functions square-summable on C with respect to Lebesgue measure; and, if $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$,

$$(T_g^{m,\varrho} f)(z) = |\beta z + \delta|^{m + i\varrho - 2} (\beta z + \delta)^{-m} f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right).$$

The representations T^σ of the supplementary series are indexed by a real number σ with $0 < \sigma < 1$. The space H_σ of T^σ is obtained as follows. Let H'_σ be the linear space of all complex measurable functions f on C such that

$$\int_C \int_C |z_1 - z_2|^{-2+2\sigma} |f(z_1)|\, |f(z_2)|\, dz_1\, dz_2 < \infty,$$

equipped with the inner product

$$(f_1, f_2) = \int_C \int_C |z_1 - z_2|^{-2+2\sigma} f_1(z_1) \overline{f_2(z_2)}\, dz_1\, dz_2.$$

Clearly H'_σ includes all continuous functions on C with compact support; in fact these are dense in H'_σ. If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, and $f \in H'_\sigma$, let

$$\left(T_g^\sigma f\right)(z) = |\beta z + \delta|^{-2-2\sigma} f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right).$$

Then T_g^σ preserves inner product on H'_σ, and so may be extended to a unitary operator (also called T_g^σ) on the completion H_σ of H'_σ. This T^σ, acting in H_σ, is then a representation of the supplementary series.

Two representations of the principal series corresponding to distinct parameter pairs (m, ϱ) are (m', ϱ') are unitarily equivalent if and only if $m' = -m$, $\varrho' = -\varrho$. Representations of the supplementary series corresponding to distinct σ are inequivalent to each other, and to all members of the principal series. In view of this, let us agree always to index the principal series with pairs (m, ϱ) for which $m \geqslant 0$, and $\varrho \geqslant 0$ when $m = 0$.

The *group C^*-algebra* $C^*(G)$ of G is defined as the completion of $L_1(G)$ with respect to its minimal regular norm. (1) It is well known that the irreducible unitary representations of G are in one-to-one correspondence with the dual space of $C^*(G)$. Hence this dual space will usually be denoted by $\hat{G}$; and corresponding representations of G and $C^*(G)$ will be denoted by the same letter.

Now the hull–kernel topology of $\hat{G}$ was worked out in Chapter 3 of [1]. The result was as follows:

THEOREM 5.1. *Let $\hat{G}_p$, $\hat{G}_s$ denote the principal and supplementary series of representations of G respectively, so that $\hat{G} = \hat{G}_p \cup \hat{G}_s \cup \{I\}$.*

(a) *The topology of $\hat{G}$ relativized to $\hat{G}_p$ (or $\hat{G}_s$) is the natural topology of the parameters (m, ϱ) (or σ).*

(b) *$\hat{G}_p$ and $\{I\}$ are closed subsets of $\hat{G}$.*

(c) *Let X be a subset of $\hat{G}_s$, with closure $\bar{X}$ (in $\hat{G}$); and set $S = \{\sigma \mid T^\sigma \in X\}$. Then* (i) $\bar{X} \subset \hat{G}_s \cup \{T^{0,0}, T^{2,0}, I\}$; (ii) $T^{0,0} \in \bar{X}$ *if and only if* 0 *is a limit point of S;* (iii) $T^{2,0} \in \bar{X}$ *if and only if* 1 *is a limit point of S;* (iv) $I \in \bar{X}$ *if and only if* 1 *is a limit point of S.*

We shall now represent $C^*(G)$ in terms of its regularized transform (see § 2.1). Let Z_1 be the space of all the parameters (m, ϱ) (m a non-negative integer, ϱ real, $\varrho \geqslant 0$ if $m = 0$), with the natural topology; Z_2 the closed unit interval $[0, 1]$ with the natural topology; Z' the disjoint union of Z_1 and Z_2; and Z the space obtained from Z' by identifying the point $(0, 0)$ in Z_1 with 0 in Z_2. Clearly Z is a locally compact Hausdorff space. To each w in Z we associate a representation T^w of $C^*(G)$ as follows: (a) if $w = (m, \varrho) \in Z_1$, T^w is the representation $T^{m,\varrho}$ of the principal series; (b) if $w = \sigma \in Z_2$, $0 < \sigma < 1$, then T^w is the representation T^σ of the supplementary series; (c) $T^1 = T^{2,0} \oplus I$. Using Theorems 2.2 and 5.1, the reader will now verify:

LEMMA 5.1. *For each x in $C^*(G)$, let $\hat{x}$ be the operator field $w \to T^w_x$ on Z.* (2) *Then the family A of all $\hat{x}$ (where $x \in C^*(G)$) is a full algebra of operator fields on Z, and is isomorphic with $C^*(G)$ under the mapping $x \to \hat{x}$.*

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(1) See [11], p. 235. The minimal regular norm of an element f of $L_1(G)$ is the supremum of the $\|T_f\|$, where T ranges over all $*$-representations T of $L_1(G)$.

(2) Z is not quite the regularized dual space of $C^*(G)$; the latter consists of Z together with the point at infinity plus one other isolated point. The $\hat{x}$ of this lemma is the restriction to Z of the regularized transform $\tilde{x}$.

In future we identify $C^*(G)$ with A. The component algebra A_w of A at w is just $T^w(C^*(G))$. Since $C^*(G)$ is a CCR algebra (see, for example, [4]), A_w consists of all completely continuous operators on $H(T^w)$ provided $w \neq 1$. For $w = 1$, we have that $H(T^1) = H(T^{2,0}) \oplus C$, where C is the one-dimensional Hilbert space; and A_1 consists of all $a \oplus \lambda$, where a is a completely continuous operator on $H(T^{2,0})$ and λ is a complex number (operating on C).

Note that A is not maximal. Indeed, the values of the $\hat{x}$ at (2, 0) and 1 are correlated:

$$T_x^1 = T_x^{2,0} \oplus I_x \quad (x \in C^*(G)). \tag{2}$$

Clearly, however, because of the inequivalence of the T^w for different $w \neq 1$, this is the *only* correlation between the values of the $\hat{x}$ at distinct points of Z. From this observation, Theorem 1.4 enables us to draw the following conclusion:

THEOREM 5.2. *Let $A_{\max}$ be the maximal full algebra of operator fields on Z (with values in the $\{A_w\}$) which contains $C^*(G)$. Then $C^*(G)$ consists precisely of all those operator fields x in $A_{\max}$ such that*

$$x(1) = x(2, 0) \oplus \lambda \tag{3}$$

for some complex λ.

