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THE CAUCHY-NEUMANN PROBLEM FOR PARABOLIC EQUATIONS IN DOMAINS WITH CONICAL POINTS

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Abstract. The purpose of this paper is to establish the well-posedness and the regularity of solutions of the Cauchy-Neumann problem for second order parabolic equations in cylinders with the base containing conical points.

1. Introduction

We are concerned with initial boundary value problems for parabolic equations in nonsmooth domains. The Cauchy - Dirichlet problem for higher order parabolic systems in domains containing conical points has been investigated in [5, 6]. The Cauchy - Neumann problem in domains with edges has been dealt with for the classical heat equation in [9]. In the present paper, we consider the Cauchy - Neumann problem for general second order linear parabolic equations in domains containing conical points.

The main goal of this paper is to obtain the regularity of the solution of the problem. We will investigate the problem by modifying the approach suggested in [2, 5]. First, we study the unique solvability and the regularity with respect to the time variable for the generalized solution in the Sobolev space $H^{1,1}(Q)$ by Galerkin's approximate method. After that, we take the term containing the derivative in time of the unknown function to the right-hand side of the equation such that the problem can be considered as an elliptic one. With the help of some auxiliary results we can apply the results for elliptic boundary value problems and our previous ones to deal with the regularity with respect to both of time and spatial variables of the solution.

Our paper is organized as follow. In Sec. 2, we introduce some notations and the formulation of the problem. The main results, Theorems 3.1 and 3.2, are stated

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in Sec. 3. Sec. 4 is devoted to establish the unique existence and the regularity in time of the generalized solution of the problem. In Sec. 5, we present some auxiliary results and the proofs of Theorems 3.1, 3.2.

2. Notation and Formulation of the Problem

Let G be a bounded domain in $\mathbb{R}^n (n \geqslant 2)$ with the boundary ∂G. We suppose that $\Gamma = \partial G \setminus \{0\}$ is a smooth manifold and G in a neighborhood of the origin 0 coincides with the cone $K = \{x : x/|x| \in \Omega\}$ where Ω is a smooth domain on the unit sphere S^{n-1} in $\mathbb{R}^n$. Set $Q_t = G \times (0, t)$ for each $t \in (0, +\infty), Q = Q_\infty = G \times (0, +\infty), \widetilde{Q} = (\overline{G} \setminus \{0\}) \times [0, +\infty), S_T = \Gamma \times [0, T]$ and $S = S_\infty = \Gamma \times [0, +\infty)$. We will use notations: $\partial_{x_j} = \partial/\partial x_j, u_{x_j} = \partial_{x_j} u, u_{t^k} = \partial_t^k u, r = |x| = \big(\sum_{k=1}^n x_k^2\big)^{\frac{1}{2}}$. For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, set $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $\partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$.

In this paper we consider the following problem

$$u_t - Lu = f \quad \text{in } Q, \tag{2.1}$$

$$Nu = 0, \text{ on } S, \tag{2.2}$$

$$u|_{t=0} = \varphi \text{ on } G, \tag{2.3}$$

where L is a formal self-adjoint differential operator of second order defined in Q with coefficients infinitely differentiable in $\overline{Q}$,

$$Lu = L(x, t, \partial)u = \sum_{j,k=1}^{n} (a_{jk}(x, t)\, u_{x_k})_{x_j} + a(x, t)u$$

$$(a_{jk} = \overline{a}_{kj}, j, k = 1, \ldots, n, a = \overline{a}),$$

and

$$N = N(x, t, \partial) = \sum_{j,k=1}^{n} a_{jk}(x, t) u_{x_k} \cos(\nu, x_j)$$

is the conormal derivative on S, ν is the unit exterior normal to S.

Let us introduce some functional spaces which will be used in this paper.

Let l be a nonnegative integer. We define the weighted space $V_{2,\gamma}^l(G)$ $(\gamma \in \mathbb{R})$ as the closure of $C_0^\infty(\overline{G} \setminus \{0\})$ with respect to the norm

$$\|u\|_{V_{2,\gamma}^l(G)} = \Big(\sum_{|\alpha| \leqslant l} \int_G r^{2(\gamma + |\alpha| - l)} |\partial^\alpha u|^2 dx\Big)^{\frac{1}{2}},$$

and the weighted space $H^l_\gamma(G)$ as the closure of $C^\infty(\overline{G})$ with respect to the norm

$$\|u\|_{H^l_\gamma(G)} = \Big(\sum_{|\alpha|\leqslant l}\int_G r^{2\gamma}|\partial^\alpha u|^2dx\Big)^{\frac{1}{2}}.$$

For $\gamma = 0$ the space $H^l_0(G)$ coincides with the usual Sobolev space $H^l(G)$ and $H^0_0(G)$ coincides with $L_2(G)$.

If $l \geqslant 1$, then $V^{l-\frac{1}{2}}_{2,\gamma}(\Gamma), H^{l-\frac{1}{2}}_\gamma(\Gamma)$ denote the spaces consisting of traces of functions from respective spaces $V^l_{2,\gamma}(G), H^l_\gamma(G)$ on Γ with the respective norms

$$\|u\|_{V^{l-\frac{1}{2}}_{2,\gamma}(\Gamma)} = \inf\big\{\|v\|_{V^l_{2,\gamma}(G)} : v \in V^l_{2,\gamma}(G), v|_\Gamma = u\big\},$$
$$\|u\|_{H^{l-\frac{1}{2}}_\gamma(\Gamma)} = \inf\big\{\|v\|_{H^l_\gamma(G)} : v \in H^l_\gamma(G), v|_\Gamma = u\big\}.$$

