CRITICAL SINGULAR PROBLEMS ON UNBOUNDED DOMAINS

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We present some results of existence for the following problem: $-\Delta u = a(x)g(u) + u|u|^{2^*-2}$, $x \in \mathbb{R}^N$ $(N \geq 3)$, $u \in D^{1,2}(\mathbb{R}^N)$, where the function a is a sign-changing function with a singularity at the origin and g has growth up to the Sobolev critical exponent $2^* = 2N/(N-2)$.

1. Introduction

Recently many works have been devoted to the study of existence of positive solutions u of the equation

$$-\Delta u = a(|x|)g(u), \quad x \in \mathbb{R}^N \ (N \geq 3),\ u \in H^1(\mathbb{R}^N), \tag{1.1}$$

with a continuous function a and a subcritical growth function g. This type of equation includes the Makutuma equation, when $a(|x|) = 1/(1+|x|^2)$ and $g(s) = |s|^{p-1}s$, with $1 < p < 2^* - 1 = (N+2)/(N-2)$, which appears in astrophysics and scalar curvature equations on $\mathbb{R}^N$ (see, *e.g.*, [15, 17, 18])

In [16] Munyamarere and Willem obtained a result of multiplicity of nodal solutions for these equations, considering the function a nonnegative and radially symmetric. The authors worked with a subspace of radial functions of $H^1(\mathbb{R}^N)$ which has the compactness properties desired to handle a problem like this modelled on an unbounded domain. In the same direction, Alama and Tarantello [2] studied the following problem when a is not radially symmetric and changes sign (see also [1, 7, 21, 22]):

$$-\Delta u - \lambda u = a(x)g(u), \quad x \in \Omega \subseteq \mathbb{R}^N(N \geq 3),\ u \in H^1(\Omega), \tag{1.2}$$

where Ω is a bounded domain and g behaves at infinity like a power function, $g(s) \simeq |s|^{p-1}s$, with $1 < p < (N+2)/(N-2)$ (subcritical case).

The above results on a bounded domain were extended, in part, by Costa and Tehrani in [12] for the whole space $\mathbb{R}^N$. They considered a weighted eigenvalue problem, namely,

$$-\Delta u = \lambda h(x)u, \quad x \in \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N)(N \geq 3), \tag{1.3}$$

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with $0 \leq h \in L^{N/2}(\mathbb{R}^N) \cap L^{\alpha}(\mathbb{R}^N)$, $\alpha > N/2$, which has the same properties as the eigenvalue problem for $-\Delta$ in a bounded domain (see, e.g., [11]). With the aid of this information, they studied the problem

$$-\Delta u - \lambda h(x)u = a(x)g(u), \quad x \in \mathbb{R}^N (N \geq 3),\ u \in H^1(\mathbb{R}^N). \tag{1.4}$$

Recently, still in the subcritical case, Tehrani in [23], considering problem (1.4) with $h = 0$ and Ω an unbounded exterior domain $\Omega = \mathbb{R}^N \setminus \overline{O}$ with $\overline{O} \neq \emptyset$, obtained similar results to those papers above.

Our main purpose in this work is to study the problem

$$-\Delta u = a(x)g(u) + u|u|^{2^*-2}, \quad x \in \mathbb{R}^N (N \geq 3),\ u \in D^{1,2}(\mathbb{R}^N), \tag{1.5}$$

where the Hilbert space $D^{1,2}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ endowed with the norm $\|u\| = (\int_{\mathbb{R}^N} |\nabla u|^2\,dx)^{1/2}$.

The above kind of problem is important since it is related to conformal deformations of Riemannian structures on noncompact manifolds (see, *e.g.*, [14]). Also, it is a physical model that appears when one describes the dynamics of galaxies (see, *e.g.*, [4]).

It is relevant to remark that our concern to study this type of problem with a function a changing sign comes from the following fact: if $u \in D^{1,2}(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ is a positive solution of (1.5), using a generalized Pohazev identity (see [8, Proposition 1]), we have

$$\int_{\mathbb{R}^N} (a(x)g(u)u + u^{2^*})\,dx = 2^* \int_{\mathbb{R}^N} \left(a(x)G(u) + \frac{1}{2^*}u^{2^*} \right) dx, \tag{1.6}$$

where $G(t) = \int_0^t g(s)ds$. Thus, for instance, if $g(s) = |s|^{p-2}s$, $2 < p < 2^*$, then a must change sign.

We would like to mention that when $a \in L^{2^*/(2^*-2)}(\mathbb{R}^N)$, Benci and Cerami in [6] studied the case $a \leq 0$ on $\mathbb{R}^N$, while in [19] Pan treated the case $a > 0$, and a case when a changes sign was handled by Ben-Naoum et al. in [5].

