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# INEXACT ORBITS OF NONEXPANSIVE MAPPINGS

Evgeniy Pustylnik, Simeon Reich and Alexander J. Zaslavski

Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

**Abstract.** We study the influence of errors on the convergence of orbits of nonexpansive mappings in Banach and metric spaces.

## 1. Introduction

Convergence analysis of iterations of nonexpansive mappings is a central topic in nonlinear functional analysis. It began with the classical Banach theorem [1] on the existence of a unique fixed point for a strict contraction. Banach's celebrated result also yields convergence of iterates to the unique fixed point. There are several generalizations of Banach's theorem which show that the convergence of iterates holds for larger classes of nonexpansive mappings. For example, Rakotch [7] introduced the class of contractive mappings and showed that their iterates also converged to their unique fixed point. Note that this situation is in some sense typical [8, 9, 10]. Earlier, De Blasi and Myjak [4, 5], also using the generic approach, showed that most (in the sense of Baire category) nonexpansive mappings possessed a unique fixed point which attracted all their powers.

In view of the above discussion, it is natural to ask if convergence of the iterates of nonexpansive mappings will be preserved in the presence of computational errors. In [2] we provide affirmative answers to this question. Related results can be found, for example, in [3, 6]. More precisely, in [2] we show that if all exact iterates of a given nonexpansive mapping converge (to fixed points), then this convergence continues to hold for inexact orbits with summable errors.

In the present paper we continue to study the influence of computational errors on the convergence of iterates of nonexpansive mappings in Banach and metric spaces. In Sections 2 and 3 we study the convergence of such iterates to attractor

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sets, while Section 4 is devoted to convergence to fixed points. In addition to providing convergence theorems, we also show that the sufficient conditions we impose on the computational errors in order to guarantee convergence are, in many cases, also necessary.

## 2. Convergence to Fixed Point Sets

Let $(X, \rho)$ be a metric space. For each $x \in X$ and each nonempty and closed subset $A \subset X$, put

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

**Theorem 2.1.** *Let $T : X \to X$ satisfy*

$$\rho(Tx, Ty) \le \rho(x, y) \text{ for all } x, y \in X. \tag{2.1}$$

*Suppose that $F$ is a nonempty and closed subset of $X$ such that for each $x \in X$,*

$$\lim_{i\to\infty} \rho(T^i x, F) = 0.$$

*Assume that $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$,*

$$\{x_n\}_{n=0}^{\infty} \subset X \text{ and } \rho(x_{n+1}, Tx_n) \le \gamma_n, \ n = 0, 1, \ldots. \tag{2.2}$$

*Then*

$$\lim_{n\to\infty} \rho(x_n, F) = 0.$$

*Proof.* Let $\epsilon > 0$. Since the series $\sum \gamma_n$ converges, there is an integer $k \ge 1$ such that

$$\sum_{i=k}^{\infty} \gamma_i < \epsilon. \tag{2.3}$$

Define a sequence $\{y_i\}_{i=k}^{\infty}$ by

$$y_k = x_k, \tag{2.4}$$

$$y_{i+1} = Ty_i \text{ for all integers } i \ge k.$$

By (2.2) and (2.4),

$$\rho(x_{k+1}, y_{k+1}) \le \gamma_k. \tag{2.5}$$

Assume that $q \geq k+1$ is an integer and that for $i = k+1, \ldots, q$,

$$\rho(x_i, y_i) \leq \sum_{j=k}^{i-1} \gamma_j. \tag{2.6}$$

(Note that in view of (2.5), inequality (2.6) is valid when $q = k+1$.)