By $H^{-1}(G)$ we denote the dual space to $H^1(G)$. We write $\langle .,.\rangle$ to denote the pairing between $H^1(G)$ and $H^{-1}(G)$, and $(.,.)$ to denote the usual inner product in $L_2(G)$. By identifying $L_2(G)$ with its dual, we have the continuous imbeddings $H^1(G) \hookrightarrow L_2(G) \hookrightarrow H^{-1}(G)$ with the equation

$$\langle f, \overline{v}\rangle = (f, v) \quad \text{for } f \in L_2(G) \subset H^{-1}(G), v \in H^1(G).$$

Let X, Y be Banach spaces. We denote by $L_2(0, T; X)$ $(0 < T \leqslant +\infty)$ the space consisting of all measurable functions $u : (0, T) \to X$ with the norm

$$\|u\|_{L_2(0,T;X)} = \Big(\int_0^T \|u(t)\|^2_X dt\Big)^{\frac{1}{2}},$$

and by $H^1(0, T; X, Y)$ the space consisting of all functions $u \in L_2(0, T; X)$ such that the generalized derivative $u_t = u'$ exists and belongs to $L_2(0, T; Y)$. The norm in $H^1(0, T; X, Y)$ is defined by

$$\|u\|_{H^1(0,T;X,Y)} = \big(\|u\|^2_{L_2(0,T;X)} + \|u_t\|^2_{L_2(0,T;Y)}\big)^{\frac{1}{2}}.$$

For shortness we set

$$H^{l,0}(Q_T) = L_2(0, T; H^l(G)), \qquad H^{1,1}(Q_T) = H^1(0, T; H^1(G), L_2(G)),$$
$$H^{-1,0}(Q_T) = L_2(0, T; H^{-1}(G)), \quad H^{-1,1}(Q_T) = H^1(0, T; H^{-1}(G), L_2(G)),$$
$$H^{l,0}_\gamma(Q_T) = L_2(0, T; H^l_\gamma(G)), \qquad H^{l+\frac{1}{2},0}_\gamma(S_T) = L_2(0, T; H^{l+\frac{1}{2}}_\gamma(\Gamma)),$$

and

$$\mathcal{H}^{1,1}(Q_T) = H^1(0, T; H^1(G), H^{-1}(G)).$$

Finally, by $H^{2l,l}_{loc}(\widetilde{Q})$ we denote the space of all functions having generalized derivatives $\partial^\alpha u_{t^k}, |\alpha|+2k \leqslant 2l$, in Q such that

$$\int_P \sum_{|\alpha|+2k\leqslant 2l} |\partial^\alpha u_{t^k}|^2 dxdt < +\infty$$

for every compact subset P of $\widetilde{Q}$, and by $H^{2l,l}_\gamma(Q_T)$ we denote the weighted Sobolev space with respect to the norm

$$\|u\|_{H^{2l,l}_\gamma(Q_T)} = \Big(\int_{Q_T} \big(r^{2\gamma} \sum_{|\alpha|+2k\leqslant 2l} |\partial^\alpha u_{t^k}|^2 + \sum_{k=0}^{l} |u_{t^k}|^2 \big) dxdt \Big)^{\frac{1}{2}}.$$

We set

$$B(t,u,v) = \int_G \Big(\sum_{j,k=1}^{n} a_{jk}\, u_{x_k} \overline{v_{x_j}} + au\overline{v} \Big) dx.$$

Throughout this paper, we assume that coefficients of L together with all their derivatives are bounded in $\overline{Q}$ and the form $B(t,.,.)$ is $H^1(G)-$coercive uniformly with respect to $t \in [0,+\infty)$, i.e.,

$$B(t,u,u) \geqslant \mu \|u\|^2_{H^1(G)} \text{ for all } u \in H^1(G), t \in [0,+\infty), \tag{2.4}$$

where μ is a positive constant independent of u and t.

Let $f \in H^{-1,0}(Q), \varphi \in L_2(G)$. A function $u \in \mathcal{H}^{1,1}(Q)$ is called a generalized solution of the problem (2.1)-(2.3) iff $u(.,0) = \varphi$ and the equality

$$\langle u_t(.,t), \overline{v} \rangle + B(t,u,v) = \langle f(.,t), \overline{v} \rangle \tag{2.5}$$

holds for a.e. $t \in (0,+\infty)$ and all $v \in H^1(G)$.

3. Formulation of the Main Results

Let $\varphi \in H^{2h+1}(G), f \in H^{2h,h}_{loc}(Q)$, where h is a positive integer. We set

$$\begin{aligned} &\varphi_0 = \varphi, \varphi_1 = f(.,0) + L(x,0,\partial)\varphi_0, \\ &\ldots, \varphi_h = f_{t^{h-1}}(.,0) + \textstyle\sum_{k=0}^{h-1} \binom{h-1}{k} L_{t^{h-1-k}}(x,0,\partial)\varphi_k, \end{aligned} \tag{3.1}$$

where

$$L_{t^k}(x,t,\partial)u = \sum_{j,k=1}^{n} (\partial_t^k a_{jk}\, u_{x_k})_{x_j} + \partial_t^k au.$$

We will say that the h^{th}-order compatibility conditions are fulfilled if $\varphi_0, \ldots, \varphi_{h-1} \in H^2(G)$, $\varphi_h \in H^1(G)$ and

$$\sum_{j=0}^{k} \binom{k}{j} N_{t^{k-j}}(x, 0, \partial)\varphi_j|_\Gamma = 0, 0 \leqslant k \leqslant h-1. \tag{3.2}$$

We note that values of $f_{t^k}(x, 0), k = 0, \ldots, h$, and the identity (3.2) are understood in the trace sense. In general, the fact that $\varphi \in H^{2h+1}(G), f \in H^{2h,h}_{loc}(Q)$ does not guarantee that $\varphi_0, \ldots, \varphi_{h-1} \in H^2(G)$, $\varphi_h \in H^1(G)$.

