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MULTIPLICATIVE SK INVARIANTS ON $\mathbf{Z}_n$-MANIFOLDS WITH BOUNDARY

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ABSTRACT. Let $\mathbf{Z}_n$ be the cyclic group of order n. In this paper, we study a map T for $\mathbf{Z}_n$-manifolds with boundary which takes values in the ring $\mathbf{Z}$ and is additive with respect to the disjoint union of $\mathbf{Z}_n$-manifolds. We call T a $\mathbf{Z}_n$-SK invariant if it is invariant under $\mathbf{Z}_n$-cuttings and pastings. Then T induces an additive homomorphism $T : SK_*^{\mathbf{Z}_n}(pt, pt) \to \mathbf{Z}$, where $SK_*^{\mathbf{Z}_n}(pt, pt)$ is a cutting and pasting group (SK group) of all $\mathbf{Z}_n$-manifolds. First we obtain a basis of a free $\mathbf{Z}$-module $\mathcal{I}_*^{\mathbf{Z}_n}$ of all these invariants by using the Euler characteristic $\overline{\chi}$ of manifolds with boundary. As a result, we determine the class of all multiplicative invariants, which includes $\overline{\chi}^{\mathbf{Z}_s}$ (and $\chi^{\mathbf{Z}_s}$) in particular.

Introduction. Let G be a finite abelian group. Throughout this paper, by a G-manifold we mean an unoriented compact smooth manifold (which may have boundary) with smooth G-action. In [**2**] and [**3**], we have studied an equivariant cutting and pasting theory (SK theory) $SK_*^G(pt, pt)$ based on G-manifolds by using the notion of G-slice types. We now consider a map T for G-manifolds which takes values in the ring $\mathbf{Z}$ of rational integers and is additive with respect to the disjoint union of G-manifolds. We call T a G-SK invariant if it is invariant under G-cuttings and pastings. Furthermore, such T is said to be multiplicative if $T(M \times N) = T(M)T(N)$ for any G-manifolds M and N. Let $\overline{\chi}(M) = \chi(M) - \chi(\partial M)$ for a pair $(M, \partial M)$ of G-manifold and its boundary, where χ is the Euler characteristic. Then $\overline{\chi}^H$ and χ^H are multiplicative G-SK invariants for any subgroup H of G, where $\overline{\chi}^H(M) = \overline{\chi}(M^H)$, $\chi^H(M) = \chi(M^H)$ and $M^H = \{x \in M \mid hx = x \text{ for any } h \in H\}$.

The main object of this paper is to study such kind of invariants when G is the cyclic group $\mathbf{Z}_n$ of order n, $n \geq 1$. Here $\mathbf{Z}_1$ is the trivial group $\{1\}$.

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For the purpose, we define in Section 1 an SK group $SK_*^G[\mathcal{F},\mathcal{F}]$ obtained by G-manifolds of type $\mathcal{F}$, where $\mathcal{F}$ is a family of G-slice types. In particular, we have $SK_*^G(pt,pt) = SK_*^G[\mathcal{F},\mathcal{F}]$ if $\mathcal{F}$ is the family $St(G)$ of all G-slice types. When $G = \mathbf{Z}_n$, there is a total ordering on $St(\mathbf{Z}_n)$ which gives a basis of $SK_*^{\mathbf{Z}_n}(pt,pt)$ as a free SK_*-module, where SK_* is the polynomial ring $\mathbf{Z}[\alpha]$ with generator α represented by the real projective plane $\mathbf{R}P^2$.