Our contribution to the study of these problems relay on the fact that we are working with a sign-changing discontinuous function a and with nonlinearities defined on the whole space $\mathbb{R}^N$ involving critical Sobolev exponent growth. These conditions imply a series of restrictions on the usual methods of dealing with these problems since the compactness of the Sobolev embedding is lost. In our case, a Hardy-type inequality is demanded. We would like to point out that our approach, with the corresponding changes, also works replacing $\mathbb{R}^N$ by a bounded or unbounded domain Ω. Finally we note that our work is precisely a version of the classical result of Brézis and Nirenberg (see [10]) considered under the aforementioned conditions.

Before stating our main theorem, we have to precise the set of assumptions on the functions g and a:

(i) $g:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying.

$$g(s)=o(|s|)\quad \text{as } s\longrightarrow 0, \tag{1.7}$$

$$\lim_{|s|\longrightarrow+\infty}\frac{sg(s)}{|s|^p}=1,\quad \text{for some } 2<p<2^*, \tag{1.8}$$

$$g(s)>0,\quad \forall s>0, \tag{1.9}$$

$$\lim_{\varepsilon\longrightarrow 0}\int_0^{\varepsilon^{-1/2}} G\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(N-2)/2} s^{N-1}ds=\infty. \tag{1.10}$$

(ii) $a:\mathbb{R}^N\to\mathbb{R}$ is a sign-changing function such that

$$a(x)=O(|x|^{-\alpha}),\quad \text{as } |x|\longrightarrow 0, \text{ for some } 0<\alpha\le\frac{2N-p(N-2)}{2}, \tag{1.11}$$

$$a(x)=O(|x|^{-2}),\quad \text{as } |x|\longrightarrow+\infty, \tag{1.12}$$

$$a \text{ is a continuous function in } 1\le|x|\le M \text{ for some } M>0, \tag{1.13}$$

$$a\in L(\mathbb{R}^N-B_\rho(0))\quad \text{for some } \rho>0, \tag{1.14}$$

($B_r(a)$ denotes a ball with radius r centered at a)

$$a(x)<0,\quad \text{for } |x|\ge R_0, \tag{1.15}$$

$$a(x)>0,\quad \text{for } |x|\le R_0-\delta, \tag{1.16}$$

where $R_0>M$ and $\delta>0$ is small.

We also require that $\Omega^0=\{x\in\mathbb{R}^N; a(x)=0\}$ have "*thick*" zero measure, that is,

$$\overline{\Omega^+}\cap\overline{\Omega^-}=\emptyset, \tag{1.17}$$

where $\Omega^+=\{x\in\mathbb{R}^N; a(x)>0\}$, and $\Omega^-=\{x\in\mathbb{R}^N; a(x)<0\}$.

Our main theorem is the following.

THEOREM 1.1. *Suppose that (1.7)–(1.17) hold. Then problem (1.5) has a positive solution.*

Remark 1.2. In addition to the hypotheses of Theorem 1.1, assuming that g is odd, problem (1.5) has infinitely many solutions. This follows by applying the classical genus theory, more exactly, a critical point theorem for even functional due to Rabinowitz (see [20]).

2. Variational framework

We are going to employ the variational methods to find a nontrivial weak solution for problem (1.5). To start, we define the Euler-Lagrange functional associated to it.

Let $\Psi : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} a(x)G(u)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx, \tag{2.1}$$

where $G(s) = \int_0^s g(t)dt$.

In order to guarantee that Ψ is well defined, we need the following Hardy-type inequality (see [13]).

Proposition 2.1. *For $N \geq 2$, there exists a constant $C = C(N)$ such that*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx \leq C \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \tag{2.2}$$

for all $u \in D^{1,2}(\mathbb{R}^N)$.

We check that the functional Ψ is well defined. Hereafter, we denote by C a generic positive constant. By (1.7) and (1.8), we have that

$$\int_{\mathbb{R}^N} a(x)G(u)dx \leq C\left(\int_{\mathbb{R}^N} |a(x)||u|^2 \, dx + \int_{\mathbb{R}^N} |a(x)||u|^p \, dx\right) \equiv I_1 + I_2. \tag{2.3}$$

We check that I_1 is finite. Since $(2N - p(N-2))/2 < 2$, we have by (1.11) and (2.2) that

$$\int_{|x|\leq 1} |a(x)||u|^2 dx \leq C\int_{|x|\leq 1} \frac{|u|^2}{|x|^\alpha} dx \leq C\int_{|x|\leq 1} \frac{|u|^2}{|x|^2} dx \leq C. \tag{2.4}$$

By (1.12) and (2.2),

$$\int_{|x|\geq M} |a(x)||u|^2 \, dx \leq \int_{|x|\geq M} \left(|a(x)||x|^2\right)\left(\frac{|u|^2}{|x|^2}\right) dx \leq C\int_{|x|\geq M} \frac{|u|^2}{|x|^2} dx \leq C. \tag{2.5}$$

Hence, by (2.4), (2.5), and (1.13), we have $I_1 < \infty$.