Let $L_0(x, t, \partial)$ be the principal homogenous part of $L(x, t, \partial)$. We can write $L_0(0, t, \partial), N(0, t, \partial)$ in the form

$$L_0(0, t, \partial) = r^{-2}\mathcal{L}(\omega, t, \partial_\omega, r\partial_r), \tag{3.3}$$

$$N(0, t, \partial) = r^{-1}\mathcal{N}(\omega, t, \partial_\omega, r\partial_r), \tag{3.4}$$

where $r = |x|$, ω is an arbitrary local coordinate system on S^{n-1}. We denote by $\mathcal{U}(\lambda, t)$ $(\lambda \in \mathbb{C}, t \in (0, +\infty))$ the operator of the parameter-depending boundary problem

$$\mathcal{L}(\omega, t, \partial_\omega, \lambda) = f \qquad \text{in } \Omega, \tag{3.5}$$

$$\mathcal{N}(\omega, t, \partial_\omega, \lambda) = g \qquad \text{on } \partial\Omega. \tag{3.6}$$

For every fixed $\lambda \in \mathbb{C}$ this operator continuously maps

$$H^l(\Omega) \text{ into } H^{l-2}(\Omega) \times H^{l-\frac{3}{2}}(\partial\Omega)\ (l \geqslant 2).$$

For each $t \in (0, +\infty)$ we have the operator pencil $\mathcal{U}(\lambda, t)$ which has the spectrum being an enumerable set of eigenvalues (see [4, Th. 5.2.1]).

Now let us give the main results of the present paper:

Theorem 3.1. *Let h be a nonnegative integer. Let $\varphi \in H^{2h+1}(G), f \in H^{2h,h}_{2h+1}(Q)$ such that the h^{th}-order compatibility conditions are fulfilled if $h \geqslant 1$. Then the problem $(2.1) - (2.3)$ has a unique generalized solution $u \in \mathcal{H}^{1,1}(Q)$ which belongs to $H^{2h+2,h+1}_{2h+1}(Q)$, moreover,*

$$\|u\|^2_{H^{2h+2,h+1}_{2h+1}(Q)} \leqslant C\Big(\sum_{j=0}^{h} \|\varphi_j\|^2_{H^1(G)} + \|f\|^2_{H^{2h,h}_{2h+1}(Q)}\Big), \tag{3.7}$$

where C is the constant independent of u, f, φ.

Theorem 3.2. *Let $\varphi \in H^{2h+1}(G), f \in H^{2h,h}_{\gamma}(Q)$ such that the h^{th}-order compatibility conditions are fulfilled if $h \geqslant 1$, where h is a nonnegative integer and $0 \leqslant \gamma \leqslant 2h+1$. Assume further that the strip $1-\epsilon-\frac{n}{2} \leqslant Re\lambda \leqslant -\gamma+2h+2-\frac{n}{2}$ does not contain any eigenvalue of $\mathcal{U}(\lambda,t)$ for all $t \in (0,+\infty)$, where $\epsilon = 0$ or $\epsilon > 0$ according as $n > 2$ or $n = 2$. Then the generalized solution of the problem $(2.1)-(2.3)$ belongs to $H^{2h+2,h+1}_{\gamma}(Q)$, and the following estimate holds*

$$\|u\|^2_{H^{2h+2,h+1}_{\gamma}(Q)} \leqslant C\Big(\sum_{j=0}^{h}\|\varphi_j\|^2_{H^1(G)} + \|f\|^2_{H^{2h,h}_{\gamma}(Q)}\Big), \tag{3.8}$$

where C is the constant independent of u, f, φ.

4. Solvability and Regularity with Respect to the Time Variable

For integer $k \geqslant 0, u, v \in H^{1,0}(Q_T), t \in [0,+\infty)$ we set

$$B_{t^k}(t,u,v) = \int_G \Big(\sum_{j,k=1}^{n} \partial_t^k a_{jk}u_{x_k}\overline{v_{x_j}} + \partial_t^k au\overline{v}\Big)dx.$$

Lemma 4.1. *Let $F(t,.,.)$ be a bilinear form on $H^1(G)\times H^1(G)$ such that*

$$|F(t,v,w)| \leqslant C\|v\|_{H^1(G)}\|w\|_{H^1(G)} \quad (C = \text{ const }) \tag{4.1}$$

for all $t \in [0,+\infty)$ and all $v,w \in H^1(G)$, and $F(.,v,w)$ is measurable on $[0,+\infty)$ for each pair $v,w \in H^1(G)$. Assume that $u \in \mathcal{H}^{1,1}(Q)$ satisfies $u(.,0)\equiv 0$ and

$$\langle u_t(.,t),\overline{v}\rangle + B(t,u(.,t),v) = \int_0^t F(\tau,u(.,\tau),\overline{v})d\tau \tag{4.2}$$

for a.e. $t \in [0,+\infty)$ and all $v \in H^1(G)$. Then $u \equiv 0$ on Q.