By (2.1) and (2.6),

$$\rho(Ty_q, Tx_q) \leq \rho(y_q, x_q) \leq \sum_{j=k}^{q-1} \gamma_j.$$

When combined with (2.4) and (2.2), this implies that

$$\rho(x_{q+1}, y_{q+1}) \leq \rho(x_{q+1}, Tx_q) + \rho(Tx_q, Ty_q) \leq \gamma_q + \sum_{j=k}^{q-1} \gamma_j = \sum_{j=k}^{q} \gamma_j,$$

so that (2.6) also holds for $i = q+1$. Thus we have shown that for all integers $q \geq k+1$,

$$\rho(y_q, x_q) \leq \sum_{j=k}^{q-1} \gamma_j < \sum_{j=k}^{\infty} \gamma_j < \epsilon, \tag{2.7}$$

by (2.3). In view of (2.4) and the hypotheses of the theorem we note that

$$\lim_{i \to \infty} \rho(y_i, F) = 0. \tag{2.8}$$

Therefore it follows from (2.7) and (2.8) that

$$\limsup_{i \to \infty} \rho(x_i, F) \leq \epsilon.$$

Since $\epsilon$ is an arbitrary positive number, we conclude that

$$\lim_{i \to \infty} \rho(x_i, F) = 0,$$

as asserted.

**Theorem 2.2.** *Let $X$ be a nonempty and closed subset of a reflexive Banach space $(E, ||\cdot||)$ and let $T : X \to X$ be such that*

$$||Tx - Ty|| \leq ||x - y|| \text{ for all } x, y \in X. \tag{2.9}$$

*Let $F$ be a nonempty and closed subset of $X$ such that for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ is bounded and all its weak limit points belong to $F$.*

*Assume that* $\{\gamma_i\}_{i=0}^{\infty} \subset (0, \infty)$, $\sum_{i=0}^{\infty} \gamma_i < \infty$, $\{x_i\}_{i=0}^{\infty} \subset X$ *and*

$$||x_{i+1} - Tx_i|| \leq \gamma_i \textit{ for all integers } i \geq 0. \tag{2.10}$$

*Then the sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *is bounded and all its weak limit points also belong to* $F$.

*Proof.* Let $\epsilon > 0$. There is an integer $k \geq 1$ such that

$$\sum_{i=k}^{\infty} \gamma_i < \epsilon. \tag{2.11}$$

Define a sequence $\{y_i\}_{i=k}^{\infty}$ by

$$y_k = x_k,\ y_{i+1} = Ty_i \text{ for all integers } i \geq k. \tag{2.12}$$

Arguing as in the proof of Theorem 2.1, we can show that for all integers $q \geq k+1$,

$$||y_q - x_q|| \leq \sum_{j=k}^{q-1} \gamma_j < \epsilon. \tag{2.13}$$

Obviously, (2.13) implies that the sequence $\{x_k\}_{k=0}^{\infty}$ is bounded.

Assume now that $z$ is a weak limit point of the sequence $\{x_k\}_{k=0}^{\infty}$. There exists a subsequence $\{x_{i_p}\}_{p=1}^{\infty}$ which weakly converges to $z$. We may assume without loss of generality that $\{y_{i_p}\}_{p=1}^{\infty}$ weakly converges to $\tilde{z} \in F$. By (2.13) and the weak lower semicontinuity of the norm,

$$||\tilde{z} - z|| \leq \epsilon.$$

Since $\epsilon$ is an arbitrary positive number, we conclude that

$$z \in F.$$

Theorem 2.2 is proved.

Note that Theorem 2.1 is an extension of the following result established in [2].

**Theorem 2.3.** *Let* $(X, \rho)$ *be a complete metric space and let* $T : X \to X$ *be such that*

$$\rho(Tx, Ty) \leq \rho(x, y) \textit{ for all } x, y \in X,$$

*and for each* $x \in X$, *the sequence* $\{T^n x\}_{n=1}^{\infty}$ *converges in* $(X, \rho)$.

*Assume that* $\{\gamma_n\}_{n=0}^{\infty} \subset (0, \infty)$ *satisfies* $\sum_{n=0}^{\infty} \gamma_n < \infty$, *and that a sequence* $\{x_n\}_{n=0}^{\infty} \subset X$ *satisfies* $\rho(x_{n+1}, Tx_n) \leq \gamma_n$, $n = 0, 1, 2, \ldots$ *Then the sequence* $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of* $T$ *in* $(X, \rho)$.