Choosing $r = 2N/(2N - p(N-2))$, by Hölder's inequality and, respectively, by (1.11) and (1.12), we have

$$\int_{|x|\leq 1} a(x)|u|^p \, dx \leq \left(\int_{|x|\leq 1} \frac{1}{|x|^{\alpha r}} dx\right)^{1/r} \left(\int_{|x|\leq 1} |u|^{2^*} dx\right)^{p/2^*} \leq C, \tag{2.6}$$

$$\int_{|x|\geq M} a(x)|u|^p \, dx \leq \left(\int_{|x|\geq M} \frac{dx}{|x|^{2r}}\right)^{1/r} \left(\int_{|x|\geq M} |u|^{2^*} dx\right)^{p/2^*} \leq C. \tag{2.7}$$

By (2.6), (2.7), and (1.13), we achieve that $I_2 < \infty$.

Therefore, Ψ is well defined and under the assumptions on the nonlinearities, a straightforward computation yields that $\Psi \in C^1(D^{1,2}(\mathbb{R}^N))$ and that for $v \in D^{1,2}(\mathbb{R}^N)$, we have

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx - \int_{\mathbb{R}^N} a(x) g(u) v \, dx - \int_{\mathbb{R}^N} v u |u|^{2^*-2} dx. \tag{2.8}$$

Hence, the critical points of Ψ are precisely the weak solutions for (1.5) and *vice versa*.

We also point out that with convenient hypotheses on the nonlinearities it is possible to obtain some regularization of the solutions.

3. Obtaining critical points Ψ

We are going to find a solution as a critical point of the functional Ψ. Before proceeding, we assure that the solution that we will find is indeed positive. Taking

$$\tilde{g}(u) = \begin{cases} g(u), & \text{if } u \geq 0, \\ g(-u), & \text{if } u < 0, \end{cases} \tag{3.1}$$

and using from now on the function $\tilde{g}(u)$, the critical point of Ψ is such that $u \geq 0$.

Now applying the maximum principle to the equation

$$-\Delta u - a^-(x)\tilde{g}(u) = a^+(x)\tilde{g}(u) + u_+^{2^*-1}, \quad x \in \mathbb{R}^N, \ u \in D^{1,2}(\mathbb{R}^N), \tag{3.2}$$

we infer that u must be positive ($a^+ = \max\{a, 0\}$ and $a^- = a - a^+$).

For simplicity, in what follows, the function $\tilde{g}$ will be denoted by g.

Returning to the functional Ψ, let $E = D^{1,2}(\mathbb{R}^N)$ and we firstly check that under our hypotheses, Ψ has the mountain pass geometry, that is,

$$\exists \beta > 0, \rho > 0 \text{ s.t. } \Psi(u) \geq \rho \quad \text{if } \|u\| = \beta, \tag{3.3}$$

$$\Psi(0) = 0, \exists e \in E, \quad \|e\| > \beta \text{ s.t. } \Psi(e) \leq 0. \tag{3.4}$$

Proposition 3.1. *If (1.7), (1.8), (1.11), (1.12), and (1.13) hold, then (3.3) and (3.4) also hold.*

Proof of (3.3). By (1.7) and (1.8), for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, p) > 0$ such that

$$|G(s)| \leq \varepsilon |s|^2 + C|s|^p, \quad s \in \mathbb{R}. \tag{3.5}$$

Hence, by estimates (2.4), (2.5), (2.6), and (2.7), together with the last inequality, we have

$$\Psi(u) \geq \frac{\|u\|^2}{2} - C\left\{\varepsilon \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{p/2^*}\right\}. \tag{3.6}$$

By (2.2) and the Sobolev embedding, for $\|u\|$ sufficiently small, we achieve that

$$\Psi(u) \geq \frac{\|u\|^2}{2} - C(\varepsilon\|u\|^2 + \|u\|^p) \geq \widetilde{C}\|u\|^2, \tag{3.7}$$

for some constant $\widetilde{C} > 0$ and $\varepsilon > 0$ small enough. Therefore, (3.3) holds. □

