GENERIC UNIQUENESS OF A MINIMAL SOLUTION FOR VARIATIONAL PROBLEMS ON A TORUS

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We study minimal solutions for one-dimensional variational problems on a torus. We show that, for a generic integrand and any rational number α, there exists a unique (up to translations) periodic minimal solution with rotation number α.

1. Introduction

In this paper, we consider functionals of the form

$$I^f(a,b,x) = \int_a^b f\left(t, x(t), x'(t)\right) dt, \tag{1.1}$$

where a and b are arbitrary real numbers satisfying $a < b$, $x \in W^{1,1}(a,b)$ and f belongs to a space of functions described below. By an appropriate choice of representatives, $W^{1,1}(a,b)$ can be identified with the set of absolutely continuous functions $x : [a,b] \to \mathbb{R}^1$, and henceforth we will assume that this has been done.

Denote by $\mathfrak{M}$ the set of integrands $f = f(t,x,p) : \mathbb{R}^3 \to \mathbb{R}^1$ which satisfy the following assumptions:

(A1) $f \in C^3$ and $f(t,x,p)$ has period 1 in t,x;
(A2) $\delta_f \le f_{pp}(t,x,p) \le \delta_f^{-1}$ for every $(t,x,p) \in \mathbb{R}^3$;
(A3) $|f_{xp}| + |f_{tp}| \le c_f(1+|p|)$, $|f_{xx}| + |f_{xt}| \le c_f(1+p^2)$,

with some constants $\delta_f \in (0,1)$, $c_f > 0$.

Clearly, these assumptions imply that

$$\tilde{\delta}_f p^2 - \tilde{c}_f \le f(t,x,p) \le \tilde{\delta}_f^{-1} p^2 + \tilde{c}_f \tag{1.2}$$

for every $(t,x,p) \in \mathbb{R}^3$ for some constants $\tilde{c}_f > 0$ and $0 < \tilde{\delta}_f < \delta_f$.

In this paper, we analyse extremals of variational problems with integrands $f \in \mathfrak{M}$. The following optimality criterion was introduced by Aubry and Le

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Daeron [2] in their study of the discrete Frenkel-Kontorova model related to dislocations in one-dimensional crystals.

Let $f \in \mathfrak{M}$. A function $x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ is called an (f)-minimal solution if

$$I^f(a,b,y) \ge I^f(a,b,x) \tag{1.3}$$

for each pair of numbers $a < b$ and each $y \in W^{1,1}(a,b)$ which satisfies $y(a) = x(a)$ and $y(b) = x(b)$ (see [2, 9, 10, 12]).

Our work follows Moser [9, 10], who studied the existence and structure of minimal solutions in the spirit of Aubry-Mather theory [2, 7].

Consider any $f \in \mathfrak{M}$. It was shown in [9, 10] that (f)-minimal solutions possess numerous remarkable properties. Thus, for every (f)-minimal solution $x(\cdot)$, there is a real number α satisfying

$$\sup\left\{\left|x(t)-\alpha t\right| : t \in \mathbb{R}^1\right\} < \infty \tag{1.4}$$

which is called the rotation number of $x(\cdot)$, and given any real α there exists an (f)-minimal solution with rotation number α. Senn [11] established the existence of a strictly convex function $E_f : \mathbb{R}^1 \to \mathbb{R}^1$, which is called the minimal average action of f such that, for each real α and each (f)-minimal solution x with rotation number α,

$$\left(T_2 - T_1\right)^{-1} I^f\left(T_1, T_2, x\right) \longrightarrow E_f(\alpha) \quad \text{as } T_2 - T_1 \longrightarrow \infty. \tag{1.5}$$

This result is an analogue of Mather's theorem about the average energy function for Aubry-Mather sets generated by a diffeomorphism of the infinite cylinder [8].

In this paper, we show that for a generic integrand f and any rational α, there exists a unique (up to translations) (f)-minimal periodic solution with rotation number α.

Let $k \ge 3$ be an integer. Set $\mathfrak{M}_k = \mathfrak{M} \cap C^k(\mathbb{R}^3)$. For $f \in \mathfrak{M}_k$ and $q = (q_1, q_2, q_3) \in \{0, \dots, k\}^3$ satisfying $q_1 + q_2 + q_3 \le k$, we set

$$|q| = q_1 + q_2 + q_3, \quad D^q f = \frac{\partial^{|q|} f}{\partial t^{q_1} \partial x^{q_2} \partial p^{q_3}}. \tag{1.6}$$