Proof. Substituting $v := u(.,t)$ into (4.2), then integrating both sides of the obtained equality with respect to t from 0 to b ($b > 0$), after all using the assumptions (2.4), (4.1), we arrive at

$$\frac{1}{2}\|u(.,b)\|^2_{L_2(G)} + \mu\|u\|^2_{H^{1,0}(Q_b)} \leqslant C\int_0^b\int_0^t \|u(.,t)\|_{H^1(G)}\|u(.,\tau)\|_{H^1(G)}d\tau dt$$
$$\leqslant \frac{1}{2}C\int_0^b\int_0^t(\|u(.,t)\|^2_{H^1(G)} + \|u(.,\tau)\|^2_{H^1(G)})d\tau dt \leqslant bC\|u\|^2_{H^{1,0}(Q_b)}.$$

Choosing $b = \frac{\mu}{2C}$, we have $\frac{1}{2}\Big(\|u(.,b)\|^2_{L_2(G)} + \mu\|u\|^2_{H^{1,0}(Q_b)}\Big) \leqslant 0$. This implies $u \equiv 0$ on $[0, \frac{\mu}{2C}]$. Repeating this argument we can show that $u \equiv 0$ on intervals $[\frac{\mu}{2C}, \frac{\mu}{C}], [\frac{\mu}{C}, \frac{3\mu}{2C}], \ldots$, and, therefore, $u \equiv 0$ on Q. ■

Lemma 4.2. *Let $f \in H^{-1,0}(Q), \varphi \in H^1(G)$. Then the problem* $(2.1) - (2.3)$ *has a unique generalized solution $u \in \mathcal{H}^{1,1}(Q)$. It satisfies the inequality*

$$\|u\|^2_{\mathcal{H}^{1,1}(Q)} \leqslant C\big(\|\varphi\|^2_{H^1(G)} + \|f\|^2_{H^{-1,0}(Q)}\big) \tag{4.3}$$

with the constant C independent of φ, f.

Proof. The uniqueness follows directly from Lemma 4.1. The existence is proved by Galerkin's approximate method. Let $\{\psi_k\}_{k=1}^{\infty}$ be a set of smooth functions which is not only an orthogonal basis of $H^1(G)$ but also an orthonormal basis of $L_2(G)$. Now for each positive integer N, we consider the function $u^N(x,t) = \sum_{k=1}^{N} C_k^N(t)\psi_k(x)$, where $\{C_k^N(t)\}_{k=1}^{N}$ is the solution of the ordinary differential system:

$$(u_t^N, \psi_l) + B(t, u^N, \psi_l) = \langle f(.,t), \overline{\psi_l}\rangle, \tag{4.4}$$

$$C_k^N(0) = C_k, \quad l, k = 1, \ldots, N. \tag{4.5}$$

Here $C_k = (\varphi, \psi_k)_{H^1(G)}, k = 1, 2, \ldots, (.,.)_{H^1(G)}$ stands for the inner product in $H^1(G)$. By the same arguments as in [7, Ch. III], [1, Ch. 7], one can receive the following estimate

$$\|u^N\|^2_{\mathcal{H}^{1,1}(Q_T)} \leqslant C\big(\|\varphi\|^2_{H^1(G)} + \|f\|^2_{H^{-1,0}(Q)}\big), \tag{4.6}$$

where C is a positive constant independent of N, f, φ and T. Sending $T \to +\infty$, we obtain

$$\|u^N\|^2_{\mathcal{H}^{1,1}(Q)} \leqslant C\big(\|\varphi\|^2_{H^1(G)} + \|f\|^2_{H^{-1,0}(Q)}\big), \tag{4.7}$$

From this estimate, by the same arguments as in [1, Ch. 7, Th. 3], we conclude that there exists a subsequence of $\{u^N\}$ which weakly converges to a generalized solution $u \in \mathcal{H}^{1,1}(Q)$ of the problem (2.1)-(2.3). The estimate (4.3) follows from (4.7). ■

Lemma 4.3. *Let $\varphi \in H^1(G)$, $f \in L_2(Q)$ or $f \in H^{-1,1}(Q)$ and $u \in \mathcal{H}^{1,1}(Q)$ be the generalized solution of the problem* $(2.1) - (2.3)$*. Then $u \in H^{1,1}(Q)$ and the following estimate*

$$\|u\|^2_{H^{1,1}(Q)} \leqslant C\ \big(\|\varphi\|^2_{H^1(G)} + \|f\|^2_X\big) \tag{4.8}$$

holds with the constant C independent of φ, f, and u. Here X is $L_2(Q)$ or $H^{-1,1}(Q)$ according as $f \in L_2(Q)$ or $f \in H^{-1,1}(Q)$.

Proof. For the case $f \in L_2(Q)$ the assertion of the lemma is proved similarly to [7, Th. 6.1] and [1, Ch. 7, Th. 5].

Now let $f \in H^{-1,1}(Q)$. Then f is continuous on $[0, +\infty)$ (as a function from $[0, +\infty)$ to $H^{-1}(G)$) and has the representation $f(., t) = f(., s) + \int_s^t f_t(., \tau)d\tau$ for all $s, t \in [0, +\infty)$ (see [1, Sec. 5.9, Th. 2]). This implies

$$\|f(.,t)\|^2_{H^{-1}(G)} \leqslant 2\|f(.,s)\|^2_{H^{-1}(G)} + 2\int_J \|f_t(.,\tau)\|^2_{H^{-1}(G)} d\tau, \tag{4.9}$$

where $J = [a, b] \subset [0, +\infty)$ such that $a \leqslant t \leqslant b$ and $b - a = 1$. Integrating both sides of (4.9) with respect to s on J, we obtain

$$\|f(.,t)\|^2_{H^{-1}(G)} \leqslant 2\|f\|^2_{H^{-1,1}(Q)} \; (t \in [0, +\infty)). \tag{4.10}$$