Theorem 5.2 embodies all the information about $C^*(G)$ that is available from an immediate application of the preceding chapters. However, the structure of $C^*(G)$ is still not determined. By Theorem 3.3 $C^*(G)$ is derived from some fibre structure; we do not yet know what kind of "twists", if any, this fibre structure has. Nor do we know just how the representation T^σ "joins on" to $T^1 = T^{2,0} \oplus I$ as $\sigma \to 1-$. Indeed, we defined T^1 as $T^{2,0} \oplus I$ only in order to satisfy the condition $\|T_x^1\| = \sup(\|T_x^{2,0}\|, \|I_x\|)$.(1) The same end would have been served by defining $T^1 = nT^{2,0} \oplus mI$ (n, m any positive integers). By what n and m is the limiting behavior of T^σ (as $\sigma \to 1-$) best described?

It is the object of the following sections to answer these questions. The answers are as simple as they could be. The fibre structure associated with $C^*(G)$ has no "twists"; it is equivalent to a "product structure" (see Theorem 5.4). And it is $T^{2,0} \oplus I$, rather than any other $nT^{2,0} \oplus mI$, which describes the limiting behaviour of T^σ as $\sigma \to 1-$.

We arrive at these answers in four steps. In the first step (§ 5.2), $T^{2,0}$ is expressed in a new form, more suitable for the definition of T^1 as the limit of T^σ as $\sigma \to 1-$. The second step (§ 5.3) consists in defining a continuity structure X for *vector* fields on Z. In the third step (§ 5.4) it is shown that $C^*(G)$ is weakly continuous with respect to X. Finally, in

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(1) This will be clear to the reader who has verified Lemma 5.1.

§ 5.5 we show that $C^*(G)$ is a subalgebra of the algebra with continuous trace derived from X, and combine this result with Theorem 5.2 to obtain the complete description of $C^*(G)$.

5.2. A new description of $T^{2,0}$

As usual, C denotes the complex plane. If f is a complex function on C, we write $D_1 f$ and $D_2 f$ for the first partial derivatives of f with respect to the real and imaginary parts of the argument, and introduce

$$D_- = \tfrac{1}{2}(D_1 + i D_2),$$
$$D_+ = \tfrac{1}{2}(D_1 - i D_2).$$

(D_+ and D_- are commonly called $\partial/\partial z$ and $\partial/\partial \bar{z}$ respectively). We denote by L the family of all complex functions h on C with compact support which are everywhere infinitely differentiable; and by L_0 the subset of L consisting of those h for which

$$\int_C h(z)\,dz = 0. \tag{4}$$

Further, E will be the set of those functions in $L_2(C)$ which are infinitely differentiable at all but finitely many points of C; and F will be the image of E under D_-. If $f \in F$ and $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, we shall define

$$(S_g f)(z) = |\beta z + \delta|^{-4} f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{5}$$

The reason for this definition lies in the following lemma:

LEMMA 5.3. *Each operator $T_g^{2,0}$ $(g \in G)$ leaves E invariant. If $f \in E$,*

$$D_- T_g^{2,0} f = S_g D_- f. \tag{6}$$

Proof. We recall that

$$(T_g^{2,0} f)(z) = (\beta z + \delta)^{-2} f\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{7}$$

It is clear that $T_g^{2,0}$ leaves E invariant. The verification of (6) is straightforward, and is left to the reader.

It follows from (6) that F is invariant under the S_g $(g \in G)$.

LEMMA 5.4. *D_- is one-to-one on E.*

Proof. Suppose that $f \in E$, $D_- f = 0$. It is enough to show $f = 0$.

By the hypotheses, f is analytic (except perhaps for finitely many singularities) and square-summable on C. For simplicity of notation, let 0 be a typical singularity of f; and expand f in a Laurent series about 0, valid in $\{z \mid 0<|z| \leqslant R\}=B$:

$$f(z)=\sum_{n=-\infty}^{\infty} a_n z^n.$$

Put $f_1(z)=\sum_{n=-\infty}^{-1} a_n z^n$. Now the different z^n are orthogonal in each annulus

$$A=\{z \mid \varrho \leqslant |z| \leqslant R\} \quad (0<\varrho<R).$$

Hence

$$\int_A |f_1(z)|^2 dz=\sum_{n=-\infty}^{-1} |a_n|^2 \int_A |z|^{2n} dz=|a_{-1}|^2\, 2\pi \log \frac{R}{\varrho}+\sum_{n=-\infty}^{-2} 2\pi |a_n|^2 \frac{\varrho^{2n+2}-R^{2n+2}}{-2-2n}. \tag{8}$$

Since f is square-summable and $f-f_1$ is bounded on B, f_1 is square-summable on B; hence

$$\infty>\lim_{\varrho \to 0+} \int_A |f_1(z)|^2 dz.$$

But by (8) this is impossible unless $a_n=0$ for $n<0$. It follows that f can have no singularities in the finite part of the plane. A similar argument shows that it has no singularity at ∞ either. Hence f is identically 0.

LEMMA 5.5. *$L_0 \subset F$. Further, the inverse image of L_0 under D_- is dense in $L_2(C)$.*

Proof. Certainly $L \subset E$, and $D_-(L) \subset L_0$. Since L is dense in $L_2(C)$, the last statement of the lemma is proved.

Now let h be a function in L_0. Its Fourier transform

$$\varphi(w)=\frac{1}{2\pi} \int_C h(z)\, e^{i \operatorname{Re}(z \bar{w})} dz$$

is infinitely differentiable and goes to 0 at ∞ faster than any $|w|^{-n}$. Also by (4) $\varphi(0)=0$. Hence, putting $w_1=\operatorname{Re} w$, $w_2=\operatorname{Im} w$,

$$\varphi(w)=w_1(c_1+\varepsilon_1(w))+w_2(c_2+\varepsilon_2(w)),$$

where the c_i are constant and $\lim_{w \to 0} \varepsilon_i(w)=0$. From this it follows that

$$\psi(w)=2i \frac{\varphi(w)}{\bar{w}} \tag{9}$$

is bounded, besides going to 0 at ∞ faster than any $|w|^{-n}$. In particular, ψ belongs to $L_1(C) \cap L_2(C)$; and we may take its inverse Fourier transform

$$f(z) = \frac{1}{2\pi} \int \psi(w)\, e^{-i\,\mathrm{Re}(z\bar{w})}\, dw.$$

Then $f \in L_2(C)$ and is infinitely differentiable (since ψ vanishes rapidly at ∞). Thus $f \in E$.