For $N, \epsilon > 0$ we set

$$\begin{aligned} E_k(N,\epsilon) = \big\{ (f,g) \in \mathfrak{M}_k \times \mathfrak{M}_k : &\left|D^q f(t,x,p) - D^q g(t,x,p)\right| \\ &\le \epsilon + \epsilon \max\left\{\left|D^q f(t,x,p)\right|, \left|D^q g(t,x,p)\right|\right\} \\ &\forall q \in \{0,1,2\}^3 \text{ satisfying } |q| \in \{0,2\},\ \forall (t,x,p) \in \mathbb{R}^3 \big\} \\ \cap \big\{ (f,g) \in \mathfrak{M}_k \times \mathfrak{M}_k : &\left|D^q f(t,x,p) - D^q g(t,x,p)\right| \le \epsilon \\ &\forall q \in \{0,\dots,k\}^3 \text{satisfying } |q| \le k,\ \forall (t,x,p) \in \mathbb{R}^3 \\ &\qquad \text{such that } |p| \le N \big\}. \end{aligned} \tag{1.7}$$

It is easy to verify that, for the set $\mathfrak{M}_k$ there exists a uniformity which is determined by the base $E_k(N,\epsilon)$, $N,\epsilon > 0$, and that the uniform space $\mathfrak{M}_k$ is metrizable and complete [3]. We establish the existence of a set $\mathcal{F}_k \subset \mathfrak{M}_k$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}_k$ such that, for each $f \in \mathcal{F}_k$ and each rational $\alpha \in \mathbb{R}^1$, there exists a unique (up to translations) (f)-minimal periodic solultion with rotation number α.

2. Properties of minimal solutions

Consider any $f \in \mathfrak{M}$. We note that, for each pair of integers j and k the translations $(t,x) \to (t+j, x+k)$ leave the variational problem invariant. Therefore, if $x(\cdot)$ is an (f)-minimal solution, so is $x(\cdot+j)+k$. Of course, on the torus, this represents the same curve as does $x(\cdot)$. This motivates the following terminology [9, 10].

We say that a function $x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ has no self-intersections if for all pairs of integers j,k the function $t \to x(t+j)+k-x(t)$ is either always positive, or always negative, or identically zero.

Denote by $\mathbb{Z}$ the set of all integers. We have the following result (see [6, Proposition 3.2] and [9, 10]).

Proposition 2.1. (i) *Let $f \in \mathfrak{M}$. Given any real α there exists a nonself-intersecting (f)-minimal solution with rotation number α.*

(ii) *For any $f \in \mathfrak{M}$ and any (f)-minimal solution x, there is the rotation number of x.*

For each $f \in \mathfrak{M}$, each rational number α, and each natural number q satisfying $q\alpha \in \mathbb{Z}$, we define

$$\begin{aligned} \mathcal{N}(\alpha,q) &= \{x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) : x(t+q) = x(t)+\alpha q,\ t \in \mathbb{R}^1\}, \\ \mathcal{M}_f(\alpha,q) &= \{x(\cdot) \in \mathcal{N}(\alpha,q) : I^f(0,q,x) \le I^f(0,q,y)\ \forall y \in \mathcal{N}(\alpha,q)\}. \end{aligned} \tag{2.1}$$

We have the following result [9, Theorems 5.1, 5.2, 5.4, and Corollaries 5.3 and 5.5].

Proposition 2.2. *Let $f \in \mathfrak{M}$, let α be a rational number, and let $p,q \ge 1$ be integers satisfying $p\alpha, q\alpha \in \mathbb{Z}$. Then $\mathcal{M}_f(\alpha,q) = \mathcal{M}_f(\alpha,p) \ne \emptyset$, each $x \in \mathcal{M}_f(\alpha,q)$ is a nonself-intersecting (f)-minimal solution with rotation number α and the set $\mathcal{M}_f(\alpha,q)$ is totally ordered, that is, if $x,y \in \mathcal{M}_f(\alpha,q)$, then either $x(t) < y(t)$ for all t, or $x(t) > y(t)$ for all t, or $x(t) = y(t)$ identically.*

For any $f \in \mathfrak{M}$ and any rational number α we set $\mathcal{M}^{\text{per}}_f(\alpha) = \mathcal{M}_f(\alpha,q)$, where q is a natural number satisfying $q\alpha \in \mathbb{Z}$.

We have the following result (see [6, Theorem 1.1]).