We may consider a $\mathbf{Z}_n$-SK invariant T to be an additive homomorphism $T : SK_*^{\mathbf{Z}_n}(pt,pt) \to \mathbf{Z}$. From this point of view, in Section 2 we have a basis of a $\mathbf{Z}$-module $\mathcal{I}_*^{\mathbf{Z}_n}$ consisting of all these invariants by using the characteristic $\overline{\chi}$ (Theorem 2.7). As a result, we see that an element $[M]$ in $SK_*^{\mathbf{Z}_n}(pt,pt)$ is determined by the class $\{\overline{\chi}(M_\sigma) \mid \sigma \in St(\mathbf{Z}_n)\}$ where each M_σ is the invariant submanifold of M with slice type contained in σ (Proposition 2.9). Further we mention some relations among $\mathbf{Z}_n$-SK invariants (Proposition 2.10). At the end of this section, we present a multiplicative structure on $SK_*^{\mathbf{Z}_n}(pt,pt)$ given by the cartesian product $M \times N$ of $\mathbf{Z}_n$-manifolds M and N (Proposition 2.12).

In Section 3, we study a multiplicative invariant which may be considered as a ring homomorphism $T : SK_*^{\mathbf{Z}_n}(pt,pt) \to \mathbf{Z}$. Such T is said to be of type (s) if $T(\mathbf{Z}_n/\mathbf{Z}_t) = 0$ for each 0-dimensional $\mathbf{Z}_n$-manifolds $\mathbf{Z}_n/\mathbf{Z}_t$ with $1 \le t < s$ and $T(\mathbf{Z}_n/\mathbf{Z}_s) \neq 0$. For example, $\overline{\chi}^{\mathbf{Z}_s}$ and $\chi^{\mathbf{Z}_s}$ are of type (s). In this case, we see that T is determined by the values on the unit disk D^1, with trivial action, and $\mathbf{Z}_n$-manifolds $\mathbf{Z}_n \times_{\mathbf{Z}_s} D(V_i)$ where $\{V_i\}$ is a complete set of nontrivial irreducible $\mathbf{Z}_s$-modules and $\{D(V_i)\}$ are the associated disks (Theorem 3.10). As an application, we study mod p multiplicative invariants on $\mathbf{Z}_2$ or $\mathbf{Z}_3$-manifolds, where p is a prime integer (Proposition 3.19).

1. Preliminaries.

Let M^k be a k-dimensional smooth G-manifold with boundary ∂M and let $(L, \partial L) \subset (M, \partial M)$ be a submanifold which satisfies the following properties:

(1) $(L, \partial L)$ is a G-invariant codimension 1 submanifold of $(M, \partial M)$. Here we admit the case $\partial L = \varnothing$ and $\partial M \neq \varnothing$, and

(2) the normal bundle of $(L, \partial L)$ in $(M, \partial M)$ is G-equivalent to the trivial bundle $(L, \partial L) \times \mathbf{R}$ with trivial action of G on the set $\mathbf{R}$ of real numbers.

We assume that L separates M, that is, $M = N_1 \cup N_2$ (pasting along the common parts L) for some G-invariant submanifolds N_i of codimension zero. It is no gain in generality to drop this condition, since the union of L with a second copy of L, suitably embedded near L, will separate M.

Definition 1.1. Let M_1 and M_2 be k-dimensional G-manifolds. We say that M_1 and M_2 are obtained from each other by a G-equivariant cutting and pasting if M_1 has been obtained from M_2 by the step as mentioned above, that is, $M_1 = N_1 \cup_\phi N_2$ and $M_2 = N_1 \cup_\psi N_2$ pasting along the common parts $L \subset M_i$ by some G-diffeomorphisms $\phi, \psi : L \to L$, $i = 1, 2$.

If H is a subgroup of G, then an H-module is a finite dimensional real vector space together with a linear action of H on it. If M is a G-manifold and $x \in M$, then there is a G_x-module U_x which is equivariantly diffeomorphic to a G_x-neighborhood of x, where $G_x = \{g \in G \mid gx = x\}$ is the isotropy subgroup at x. This module U_x decomposes as $U_x = \mathbf{R}^p \oplus V_x$ where G_x acts trivially on $\mathbf{R}^p$ and $V_x^{G_x} = \{0\}$. We refer to the pair $[G_x; V_x]$ as the slice type of $x \in M$. By a G-slice type, we mean a pair $[H; V]$ of a subgroup H and an H-module V such that $V^H = \{0\}$ in general. A family $\mathcal{F}$ of G-slice types is a collection of G-slice types satisfying the condition that if $[H; V] \in \mathcal{F}$ and $x \in G \times_H V$ then the slice type $[G_x; V_x]$ of x belongs to $\mathcal{F}$.