The lemma will be proved if we show

$$D_- f = h. \tag{10}$$

Let m be an arbitrary function in L. Then (10) will be proved if we show

$$\int_C (D_- f)(z)\, \overline{m(z)}\, dz = \int_C h(z)\, \overline{m(z)}\, dz. \tag{11}$$

Integrating by parts, we obtain

$$\int_C (D_- f)(z)\, \overline{m(z)}\, dz = -\int_C f(z)\, \overline{(D_+ m)(z)}\, dz. \tag{12}$$

Now, if the Fourier transform of m is μ, that of $D_+ m$ is $\frac{1}{2} i w \mu(w)$. Combining this with (12) and (9), we obtain from the Plancherel formula

$$\int_C (D_- f)(z)\, \overline{m(z)}\, dz = -\frac{i}{2} \int \psi(w)\, \bar{w}\, \overline{\mu(w)}\, dw$$

$$= \int_C \varphi(w)\, \overline{\mu(w)}\, dw = \int_C h(z)\, \overline{m(z)}\, dz,$$

which is (11). The proof is complete.

In view of Lemma 5.4, the inner product $(\ ,\ )$ in $L_2(C)$ can be transferred via D_- to F. In fact, if $h_i = D_- f_i\ (f_i \in E)$,

we define
$$(h_1, h_2)_0 = (f_1, f_2), \tag{13}$$

$$\|h_1\|_0 = \|f_1\|. \tag{14}$$

Then F is an incomplete Hilbert space under $\|\ \ \|_0$; its completion will be called K. By Lemma 5.5, L_0 is dense in K. By Lemma 5.3, the operators $S_g\,(g \in G)$ are linear isometries of F into itself, which can be extended to unitary operators, also called S_g, on K. Thus we have:

LEMMA 5.6. *S is a unitary representation of G acting in K; and $S \cong T^{2,0}$. In fact, extending D_- to an isometry (also called D_-) of $L_2(C)$ onto K, we have $T^{2,0}_g = D_-^{-1} S_g D_-\ (g \in G)$.*

In Lemma 5.8 we shall obtain an explicit expression for $(h_1, h_2)_0$ in case $h_1, h_2 \in L_0$. A fundamental tool for this and much of what follows is the following known result: (1)

LEMMA 5.7. If $h_1, h_2 \in L$, and φ_i is the Fourier transform of h_i:

$$\varphi_i(w) = \frac{1}{2\pi}\int_C h_i(z)\, e^{i\,\mathrm{Re}(z\bar{w})}\, dz,$$

then for all $0 < \sigma < 1$,

$$\int_C \int_C |z - z'|^{-2+2\sigma} h_1(z)\, \overline{h_2(z')}\, dz\, dz' = 2^{2\sigma}\pi \frac{\Gamma(\sigma)}{\Gamma(1-\sigma)}\int_C |w|^{-2\sigma} \varphi_1(w)\, \overline{\varphi_2(w)}\, dw. \tag{15}$$

LEMMA 5.8. *If $h_1, h_2 \in L_0$, then*

$$(h_1, h_2)_0 = -\frac{2}{\pi}\int_C \int_C \log|z - z'|\, h_1(z)\, \overline{h_2(z')}\, dz\, dz'. \tag{16}$$

Proof. The proof of (16) consists essentially in passing to the limit $\sigma \to 1-$ in (15). Let us denote either side of (15) by I_σ. In view of (4), we have

$$\frac{I_\sigma}{1-\sigma} = \int_C \int_C \left\{\frac{|z - z'|^{-2+2\sigma} - 1}{1-\sigma}\right\} h_1(z)\, \overline{h_2(z')}\, dz\, dz'. \tag{17}$$

Now, if $z \neq z'$, $(|z - z'|^{-2+2\sigma} - 1)/(1-\sigma) \to -2\log|z - z'|$ as $\sigma \to 1-$. An easy dominated-convergence argument applied to (17) therefore gives

$$\lim_{\sigma\to 1-} \frac{I_\sigma}{1-\sigma} = -2\int_C \int_C \log|z - z'|\, h_1(z)\, \overline{h_2(z')}\, dz\, dz'. \tag{18}$$

As in Lemma 5.7, let φ_i be the Fourier transform of h_i. By (4) we have $\varphi_i(0) = 0$, so that the function $|w|^{-2}\varphi_1(w)\,\overline{\varphi_2(w)}$ is bounded. Applying the dominated-convergence argument to the right side of (15), we have

$$\lim_{\sigma\to 1-} \frac{I_\sigma}{1-\sigma} = 4\pi\int_C |w|^{-2} \varphi_1(w)\, \overline{\varphi_2(w)}\, dw. \tag{19}$$

Now suppose (see Lemma 5.5) that $h_i = D_- f_i$, where $f_i \in E$. If ψ_i is the Fourier transform of f_i, we see from the proof of Lemma 5.5 that

$$\psi_i(w) = 2i\frac{\varphi_i(w)}{\bar{w}}.$$

(1) This is the Lemma on p. 454 of [3].

Hence $$\int_C |w|^{-2} \varphi_1(w) \overline{\varphi_2(w)}\, dw = \tfrac{1}{4} \int_C \psi_1(w) \overline{\psi_2(w)}\, dw = \tfrac{1}{4}(f_1, f_2) = \tfrac{1}{4}(h_1, h_2)_0. \qquad (20)$$

Combining (18), (19), and (20), we get (16).

Observe that, if $h \in L_0$, $S_g h$ need not be in L_0. We shall need to know that (16) is valid in the more general case that h_i is replaced by $S_{g_i} h_i$. To see this we note the following easy consequence of Lemma 5.8:

LEMMA 5.9. *Let $\{f_n\}$ be a sequence of functions in L_0, all vanishing outside the same compact set, all bounded in absolute value by the same number, and such that*

$$\lim_{n\to\infty} \int_C |f_n(z)|\, dz = 0.$$

Then $\lim_{n\to\infty} \|f_n\|_0 = 0$.

LEMMA 5.10. *If $h, h' \in L_0$, and $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, then*

$$(S_g h, h')_0 = -\frac{2}{\pi} \int_C \int_C \log |z - z'|\, (S_g h)(z)\, \overline{h'(z')}\, dz\, dz'.$$

Proof. If $\beta = 0$, then $S_g h \in L_0$, and Lemma 5.8 applies immediately.