Proposition 2.3. *Let $f \in \mathfrak{M}$. Then there exist a strictly convex function $E_f : \mathbb{R}^1 \to \mathbb{R}^1$ satisfying $E_f(\alpha) \to \infty$ as $|\alpha| \to \infty$ and a monotonically increasing function $\Gamma_f : (0,\infty) \to [0,\infty)$ such that for each real α, each (f)-minimal solution x with* *rotation number α and each pair of real numbers S and T,*

$$\left|I^f(S,S+T,x)-E_f(\alpha)T\right| \le \Gamma_f(|\alpha|). \tag{2.2}$$

By Proposition 2.3 for each $f \in \mathfrak{M}$ there exists a unique number $\alpha(f)$ such that

$$E_f(\alpha(f)) = \min\{E_f(\beta) : \beta \in \mathbb{R}^1\}. \tag{2.3}$$

Note that assumptions (A1), (A2), and (A3) play an important role in the proofs of Propositions 2.1, 2.2, and 2.3 (see [9, 10]).

3. The main results

Theorem 3.1. *Let $k \ge 3$ be an integer and α be a rational number. Then there exists a set $\mathcal{F}_{k\alpha} \subset \mathfrak{M}_k$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}_k$ such that for each $f \in \mathfrak{M}_k$ the following assertions hold:*

(1) If $x, y \in \mathcal{M}_f^{(\text{per})}(\alpha)$, then there are integers p, q such that $y(t) = x(t+p)-q$ for all $t \in \mathbb{R}^1$.

(2) Let $x \in \mathcal{M}_f^{(\text{per})}(\alpha)$ and $\epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathfrak{M}_k$ such that for each $g \in \mathcal{U}$ and each $y \in \mathcal{M}_g^{(\text{per})}(\alpha)$ there are integers p, q such that $|y(t)-x(t+p)+q| \le \epsilon$ for all $t \in \mathbb{R}^1$.

It is not difficult to see that Theorem 3.1 implies the following result.

Theorem 3.2. *Let $k \ge 3$ be an integer. Then there exists a set $\mathcal{F}_k \subset \mathfrak{M}_k$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{M}_k$ such that, for each $f \in \mathfrak{M}_k$ and each rational number α the assertions (1) and (2) of Theorem 3.1 hold.*

Note that minimal solutions with irrational rotation numbers were studied in [2, 7, 9, 10, 12].

4. An auxiliary result

Let $k \ge 3$ be an integer and $\beta \in \mathbb{R}^1$. For each $f \in \mathfrak{M}_k$, define $\mathcal{A}f \in C^3(\mathbb{R}^3)$ by

$$(\mathcal{A}f)(t,x,u) = f(t,x,u)-\beta u, \quad (t,x,u) \in \mathbb{R}^3. \tag{4.1}$$

Clearly $\mathcal{A}f \in \mathfrak{M}_k$ for each $f \in \mathfrak{M}_k$.

Proposition 4.1. *The mapping $\mathcal{A} : \mathfrak{M}_k \to \mathfrak{M}_k$ is continuous.*

Proof. Let $f \in \mathfrak{M}_k$ and let $N, \epsilon > 0$. In order to prove the proposition, it is sufficient to show that there exists $\epsilon_0 \in (0,\epsilon)$ such that

$$\mathcal{A}\left(\{g \in \mathfrak{M}_k : (f,g) \in E_k(N,\epsilon_0)\}\right) \subset \{h \in \mathfrak{M}_k : (h,\mathcal{A}f) \in E_k(N,\epsilon)\}. \tag{4.2}$$

Set

$$\Delta_0 = 2(|\beta|+1). \tag{4.3}$$

Equation (1.2) implies that there exists $c_0 > 0$ such that

$$\Delta_0|u| - c_0 \le f(t,x,u) \quad \forall (t,x,u) \in \mathbb{R}^3. \tag{4.4}$$

Choose a number ϵ_0 such that

$$0 < \epsilon_0 < \min\{1,\epsilon\}, \quad 4\epsilon_0 + 4\epsilon_0(1-\epsilon_0)^{-1}(4+c_0) < \epsilon. \tag{4.5}$$

It follows from (4.3) and (4.4) that for each $(t,x,u) \in \mathbb{R}^3$,

$$\begin{aligned}
|f(t,x,u) - \beta u| &\ge |f(t,x,u)| - |\beta u| \ge |f(t,x,u)| - |\beta|\Delta_0^{-1}(f(t,x,u) + c_0) \\
&\ge |f(t,x,u)|(1 - |\beta|\Delta_0^{-1}) - |\beta|\Delta_0^{-1}c_0 \\
&\ge 2^{-1}|f(t,x,u)| - 2^{-1}c_0.
\end{aligned} \tag{4.6}$$

