TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 5, pp. 2351-2356, October 2011
This paper is available online at http://tjm.math.ntu.edu.tw/index.php/TJM

# OSCILLATION'S THEOREM FOR ONE BOUNDARY VALUE PROBLEM

G. G. Sahakyan

**Abstract.** A theorem is proved on oscillation of the components of the eigenvector-functions of a one boundary value problem for the canonical one-dimentional Dirac system.

## 1. Introduction

We consider the following value problem for Dirac's canonical system (e.g., [1, p. 236])

$$\begin{cases} y_2' + p(t)y_1 + q(t)y_2 = \lambda y_1, \\ -y_1' + q(t)y_1 - p(t)y_2 = \lambda y_2, \end{cases} \tag{1}$$

$$y_1(0)\sin\alpha + y_2(0)\cos\alpha = 0, \tag{2}$$

$$y_1(\pi)\sin\beta + y_2(\pi)\cos\beta = 0, \tag{3}$$

where $p, q \in C_R(0,\pi)$, $\alpha$ and $\beta$ are real numbers and $\lambda$ is a parameter.

If the boundary value problem (1)-(3) has a non-trivial solution

$$\overline{y}(t) = \begin{pmatrix} y_1(t,\lambda) \\ y_2(t,\lambda) \end{pmatrix}$$

for some $\lambda = \lambda_0$, then the number $\lambda_0$ is an *eigenvalue* and corresponding solution $\overline{y}(t,\lambda_0)$*is an eigenvector-function* of the problem.

It is known (e.g.,[1,p.243]) that eigenvalues of the boundary value problem (1)-(3) are real, the values range from $-\infty$ to $+\infty$ and can be numerated in increasing order.

Received May 4, 2010, accepted July 16, 2010.
Communicated by Aram Arutyunov.
2010 *Mathematics Subject Classification*: 34C10, 35Q40, 34L05, 34K10, 34K11.
*Key words and phrases*: Dirac system, Boundary value problem, Oscillation.

Among papers generalizing Sturm's theorem on oscillation properties of the solutions of the Sturm-Liouville equation we mention [2]-[4]. However, the oscillation properties of the solutions of Dirac system require more investigation.

This paper is mainly aimed at investigation of the problem (1)-(3), using the Theorems 1 and 2 from [6]. The following systems of the differential equations will be referred to:

$$\begin{cases} y_1' = p_i(t)y_2, \\ y_2' = r_i(t)y_1, \end{cases} \tag{4}$$

where $p_i, r_i \in C_R[0,\pi](i = 1,2)$.

**Theorem 1.** (On comparison). *Let*

$$\overline{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad and \qquad \overline{v}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

*be some non-trivial solutions of the system* (4) *for* $i = 1$ *and* $i = 2$ *respectively, which satisfy the same initial conditions*

$$u_1(a) = v_1(a) \quad \textit{and} \qquad u_2(a) = v_2(a)$$

*and let*

$$p_1(t)p_2(t) > 0, \qquad r_1(t)r_2(t) > 0,$$

$$p_i(t)r_i(t) < 0, \qquad (i = 1,2),$$

$$|p_2(t)| \geq |p_1(t)|, \qquad |r_2(t)| \geq |r_1(t)|.$$

*Under these assumptions, if one of the components* $\overline{u}(t)$ *has* $l$ *zeros in an interval* $[a,b]$, *then one of the components* $\overline{v}(t)$ *has not less than* $l$ *zeros in the same interval, and the* $k$*-th zero of this component* $\overline{v}(t)$ *is not greater than the* $k$*-th zero of the component* $\overline{u}(t)$.

**Theorem 2.** (On alternation of zeros). *If in* (4) $p_i(t)r_i(t) \neq 0, t \in [a,b]$, *then only one zero of a component of a non-trivial solution of* (4) *lies between any neighboring zeros of the other component of the same solution.*

*The main result of this paper is the following theorem on the problem (1)-(3).*

**Theorem 3.** *For any natural number* $n$, *there exist a number* $\mu_n$ *such that if an eigenvalue of the problem* (1)-(3) *satisfies the inequality* $|\lambda| \geq \mu_n$, *then any component of the eigenvector-function of the problem* (1)-(3) *has at least* $n$ *zeros in the interval* $[0,\pi]$.