Let $\{u^N\}_{N=1}^{\infty}$ be the functions defined as in the proof of Lemma 4.2. Multiplying both sides of (4.4) by $\dfrac{d\overline{C_l^N}}{dt}$, then taking sum with respect to l from 1 to N, after that integrating with respect to t from 0 to T $(0 < T < +\infty)$, and adding the obtained equality with its complex conjugate, we arrive at

$$2\|u_t^N\|^2_{L_2(Q_T)} + \int_{Q_T} \Big(\sum_{j,k=1}^{n} a_{jk}\partial_t(u^N_{x_k}\overline{u^N_{x_j}}) + a\partial_t(u^N\overline{u^N}) \Big) dxdt = 2\mathrm{Re}\int_0^T \langle f, u_t^N\rangle dt.$$

By the integration by parts, we get

$$\begin{aligned} &2\|u_t^N\|_{L_2(Q_T)} + B(T, u^N, u^N)|_0^T \\ &= \int_0^T B_t(t, u^N, u^N)dt + 2\mathrm{Re}\int_0^T \langle f, u_t^N\rangle dt. \end{aligned} \tag{4.11}$$

Since $a_{jk}, a, \partial_t a_{jk}, \partial_t a$ are bounded on $\overline{Q}$, using Cauchy's inequality, we get

$$|B(0, u^N, u^N)| \leqslant C\|u^N(x,0)\|^2_{H^1(G)} \leqslant C\|\varphi\|^2_{H^1(G)}, \tag{4.12}$$

$$|\int_0^T B_t(t, u^N, u^N)dt| \leqslant C\|u^N\|^2_{H^{1,0}(Q_T)}. \tag{4.13}$$

Noting that $\int_0^T \langle f, u_t^N\rangle dt = -\int_0^T \langle f_t, u^N\rangle dt + \langle f, u^N\rangle\Big|_0^T$, and using (4.10), we obtain

$$
\begin{aligned}
(4.14)\qquad &\left|\int_0^T \langle f, u_t^N\rangle dt\right| \leqslant \|f_t\|_{H^{-1,0}(Q)}\|u^N\|_{H^{1,0}(Q)}\\
&+\|f(.,t)\|_{H^{-1}(G)}\|u^N(.,t)\|_{H^1(G)}\\
&+\|f(.,0)\|_{H^{-1}(G)}\|u^N(.,0)\|_{H^1(G)} \leqslant C(\epsilon)\|f\|^2_{H^{-1,1}(Q)}\\
&+\epsilon\big(\|u^N\|^2_{H^{1,0}(Q_T)} + \|u^N(.,t)\|^2_{H^1(G)} + \|u^N(.,0)\|^2_{H^1(G)}\big)(\epsilon > 0).
\end{aligned}
$$

Using (2.4), (4.6), (4.12), (4.13) and (4.14) for $0 < \epsilon < \mu$, we get from (4.11) that

$$\|u_t^N\|^2_{L_2(Q_T)} \leqslant C\big(\|\varphi\|^2_{H^1(G)} + \|f\|^2_{H^{-1,1}(Q)}\big).$$

Sending $T \to +\infty$, we can see

$$\|u_t^N\|^2_{L_2(Q)} \leqslant C\big(\|\varphi\|^2_{H^1(G)} + \|f\|^2_{H^{-1,1}(Q)}\big). \tag{4.15}$$

Combining (4.7) and (4.15), we have

$$\|u^N\|^2_{H^{1,1}(Q)} \leqslant C\big(\|\varphi\|^2_{H^1(G)} + \|f\|^2_{H^{-1,1}(Q)}\big). \tag{4.16}$$

This implies that the sequence $\{u^N\}$ contains a subsequence which weakly converges to a function $v \in H^{1,1}(Q)$. Passing to the limit of the subsequence, we can see that v is also a generalized solution of the problem (2.1)-(2.2). Thus, $u = v \in H^{1,1}(Q)$. The estimate (4.8) with $X = H^{-1,1}(Q)$ follows from (4.16). ■

Remark. It follows from the proof of Lemma 4.3 that if $\varphi \in H^1(G)$ and $f = f_1 + f_2$, where $f_1 \in L_2(Q), f_2 \in H^{-1,1}(Q)$ then the assertion of the lemma is also true with $\|f\|^2_X$ replaced by $\|f_1\|^2_{L_2(Q)} + \|f_2\|^2_{H^{-1,1}(Q)}$.

Theorem 4.1. *Let h be a nonnegative integer. Let $\varphi \in H^{2h+1}(G), f \in H^{2h,h}_{loc}(Q)$ such that h^{th}-order compatibility conditions are fulfilled if $h \geqslant 1$ and $f, f_t, \ldots, f_{t^h} \in L_2(Q)$. Then the problem $(2.1) - (2.3)$ has a unique generalized solution $u \in \mathcal{H}^{1,1}(Q)$, moreover,*

$$u_{t^k} \in H^{1,1}(Q) \text{ for } k = 0, \ldots, h, \tag{4.17}$$

and the following estimate holds

$$\sum_{k=0}^{h} \|u_{t^k}\|^2_{H^{1,1}(Q)} \leqslant C \sum_{k=0}^{h} \big(\|\varphi_k\|^2_{H^1(G)} + \|f_{t^k}\|^2_{L_2(Q)}\big), \tag{4.18}$$

where C is a constant independent of u, f, φ.

Proof. We will show by induction that not only the assertions (4.17), (4.18) but also the following equalities hold:

$$u_{t^k}(.,0) = \varphi_k,\ k = 1, \dots, h, \tag{4.19}$$

and

$$(u_{t^{h+1}}, \eta) + \sum_{k=0}^{h} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta) = (f_{t^h}, \eta) \text{ for all } \eta \in H^1(G). \tag{4.20}$$

The case $h = 0$ follows from Lemmas 4.2, 4.3. Assuming now that they hold for $h-1$, we will prove them for $h\,(h \geqslant 1)$. We consider first the following problem: find a function $v \in \mathcal{H}^{1,1}(Q)$ satisfying $v(.,0) = \varphi_h$ and

$$\langle v_t, \overline{\eta}\rangle + B(t, v, \eta) = (f_{t^h}, \eta) - \sum_{k=0}^{h-1} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta) \tag{4.21}$$

for all $\eta \in H^1(G)$ and a.e. $t \in (0, +\infty)$.