Assume that

$$g \in \mathfrak{M}_k, \quad (f,g) \in E_k(N,\epsilon_0). \tag{4.7}$$

By (1.7) and (4.7) for each $(t,x,u) \in \mathbb{R}^3$,

$$\begin{gathered}
|f(t,x,u) - g(t,x,u)| \le \epsilon_0 + \epsilon_0 \max\{|f(t,x,u)|, |g(t,x,u)|\}, \\
\max\{|f(t,x,u)|, |g(t,x,u)|\} - \min\{|f(t,x,u)|, |g(t,x,u)|\} \\
\le \epsilon_0 + \epsilon_0 \max\{|f(t,x,u)|, |g(t,x,u)|\}, \\
(1-\epsilon_0)\max\{|f(t,x,u)|, |g(t,x,u)|\} \le \min\{|f(t,x,u)|, |g(t,x,u)|\} + \epsilon_0, \\
|g(t,x,u)| \le (1-\epsilon_0)^{-1}|f(t,x,u)| + (1-\epsilon_0)^{-1}\epsilon_0.
\end{gathered} \tag{4.8}$$

We show that $(\mathcal{A}f, \mathcal{A}g) \in E_k(N,\epsilon)$. It follows from (1.7), (4.1), (4.5), and (4.7) that, for each $q = (q_1,q_2,q_3) \in \{0,\dots,k\}^3$ satisfying $|q| \le k$ and each $(t,x,p) \in \mathbb{R}^3$ satisfying $|p| \le N$,

$$|D^q(\mathcal{A}f)(t,x,p) - D^q(\mathcal{A}g)(t,x,p)| = |D^q f(t,x,p) - D^q g(t,x,p)| \le \epsilon_0 < \epsilon. \tag{4.9}$$

Let $q \in \{0,1,2\}^3$, $|q| \in \{0,2\}$, and $(t,x,p) \in \mathbb{R}^3$. Equation (4.1) implies that

$$|D^q(\mathcal{A}f)(t,x,p) - D^q(\mathcal{A}g)(t,x,p)| = |D^q f(t,x,p) - D^q g(t,x,p)|. \tag{4.10}$$

If $|q| = 2$, then by (1.7), (4.1), (4.5), (4.7), and (4.10),

$$\begin{aligned}
&|D^q(\mathcal{A}f)(t,x,p) - D^q(\mathcal{A}g)(t,x,p)| \\
&\quad \le \epsilon_0 + \epsilon_0 \max\{|D^q f(t,x,p)|, |D^q g(t,x,p)|\} \\
&\quad < \epsilon + \epsilon \max\{|D^q(\mathcal{A}f)(t,x,p)|, |D^q(\mathcal{A}g)(t,x,p)|\}.
\end{aligned} \tag{4.11}$$

Assume that $q = 0$. By (1.7), (4.1), (4.5), (4.6), (4.7), and (4.8),

$$\begin{aligned}
&\left|D^q(\mathcal{A}f)(t,x,p)-D^q(\mathcal{A}g)(t,x,p)\right| \\
&\quad= \left|f(t,x,p)-g(t,x,p)\right| \le \epsilon_0+\epsilon_0 \max\left\{\left|f(t,x,p)\right|,\left|g(t,x,p)\right|\right\} \\
&\quad\le \epsilon_0+\epsilon_0 \max\left\{\left|f(t,x,p)\right|,\left(1-\epsilon_0\right)^{-1}\left|f(t,x,p)\right|+\left(1-\epsilon_0\right)^{-1}\epsilon_0\right\} \\
&\quad= \epsilon_0+\epsilon_0\left(1-\epsilon_0\right)^{-1}\left|f(t,x,p)\right|+\epsilon_0^2\left(1-\epsilon_0\right)^{-1} \\
&\quad\le \epsilon_0+\epsilon_0^2\left(1-\epsilon_0\right)^{-1}+\epsilon_0\left(1-\epsilon_0\right)^{-1}\left[2\left|f(t,x,p)-\beta p\right|+2c_0\right] \\
&\quad\le \epsilon_0+\epsilon_0^2\left(1-\epsilon_0\right)^{-1}+2\epsilon_0\left(1-\epsilon_0\right)^{-1}c_0+2\epsilon_0\left(1-\epsilon_0\right)^{-1}\left|f(t,x,p)-\beta p\right| \\
&\quad\le 2\epsilon_0(1-\epsilon_0)^{-1}\left|(\mathcal{A}f)(t,x,p)\right|+\epsilon \le \epsilon+\epsilon\left|(\mathcal{A}f)(t,x,p)\right|.
\end{aligned} \tag{4.12}$$

Equations (4.9), (4.11), and (4.12) imply that $(\mathcal{A}f,\mathcal{A}g) \in E_k(N,\epsilon)$. Proposition 4.1 is proved. □

Let $-\infty < T_1 < T_2 < \infty$ and $x \in W^{1,1}(T_1,T_2)$. By (4.1) we have

$$\begin{aligned}
I^{\mathcal{A}f}(T_1,T_2,x) &= \int_{T_1}^{T_2}\left(f\left(t,x(t),x'(t)\right)-\beta x'(t)\right)dt \\
&= I^f(T_1,T_2,x)-\beta x(T_2)+\beta x(T_1).
\end{aligned} \tag{4.13}$$

Therefore, each $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^1)$ is an $(\mathcal{A}f)$-minimal solution if and only if $x(\cdot)$ is an (f)-minimal solution.