Proof of (3.4). Hypothesis (1.8) implies that

$$0 < \theta G(s) \leq sg(s), \quad |s| \geq s_0, \text{ for some } s_0 > 0,\ 2 < \theta < 2^*, \tag{3.8}$$

which, on its turn, implies that there exists $A > 0$ such that

$$|G(s)| \geq A|s|^\theta \quad \text{for } |s| \geq s_0. \tag{3.9}$$

Therefore, if $0 \leq \xi \in C_0^\infty(\Omega^+)$, by (3.9), we have that

$$\Psi(t\xi) = \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla\xi|^2\,dx - \int_{\mathbb{R}^N} a(x)G(t\xi)dx - \frac{t^{2^*}}{2}\int_{\mathbb{R}^N} |\xi|^{2^*}\,dx \longrightarrow -\infty, \tag{3.10}$$

as $t \to \infty$. □

Since (3.3) and (3.4) hold, by the mountain pass theorem without the Palais-Smale condition ((*PS*) *condition*, for short) (see [3]), if

$$\Gamma = \{\gamma \in C([0,1],E);\ \gamma(0) = 0,\ \gamma(1) = e\}, \tag{3.11}$$

$$c := \inf_{P\in\Gamma}\max_{w\in P}\Psi(w) \geq \rho, \tag{3.12}$$

then there exists a sequence $(u_n) \subset E$ such that

$$\Psi(u_n) \longrightarrow c \quad \text{in } \mathbb{R}, \text{ as } n \longrightarrow \infty, \tag{3.13}$$

$$\Psi'(u_n) \longrightarrow 0 \quad \text{in } E', \text{ as } n \longrightarrow \infty, \tag{3.14}$$

where Ψ' is the Frechet derivative of Ψ and E' is the dual space of E.

We define

$$S = \inf_{u\in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^N} |u|^{2^*}\,dx)^{2/2^*}}. \tag{3.15}$$

In the following result, we are going to prove that there exists $w \in E$ such that the constant c in (3.12) may be chosen is such a way that $c < (1/N)S^{N/2}$.

Proposition 3.2. *Suppose that (1.7)–(1.15) hold. Then there exists $u_0 \in E\setminus\{0\}$ such that*

$$\sup_{t\geq 0}\Psi(tu_0) < \frac{1}{N}S^{N/2} \tag{3.16}$$

Proof. Some ideas that follow in this proof were borrowed from [10]. We present them for completeness of the work.

We have that $a(x) > 0$ in $B_{R_0-\delta}(0)$. We choose a cutoff function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi \subset B_{2R}(x_0) \subset (B_{R_0-\delta}(0) - \{0\})$, $\varphi \equiv 1$ on $B_R(x_0)$ and $0 \le \varphi \le 1$ on $B_{2R}(x_0)$, for some convenient open ball $B_{2R}(x_0)$. For $\varepsilon > 0$, if

$$U_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{[\varepsilon + |x - x_0|^2]^{(N-2)/2}}, \tag{3.17}$$

it is well known that

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 \, dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*} \, dx = S^{N/2}, \tag{3.18}$$

$$\int_{B_R(x_0)} |\nabla U_\varepsilon|^2 \, dx \le \int_{B_R(x_0)} |U_\varepsilon|^{2^*} \, dx. \tag{3.19}$$

If we define $\eta_\varepsilon = \varphi U_\varepsilon$, it is easy to prove that

$$\int_{\mathbb{R}^N - B_R(x_0)} |\nabla \eta_\varepsilon|^2 \, dx = O(\varepsilon^{(N-2)/2}), \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.20}$$

To rewrite Ψ in a convenient way, let

$$v_\varepsilon = \frac{\eta_\varepsilon}{\left(\int_{B_{2R}(x_0)} |\eta_\varepsilon|^{2^*} \, dx\right)^{1/2^*}}, \qquad \chi_\varepsilon = \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 \, dx. \tag{3.21}$$

With this notation, it is forward to check that Ψ is bounded from above and that $\lim_{t\to\infty} \Psi(tv_\varepsilon) = -\infty$, for all $\varepsilon > 0$. So there exists $t_\varepsilon \ge 0$ such that

$$\sup_{t\ge 0} \Psi(tv_\varepsilon) = \Psi(t_\varepsilon v_\varepsilon). \tag{3.22}$$

Then differentiating $\Psi(tv_\varepsilon)$, we achieve that

$$t_\varepsilon \chi_\varepsilon - t_\varepsilon^{2^*} - \int_{B_{2R}(x_0)} a(x) g(t_\varepsilon v_\varepsilon) dx = 0 \tag{3.23}$$

and hence that

$$t_\varepsilon \le \chi_\varepsilon^{1/(2^*-1)}. \tag{3.24}$$

Also note that by (3.18), (3.19), (3.20), and (3.24), it follows that

$$\chi_\varepsilon \le S + O(\varepsilon^{(N-2)/2}). \tag{3.25}$$

On the other hand, the function $t \to t^2 t_0^{2^*-2}/2 - t^{2^*}/2^*$ is increasing on the interval $[0, t_0]$, where $t_0 = \chi_\varepsilon^{1/(2^*-2)}$. Then assertion (3.22) together with the above inequalities implies that

$$\Psi(t_\varepsilon u_\varepsilon) \le \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) - \int_{B_{2R}(x_0)} a(x) G(t_\varepsilon v_\varepsilon) dx. \tag{3.26}$$

By (1.7) and (1.8), for all $\tau > 0$ sufficiently small, there exists $C > 0$ satisfying $|a(x)g(u)| \le C|u| + \tau|u|^{2^*-1}$, for all $x \in B_{2R}(x_0)$.