Assume $\beta \neq 0$. By the definition of S_g, it is clearly possible to pick a sequence $\{f_n\}$ of functions in L_0 satisfying the hypotheses of Lemma 5.9, and such that, for each n, $h(z) + f_n(z) = 0$ in some neighborhood (depending on n) of $z_0 = \alpha/\beta$. Define $h_n = h + f_n$. Then, by Lemma 5.9,

$$h_n \to h \text{ in both } L_1(C) \text{ and in } K. \qquad (21)$$

Now let φ_n, φ, and φ' be the inverse images of h_n, h, and h' under D_-. By (21)

$$\varphi_n \to \varphi \text{ in } L_2(C). \qquad (22)$$

Now, by the definition of h_n, $S_g h_n$ has compact support. By (1) and (4), $\int_C (S_g h_n)(z)\, dz = 0$. It follows that

$$S_g h_n \in L_0. \qquad (23)$$

Since by (1) S_g is an isometry in $L_1(C)$, (21) gives

$$S_g h_n \to S_g h \text{ in } L_1(C). \qquad (24)$$

By (22) $$T_g^{2,0} \varphi_n \to T_g^{2,0} \varphi. \qquad (25)$$

Now, by (23), (24), (25), and Lemma 5.8,

$$
\begin{aligned}
(S_g h, h')_0 &= (T_g^{2,0}\varphi, \varphi') \\
&= \lim_n (T_g^{2,0}\varphi_n, \varphi') = \lim_n (S_g h_n, h')_0 \\
&= \lim_n \left(-\frac{2}{\pi}\right) \int_C \int_C \log|z-z'|\,(S_g h_n)(z)\,\overline{h'(z')}\,dz\,dz' \\
&= -\frac{2}{\pi} \int_C \int_C \log|z-z'|\,(S_g h)(z)\,\overline{h'(z')}\,dz\,dz'.
\end{aligned}
$$

(For the last step, we use (24), and observe that $\int_C \log|z-z'|\,\overline{h'(z')}\,dz'$ is bounded in z because $h' \in L_0$.)

5.3. A continuity structure for vector fields on Z

Let Z_1, Z_2, Z be as in § 5.1, and L, L_0, K, S as in § 5.2. For $w \in Z$, $w \neq 1$, let T^w be as in § 5.1, and put $H_w = H(T^w)$. For the case $w = 1$, we shall put $T^1 = S \oplus I$,(1) and $H_1 = H(T^1) = K \oplus C$ (C being the one-dimensional Hilbert space).

For the rest of this paper let us fix an element h_1 of L satisfying

$$\int_C h_1(z)\,dz = 1. \tag{26}$$

For each complex number λ and each h in L_0, we define a vector field $x_{\lambda,h}$ on Z, with values in the H_w, as follows:

$$
\begin{aligned}
&\text{If } (m,\varrho) \in Z_{z_1}, \quad x_{\lambda,h}(m,\varrho) = \lambda h_1 + h; \\
&\text{if } 0<\sigma<1, \quad x_{\lambda,h}(\sigma) = \frac{1}{\sqrt{\pi}}\left[\sqrt{\sigma}\,\lambda h_1 + h\sqrt{\left(\frac{\sigma}{1-\sigma}\right)}\right]; \\
&\qquad x_{\lambda,h}(1) = h \oplus \frac{\lambda}{\sqrt{\pi}}.
\end{aligned}
$$

(Since $L_0 \subset K$, $h \oplus \frac{\lambda}{\sqrt{\pi}} \in K \oplus C = H_1$.)

DEFINITION. We shall denote by X the family of all $x_{\lambda,h}$ where λ ranges over C and h over L_0.

Clearly X is a linear space of vector fields.

LEMMA 5.11. *For each w in Z, $\{x_{\lambda,h}(w) \mid x_{\lambda,h} \in X\}$ is dense in H_w.*

This follows from the fact that L is dense in H_w for each $w \in Z$, $w \neq 1$, while L_0 is dense in K.

Thus, X will be a continuity structure for Z, $\{H_w\}$, if the following lemma holds:

(1) In § 5.1 T^1 was defined as $T^{2,0} \oplus I$. The present definition (which will be maintained throughout the rest of the paper) is unitarily equivalent to the former one by Lemma 5.6.

LEMMA 5.12. *The function* $w \to \| x_{\lambda, h}(w) \|_{H_w}$ *is continuous on* Z *for each* $x_{\lambda, h}$ *in* X.

Proof. This is evident for all points of Z except 0 and 1. We consider first the point 1. Let $x = x_{\lambda, h} \in X$. Now

$$\| x(1) \|^2 = \frac{|\lambda|^2}{\pi} + \| h \|_0^2 = \frac{1}{\pi} \left[|\lambda|^2 - 2 \int_C \int_C \log |z - z'| \, h(z) \overline{h(z')} \, dz \, dz' \right]. \tag{27}$$

On the other hand, for $0 < \sigma < 1$,

$$\| x(\sigma) \|^2 = \frac{1}{\pi} (I_1 + I_2 + I_3 + \bar{I}_3), \tag{28}$$

where

$$I_1 = \sigma |\lambda|^2 \int_C \int_C |z - z'|^{-2+2\sigma} h_1(z) \overline{h_1(z')} \, dz \, dz', \tag{29}$$

$$I_2 = \frac{\sigma}{1 - \sigma} \int_C \int_C |z - z'|^{-2+2\sigma} h(z) \overline{h(z')} \, dz \, dz', \tag{30}$$

$$I_3 = \frac{\sigma \lambda}{\sqrt{(1 - \sigma)}} \int_C \int_C |z - z'|^{-2+2\sigma} h_1(z) \overline{h(z')} \, dz \, dz'. \tag{31}$$

For $z \neq z'$, $|z - z'|^{-2+2\sigma} \to 1$ as $\sigma \to 1-$; also, for all σ near to 1, the integrand in (29) is uniformly majorized by a summable function. It follows from (26) that

$$\lim_{\sigma \to 1-} I_1 = |\lambda|^2 \int_C \int_C h_1(z) \overline{h_1(z')} \, dz \, dz' = |\lambda|^2. \tag{32}$$

Also, it was shown in the proof of Lemma 5.8 that

$$\lim_{\sigma \to 1-} I_2 = -2 \int_C \int_C \log |z - z'| \, h(z) \overline{h(z')} \, dz \, dz'. \tag{33}$$