Let $F(t), t \in [0, +\infty)$, be functionals defined by

$$\langle F(t), \overline{\eta}\rangle = (f_{t^h}, \eta) - \sum_{k=0}^{h-1} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta), \eta \in H^1(G). \tag{4.22}$$

Then $F \in H^{-1,0}(Q)$ by the inductive assumption. Hence, according to Lemma 4.2, the problem (4.21) has a solution $v \in \mathcal{H}^{1,1}(Q)$. We put now

$$w(x,t) = \varphi_{h-1}(x) + \int_0^t v(x,\tau)d\tau,\ x \in G, t \in [0, +\infty).$$

Then we have $w(.,0) = \varphi_{h-1}, w_t = v, w_t(.,0) = \varphi_h$. It follows from (5.21) that

$$\begin{aligned} \langle w_{tt}, \overline{\eta}\rangle\ &+ \frac{\partial}{\partial t} B(t, w, \eta) = (f_{t^h}, \eta) + B_t(t, w - u_{t^{h-1}}, \eta) \\ &- \frac{\partial}{\partial t} \sum_{k=0}^{h-2} \binom{h-1}{k} B_{t^{h-1-k}}(t, u_{t^k}, \eta). \end{aligned} \tag{4.23}$$

One knows that

$$(L\psi, \eta) + B(t, \psi, \eta) = (N\psi, \eta)_\Gamma, \psi \in H^2(G), \eta \in H^1(G),$$

and therefore,

$$(L_{t^k}\psi, \eta) + B_{t^k}(t, \psi, \eta) = (N_{t^k}\psi, \eta)_\Gamma, t \in [0, +\infty) \tag{4.24}$$

where $(.,.)_\Gamma$ stands for the usual inner product in $L_2(\Gamma)$, k is an arbitrary nonnegative integer. From (4.24) and the conditions (3.2), we have

$$\sum_{k=0}^{h-1} \binom{h-1}{k} \Big(L_{t^{h-1-k}}(x,0,\partial)\varphi_k, \eta \Big) = -\sum_{k=0}^{h-1} \binom{h-1}{k} B_{t^{h-1-k}}(0,\varphi_k,\eta).$$

From this we have

$$(\varphi_h, \eta) = (f_{t^{h-1}}(.,0), \eta) - \sum_{k=0}^{h-1} \binom{h-1}{k} B_{t^{h-1-k}}(0,\varphi_k,\eta). \tag{4.25}$$

Now integrating equality (4.23) with respect to t from 0 to t and using (4.25), we arrive at

$$\begin{aligned} \langle w_t, \overline{\eta}\rangle \ &+ B(t,w,\eta) = (f_{t^{h-1}}, \eta) \\ &+ \int_0^t B_t(\tau, w - u_{t^{h-1}}, \eta) d\tau - \sum_{k=0}^{h-1} \binom{h-1}{k} B_{t^{h-1-k}}(t, u_{t^k}, \eta). \end{aligned} \tag{4.26}$$

Put $z = w - u_{t^{h-1}}$. Then $z(.,0) = 0$ since $u(.,0) = w(.,0) = \varphi_{h-1}$. It follows from the inductive assumption (4.20) with h replaced by $h-1$ and (4.26) that

$$\langle z_t(.,t), \overline{\eta}\rangle + B(t, z(.,t), \eta) = \int_0^t B_t(\tau, z(.,\tau), \eta) d\tau. \tag{4.27}$$

Applying Lemma 4.1, we can see from (4.27) that $z \equiv 0$ on Q. Therefore, $u_{t^h} = w_t = v \in \mathcal{H}^{1,1}(Q)$.

Now we show that in fact $u_{t^h} \in H^{1,1}(Q)$. We rewrite (4.21) in the form

$$\langle v_t, \overline{\eta}\rangle + B(t,v,\eta) = (f_{t^h}, \eta) + \langle \widehat{F}(t), \eta\rangle, \tag{4.28}$$

where $\widehat{F}(t), t \in [0,+\infty)$, are functionals on $H^1(G)$ defined by

$$\langle \widehat{F}(t), \overline{\eta}\rangle = -\sum_{k=0}^{h-1} \binom{h}{k} B_{t^{h-k}}(t, u_{t^k}, \eta), \eta \in H^1(G). \tag{4.29}$$

Since $u_{t^k} \in H^{1,0}(Q)$ for $k = 0,\dots,h$, then $\widehat{F}_t$ exists and belongs to $H^{-1,0}(G)$. Then, according to the remark below Lemma 4.3, we obtain from (4.28) that $u_{t^h} = v \in H^{1,1}(Q)$. The desired estimate holds since $\|f_{t^h}\|_{L_2(Q)}$ and $\|\widehat{F}_t\|_{H^{1,0}(Q)}$ can be estimated by the right-hand side of (4.18). The proof is completed. ■

5. Proof of Theorems 3.1, 3.2

The following lemma can be proved similarly to Lemma 3 of [3], Theorems 4.1, 4.1' of [8] and Lemma 3.1 of [5].