## 3. Nonconvergence to Fixed Point Sets

In this section we show that both Theorems 2.1 and 2.3 cannot, in general, be improved. We begin with Theorem 2.3 [2].

**Proposition 3.1.** *For any normed space $X$, there exists a mapping $T : X \to X$ such that $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges for each $x \in X$ and, for any sequence of positive numbers $\{\gamma_n\}_{n=0}^{\infty}$, there exists a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ with $\|x_{n+1} - Tx_n\| \le \gamma_n$ for all nonnegative integers $n$, which converges if and only if the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is summable, i.e., $\sum_{n=0}^{\infty} \gamma_n < \infty$.*

*Proof.* This is a simple fact because we may take $T$ to be the identity mapping: $Tx = x$, $\forall x$. Then we may take $x_0$ to be an arbitrary element of $X$ with $\|x_0\| = 1$, and define by induction

$$x_{n+1} = Tx_n + \gamma_n x_0, \qquad n = 0, 1, 2, \ldots.$$

Evidently, $\|x_{n+1} - Tx_n\| = \gamma_n$ and $x_{n+1} = x_0(1 + \sum_{i=0}^{n} \gamma_i)$ for all integers $n \ge 0$, so that the convergence of $\{x_n\}_{n=0}^{\infty}$ is equivalent to the summability of the sequence $\{\gamma_n\}_{n=0}^{\infty}$.

Counterexamples to possible improvements of Theorem 2.1 are more difficult to construct because this theorem deals with convergence to attractors. For simplicity, we assume that the non-summable sequence $\{\gamma_n\}_{n=0}^{\infty}$ decreases to 0 and that $\gamma_1 \le 1$.

**Proposition 3.2.** *Let $X$ be an arbitrary (but not one-dimensional) normed space and let a non-summable sequence of positive numbers $\{\gamma_n\}_{n=0}^{\infty}$ decrease to 0. Then there exist a closed subspace $F \subset X$ and a nonexpansive (with respect to an equivalent norm on $X$) mapping $T : X \to X$ such that $\rho(T^n u, F) \to 0$ as $n \to \infty$ for any $u \in X$ and there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset X$ such that $\|u_{n+1} - Tu_n\| \le \gamma_n$ for all integers $n \ge 0$, but the sequence $\{\rho(u_n, F)\}$ does not tend to $0$ as $n \to \infty$.*

*Proof.* We take any 2-dimensional subspace of $X$, identify it with $R^2$ (with coordinates $(x, y)$), and restrict all constructions and arguments to this subspace, taking as $F$ the one-dimensional space $L := \{(x, y) \in R^2 : \ y = 0\}$. The same counterexample may be then applied to the whole space $X$ if we take $F$ to be a complement of the one-dimensional space $\{(x, y) \in R^2 : \ x = 0\}$ which includes $L$.

So, consider a plane with orthogonal axes $x, y$ and the norm $\|u\| = \|(x, y)\| = \max(|x|, |y|)$ (recall that in a finite dimensional space all norms are equivalent). At the first stage we only consider the case where $\gamma_{n+1}/\gamma_n \ge 1/2$ for all $n$ and we define a decreasing function $y = \gamma(x)$ which equals $\gamma_n$ at $x = 2n$, $n = 1, 2, \ldots$, and is linear on the intermediate segments. Finally, we define the mapping $T$ as the superposition $T = T_4T_3T_2T_1$ of the following four mappings: (a) $T_1 : (x, y) \mapsto (|x|, |y|)$; (b) $T_2 : (x, y) \mapsto \big(x, \min(1, y)\big)$; (c) $T_3 : (x, y) \mapsto (x + 2, y)$; (d) $T_4 : (x, y) \mapsto \big(x, [1 - \gamma(x)]y\big)$.

The principal point of the proof is to show that the mapping $T$ is nonexpansive.

Since this is obviously true for the first three mappings $T_1, T_2$ and $T_3$, we need only consider the fourth mapping $T_4$. For simplicity, we may assume from the very beginning that $T = T_4$.