Now, since $\int_C h(z) \, dz = 0$,

$$\frac{I_3}{\sigma \lambda} = \sqrt{(1 - \sigma)} \int_C \int_C \left\{ \frac{|z - z'|^{-2+2\sigma} - 1}{1 - \sigma} \right\} h_1(z) \overline{h(z')} \, dz \, dz'. \tag{34}$$

It follows as in the proof of Lemma 5.8 that the integral in (34) approaches a finite limit as $\sigma \to 1-$. Therefore

$$\lim_{\sigma \to 1-} I_3 = 0. \tag{35}$$

Combining (27), (28), (32), (33), and (35), we have

$$\lim_{\sigma \to 1-} \| x(\sigma) \| = \| x(1) \|. \tag{36}$$

Next we consider the point $0=(0, 0)$ in Z. Let ψ_σ be the Fourier transform of $\frac{1}{\sqrt{\sigma}}x(\sigma)$. Then by Lemma 5.7, for $0<\sigma<1$,

$$\lim_{\sigma\to0+}\|x(\sigma)\|^2=\lim_{\sigma\to0+}2^{2\sigma}\pi\frac{\sigma\Gamma(\sigma)}{\Gamma(1-\sigma)}\int_C|w|^{-2\sigma}|\psi_\sigma(w)|^2\,dw=\pi\int_C|\psi(w)|^2\,dw, \tag{37}$$

where ψ is the Fourier transform of $\lim\limits_{\sigma\to0+}x(\sigma)/\sqrt{\sigma}=x(0)/\sqrt{\pi}$. Thus (37) gives

$$\lim_{\sigma\to0+}\|x(\sigma)\|=\|x(0)\|. \tag{38}$$

Now (36) and (38) complete the proof.

As we have already mentioned, this lemma implies the following consequence:

LEMMA 5.13. *X is a continuity structure for vector fields on Z with values in the $\{H_w\}$.*

5.4. The weak continuity of $C^*(G)$ with respect to X

In Lemma 5.1 $C^*(G)$ was identified with the algebra of operator fields $w\to T_x^w$ on Z. We continue to make this identification, reminding the reader of the slight alteration in the definition of T^1 made in § 5.3. In this section we prove that each operator field in $C^*(G)$ is weakly continuous with respect to X. For this purpose it is enough to consider only those which arise from continuous complex functions a on G with compact support; for these are dense in $C^*(G)$.

Let a be a continuous complex function on G with compact support; and let $x=x_{\lambda,h}$, $x'=x_{\lambda',h'}$ be elements of X. We shall prove that the function

$$w\to(T_a^w(x(w)), x'(w)) \tag{39}$$

is continuous on Z.

If $w=(m,\varrho)\in Z_1$, we have, setting $f=\lambda h_1+h$, $f'=\lambda' h_1+h'$, $g=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$,

$$(T_a^w(x(w)), x'(w))=\int_G\int_C a(g)\,|\beta z+\delta|^{m+i\varrho-2}\,|\beta z+\delta|^{-m}f\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\overline{f'(z)}\,dz\,dg.$$

Since this integral converges absolutely uniformly in m and ϱ,

$$(39) \text{ is continuous on } Z_{z_1}. \tag{40}$$

Next we shall prove the continuity of (39) at points $w=\sigma$, where $0<\sigma<1$. For this purpose it is sufficient to show that, for each f, f' in L, the integral

$$\int_C \int_C \int_G a(g)\,|z-z'|^{-2+2\sigma}\,|\beta z+\delta|^{-2-2\sigma} f\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\overline{f'(z')}\,dg\,dz\,dz' \tag{41}$$

is continuous in σ for $0<\sigma<1$. It is convenient to define an auxiliary function

$$J(\sigma;\,z)=\int_C |z-z'|^{-2+2\sigma}\,\overline{f'(z')}\,dz'$$

($z\in C$, $0<\sigma\leqslant 1$). For each fixed z, the function $J(\sigma;\,z)$ is clearly continuous in σ for $0<\sigma\leqslant 1$. We may write (41) in terms of J as follows:

$$\int_C \int_G a(g)\,|\beta z+\delta|^{-2-2\sigma} J(\sigma;\,z) f\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right) dg\,dz. \tag{42}$$

LEMMA 5.14. *There exist positive constants M and p such that, for all z and all $0<\sigma\leqslant 1$,*

$$|J(\sigma;\,z)|\leqslant \frac{1}{\sigma}\,\frac{M}{1+p|z|^{2-2\sigma}}.$$

Proof. Let f' have upper bound N; and let the support of f' be contained in a circle about 0 of radius $R\geqslant 1$. Then

$$|J(\sigma;\,z)|\leqslant N\int_{|z'|\leqslant R} |z-z'|^{-2+2\sigma}\,dz' = K(\sigma;\,z). \tag{43}$$

An easy geometrical argument, which we omit, shows that, for fixed σ, $K(\sigma;\,z)$ attains its maximum value when $z=0$. Now

$$K(\sigma;\,0)=\frac{\pi N R^{2\sigma}}{\sigma};$$

hence

$$K(\sigma;\,z)\leqslant \frac{\pi N R^{2\sigma}}{\sigma}. \tag{44}$$

On the other hand, if $|z|\geqslant 2R$, we have $|z-z'|\geqslant \frac{1}{2}|z|$ for $|z'|\leqslant R$, so that

$$K(\sigma;\,z)\leqslant N\left(\frac{|z|}{2}\right)^{-2+2\sigma}\pi R^2=\pi N R^{2\sigma}\left(\frac{|z|}{2R}\right)^{-2+2\sigma}. \tag{45}$$

Now we verify that, if $|z|/2R\leqslant 1$, then

$$\frac{2\pi N R^{2\sigma}}{\sigma\left(1+\left(\frac{|z|}{2R}\right)^{2-2\sigma}\right)}\geqslant \frac{\pi N R^{2\sigma}}{\sigma}; \tag{46}$$

while, if $|z|/2R \geqslant 1$,

$$\frac{2\pi N R^{2\sigma}}{\sigma\left(1+\left(\frac{|z|}{2R}\right)^{2-2\sigma}\right)} \geqslant \pi N R^{2\sigma}\left(\frac{|z|}{2R}\right)^{-2+2\sigma}. \tag{47}$$

The lemma now follows from inequalities (43) to (47).