Lemma 5.1. *For every fixed* $t_0 \in [0, +\infty)$ *let* $u \in H^{l+2}_{loc}(\overline{G}\backslash\{0\}) \cap V^0_{2,\gamma-l-2}(G)$ *be a solution of the problem*

$$L(x, t_0, \partial)u = f \quad \textit{in } G, \tag{5.1}$$

$$N(x, t_0, \partial)u = 0 \quad \textit{on } \Gamma, \tag{5.2}$$

where $f \in V^l_{2,\gamma}(G)$, l *is a nonnegative integer. Then* $u \in V^{l+2}_{2,\gamma}(G)$ *and the following estimate*

$$\|u\|^2_{V^{l+2}_{2,\gamma}(G)} \leqslant C\big(\|f\|^2_{V^l_{2,\gamma}(G)} + \|u\|^2_{V^0_{2,\gamma-l-2}(G)}\big) \tag{5.3}$$

holds with the constant C *independent of* u, f *and* t_0 .

Lemma 5.2. *For every fixed* $t_0 \in (0, +\infty)$ *let* $f \in H^0_1(G)$ *and* $u \in H^1(G)$ *be a generalized solution of the problem* (5.1), (5.2)*, i.e* u *satisfies the identity*

$$B(t_0, u, \eta) = (f, \eta) \textit{ for all } \eta \in H^1(G).$$

Then $u \in H^2_1(G)$ *and*

$$\|u\|^2_{H^2_1(G)} \leqslant C\big(\|f\|^2_{H^0_1(G)} + \|u\|^2_{H^1(G)}\big), \tag{5.4}$$

where the constant C *is independent of* u, f *and* t_0.

Proof. According to results for elliptic boundary value problem in domains with smooth boundaries, we have $u \in H^2_{\text{loc}}(\overline{G} \setminus \{0\})$. If $n \geqslant 3$, then $H^1(G) = V^1_{2,0}(G)$ by [4, Th. 7.1.1], and therefore, the assertion of the lemma follows from Lemma 5.1.

Now let $n = 2$. According to [4, Th. 7.1.1], $u \in H^1(G)$ can be written in the form $u = v + w$, where $v \in V^1_{2,0}(G)$, $w \in H^2_1(G)$, and

$$\|v\|^2_{V^1_{2,0}(G)} + \|w\|^2_{H^2_1(G)} \leqslant C\|u\|^2_{H^1(G)} \;\; (C = const). \tag{5.5}$$

We rewrite (5.1), (5.2) in the form

$$L(x, t_0, \partial)v = f - L(x, t_0, \partial)w \in H^0_1(G) \equiv V^0_{2,1}(G), \tag{5.6}$$

$$N(x,t_0,\partial)v = -N(x,t_0,\partial)w \in H_1^{\frac{1}{2}}(\Gamma) \equiv V_{2,1}^{\frac{1}{2}}(\Gamma). \tag{5.7}$$

Here we note that, according to [4, Th. 7.1.1], for $n=2$, $H_1^1(G) = V_{2,1}^1(G)$, and therefore, $H_1^{\frac{1}{2}}(\Gamma) = V_{2,1}^{\frac{1}{2}}(\Gamma)$. Now applying Lemma 5.1, we can see from (5.6), (5.7) that $v \in V_{2,1}^2(G)$. Therefore, $u = v + w \in H_1^2(G)$. The estimate (5.4) follows from (5.3) and (5.5). ■

Lemma 5.3. *Let l, k, s be nonnegative integers and γ, δ be real numbers, $k-\delta > l-\gamma$. Let $u \in H_\gamma^{l+2,0}(Q)$ be a solution of the following problem*

$$L(x,t,\partial)u = f \quad \textit{in } Q, \tag{5.8}$$

$$N(x,t,\partial)u = g \quad \textit{on } S. \tag{5.9}$$

(*i*) *If $f \in H_{\gamma+s}^{l+s,0}(Q), g \in H_{\gamma+s}^{l+s+\frac{1}{2},0}(S)$, then $u \in H_{\gamma+s}^{l+s+2,0}(Q)$ and*

$$\|u\|^2_{H_{\gamma+s}^{l+s+2,0}(Q)} \leqslant C\big(\|f\|^2_{H_{\gamma+s}^{l+s,0}(Q)} + \|g\|^2_{H_{\gamma+s}^{l+s+\frac{1}{2},0}(S)} + \|u\|^2_{H_\gamma^{l+2,0}(Q)}\big) \tag{5.10}$$

with the constant C independent of u, f, g.

(*ii*) *Suppose that $f \in H_\delta^{k,0}(Q), g \in H_\delta^{k+\frac{1}{2},0}(S)$ and the strip $-\gamma + l + 2 - \frac{n}{2} \leqslant Re\lambda \leqslant -\delta + k + 2 - \frac{n}{2}$ does not contain any eigenvalue of $\mathcal{U}(\lambda, t)$ for all $t \in (0, +\infty)$ and $\gamma + \frac{n}{2} \notin \{1, \dots, l\}$. Then $u \in H_\delta^{k+2,0}(Q)$ and*

$$\|u\|^2_{H_\delta^{k+2,0}(Q)} \leqslant C\big(\|f\|^2_{H_\delta^{k,0}(Q)} + \|g\|^2_{H_\delta^{k+\frac{1}{2},0}(S)} + \|u\|^2_{H_\gamma^{l+2,0}}\big) \tag{5.11}$$

with the constant C independent of u, f, g.

Proof. First, we prove the part (i). We fix $t \in (0, +\infty)$ and consider (5.8), (5.9) as an elliptic boundary value problem. Since coefficients of $L(x,t,\partial)$ are bounded smooth functions, we can apply Theorems 7.2.2, 7.2.3 and 7.3.5 of [4] to conclude that $u(.,t) \in H_\delta^{k+2}(G)$ and

$$\begin{aligned} \|u(.,t)\|_{H_\delta^{k+2}(G)} \leqslant C\Big(&\|f(.,t)\|_{H_\delta^k(G)} \\ &+ \|g(.,t)\|_{H_\delta^{k+\frac{1}{2}}(\Gamma)} + \|u(.,t)\|^2_{H_\gamma^{l+2}(G)}\Big), \end{aligned} \tag{5.12}$$

where the constant C is independent of u, f, g and t. Now integrating both sides of (5.12) with respect to t from 0 to $+\infty$, we get the assertion (i) of the lemma.