For arbitrary $x_1 < x_2$, let $u_1 = (x_1, y_1)$ and $u_2 = (x_2, y_2)$. Then $Tu_1 = (x_1, [1 - \gamma(x_1)]y_1)$ and $Tu_2 = (x_2, [1 - \gamma(x_2)]y_2)$. Our aim is to show that $\|Tu_1 - Tu_2\| \le \|u_1 - u_2\|$, where $\|u_1 - u_2\| = \max(x_2 - x_1, |y_2 - y_1|)$ and $\|Tu_1 - Tu_2\| = \max(x_2 - x_1, |[1 - \gamma(x_2)]y_2 - [1 - \gamma(x_1)]y_1|)$. Since after the application of the first two mappings $T_1$ and $T_2$, the second coordinate $y$ already belongs to $[0, 1]$, the case where $x_2 - x_1 \ge 1$ is trivial because then $\|Tu_1 - Tu_2\| = \|u_1 - u_2\| = x_2 - x_1$. Hence we may assume in what follows that $x_2 - x_1 < 1$ and thus we need only consider one of the two following possibilities: either both $x_1$ and $x_2$ belong to the same interval $[2n, 2(n + 1)]$ or they belong to two adjoining intervals $[2n, 2(n + 1)]$ and $[2(n + 1), 2(n + 2)]$ for some $n = 1, 2, \ldots$. We claim that in both cases,

$$\gamma(x_1) - \gamma(x_2) \le (x_2 - x_1)\gamma(x_1). \tag{3.1}$$

If $2n \le x_1 < x_2 \le 2(n + 1)$, then the points $u_1$ and $u_2$ lie on the straight line connecting the points $(2n, 1 - \gamma_n)$ and $(2(n + 1), 1 - \gamma_{n+1})$, so that the ratio $\big(\gamma(x_1) - \gamma(x_2)\big)/(x_2 - x_1)$ coincides with the slope $k_n$ of this line:

$$k_n = (\gamma_n - \gamma_{n+1})/2 \le \gamma_n/2 \le \gamma_{n+1} \le \gamma(x_1).$$

In the second case the same ratio is less or equal to $\max(k_n, k_{n+1})$, where

$$k_{n+1} = (\gamma_{n+1} - \gamma_{n+2})/2 \le \gamma_{n+1} \le \gamma(x_1),$$

and therefore inequality (3.1) is proved in both cases.

Note that in order to compare the distances between $u_1$ and $u_2$, and between $Tu_1$ and $Tu_2$, it is enough to show that

$$|y_2[1 - \gamma(x_2)] - y_1[1 - \gamma(x_1)]| \le \max(x_2 - x_1, |y_2 - y_1|). \tag{3.2}$$

If $y_1 \ge y_2$, then

$$y_1[1 - \gamma(x_1)] - y_2[1 - \gamma(x_2)] = (y_1 - y_2) - [y_1\gamma(x_1) - y_2\gamma(x_2)] \le y_1 - y_2$$

because $\gamma(x_1) \geq \gamma(x_2)$. On the other hand,

$$y_1[1-\gamma(x_1)] - y_2[1-\gamma(x_2)] = (y_1 - y_2)[1-\gamma(x_2)] + y_1[\gamma(x_2) - \gamma(x_1)]$$
$$\geq -(x_2 - x_1)\gamma(x_1)y_1$$

by (3.1). Now inequality (3.2) follows because $\gamma(x_1)y_1 < 1$.

If $y_2 - y_1 \geq 0$, then also $y_2[1-\gamma(x_2)] - y_1[1-\gamma(x_1)] \geq 0$ and it suffices to estimate this difference only from above. Bearing in mind that all $y \leq 1$, we obtain by (3.1) that

$$\begin{aligned} &y_2[1-\gamma(x_2)] - y_1[1-\gamma(x_1)] = (y_2 - y_1)[1-\gamma(x_1)] + y_2[\gamma(x_1) - \gamma(x_2)] \\ &\leq (y_2 - y_1)[1-\gamma(x_1)] + \gamma(x_1)(x_2 - x_1) \leq \max(x_2 - x_1, y_2 - y_1), \end{aligned}$$

as needed.