Let $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^1)$ be an (f)-minimal solution with rotation number r. By Proposition 2.1 there exists $c_1 > 0$ such that for all $s,t \in \mathbb{R}^1$,

$$\left|x(t+s)-x(t)-rs\right| \le c_1. \tag{4.14}$$

Proposition 2.3 implies that there exists a constant $c_2 > 0$ such that for each $s \in \mathbb{R}^1$ and each $t > 0$,

$$\left|I^f(s,s+t,x)-E_f(r)t\right| \le c_2, \tag{4.15}$$

$$\left|I^{\mathcal{A}f}(s,s+t,x)-E_{\mathcal{A}f}(r)t\right| \le c_2. \tag{4.16}$$

It follows from (4.13), (4.14), (4.15), and (4.16) that, for each $s \in \mathbb{R}^1$ and each $t > 0$,

$$\begin{aligned}
&\left|E_{\mathcal{A}f}(r)t+\beta tr-E_f(r)t\right| \\
&\quad\le \left|E_{\mathcal{A}f}(r)t-I^{\mathcal{A}f}(s,s+t,x)\right|+\left|I^{\mathcal{A}f}(s,s+t,x)+\beta tr-I^f(s,s+t,x)\right| \\
&\qquad+\left|I^f(s,s+t,x)-E_f(r)t\right| \\
&\quad\le c_2+\left|\beta tr-\beta\left[x(t+s)-x(s)\right]\right|+c_2 \le 2c_2+|\beta|c_1.
\end{aligned} \tag{4.17}$$

These inequalities imply that

$$E_{\mathcal{A}f}(r) = E_f(r)-\beta r \quad \forall r \in \mathbb{R}^1. \tag{4.18}$$

5. Proof of Theorem 3.1

Let $g \in \mathfrak{M}$. We define

$$\mu(g) = \inf\left\{\liminf_{T\to\infty} T^{-1} I^g(0,T,x) : x(\cdot) \in W^{1,1}_{\mathrm{loc}}([0,\infty))\right\}. \tag{5.1}$$

In [13, Section 5] we showed that the number $\mu(g)$ is well defined and proved the following result [13, Theorem 5.1].

Proposition 5.1. *Let $f \in \mathfrak{M}$. Then there exists a constant $M_0 > 0$ such that:*
(i) *$I^f(0,T,x) - \mu(f)T \ge -M_0$ for each $x \in W^{1,1}_{\mathrm{loc}}([0,\infty))$ and each $T > 0$.*
(ii) *For each $a \in \mathbb{R}^1$ there exists $x \in W^{1,1}_{\mathrm{loc}}([0,\infty))$ such that $x(0) = a$ and*

$$\left|I^f(0,T,x) - \mu(f)T\right| \le 4M_0 \quad \forall T > 0. \tag{5.2}$$

Note that assertion (ii) of Proposition 5.1 holds by the periodicity of f in x.
Let $f \in \mathfrak{M}$. A function $x \in W^{1,1}_{\mathrm{loc}}([0,\infty))$ is called (f)-good (see [5]) if

$$\sup\left\{\left|I^f(0,T,x) - \mu(f)T\right| : T \in (0,\infty)\right\} < \infty. \tag{5.3}$$

By [6, Theorem 4.1],

$$E_f(\alpha(f)) = \mu(f) \quad \forall f \in \mathfrak{M}. \tag{5.4}$$

For $f \in \mathfrak{M}$, $x, y, T_1 \in \mathbb{R}^1$, and $T_2 > T_1$ we set

$$U^f(T_1,T_2,x,y) = \inf\left\{I^f(T_1,T_2,v) : v \in W^{1,1}(T_1,T_2), v(T_1) = x, v(T_2) = y\right\}. \tag{5.5}$$

It is not difficult to see that for each $x, y, T_1 \in \mathbb{R}^1$, $T_2 > T_1$,

$$\begin{gathered} U^f(T_1,T_2,x+1,y+1) = U^f(T_1,T_2,x,y), \\ U^f(T_1+1,T_2+1,x,y) = U^f(T_1,T_2,x,y), \quad -\infty < U^f(T_1,T_2,x,y) < \infty, \\ \inf\left\{U^f(T_1,T_2,a,b) : a,b \in \mathbb{R}^1\right\} > -\infty. \end{gathered} \tag{5.6}$$