The part (ii) of the lemma is proved by the same procedure as in the proof the part (i), but we apply Theorems 7.2.2, 7.2.4 and the note below Theorem 7.3.5 of [4] instead of ones above. ∎

Proof of Theorem 3.1. The proof is an induction on h. Let us consider first the case $h = 0$. We rewrite the equation (2.1), (2.2) in the form

$$L(x,t,\partial)u = f_1 := u_t - f \quad \text{in } Q, \tag{5.13}$$

$$N(x,t,\partial)u = 0 \quad \text{on } S. \tag{5.14}$$

According to Theorem 4.1, we have $u_t \in L_2(Q) \subset H_1^{0,0}(Q)$. Thus, $f_1 \in H_1^{0,0}(Q)$. By Lemma 5.2, it follows from (5.13), (5.14) that for a.e. $t \in (0, +\infty)$ $u(.,t) \in H_1^2(G)$ and

$$\|u(.,t)\|^2_{H_1^2(G)} \leqslant C\big(\|f_1(.,t)\|^2_{H_1^0} + \|u(.,t)\|^2_{H^1(G)}\big) \tag{5.15}$$

for a.e. $t \in (0, +\infty)$, where C is a constant independent of u, f_1 and t. Integrating both sides of (5.15) with respect to t from 0 to $+\infty$, we obtain $u \in H_1^{2,0}(Q)$. This and the fact that $u_t \in L_2(Q)$ imply $u \in H_1^{2,1}(Q)$. Hence, the theorem is valid for $h = 0$.

Assume that it is true for $h-1, h \geqslant 1$. Then we have $u \in H^{2h,h}_{2h-1}(Q)$, and therefore,

$$u_{t^s} \in H^{2h-2s,0}_{2h-1}(Q),\ s \leqslant h. \tag{5.16}$$

We prove now the theorem for h. We have to show that $u \in H^{2h+2,h+1}_{2h+1}(Q)$. To this end, it is only needed to make clear that

$$u_{t^k} \in L_2(Q), k \leqslant h+1, \tag{5.17}$$

and

$$u_{t^k} \in H^{2h-2k+2,0}_{2h+1}(Q) \tag{5.18}$$

for $k \leqslant h+1$. (5.17) is true according to Theorem 4.1. We will prove (5.18) by induction on k. By Theorem 4.1, $u_{t^{h+1}} \in L_2(Q) \subset H^{0,0}_{2h+1}(Q)$. This means that (5.18) holds for $k = h+1$. Assume that it holds for $k = h+1, h, \ldots, p+1$ $(0 < p < h)$. Differentiating both sides of (5.13), (5.14) with respect to t p times, we have

$$L(x,t,\partial)u_{t^p} = u_{t^{p+1}} - f_{t^p} - \sum_{s=0}^{p-1} \tbinom{p}{s} L_{t^{p-s}}(x,t,\partial)u_{t^s} \quad \text{in } Q, \tag{5.19}$$

(5.20) $$N(x,t,\partial)u_{t^p} = -\sum_{s=0}^{p-1} \binom{p}{s} N_{t^{p-s}}(x,t,\partial)u_{t^s} \quad \text{on } S.$$

By (5.16), we see that $u_{t^s} \in H^{2h-2s,0}_{2h-1}(Q) \subset H^{2h-2p+2,0}_{2h-1}(Q) \subset H^{2h-2p+2,0}_{2h+1}(Q)$, $s \leqslant p-1$. Moreover, $u_{t^{p+1}} \in H^{2h-2p,0}_{2h+1}(Q)$ by the inductive assumption, and $f_{t^p} \in H^{2h-2p,0}_{2h+1}(Q)$ by the assumption of the theorem. Therefore, the right-hand side of (5.19) belongs to $H^{2h-2p,0}_{2h+1}(Q)$. From $u_{t^s} \in H^{2h-2p+2,0}_{2h+1}(Q)$ we also have $N_{t^{p-s}}(x,t,\partial)u_{t^s} \in H^{2h-2p+\frac{1}{2},0}_{2h+1}(S)$, $s \leqslant p-1$. Now we can apply the part (i) of Lemma 5.3 to the problem (5.19), (5.20) to conclude that $u_{t^p} \in H^{2h-2p+2,0}_{2h+1}(Q)$. Thus, (5.18) holds for all $k \leqslant h+1$. The inequality (3.7) is obtained by the estimates in Theorem 4.111 and Lemma 5.3. The proof is completed.

Proof of Theorem 3.2. The theorem is proved by a similar procedure as in the proof of Theorem 3.1 above. However, instead of applying the assertion (i) of Lemma 5.3, which we made use to prove Theorem 3.1, we use the assertion (ii). For example, in the case $h = 0$, from (5.13), (5.14) with $f_1 \in H^{0,0}_{\gamma}(Q)$, first we have $u \in H^{2,0}_{1}(Q)$ by Lemma 5.2, and after that we can conclude $u \in H^{2,1}_{\gamma}(Q)$ thanks to the assumption that the strip $1-\epsilon-\dfrac{n}{2} \leqslant \mathrm{Re}\lambda \leqslant -\gamma+2-\dfrac{n}{2}$ is free of eigenvalues of $\mathcal{U}(\lambda,t)$ for all $t \in (0,+\infty)$.

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