Let $u$ be an arbitrary point in $R^2$. Then $T_2T_1u$ belongs to the set $\{(x, y) \in R^2 : x \geq 0,\ 0 \leq y \leq 1\}$ and thereafter the mappings $T_1$ and $T_2$ coincide with the identity mapping. Defining the integer $k$ by $2k \leq x < 2(k+1)$, we see that

$$\rho(T^n u, F) = y\prod_{i=1}^{n}[1-\gamma(x+2i)] \leq y \prod_{i=k+1}^{k+n}(1-\gamma_i) \ \longrightarrow\ 0$$

as $n \to \infty$ because the series $\sum_{i=1}^{\infty}\gamma_i$ is divergent.

To finish the proof for the case where $\gamma_{n+1}/\gamma_n \geq 1/2$ for all natural numbers $n$, we define $u_n = (2(n-1), 1)$ for $n = 1, 2, \ldots$. Then $Tu_n = T_4T_3u_n = (2n, 1-\gamma_n)$ and $\|u_{n+1} - Tu_n\| = \gamma_n$. At the same time, $\rho(u_n, F) = 1$ for all $n$ and so the sequence $\{\rho(u_n, F)\}$ does not tend to 0.

We now proceed to the general case where the given sequence $\{\gamma_n\}_{n=0}^{\infty}$ does not necessarily satisfy the condition $\gamma_{n+1}/\gamma_n \geq 1/2$ for all $n \geq 0$. We then define by induction a new sequence:

$$\gamma_1' = \gamma_1, \qquad \gamma_{n+1}' = \max\{\gamma_{n+1},\ \gamma_n'/2\}, \quad n = 1, 2, \ldots,$$

so that $\gamma_{n+1}'/\gamma_n' \geq 1/2$. Using the new sequence $\{\gamma_n'\}_{n=0}^{\infty}$, we construct the mapping $T$ as before, replacing each $\gamma_n$ by $\gamma_n'$. The sequence $\{u_n\}_{n=0}^{\infty}$ will be defined by induction. Let $u_1 = (0, 1)$. If the point $u_n = (x_n, y_n)$ has already been defined, then to obtain the next point $u_{n+1} = (x_{n+1}, y_{n+1})$, we set $x_{n+1} = x_n+2$, $y_{n+1} = y_n$ if $\gamma_n' = \gamma_n$, and $y_{n+1} = y_n[1-\gamma_n']$ if $\gamma_n' > \gamma_n$. Since $Tu_n = (x_{n+1}, y_n[1-\gamma_n'])$ for each $n$, we find that $\|u_{n+1} - Tu_n\| \leq \gamma_n$ for all $n$, as needed.

It is easy to see that

$$y_{n+1} = \prod_{k=1}^{n}(1-\sigma_k\gamma_k'),$$

where $\sigma_k = 1$ when $\gamma'_n > \gamma_n$ and $\sigma_k = 0$ otherwise. But the series $\sum_{k=1}^{\infty} \sigma_k \gamma'_k$ converges, since the ratio of any two consecutive nonzero terms of this series is not greater than $1/2$. Therefore

$$\rho(u_n, F) \geq \prod_{k=1}^{\infty} (1 - \sigma_k \gamma'_k) > 0.$$

That is, the sequence $\{\rho(u_n, F)\}$ again does not tend to 0, as claimed.

## 4. Convergence and Nonconvergence to Fixed Points

In Section 3 we have shown that Theorems 2.1 and 2.3 cannot be, in general, improved. However, in Proposition 3.1 every point of the space is a fixed point of the mapping $T$ and the inexact orbits tend to infinity. In Proposition 3.2 the attractor $F$ is unbounded and the mapping $T$ depends on the sequence of errors. In this section we construct a mapping $T$ on a complete metric space $X$ such that all of its orbits converge to its unique fixed point, and for any nonsummable sequence of errors and any initial point, there exists a divergent inexact orbit with a convergent subsequence. On the other hand, we emphasize that while the example of the present section is for a particular subset of an infinite-dimensional Banach space, the examples in Section 3 apply to general normed spaces, even finite-dimensional ones.