Denote by $\mathfrak{M}_{\mathrm{per}}$ the set of all $f \in \mathfrak{M}$ such that $\alpha(f)$ is rational and denote by $\mathfrak{M}^0_{\mathrm{per}}$ the set of all $g \in \mathfrak{M}_{\mathrm{per}}$ for which there exist an (g)-minimal solution $w \in C^2(\mathbb{R}^1)$, a continuous function $\pi : \mathbb{R}^1 \to \mathbb{R}^1$, and integers m, n such that the following properties hold:

(P1) $\pi(x+1) = \pi(x)$, $x \in \mathbb{R}^1$;
(P2) $n \ge 1$ and $\alpha(g) = mn^{-1}$ is an irreducible fraction;
(P3) $w(t+n) = w(t) + m$ for all $t \in \mathbb{R}^1$;
(P4) $U^g(0,1,x,y) - \mu(g) - \pi(x) + \pi(y) \ge 0$ for each $x, y \in \mathbb{R}^1$;
(P5) for any $u \in W^{1,1}(0,n)$, the equality

$$I^g(0,n,u) = n\mu(g) + \pi(u(0)) - \pi(u(n)) \tag{5.7}$$

holds if and only if there are integers i, j such that $u(t) = w(t+i) - j$ for all $t \in [0,n]$.

Consider the manifold $(\mathbb{R}^1/\mathbb{Z})^2$ and the canonical mapping $P:\mathbb{R}^2 \to (\mathbb{R}^1/\mathbb{Z})^2$. We have the following result [13, Proposition 6.2].

Proposition 5.2. *Let Ω be a closed subset of $(\mathbb{R}^1/\mathbb{Z})^2$. Then there exists a bounded nonnegative function $\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)$ such that*

$$\Omega = \{x \in (\mathbb{R}^1/\mathbb{Z})^2 : \phi(x) = 0\}. \tag{5.8}$$

Proposition 5.2 is proved by using [1, Chapter 2, Section 3, Theorem 1] and the partition of unity (see [4, Appendix 1]).

We also have the following result (see [13, Proposition 6.3]).

Proposition 5.3. *Suppose that $f \in \mathfrak{M}_{\text{per}}$, $\alpha(f) = mn^{-1}$ is an irreducible fraction (m, n are integers, $n \geq 1$) and $w \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ is an (f)-minimal solution satisfying $w(t+n) = w(t)+m$ for all $t \in \mathbb{R}^1$. Let $\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)$ be as guaranteed in Proposition 5.2 with*

$$\Omega = \{P(t, w(t)) : t \in [0,n]\}, \tag{5.9}$$

and let

$$g(t,x,p) = f(t,x,p) + \phi(P(t,x)), \quad (t,x,p) \in \mathbb{R}^3. \tag{5.10}$$

Then $g \in \mathfrak{M}^0_{\text{per}}$ and there is a continuous function $\pi : \mathbb{R}^1 \to \mathbb{R}^1$ such that the properties (P1), (P2), (P3), (P4), and (P5) hold with g, w, π, m, n and $\alpha(g) = \alpha(f)$.

In the sequel we need the following two lemmas proved in [13].

Lemma 5.4 [13, Lemma 6.6]. *Assume that $k \geq 3$ is an integer, $g \in \mathfrak{M}^0_{\text{per}} \cap \mathfrak{M}_k$, and properties (P1), (P2), (P3), (P4), and (P5) hold with a g-minimal solution $w(\cdot) \in C^2(\mathbb{R}^1)$, a continuous function $\pi : \mathbb{R}^1 \to \mathbb{R}^1$ and integers m, n. Then for each $\epsilon \in (0,1)$, there exists a neighborhood $\mathcal{U}$ of g in $\mathfrak{M}_k$ such that for each $h \in \mathcal{U}$ and each (h)-good function $v \in W^{1,1}_{\text{loc}}([0,\infty))$ there are integers p, q such that*

$$|v(t) - w(t+p) - q| \leq \epsilon \quad \text{for all large enough } t. \tag{5.11}$$

Lemma 5.5 [13, Corollary 6.1]. *Assume that $k \geq 3$ is an integer, $g \in \mathfrak{M}^0_{\text{per}} \cap \mathfrak{M}_k$, and properties (P1), (P2), (P3), (P4), and (P5) hold with a g-minimal solution $w(\cdot) \in C^2(\mathbb{R}^1)$, a continuous function $\pi : \mathbb{R}^1 \to \mathbb{R}^1$ and integers m, n. Then there exist a neighborhood $\mathcal{U}$ of g in $\mathfrak{M}_k$ and a number $L > 0$ such that for each $h \in \mathcal{U}$ and each (h)-good function $v \in W^{1,1}_{\text{loc}}([0,\infty))$, the following property holds.*

There is a number $T_0 > 0$ such that

$$|v(t_2) - v(t_1) - \alpha(g)(t_2 - t_1)| \leq L \tag{5.12}$$

for each $t_1 \geq T_0$ and each $t_2 > t_1$.