Thus

$$\left| \int_{B_{2R}(x_0)} \frac{a(x)g(t_\varepsilon v_\varepsilon)}{t_\varepsilon} dx \right| \le \tau t_\varepsilon^{2^*-1} |v_\varepsilon|_{2^*}^{2^*} + C|v_\varepsilon|_2^2, \tag{3.27}$$

for τ sufficiently small.

Recalling that

$$|v_\varepsilon|_2^2 = \begin{cases} O(\varepsilon) & \text{if } N \ge 5, \\ O(\varepsilon \log \varepsilon) & \text{if } N = 4, \\ O(\varepsilon^{1/2}) & \text{if } N = 3, \end{cases} \tag{3.28}$$

from (3.27), we obtain

$$\int_{B_{2R}(x_0)} \frac{a(x)g(t_\varepsilon v_\varepsilon)}{t_\varepsilon} dx \longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.29}$$

Using this fact in (3.23), we conclude that

$$t_\varepsilon \longrightarrow S^{1/(2^*-2)}, \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.30}$$

Now, using (3.18)–(3.20) and (3.30), we have

$$\int_{\mathbb{R}^N} a(x)G(t_\varepsilon v_\varepsilon) dx \ge C \int_{B_{2R}(x_0)} G\left(\frac{c\varepsilon^{(N-2)/4}}{[\varepsilon + |x - x_0|^2]^{(N-2)/2}} \right) dx, \tag{3.31}$$

for some positive constants c and C. Substituting (3.31) in (3.26), we get

$$\Psi(t_\varepsilon u_\varepsilon) \le \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) - C \int_{B_{2R}(x_0)} G\left(\frac{c\varepsilon^{(N-2)/4}}{[\varepsilon + |x - x_0|^2]^{(N-2)/2}} \right) dx. \tag{3.32}$$

But

$$J_\varepsilon \equiv \frac{1}{\varepsilon^{(N-2)/2}} \int_{B_{2R}(x_0)} G\left(\frac{c\varepsilon^{(N-2)/4}}{[\varepsilon + |x - x_0|^2]^{(N-2)/2}} \right) dx \longrightarrow \infty, \quad \text{as } \varepsilon \longrightarrow 0. \tag{3.33}$$

In fact, since

$$J_\varepsilon = \frac{\omega_N}{\varepsilon^{(N-2)/2}} \int_0^R G\left(\frac{c\varepsilon^{(N-2)/4}}{[\varepsilon + r^2]^{(N-2)/2}} \right) r^{N-1} dr, \tag{3.34}$$

(ω_N is the area of S^{N-1}) making the change of variables $r = \varepsilon^{1/2}s$ and rescaling ε, we get

$$J_\varepsilon = \varepsilon\omega_N \int_0^{R\varepsilon^{-1/2}} G\left(\left[\frac{\varepsilon^{-1/2}}{1+s^2}\right]^{(N-2)/2}\right) s^{N-1}\, ds \tag{3.35}$$

for some constant $R > 0$.

Then, if $R \geq 1$, using (3.35), assertion (3.33) follows directly from hypothesis (1.10). If $R < 1$, consider

$$Z_\varepsilon = \varepsilon \int_{R\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} G\left(\left[\frac{\varepsilon^{-1/2}}{1+s^2}\right]^{(N-2)/2}\right) s^{N-1}\, ds. \tag{3.36}$$

Hence, there is $c > 0$ such that

$$|Z_\varepsilon| \leq c\varepsilon G\big(c\varepsilon^{(N-2)/4}\big)\varepsilon^{-N/2} \tag{3.37}$$

which implies, due to the growth of g, that $|Z_\varepsilon|$ is bounded as $\varepsilon \to 0$. Consequently, in the case $R < 1$, since

$$\int_0^{R\varepsilon^{-1/2}} = \int_0^{\varepsilon^{-1/2}} - \int_{R\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}, \tag{3.38}$$

and the last integral is bounded, as $\varepsilon \to 0$, it follows that (3.33) is a consequence of (3.37) and, again, of hypothesis (1.10).