Let $X$ be the set of all sequences $x = \{x_i\}_{i=1}^{\infty}$ of nonnegative numbers such that $\sum_{i=1}^{\infty} x_i \leq 1$. For $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty} \in X$, set

$$\rho(\{x_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} |x_i - y_i|. \tag{4.1}$$

Clearly, $(X, \rho)$ is a complete metric space.

Define a mapping $T : X \to X$ as follows:

$$T(\{x_i\}_{i=1}^{\infty}) = (x_2, x_3, \ldots, x_i, \ldots),\ \{x_i\}_{i=1}^{\infty} \in X. \tag{4.2}$$

In other words, for any $\{x_i\}_{i=1}^{\infty} \in X$,

$$T(\{x_i\}_{i=1}^{\infty}) = \{y_i\}_{i=1}^{\infty},\ \text{where } y_i = x_{i+1} \text{ for all integers } i \geq 1. \tag{4.3}$$

Set $T^0 x = x$ for all $x \in X$. Clearly,

$$\rho(Tx, Ty) \leq \rho(x, y) \text{ for all } x, y \in X \tag{4.4}$$

and

$$T^n x \text{ converges to } (0, 0, \ldots, \ldots) \text{ as } n \to \infty \tag{4.5}$$

for all $x \in X$.

**Theorem 4.1.** *Let* $\{r_i\}_{i=0}^{\infty} \subset [0, \infty)$*,*

$$\sum_{i=0}^{\infty} r_i = \infty, \tag{4.6}$$

*and* $x = \{x_i\}_{i=1}^{\infty} \in X$*. Then there exists a sequence* $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ *such that*

$$y^{(0)} = x,\ \rho(Ty^{(i)}, y^{(i+1)}) \le r_i,\ \ i = 0, 1, \dots,$$

*the sequence* $\{y^{(i)}\}_{i=0}^{\infty}$ *does not converge in* $(X, \rho)$*, but* $(0, 0, \dots)$ *is a limit point of* $\{y^{(i)}\}_{i=0}^{\infty}$*.*

In the proof of this theorem we may assume without loss of generality that

$$r_i \le 16^{-1} \text{ for all integers } i \ge 0. \tag{4.7}$$

We precede the proof of Theorem 4.1 with the following lemma.

**Lemma 4.1.** *Let* $z^{(0)} = \{z_i^{(0)}\}_{i=1}^{\infty} \in X$ *and let* $k \ge 0$ *be an integer. Then there exist an integer* $n \ge 4$ *and a sequence* $\{z^{(i)}\}_{i=0}^{n} \subset X$ *such that*

$$\rho(z^{(i+1)}, Tz^{(i)}) \le r_{k+i},\ i = 0, \dots, n-1,$$

*and*

$$\rho(z^{(n)}, (0, 0, \dots)) \ge 4^{-1}.$$

*Proof.* There is a natural number $m > 4$ such that

$$\sum_{i=m}^{\infty} z_i^{(0)} < 16^{-1}. \tag{4.8}$$

Set

$$z^{(i+1)} = Tz^{(i)},\ i = 0, \dots, m-1. \tag{4.9}$$

Clearly,

$$z^{(m)} = (z_{m+1}^{(0)}, z_{m+2}^{(0)}, \dots, z_i^{(0)}, \dots). \tag{4.10}$$