*Proof.* Rewrite the system (1) in the form

$$(5)\qquad \begin{cases} y_1' = q(t)y_1 - [p(t)+\lambda]y_2, \\ y_2' = [\lambda - p(t)]y_1 - q(t)y_2. \end{cases}$$

Using the

$$(6a)\qquad z_1(t) = y_1(t)e^{-2\int_0^t q(\tau)d\tau},$$

$$(6b)\qquad z_2(t) = y_2(t)e^{2\int_0^t q(\tau)d\tau},$$

we can rewrite the system (5) in the form

$$(7)\qquad \begin{cases} z_1' = p_0(t)z_2, \\ z_2' = r_0(t)z_1, \end{cases}$$

where

$$(8)\qquad p_0(t) = -[p(t)+\lambda]e^{-2\int_0^t q(\tau)d\tau}, \qquad r_0(t) = [\lambda - p(t)]e^{2\int_0^t q(\tau)d\tau}.$$

Note that according to (6a) and (6b) the zeros of $y_i(t)$ and $z_i(t)$ coinside.

For a given natural number $n$ we choose another natural number $s$ so that $s \geq n+1$ and then choose a pair of natural numbers $m$ and $k$ so that $s = mk$. If we define

$$\varphi(t) = e^{-2\int_0^t q(\tau)d\tau},$$

then we note that $\varphi(t) \neq 0$. Further, functions $p(t),\ \varphi(t)$ and $\frac{1}{\varphi(t)}$ are continuous on interval $[0,\pi]$, then the functions $\frac{m^2}{\varphi(t)} - p(t)$ and $p(t) + k^2\varphi(t)$ are also continuous too on interval $[0,\pi]$, and, that implies their boundedness, i.e. there exist some numbers $l_1, l_2, L_1$ and $L_2$, so that

$$l_1 \leq \frac{m^2}{\varphi(t)} - p(t) \leq L_1$$

and

$$l_2 \leq p(t) + k^2\varphi(t) \leq L_2.$$

In accordance with the properties of the eigenvalues of the problem (1)-(3) one can choose an eigenvalue $\lambda_n$ of the problem (1)-(3) to have

$$(9)\qquad \lambda_n \geq \frac{m^2}{\varphi(t)} - p(t)$$

and

$$(10)\qquad \lambda_n \geq p(t) + k^2\varphi(t).$$

For example, we can choose the value $\lambda_n$ which satisfies the following condition

$$\lambda_n \geq \max\left(L_1, L_2\right).$$

Such a selection is always possible due to peculiarity of arrangement of proper values on numerical line. Then, taking into account that $\varphi(t) > 0$, and according to (9) and (10), we obtain

$$-[p(t)+\lambda_n]\varphi(t) \leq -m^2 < 0, \quad \frac{\lambda_n - p(t)}{\varphi(t)} \geq k^2 > 0. \tag{11}$$

Now we compare (7) with the system

$$\begin{cases} u_1' = -m^2 u_2, \\ u_2' = k^2 u_1. \end{cases} \tag{12}$$

It is known (e.g., [7]) that the general solution

$$\overline{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

of the system (12) can be written in the form

$$u_1(t) = A\cos(mkt+\phi),$$

$$u_2(t) = \frac{Ak}{m}\sin(mkt+\phi),$$

where $A$ and $\phi$ are arbitrary constants.

Suppose that

$$\overline{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

is an eigenvector-function of the problem (1)-(3), which corresponds to the eigenvalue $\lambda_n$, and hence $\overline{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$ is the solution of the corresponding system (7), and let

$$z_1(0) = z_{10}, \quad z_2(0) = z_{20}, \tag{13}$$

where $z_{10}$ and $z_{20}$ are real numbers. It is obvious that one can choose some values of $A$ and $\phi$ for which corresponding particular solution $\overline{u}$ of the system (12) satisfies the condition

$$u_1(0) = z_{10}, \quad u_2(0) = z_{20}$$

(such a selection is always possible according to the theorem on the existence and unity of the Cashi problem for linear homogeneous systems). It is obvious that each of components of this solution on interval $[0, \pi]$ will have at least $mk = s \geq n+1$ zeros. Further, if we define

$$p_1(t) = -m^2, \quad r_1(t) = k^2, \quad p_2(t) = p_0(t), \quad r_2 = r_0(t),$$

then, according to (8) and (11), we obtain that

$$p_2(t) = -(p(t) + \lambda_n)e^{-2\int_0^t q(\tau)d\tau} = -(p(t) + \lambda_n)\varphi(t) \leq -m^2 = p_1(t) < 0,$$

$$r_2(t) = [\lambda_n - p(t)]e^{2\int_0^t q(\tau)d\tau} = \frac{\lambda_n - p(t)}{\varphi(t)} \geq k^2 = r_1(t) > 0.$$