Finally, applying (3.33) in (3.32), we see that

$$\Psi(t_\varepsilon u_\varepsilon) \leq \frac{1}{N}S^{N/2} \tag{3.39}$$

for small $\varepsilon > 0$, as desired. □

Next we prove the following.

Proposition 3.3. *If $(u_n) \subset D^{1.2}(\mathbb{R}^N)$ is a sequence such that (3.13) and (3.14) hold, then there exists a subsequence $u_n \rightharpoonup u_0$ weakly in $D^{1.2}(\mathbb{R}^N)$, as $n \to \infty$, for some $u_0 \in D^{1.2}(\mathbb{R}^N)$.*

Proof. The proof finishes if we prove that (u_n) is bounded. Suppose, on the contrary, that (u_n) is not bounded in $D^{1.2}(\mathbb{R}^N)$. We may assume that

$$\|u_n\| \equiv t_n \longrightarrow +\infty, \quad \text{as } n \longrightarrow \infty. \tag{3.40}$$

Define $v_n = u_n/t_n$. By (3.13) and (3.14), we achieve that

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\, dx - \int_{\mathbb{R}^N} \frac{aG(u_n)}{t_n^2}dx - \frac{1}{2^*}\int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2}dx = o_n(1) \tag{3.41}$$

and for all $v \in D^{1.2}(\mathbb{R}^N)$, we get that

$$\int_{\mathbb{R}^N} \nabla v_n \nabla v\, dx - \int_{\mathbb{R}^N} a\frac{g(u_n)}{t_n}v\, dx - \int_{\mathbb{R}^N} \frac{|u_n|^{2^*-2}u_n v}{t_n}dx = \frac{o_n(1)\|v\|}{t_n}. \tag{3.42}$$

Since $\|v_n\| = 1$, by (3.41), we have

$$\int_{\mathbb{R}^N} \frac{aG(u_n)}{t_n^2}dx + \frac{1}{2^*}\int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2}dx = \frac{1}{2} + o_n(1), \tag{3.43}$$

and taking $v = v_n$ in (3.42), we infer that

$$\int_{\mathbb{R}^N} \frac{ag(u_n)u_n}{t_n^2}dx + \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2}dx = 1 + o_n(1). \tag{3.44}$$

Observe that, combining (3.43) and (3.44) together with (1.9), we may assume that

$$\operatorname{supp} a \cap \operatorname{supp} g(u_n) \neq \emptyset, \qquad \operatorname{supp} a \cap \operatorname{supp} G(u_n) \neq \emptyset. \tag{3.45}$$

From (3.43) and (3.44), we get that

$$\left(\frac{2}{2^*} - 1\right)\int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2}dx = \int_{\mathbb{R}^N} \frac{a(g(u_n)u_n - 2G(u_n))}{t_n^2}dx + o_n(1). \tag{3.46}$$

Observe that, by (1.7) and (1.8), we have

$$|g(s)| \le C_1|s| + C_1|s|^q, \quad s \in \mathbb{R},\ 1 < q < 2^* - 1, \tag{3.47}$$

and hence, for a given ε, there exists a $K > 0$ such that

$$|g(t)t - 2G(t)| < \varepsilon|t|^{2^*} \quad \text{for } |t| \ge K. \tag{3.48}$$

The last integral in (3.46) may be split as

$$\begin{aligned} &\int_{|u_n|\le K} \frac{a(g(u_n)u_n - 2G(u_n))}{t_n^2}dx + \int_{|u_n|\ge K} \frac{a^+(g(u_n)u_n - 2G(u_n))}{t_n^2}dx \\ &\quad - \int_{|u_n|\ge K} \frac{a^-(g(u_n)u_n - 2G(u_n))}{t_n^2}dx. \end{aligned} \tag{3.49}$$

We bound these integrals. Since (1.14) holds, the first integral is $o_n(1)$; the second, taking $K > s_0$ in (3.9), is nonnegative, and the last one, by (3.48), is bounded as follows:

$$\int_{|u_n|\ge K} \frac{a^-(g(u_n)u_n - 2G(u_n))}{t_n^2}dx \le \varepsilon\|a^-\|_\infty \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2}. \tag{3.50}$$

Using these facts in (3.46), we have

$$\left(\left(\frac{2}{2^*}-1\right)+\varepsilon\|a^-\|_\infty\right)\int_{\mathbb{R}^N}\frac{|u_n|^{2^*}}{t_n^2}dx \geq o_n(1). \tag{3.51}$$

Thereafter, picking a small ε, we conclude that

$$\int_{\mathbb{R}^N}\frac{|u_n|^{2^*}}{t_n^2}dx \longrightarrow 0. \tag{3.52}$$

We use this limit to contradict the fact that $\|u_n\| \to \infty$.