In view of the last lemma, the integral (42) is majorized by

$$\frac{M}{\sigma}\int_C \int_G |a(g)| \left(\frac{|\beta z+\delta|}{1+\frac{|z|}{2R}}\right)^{2-2\sigma} \frac{1}{|\beta z+\delta|^4}\left| f\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\right| dg\, dz. \tag{48}$$

Now an easy calculation shows that the expression $\{|\beta z+\delta|/(1+|z|/2R)\}^{2-2\sigma}$ is bounded uniformly for all $0<\sigma<1$, all z in C, and all g in the support of a. Thus, in view of (1), the integral (48), and hence (42) also, is majorized by a summable function independent of σ. It follows that (42) is continuous in σ, and hence that

$$(39) \text{ is continuous for } 0<w<1. \tag{49}$$

We next show that

$$(39) \text{ is continuous as } w=\sigma\to 0+. \tag{50}$$

For $0<w=\sigma<1$,

$$(T_a^w(x(w)),\, x'(w)) = \int_C \int_G a(g)\,|\beta z+\delta|^{-2-2\sigma}\left(\lambda h_1+\frac{h}{\sqrt{(1-\sigma)}}\right)\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right) Q(\sigma;z)\, dg\, dz, \tag{51}$$

where

$$Q(\sigma;z) = \frac{\sigma}{\pi}\int_C |z-z'|^{-2+2\sigma}\, \overline{(\lambda' h_1+h'(1-\sigma)^{-\frac{1}{2}})(z')}\, dz' \tag{52}$$

and $g=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$. Exactly as in the proof of (49), we show that the integral (51) is majorized, uniformly for (say) $0<\sigma<\frac{1}{2}$, by the convergent integral

$$k\int_C \int_G |a(g)|\cdot|\beta z+\delta|^{-4}\cdot\left|\left(\lambda h_1+\frac{h}{\sqrt{(1-\sigma)}}\right)\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\right| dg\, dz,$$

where k is some constant.

Now it is a routine matter to verify that, for any continuous function f on C with compact support,

$$\lim_{\sigma\to 0+}\frac{\sigma}{\pi}\int_C |z-z'|^{-2+2\sigma} f(z')\, dz' = f(z).$$

Applying this to Q, we obtain

$$\lim_{\sigma\to 0+} Q(\sigma;\, z) = \overline{(\lambda' h_1 + h')\,(z)}.$$

Hence, by the Lebesgue dominated-convergence theorem, (51) gives

$$\lim_{\sigma\to 0+} (T_a^\sigma (x(\sigma)),\, x'(\sigma)) = \int_C \int_G a(g)\,|\beta z+\delta|^{-2}\,(\lambda h_1 + h)\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\overline{(\lambda' h_1 + h')\,(z)}\,dg\,dz$$

$$= (T_a^{0,\,0}(x(0,\,0)),\, x'(0,\,0));$$

and this proves (50).

Finally we shall show that

$$(39) \text{ is continuous at } 1. \tag{53}$$

We begin by observing

$$(T_a^1(x(1)),\, x'(1)) = \frac{\lambda\overline{\lambda'}}{\pi}\int_G a(g)\,dg + (S_a h,\, h'). \tag{54}$$

For $0<\sigma<1$, by (51) and (52)

$$(T_a^\sigma(x(\sigma)),\, x'(\sigma)) = I_1 + I_2 + I_3 + I_4, \tag{55}$$

where $$I_1 = \frac{\sigma}{\pi}\lambda\overline{\lambda'}\int_C\int_C\int_G a(g)\,|z-z'|^{-2+2\sigma}\,|\beta z+\delta|^{-2-2\sigma}\,h_1\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\overline{h_1(z')}\,dg\,dz\,dz',$$

$$I_2 = \frac{\sigma}{\pi(1-\sigma)}\int_C\int_C\int_G a(g)\,|z-z'|^{-2+2\sigma}\,|\beta z+\delta|^{-2-2\sigma}\,h\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\overline{h'(z')}\,dg\,dz\,dz',$$

$$I_3 = \frac{\sigma\overline{\lambda'}}{\pi\sqrt{(1-\sigma)}}\int_C\int_C\int_G a(g)\,|z-z'|^{-2+2\sigma}\,|\beta z+\delta|^{-2-2\sigma}\,h\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\overline{h_1(z')}\,dg\,dz\,dz',$$

$$I_4 = \frac{\sigma\lambda}{\pi\sqrt{(1-\sigma)}}\int_C\int_C\int_G a(g)\,|z-z'|^{-2+2\sigma}\,|\beta z+\delta|^{-2-2\sigma}\,h_1\left(\frac{\alpha z+\gamma}{\alpha z+\delta}\right)\overline{h'(z')}\,dg\,dz\,dz'.$$

Now I_1 is a particular case of the integral (41). We showed earlier that that integral is majorized uniformly for $0<\sigma\leqslant 1$ by a summable function. Hence we may pass to the limit $\sigma\to 1-$ under the integral sign, getting by (1) and (26)

$$\lim_{\sigma\to 1-} I_1 = \frac{\lambda\overline{\lambda'}}{\pi}\int_C\int_C\int_G a(g)\,|\beta z+\delta|^{-4}\,h_1\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)\overline{h_1(z')}\,dg\,dz\,dz' = \frac{\lambda\overline{\lambda'}}{\pi}\int_G a(g)\,dg. \tag{56}$$

To deal with I_2, I_3, and I_4, we introduce another auxiliary function $R(\sigma; z)$ $(z \in C,\ 0<\sigma\leqslant 1)$:

$$R(\sigma; z)=\begin{cases}\displaystyle\int_C\left\{\frac{|z-z'|^{-2+2\sigma}-1}{1-\sigma}\right\}\overline{h'(z')}\,dz' & \text{for } 0<\sigma<1,\\ \displaystyle -2\int_C \log|z-z'|\,\overline{h'(z')}\,dz' & \text{for } \sigma=1.\end{cases}$$

By the dominated-convergence argument used in the proof of Lemma 7.8, $R(\sigma; z)$ is continuous in σ for $0<\sigma\leqslant 1$. In view of $\int_C h'(z)\,dz=0$, we have for $0<\sigma<1$,

$$I_2=\frac{\sigma}{\pi}\int_C\int_G a(g)\,R(\sigma; z)\,|\beta z+\delta|^{2-2\sigma}\,|\beta z+\delta|^{-4}\,h\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right)dg\,dz. \tag{57}$$

LEMMA 5.15. *$|R(\sigma; z)|\,|\beta z+\delta|^{2-2\sigma}$ is bounded uniformly for z in C, $\frac{1}{2}<\sigma<1$, and g in any compact subset of G.*