By (4.6), there is a natural number $n > m$ such that

$$\sum_{j=k+m}^{k+n} r_j \ge 2^{-1}. \tag{4.11}$$

By (4.11) and (4.7), $n \geq m+7$ and we may assume without loss of generality that

$$\sum_{j=k+m}^{k+n-1} r_j < 1/2. \tag{4.12}$$

In view of (4.1) and (4.7),

$$\sum_{j=k+m}^{k+n-1} r_j = \sum_{j=k+m}^{k+n} r_j - r_{k+n} \geq 2^{-1} - 16^{-1}. \tag{4.13}$$

For $i = m+1, \ldots, n$, define $z^{(i)} = \{z_j^{(i)}\}_{j=1}^{\infty}$ as follows:

$$\begin{aligned} z_j^{(i)} &= z_{j+i}^{(0)},\ j \in \{1,2,\ldots\} \setminus \{n+1-i\}, \\ z_{n+1-i}^{(i)} &= z_{n+1}^{(0)} + \sum_{j=k+m}^{k+i-1} r_j. \end{aligned} \tag{4.14}$$

Clearly, for $i = m+1, \ldots, n$, $z^{(i)}$ is well-defined and by (4.14), (4.8) and (4.12),

$$\sum_{j=1}^{\infty} z_j^{(i)} = \sum_{j=i+1}^{\infty} z_j^{(0)} + \sum_{j=k+m}^{k+i-1} r_j \leq \sum_{j=m}^{\infty} z_j^{(0)} + \sum_{j=k+m}^{k+n-1} r_j \leq 16^{-1} + 2^{-1} < 1.$$

Thus $z^{(i)} \in X$, $i = m+1, \ldots, n$.

Let $i \in \{m, \ldots, n-1\}$. In order to estimate $\rho(z^{(i+1)}, Tz^{(i)})$, we first set

$$\{\tilde{z}_j\}_{j=1}^{\infty} = Tz^{(i)}. \tag{4.15}$$

In view of (4.15), (4.2) and (4.3), $\tilde{z}_j = z_{j+1}^{(i)}$ for all integers $j \geq 1$. When combined with (4.14), this implies that

$$\tilde{z}_j = z_{j+1+i}^{(0)} \text{ for all } j \in \{1,2\ldots\} \setminus \{n-i\} \tag{4.16}$$

and

$$\tilde{z}_{n-i} = z_{n+1-i}^{(i)} = z_{n+1}^{(0)} + \sum_{j=k+m}^{k+i-1} r_j.$$

By (4.16), $\tilde{z}_j = z_j^{(i+1)}$ for all $j \in \{1,2,\ldots\} \setminus \{n-i\}$. Together with (4.15), (4.1), (4.16) and (4.14), this equality implies that

$$\rho(z^{(i+1)}, Tz^{(i)}) = \rho(z^{(i+1)}, \{\tilde{z}_j\}_{j=1}^{\infty}) = |z_{n-i}^{(i+1)} - \tilde{z}_{n-i}| = r_{k+i}.$$

It follows from this relation, which holds for all $i \in \{m, \ldots, n-1\}$, and from (4.9) that

$$\rho(z^{(i+1)}, Tz^{(i)}) \le r_{k+i},\ i = 0, \ldots, n-1.$$

By (4.1), (4.14) and (4.13),

$$\rho(z^{(n)}, (0, 0, \ldots)) \ge z_1^{(n)} = z_{n+1}^{(0)} + \sum_{j=k+m}^{k+n-1} r_j \ge 2^{-1} - 16^{-1}.$$

This completes the proof of Lemma 4.1.

*Proof of Theorem 4.1.* In order to prove the theorem we construct by induction, using Lemma 4.1, sequences of nonnegative integers $\{t_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=0}^{\infty}$, and a sequence $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ such that

$$y^{(0)} = x, \tag{4.17}$$

$$\rho(y^{(i+1)}, Ty^{(i)}) \le r_i \text{ for all integers } i \ge 0, \tag{4.18}$$

$$t_0 = s_0 = 0,\ t_k < s_{k+1} < t_{k+1} \text{ for all integers } k \ge 0, \tag{4.19}$$

and for all integers $k \ge 1$,

$$\rho(y^{(s_k)}, (0, 0, \ldots)) \le 1/k \text{ and } \rho(y^{(t_k)}, (0, 0, \ldots)) \ge 1/4. \tag{4.20}$$

In the sequel we use the notation $y^{(i)} = \{y_j^{(i)}\}_{j=1}^{\infty}$, $i = 0, 1, \ldots$.