We also may consider that there exists $v \in D^{1,2}(\mathbb{R}^N)$ such that

$$v_n \longrightarrow v \quad \text{a.e. in } \mathbb{R}^N, \tag{3.53}$$

and for all bounded sets $U \subset \mathbb{R}^N$ and for $1 \leq t < 2^*$,

$$\begin{gathered} v_n \longrightarrow v \quad \text{in } L^t(U), \\ v_n(x) \longrightarrow v(x), \quad \text{for } x \in U \text{ a.e.}, \\ |v_n(x)| \leq h(x) \quad \text{for } h \in L^t(U), \text{ and a.e. in } U, \end{gathered} \tag{3.54}$$

as $n \to \infty$.

In the sequel, we need the following claim which will be proved at the end of this proof.

Claim. $v \equiv 0$.

Proceeding, we take $\xi \in C_0^\infty(\mathbb{R}^N)$. Inserting $v = v_n\xi$ in (3.42) and using the claim, we get

$$\int_{\mathbb{R}^N}|\nabla v_n|^2\xi\,dx - \int_{\mathbb{R}^N}a\frac{g(u_n)u_n}{t_n^2}\xi\,dx - \int_{\mathbb{R}^N}|u_n|^{2^*-2}v_n^2\xi\,dx = o_n(1). \tag{3.55}$$

We choose the cutoff function $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \equiv 1$ on Ω^+ and $\xi \equiv 0$ on Ω^-. Using (3.8) and (3.55), together with (3.45), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N}\frac{aG(u_n)\xi}{t_n^2}dx &= \int_{|u_n|\leq s_0}\frac{aG(u_n)}{t_n^2}\xi\,dx + \int_{|u_n|>s_0}\frac{a(G(u_n))}{t_n^2}\xi\,dx \\ &\leq o_n(1) + \frac{1}{\theta}\int_{|u_n|>s_0}\frac{ag(u_n)u_n\xi}{t_n^2}dx \\ &= o_n(1) + \frac{1}{\theta}\left[\int_{\mathbb{R}^N}|\nabla v_n|^2\xi\,dx - \int_{\mathbb{R}^N}|u_n|^{2^*-2}v_n^2\xi\,dx\right] \\ &\leq o_n(1) + \frac{1}{\theta} - \frac{1}{\theta}\int_{\mathbb{R}^N}\frac{u_n^{2^*}\xi}{t_n^2}dx. \end{aligned} \tag{3.56}$$

The above inequality together with (3.8) and (3.42) yields

$$\left(\frac{1}{2}-\frac{1}{\theta}\right) \leq \frac{1}{2^*}\int_{\mathbb{R}^N}\frac{|u_n|^{2^*}}{t_n^2}dx+o_n(1), \tag{3.57}$$

which contradicts (3.52).

Proof of the claim. We are going to prove that $v(x)=0$ a.e. for $x\in\Omega^+$, arguing by contradiction. Let $F=\{x\in\mathbb{R}^N; v(x)\neq 0\}$ and we suppose that there exists $B_r(x_0)$ such that

$$|F\cap B_{2r}(x_0)|>0, \tag{3.58}$$

where $|\cdot|$ denotes the Lebesgue measure defined in $\mathbb{R}^N$.

Pick $\xi\in C_0^\infty(B_{2r}(x_0))$ such that $\xi(x)=1$ if $x\in B_r(x_0)$. Replacing $v=v_n\xi$ in (3.42), we have

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla v_n|^2\xi\,dx&+\int_{\mathbb{R}^N}v_n\nabla v_n\nabla\xi\,dx-\int_{\mathbb{R}^N}|u_n|^{2^*-2}v_n^2\xi\,dx\\&=t_n^{p-2}\left[\int_{\mathbb{R}^N}a|v_n|^p\frac{g(t_nv_n)}{|t_nv_n|^p}t_nv_n\xi\,dx\right]+o_n(1).\end{aligned} \tag{3.59}$$

But, since $|t_nv_n(x)|\to\infty$, for $x\in F$, as $n\to\infty$, using (1.8), the growth conditions of g, (3.54), and the Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}^N}a|v_n|^p\frac{g(t_nv_n)}{|t_nv_n|^p}t_nv_n\xi\,dx\longrightarrow\int_{\mathbb{R}^N}a|v|^p\xi\,dx\geq\int_{\operatorname{supp}\xi}a|v|^p\,dx>0, \tag{3.60}$$

as $n\to\infty$. Observe that the left-hand side integrals in equality (3.59) are all bounded, but on the other hand, passing to the limit as $n\to\infty$ in (3.59), the right-hand side goes to ∞, since (3.60) holds. This is a contradiction. Hence, $v\equiv 0$ on Ω^+. A similar reasoning yields that $v\equiv 0$ on Ω^-. □

This completes the proof of the proposition. □

4. Proof of Theorem 1.1

By Proposition 3.3, we may assume that $u_n\rightharpoonup u_0$. Before proceeding further in order to prove that u_0 is the wanted positive solution, we firstly assume for a while three facts that we will prove later.