Proof.

$$\begin{aligned}R(\sigma; z)&=\frac{1}{1-\sigma}\int_C|z'|^{-2+2\sigma}\,\overline{h'(z'+z)}\,dz'\\&=\frac{1}{1-\sigma}\int_0^\infty r^{-2+2\sigma}H(z; r)\,dr,\end{aligned} \tag{58}$$

where $z'=re^{i\theta}$, and $H(z; r)=r\int_0^{2\pi}h'(z'+z)\,d\theta$. Denote $\int_0^r H(z; r)\,dr$ by $G(z; r)$, and observe that, for fixed z,

$$G(z; r)=0 \quad \text{for large } r. \tag{59}$$

Integrating (58) by parts

$$R(\sigma; z)=2\int_0^\infty r^{-3+2\sigma}G(z; r)\,dr. \tag{60}$$

Let ϱ be the radius of a circle around the origin which contains the support of h'; then by (59)

$$G(z; r)=0 \text{ if } r>|z|+\varrho \text{ or } r<|z|-\varrho. \tag{61}$$

Now it is evident that

$$|G(z; r)|\leqslant\int_C|h'(z)|\,dz=M \tag{62}$$

for all z and r. Also there is a positive b such that for all z and r

$$|G(z; r)|\leqslant br^2. \tag{63}$$

Combining (60), (61), and (63), we obtain for $|z| \leqslant 2\varrho$

$$|R(\sigma; z)| \leqslant 2b \int_0^{3\varrho} r^{-3+2\sigma} r^2 \, dr \leqslant N < \infty. \tag{64}$$

On the other hand, if $|z| \geqslant 2\varrho$ and $\frac{1}{2} < \sigma < 1$, (60), (61), and (62) combine to give

$$|R(\sigma; z)| \leqslant 2M \int_{|z|-\varrho}^{|z|+\varrho} r^{-3+2\sigma} \, dr \leqslant \frac{8M\varrho}{|z|}. \tag{65}$$

From (64) and (65) it is apparent that there is a positive number N' such that

$$|R(\sigma; z)| \leqslant \frac{N'}{1 + \dfrac{|z|}{2\varrho}} \tag{66}$$

for all z in C and $\frac{1}{2} < \sigma < 1$. Also, an easy calculation shows that, for $\frac{1}{2} < \sigma < 1$,

$$|\beta z + \delta|^{2-2\sigma} \leqslant |\beta| \, |z| + |\delta| + 1. \tag{67}$$

Combining (66) and (67), we obtain the conclusion of the lemma.

In view of this lemma, we can pass to the limit $\sigma \to 1-$ under the integral sign in (57), obtaining by Lemma 7.10

$$\begin{aligned} \lim_{\sigma \to 1-} I_2 &= \frac{1}{\pi} \int_C \int_G a(g) \, R(1; z) \, |\beta z + \delta|^{-4} \, h\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) dg \, dz \\ &= \int_G a(g) \, (S_g h, h') \, dg \\ &= (S_a h, h'). \end{aligned} \tag{68}$$

Finally, we must evaluate $\lim_{\sigma \to 1-} I_3$ and $\lim_{\sigma \to 1-} I_4$. Note that in the calculation of $\lim_{\sigma \to 1-} I_2$ no use was made of the fact that $\int_C h(z) \, dz = 0$. Hence the expression obtained from I_2 on replacing h by h_1, namely $I_4 / \lambda \sqrt{(1-\sigma)}$, also approaches a finite limit as $\sigma \to 1-$. Thus

$$\lim_{\sigma \to 1-} I_4 = 0. \tag{69}$$

Now
$$I_4 = \sigma \lambda \, (T_a^\sigma(h_1), h') / \pi \sqrt{(1-\sigma)};$$

and
$$\bar{I}_3 = \sigma \lambda' \, \overline{(T_a^\sigma(h), h_1)} / \pi \sqrt{(1-\sigma)} = \sigma \lambda' \, (T_{a^*}^\sigma(h_1) \, h) / \pi \sqrt{(1-\sigma)}.$$

Thus $\bar{I}_3$ is obtained from I_4 on replacing λ, a, and h', by λ', a^*, and h. Hence from (69) follows

$$\lim_{\sigma\to 1-} I_3 = 0. \tag{70}$$

Combining (54), (55), (56), (68), (69), and (70), we obtain (53).

Now (40), (49), (50), and (53) give the following lemma:

LEMMA 5.16. *Each operator field in $C^*(G)$ is weakly continuous with respect to the continuity structure X.*

5.5. The structure of $C^*(G)$

LEMMA 5.17. *$C^*(G)$ is a subalgebra of the algebra A with continuous trace derived from the continuity structure X.*

Proof. Let y be of the form a^*a, where a is a complex continuous function on G with compact support. By Lemmas 3.6, 3.8, and 3.9 of [1], the function

$$w \to \operatorname{Tr}(T_y^w)$$

is continuous on Z and vanishes at infinity. Since $\|T_y^w\| \leqslant \operatorname{Tr}(T_y^w)$, T_y^w vanishes at infinity (in w). Combining these remarks with Lemma 5.16 and Theorem 4.6, we see that $w \to T_y^w$ belongs to A. Since linear combinations of such y are dense in $C^*(G)$, the conclusion of the lemma follows.

Combining Lemma 5.17 with Theorem 5.2, we obtain the following structure theorem for $C^*(G)$: (1)

THEOREM 5.3. *$C^*(G)$ consists of all those operator fields a on Z such that*

(i) *for each w in Z, $a(w)$ is a completely continuous operator on H_w;*

(ii) *a belongs to the algebra with continuous trace derived from X (see § 5.2);*

(iii) *the values of a at $(2, 0)$ and at 1 are correlated as follows:*

$$a(1) = D_- a(2, 0) D_-^{-1} \oplus \lambda_a,$$

where λ_a is a complex number (depending on a).

This theorem gives complete information about the structure of $C^*(G)$, but only in terms of the rather complicated continuity structure X. It is desirable to have a simple description of the isomorphism type of $C^*(G)$, without losing sight, however, of the underlying space Z. For this purpose we introduce the general notion of a field of isometries.

(1) This theorem strengthens the Lemma on p. 4 of [10] for the case of the 2×2 complex unimodular group.

Let T be a locally compact Hausdorff space, and for each t in T let Hilbert spaces H_t and H'_t be given.

DEFINITION. By a *field* of *isometries* (*of the* $\{H_t\}$ *onto the* $\{H'_t\}$) we understand a function U on T which associates to each t a linear isometry U_t of H_t onto H'_t.