Completion of the proof of Theorem 3.1. Let $k \geq 3$ be an integer and let $\alpha = mn^{-1}$ be an irreducible fraction ($n \geq 1$ and m are integers). Let $f \in \mathfrak{M}_k$. By Proposition 2.2 there exists an (f)-minimal solution $w_f(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ such that

$$w_f(t+n) = w_f(t)+m \quad \forall t \in \mathbb{R}^1. \tag{5.13}$$

Choose

$$\beta \in \partial E_f(\alpha). \tag{5.14}$$

Consider a mapping $\mathcal{A} : \mathfrak{M}_k \to \mathfrak{M}_k$ defined by (4.1). By Proposition 4.1 the mapping $\mathcal{A}$ is continuous. Clearly there exists a continuous $\mathcal{A}^{-1} : \mathfrak{M}_k \to \mathfrak{M}_k$. Equations (5.14) and (4.18) imply that

$$0 \in \partial E_{\mathcal{A}f}(\alpha), \quad E_{\mathcal{A}f}(\alpha) = \min\{E_{\mathcal{A}f}(r) : r \in \mathbb{R}^1\} = \mu(\mathcal{A}f) \tag{5.15}$$

and that $\mathcal{A}f \in \mathfrak{M}_{\text{per}}$. It follows from Proposition 5.2 that there exists a bounded nonnegative function $\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)$ such that

$$\{x \in (\mathbb{R}^1/\mathbb{Z})^2 : \phi(x) = 0\} = \{P(t, w_f(t)) : t \in [0,n]\}. \tag{5.16}$$

Set $f^{(\beta)} = \mathcal{A}f$ and for each $\gamma \in (0,1)$ define

$$f_\gamma(t,x,u) = f(t,x,u)+\gamma\phi(P(t,x)), \quad (t,x,u) \in \mathbb{R}^3, \quad f_\gamma^{(\beta)} = \mathcal{A}(f_\gamma). \tag{5.17}$$

Proposition 5.3 implies that for each $\gamma \in (0,1)$,

$$f_\gamma^{(\beta)} \in \mathfrak{M}^0_{\text{per}} \cap \mathfrak{M}_k,$$
$$f_\gamma \longrightarrow f \quad \text{as } \gamma \longrightarrow 0^+, \qquad f_\gamma^{(\beta)} \longrightarrow f^{\beta)} \quad \text{as } \gamma \longrightarrow 0^+ \text{ in } \mathfrak{M}_k. \tag{5.18}$$

Fix $\gamma \in (0,1)$ and an integer $n \geq 1$. By Proposition 5.3 the properties (P1), (P2), (P3), (P4), and (P5) hold with $g = f_\gamma^{(\beta)}$, $\alpha(g) = \alpha$ and $w(\cdot) = w_f$.

By Lemmas 5.4 and 5.5, there exists an open neighborhood $V(f,\gamma,n)$ of $f_\gamma^{(\beta)}$ in $\mathfrak{M}_\gamma$ and a number $L(f,\gamma,n) > 0$ such that the following properties hold:

(i) for each $h \in V(f,\gamma,n)$ and each (h)-good function $v \in W^{1,1}_{\text{loc}}([0,\infty))$, there are integers p,q such that

$$|v(t)-w_f(t+p)-q| \leq \frac{1}{n} \tag{5.19}$$

for all large enough t;

(ii) for each $h \in V(f,\gamma,n)$ and each (h)-good function $v \in W^{1,1}_{\text{loc}}([0,\infty))$, there is a number T_0 such that

$$|v(t_2)-v(t_1)-\alpha(f_\gamma^{(\beta)})(t_2-t_1)| \leq L \tag{5.20}$$

for each $t_1 \geq T_0$ and each $t_2 > t_1$.