(1)

$$\int_{\mathbb{R}^N}ag(u_n)v\,dx\longrightarrow\int_{\mathbb{R}^N}ag(u)v\,dx,\quad\forall v\in E,\text{ as }n\longrightarrow\infty. \tag{4.1}$$

(2) and (3) If $u_0\equiv 0$, then

$$\int_{\mathbb{R}^N}ag(u_n)u_n\,dx\longrightarrow 0,\quad\text{as }n\longrightarrow\infty, \tag{4.2}$$

$$\int_{\mathbb{R}^N}aG(u_n)dx\longrightarrow 0,\quad\text{as }n\longrightarrow\infty. \tag{4.3}$$

By a Brézis-Lieb result (see [9]), we have that

$$\int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n v\, dx \longrightarrow \int_{\mathbb{R}^N} |u_0|^{2^*-2} u_0 v\, dx, \quad \text{as } n \longrightarrow \infty. \tag{4.4}$$

Hence, by (4.1), passing to the limit in (3.14), we achieve that

$$\langle \Psi'(u_0), v\rangle = 0, \quad \forall v \in E, \tag{4.5}$$

that is, u_0 is a weak solution for (1.5).

To see that $u_0 \neq 0$, suppose on the contrary, that $u_0 \equiv 0$. If

$$\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \longrightarrow l, \quad \text{as } n \longrightarrow \infty, \tag{4.6}$$

then by (4.2) and (3.14) with $v = u_n$, we get

$$\int_{\mathbb{R}^N} |u_n|^{2^*}\, dx \longrightarrow l, \quad \text{as } n \longrightarrow \infty. \tag{4.7}$$

Using (4.3), (4.6), (4.7) and passing to the limit in (3.13) yields

$$l = Nc > 0 \tag{4.8}$$

with the choice that

$$c < \frac{1}{N} S^{N/2}, \tag{4.9}$$

since (3.16) holds.

Passing to the limit in definition (3.15) with u_n, and regarding (4.6) and (4.7), we get that $l \geq S^{N/2}$. But this inequality contradicts (4.9).

Proof of (4.1). Using (1.7) and (1.8), we see that for a given $\varepsilon > 0$,

$$|g(s)| \leq \varepsilon |s| + C|s|^{p-1}, \quad \forall s \in \mathbb{R}, \tag{4.10}$$

for some $C > 0$. Hence, combining (4.10), (1.11), and a similar reasoning for u_n such as that made in (3.54), there exists $h \in L^t(U)$, $U \subset \mathbb{R}^N$, $1 \leq t < 2^*$, such that

$$|ag(u)| \leq C\left(\frac{|h|v}{|x|^\alpha} + \frac{|h|^{p-1}v}{|x|^\alpha}\right) \in L^1(B_R(0)), \quad \text{for some } R, C > 0. \tag{4.11}$$

Thus, applying the Lebesgue dominated convergence theorem yields

$$\int_{|x|\leq R} ag(u_n) v\, dx \longrightarrow \int_{|x|\leq R} ag(u) v\, dx, \quad \text{as } n \longrightarrow \infty. \tag{4.12}$$

The proof finishes if we prove that

$$\lim_{R\to\infty}\int_{|x|>R}|ag(u_n)v|\,dx=0,\quad \text{uniformly in } n. \tag{4.13}$$

By (1.12), (2.2), (4.10), and Hölder's inequality, we have

$$\begin{aligned}\int_{|x|>R}|ag(u_n)v|\,dx &\le \varepsilon\int_{|x|>R}|a||u_n||v|dx+\int_{|x|>R}|a||u_n|^{p-1}|v|dx\\ &\le \varepsilon C\|u_n\|\,\|v\|+C\Big(\int_{|x|>R}|a|^r\,dx\Big)^{1/r}\|u_n\|^{p-1}\,\|v\|^{2^*},\end{aligned} \tag{4.14}$$

where $r=2^*/(2^*-p)$. Since the sequence (u_n) is bounded in E norm, if $R>0$ is chosen in the above inequality, such that

$$\Big(\int_{|x|>R}|a|^r\,dx\Big)^{1/r}<\varepsilon, \tag{4.15}$$

we assure that (4.13) holds. □

Proof of (4.2) and (4.3). The proof is made using similar reasoning as those made in the previous proof. □

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