DEFINITION. Let F and F' be families of vector fields (or operator fields) on T, whose values at t are vectors in H_t and H'_t respectively (or bounded operators on H_t and H'_t respectively). We shall say that F and F' are *isomorphic under* a field of isometries U if F' consists precisely of those a' which are of the form

$$a'(t) = U_t(a(t))\,(a'(t) = U_t a(t) U_t^{-1})$$

for some a in F.

DEFINITION. Let F and F' be continuity structures for vector fields on T with values in the $\{H_t\}$ and $\{H'_t\}$ respectively; and let F_c and F'_c be the families of all vector fields which are continuous on T with respect to F and F' respectively. Then F and F' are *equivalent* if F_c and F'_c are isomorphic under some field of isometries.

DEFINITION. If all H_t are the same H, the continuity structure consisting of all constant functions on T to H is the *product structure.*

LEMMA 5.18. *Let F be a continuity structure for T, $\{H_t\}$, and $\{x_1, x_2, \ldots\}$ a countable family of vector fields on T which are continuous with respect to F and such that, for each t, the set $\{x_1(t), x_2(t), \ldots\}$ is linearly independent in H_t and spans a dense subspace of H_t. Then F is equivalent to a product structure.*

Proof. By the Gram–Schmidt orthogonalization process, the x_i may be replaced by a countable set $\{y_1, y_2, \ldots\}$ of vector fields continuous with respect to F such that, for each t, the $y_i(t)$ form an orthonormal basis of H_t. It follows that the H_t are all of the same dimension. Let H be a fixed Hilbert space of this dimension, with an orthonormal basis $\{e_1, e_2, \ldots\}$. If U_t is the isometry of H onto H_t which sends e_i into $y_i(t)$, it is clear that the product structure (for vector fields on T to H) is equivalent with F under U.

We now apply these concepts to the continuity structure X for vector fields on Z (see § 5.3); and show that X is equivalent to a product structure.

The following lemma is easily verified:

LEMMA 5.19. *There exists a sequence $\{h'_1, h'_2, \ldots\}$ of elements of L_0 such that, for each h in L_0, we can find a subsequence $\{h'_{n_j}\}$ of $\{h'_n\}$ which converges to h uniformly with bounded support (that is, $h'_{n_j} \to h$ uniformly on C, and the supports of the h'_{n_j} are all contained in the same bounded set).*

Now let us pick out from the h'_n a subsequence $\{h_2, h_3, h_4, \ldots\}$ which is linearly independent in L_0 and spans the same subspace as the h'_n. Further, let h_1 have the same meaning as in § 5.3. Then the $h_n (n = 1, 2, 3, \ldots)$ are linearly independent in L, and are dense in L in the sense of uniform convergence with bounded support. Recalling from § 5.3 the definition of the $x_{\lambda, h}$, let us put:

$$x^{(1)} = x_{1, 0},\ x^{(n)} = x_{0, h_n} \text{ for } n > 1.$$

LEMMA 5.20. *For each w in Z, the $x^{(n)}(w) (n = 1, 2, \ldots)$ are linearly independent in* H_w

Proof. Assume first that $w \neq 1$; and let

$$\sum_{i=1}^{n} \lambda_i x^{(i)}(w) = 0 \text{ in } H_w (\lambda_i \text{ complex}). \tag{71}$$

Now a non-zero element of L is also non-zero in H_w. Hence

$$\sum_{i=1}^{n} \lambda_i x^{(i)}(w) = 0 \text{ in } L. \tag{72}$$

If $w \in Z_1$, (72) states that $\sum_{i=1}^{n} \lambda_i h_i = 0$; so that the $\lambda_i = 0$, since the h_n are independent in L. A similar argument holds for $0 < w < 1$. Thus, for all $w \neq 1$, (71) implies $\lambda_i = 0$.

Now let $w = 1$. Then (71) becomes

$$\left(\sum_{i=2}^{n} \lambda_i h_i\right) \oplus \frac{\lambda_1}{\sqrt{\pi}} = 0;$$

from which we have $\lambda_1 = 0$ and $\Sigma_{i=2}^{n} \lambda_i h_i = 0$ in K. But the latter clearly implies $\Sigma_{i=2}^{n} \lambda_i h_i = 0$ in L_0; so that again $\lambda_i = 0$.

LEMMA 5.21. *For each w in Z, the $x^{(n)}(w)$ $(n = 1, 2, \ldots)$ span a dense subspace of H_w.*

Proof. Let $w \in Z$, $w \neq 1$. Since the $\lambda h_1 + h$ (λ complex, $h \in L_0$) are dense in H_w, it is enough to show that h_1 and each h in L_0 can be approximated in H_w by linear combinations of the $x^{(n)}(w)$.

But
$$x^{(1)}(w) = \begin{cases} h_1 & \text{if } w \in Z_1, \\ \sqrt{\dfrac{\sigma}{\pi}}\, h_1 & \text{if } 0 < w = \sigma < 1. \end{cases}$$

Thus h_1 can be so approximated. That each h in L_0 can be so approximated follows from the definition of the h_n (see Lemma 5.19), together with the fact that, if $f_i \to f$ in L_0 uniformly with bounded support, then $f_i \to f$ in H_w.

Now let $w=1$. Since $x^{(1)}(1)=0\oplus 1/\sqrt{\pi}$, it is enough to show that each h in L_0 can be approximated in K by linear combinations of the h_n. But this is possible by Lemma 5.19 as before.

LEMMA 5.22. *X is equivalent to a product structure.*

Proof. Combine Lemmas 5.18, 5.20, and 5.21.

THEOREM 5.4. *Let H be a fixed separable infinite-dimensional Hilbert space and A the algebra of completely continuous operators on H. Suppose that $H=K\oplus C$, where C is the one-dimensional Hilbert space, and K is a closed subspace of H of co-dimension* 1. *Let M be any isometry of K onto H.*

Then $C^*(G)$, the group C^*-algebra of the 2×2 complex unimodular group, is isomorphic, under a field of isometries on Z, to the algebra of all those norm-continuous functions a on Z to A such that a vanishes at infinity and

$$a(1)=(M^{-1}a(2,0)M)\oplus\lambda$$

(where λ is complex and depends on a).

Proof. By Theorem 5.3 and Lemma 5.22 there exists an M for which this is true. But it is obvious that the particular choice of M does not affect the validity of the result.

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