Set

$$y^{(0)} = x \text{ and } t_0, s_0 = 0. \tag{4.21}$$

Assume that $q \ge 0$ is an integer and that we have already defined sequences of nonnegative numbers $\{t_k\}_{k=0}^{q}$ and $\{s_k\}_{k=0}^{q}$, and a sequence $\{y^{(i)}\}_{i=0}^{t_q} \subset X$ such that (4.18) holds for all integers $i$ satisfying $0 \le i < t_q$, (4.21) holds,

$$t_k < s_{k+1} < t_{k+1} \text{ for all integers } k \text{ satisfying } 0 \le k < q,$$

and (4.20) holds for all integers $k$ satisfying $0 < k \le q$. (Note that for $q = 0$ this assumption indeed holds.)

Now we show that this assumption also holds for $q + 1$.

Indeed, there is a natural number $s_{q+1} > t_q + 1$ such that

$$\sum_{j=s_{q+1}-1-t_q}^{\infty} y_j^{(t_q)} < (q+1)^{-1}. \tag{4.22}$$

Set

$$y^{(i+1)} = Ty^{(i)},\ i = t_q, \ldots, s_{q+1} - 1. \tag{4.23}$$

By (4.23), (4.1), (4.2), (4.3) and (4.22),

$$\rho(y^{(s_{q+1})}, (0, 0, \ldots)) = \sum_{j=1}^{\infty} y_j^{(s_{q+1})} = \sum_{j=s_{q+1}-t_q+1}^{\infty} y_j^{(t_q)} < (q+1)^{-1}. \tag{4.24}$$

Applying Lemma 4.1 with

$$z^{(0)} = y^{(s_{q+1})} \text{ and } k = s_{q+1}, \tag{4.25}$$

we obtain that there exist an integer $n \geq 4$ and a sequence $\{y^{(i)}\}_{i=s_{q+1}}^{s_{q+1}+n} \subset X$ such that

$$\rho(y^{(i+1))}, Ty^{(i)}) \leq r_i,\ i = s_{q+1}, \ldots, s_{q+1} + n - 1, \tag{4.26}$$

and

$$\rho(y^{(s_{q+1}+n)}, (0, 0, 0 \ldots, )) \geq 1/4. \tag{4.27}$$

Put

$$t_{q+1} = s_{q+1} + n.$$

In this way we have constructed a sequence $\{y^{(i)}\}_{i=0}^{t_{q+1}} \subset X$, and sequences of nonnegative integers $\{t_k\}_{k=0}^{q+1}$ and $\{s_k\}_{k=0}^{q+1}$ such that (4.21) holds, (4.18) holds for all integers $i$ satisfying $0 \leq i < t_{q+1}$ (see (4.23) and (4.26)), $t_k < s_{k+1} < t_{k+1}$ for all integers $k$ satisfying $0 \leq k < q+1$, and (4.20) holds for all integers $k$ satisfying $0 < k \leq q+1$ (see (4.24), (4.26) and (4.27)).

In other words, the assumption made concerning $q$ also holds for $q + 1$. This means that we have indeed constructed sequences of nonnegative integers $\{t_k\}_{k=0}^{\infty}$ and $\{s_k\}_{k=0}^{\infty}$, and a sequence $\{y^{(i)}\}_{i=0}^{\infty} \subset X$ which satisfy (4.17)-(4.20). This completes the proof of Theorem 4.1.

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Evgeniy Pustylnik, Simeon Reich and Alexander J. Zaslavski
Department of Mathematics,
The Technion-Israel Institute of Technology,
32000 Haifa, Israel
E-mail: evg@tx.technion.ac.il
sreich@tx.technion.ac.il
ajzasl@tx.technion.ac.il