Let us note that with respect to the systems (7) and (12 ) and the corresponding solutions $\overline{z}(t)$ and $\overline{u}(t)$ the conditions of the theorem 1 take place and as each of the component of solution $\overline{u}(t)$ has on interval $[0, \pi]$ at least $n+1$ zeros, then applying Theorem 1, we conclude that one of the components of the eigenvector-function $\overline{z}(t, \lambda_n)$ and corresponding components of $\overline{y}(t, \lambda_n)$ has at least $n+1$ zeros in $[0, \pi]$. Consequently (Theorem 2), the other component of the same eigenvector-function has at least $n$ zeros in $[0, \pi]$, and the quantity of zeros can only grow for $\lambda \geq \lambda_n$.

Further, we choose an eigenvalue $\lambda'_n$ of the problem (1)-(3) such that

$$\lambda'_n \leq \min_{0 \leq t \leq \pi} \left( -\frac{m^2}{\varphi(t)} - p(t),\ p(t) - k^2\varphi(t) \right). \tag{14}$$

We set

$$p_1(t) = m^2, \quad r_1(t) = -k^2, \quad p_2(t) = p_0(t), \quad r_2(t) = r_0(t).$$

Then, according to (14), we have

$$\begin{aligned} p_2(t) &= -[p(t) + \lambda_n^{'}]\varphi(t) \leq -m^2 = p_1(t) < 0, \\ r_2(t) &= \frac{\lambda'_n - p(t)}{\varphi(t)} \geq k_2 = r_2(t) > 0. \end{aligned}$$

Now comparing (7) with system

$$\begin{cases} y'_1 &= m^2 y_2, \\ y'_2 &= -k^2 y_1, \end{cases} \tag{15}$$

general solution of which may be written in form

$$u_1(t) = A\sin(mkt + \phi),$$

$$u_2(t) = \frac{Ak}{m}\cos(mkt + \phi).$$

From that solution let us select now the particular solution $\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ that satisfies the conditions

$$u_1(0) = y_{10}, \quad u_2(0) = y_{20}.$$

Applying again Theorems 1 and 2 and taking into account that the components of a solution of the system (15) have at least $n+1$ zeros in $[0, \pi]$, we obtain that any component of the eigenvector-function of the problem (1)-(3) has at least $n$ zeros in $[0, \pi]$, and the quantity of zeros can only increase when $\lambda \leq \lambda'_n$. Thus, putting

$$\mu_n = max\{|\lambda_n|, |\lambda'_n|\},$$

we complete the proof.

## References


1. B. M. Levitan and I. S. Sargsian, *Sturm-Liouville and Dirac Operators*, Nauka, Moscow, 1988.
2. Don Hinton, *Sturm's 1836 oscillation results evolution of the theory*, Sturm-Liouville Theory: Past and Present, Birkhäuser, Basel, 2005, 1-27.
3. B. Simon, *Sturm Oscillation and Comparison Theorems*, Sturm-Liouville Theory: Past and Present, Birkhäuser, Basel, 2005, pp. 29-43.
4. H. Krüger and G. Teschl, *Relative Oscillation Theory, Weighted Zeros of the Wronskian, and the Spectral Shift Function,* Arxiv. math/0703574v1 [math.SP], 20, Mar. 2007.
5. E. V. Foliadova, Oscillation Theorem for Dirac's System with Parameter in the Boundary Condition, in: *Functional Analysis, Spectral Theory*, Mejvuzowskii Sbornik Nauchnikh Trudov, Uljanovsk, 1984, pp. 111-122.
6. G. G. Sahakyan, On some properties of solutions of Dirac's canonical system, *Uchenie Zapiski ArGU*, **2** (2007), 3-11.
7. J. Sansone, *Ordinary Differential Equations 1*, IL, Moscow, 1953.


G. G. Sahakyan
The Department of the Applied Mathematics and Informatics
Artsakh State University
st. M. Gosh 5
Stepanakert 375009
Armenia
E-mail: ter_saak_george@mail.ru