Let $h \in V(f,\gamma,n)$ and let $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^1)$ be an (h)-minimal solution with rotation number $\alpha(h)$. Then by Proposition 2.3, (2.3), (5.4), and property (ii), $v|_{[0,\infty)}$ is an (h)-good function and there is T_0 such that (5.20) holds for each $t_1 \geq T_0$ and each $t_2 > t_1$. Since $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^1)$ has rotation number $\alpha(h)$ it follows from Proposition 2.1 that there exists $c_1 > 0$ such that

$$\left|v(t+s) - v(t) - \alpha(h)s\right| \leq c_1 \quad \forall s,t \in \mathbb{R}. \tag{5.21}$$

Equations (5.15), (5.17), (5.20), and (5.21) imply that

$$\alpha(h) = \alpha\left(f_\gamma^{(\beta)}\right) = \alpha\left(f^{(\beta)}\right) = \alpha. \tag{5.22}$$

Thus we have shown that

$$\alpha(h) = \alpha \quad \forall h \in V(f,\gamma,n). \tag{5.23}$$

Let $h \in V(f,\gamma,n)$ and let $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^1)$ be an (h)-minimal solution with rotation number α. It follows from Proposition 2.3, (2.3), and (5.4) that $v|_{[0,\infty)}$ is an (h)-good function. By property (i) there exist integers p, q such that

$$\left|v(t) - w_f(t+p) - q\right| \leq \frac{1}{n} \quad \text{for all large enough } t. \tag{5.24}$$

Therefore we proved the following property:

(iii) for each $h \in V(f,\gamma,n)$ and each (h)-minimal solution $v \in \mathcal{M}_h^{\mathrm{per}}(\alpha)$, there exist integers p, q such that

$$\left|v(t) - w_f(t+p) - q\right| \leq \frac{1}{n} \quad \forall t \in \mathbb{R}^1. \tag{5.25}$$

Define

$$\mathcal{U}(f,\gamma,n) = \mathcal{A}^{-1}\left(V(f,\gamma,n)\right). \tag{5.26}$$

Clearly $\mathcal{U}(f,\gamma,n)$ is an open neighborhood of f_γ in $\mathfrak{M}_k$. By property (iii) the following property holds:

(iv) for each $\xi \in \mathcal{U}(f,\gamma,n)$ and each (ξ)-minimal solution $v \in \mathcal{M}_\xi^{\mathrm{per}}(\alpha)$, there exist integers p, q such that (5.25) holds.

Define

$$\mathcal{F}_{k\alpha} = \cap_{n=1}^{\infty} \cup \left\{\mathcal{U}(f,\gamma,i) : f \in \mathfrak{M}_k,\ \gamma \in (0,1),\ i \geq n\right\}. \tag{5.27}$$

It is not difficult to see that $\mathcal{F}_{k\alpha}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{M}_k$.

Let $g \in \mathcal{F}_{k\alpha}$, $\epsilon \in (0,1)$ and $x, y \in \mathcal{M}_g^{(\text{per})}(\alpha)$. Choose a natural number $n > 8\epsilon^{-1}$. By (5.27) there exist $f \in \mathfrak{M}_k$, $\gamma \in (0,1)$ and an integer $i \geq n$ such that

$$g \in \mathcal{U}(f, \gamma, i). \tag{5.28}$$

It follows from (5.28) and property (iv) that there exist integers p_1, q_1, p_2, q_2 such that

$$\left|x(t) - w_f(t + p_1) - q_1\right| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1, \tag{5.29}$$

$$\left|y(t) - w_f(t + p_2) - q_2\right| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1, \tag{5.30}$$

where $w_f \in \mathcal{M}_f^{(\text{per})}(\alpha)$.

It follows from (5.29) and (5.30) that for all $t \in \mathbb{R}^1$,

$$\begin{aligned} \left|x(t - p_1) - w_f(t) - q_1\right| &\leq \frac{1}{i}, \\ \left|y(t - p_2) - w_f(t) - q_2\right| &\leq \frac{1}{i}, \\ \left|x(t - p_1 - q_1) - (y(t - p_2) - q_2)\right| &\leq \frac{2}{i}, \\ \left|x(t + p_2 - p_1) - y(t) - q_1 + q_2\right| &\leq \frac{2}{i} \leq \frac{2}{n} < \epsilon. \end{aligned} \tag{5.31}$$

Since ϵ is any number in $(0,1)$, we conclude that there exist integers p, q such that

$$x(t + p) - q = y(t) \quad \forall t \in \mathbb{R}^1. \tag{5.32}$$

Assume that $h \in \mathcal{U}(f, \gamma, i)$ and $z \in \mathcal{M}_h^{(\text{per})}(\alpha)$. By the property (iv) there exist integers p_3, q_3 such that

$$\left|z(t) - w_f(t + p_3) - q_3\right| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1. \tag{5.33}$$

Combined with (5.29) this inequality implies that

$$\left|z(t - p_3) - q_3 - x(t - p_1) + q_1\right| \leq \frac{2}{i} \leq \frac{2}{n} < \epsilon \tag{5.34}$$

for all $t \in \mathbb{R}^1$. This completes the proof of Theorem 3.1.

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