Definition 1.2. Let $\mathcal{F}$ be a family of G-slice types. Let M_1 and M_2 be k-dimensional G-manifolds of type $\mathcal{F}$, that is, $[G_x; V_x] \in \mathcal{F}$ for each $x \in M_i$. We say that M_1 and M_2 are G-SK equivalent if M_1 can be obtained from M_2 by a finite sequence of G-equivariant cuttings and pastings. This is an equivalence relation on the set of k-dimensional G-manifolds of type $\mathcal{F}$. The set of all equivalence classes $\mathcal{M}_k^G[\mathcal{F}, \mathcal{F}]$ forms an abelian semigroup if we use disjoint union as addition. The class containing the G-manifold M is denoted by $[M]$. We define by $SK_k^G[\mathcal{F}, \mathcal{F}]$ the Grothendieck group of this semigroup. By defining $SK_*^G[\mathcal{F}, \mathcal{F}] = \bigoplus_{k \geq 0} SK_k^G[\mathcal{F}, \mathcal{F}]$ we have a graded SK_*-module with multiplication given by cartesian product of manifolds. When $\mathcal{F} = St(G)$, we write $SK_*^G[\mathcal{F}, \mathcal{F}] = SK_*^G(pt, pt)$. In case G is the trivial group $\{1\}$, $SK_*^{\{1\}}(pt,pt)$ is the theory $SK_*(pt,pt)$ studied in [**6**].

In general, let $SK_*(X,X)$ be an SK theory for singular manifolds in arcwise connected space X. Further, let $\overline{\chi}(M) = \chi(M) - \chi(\partial M)$ be a characteristic of a pair $(M, \partial M)$, cf. [**1**, Chap. V, Proposition 5.7]. If $\dim(M) = k$ is odd, then $\chi(\partial M) = 2\chi(M)$ by applying χ to the double $\mathcal{D}M = M \cup M$. Hence $\overline{\chi} = (-1)^k\chi$ for any $k \geq 0$. It is easy to see that $\overline{\chi}$ is invariant under cuttings and pastings. We then have the following lemma.

Lemma 1.3 (cf. [**6**, Theorem 1.2] and [**3**, Lemma 1.10]). *For any $k \geq 0$, $\overline{\chi} = (-1)^k\chi : SK_k(X,X) \cong SK_k(pt,pt) \cong \mathbf{Z}$, and $SK_k(X,X)$ is generated by $[D^k; *]$ where D^k is the k-disk and $*; D^k \to X$ is the constant map. Further $SK_*(X,X) = \bigoplus_{k\geq 0} SK_k(X,X)$ is a free SK_*-module with basis $[D^0; *]$ and $[D^1; *]$.*

Example 1.4. We see that $[D^2] = \alpha$ in $SK_2(pt,pt)$ by the above lemma. This is also obtained by an SK process as follows. Put $N_i = A_i + B_i$, $i = 1, 2$, where $A_1 = D^2$, $B_1 = [a,b] \times S^1$, $A_2 = [-1,1] \times_{\mathbf{Z}_2} S^1$ and $B_2 = [b,c] \times S^1$, $a < b < c$. We further consider $L = L' + L''$ where $L' = S^1 = \partial A_i$ and $L'' = \{b\} \times S^1 \subset B_i$. Let ϕ and $\psi : L \to L$ be identifications:

$$\begin{aligned} \phi &: A_1 \supset L' \to L' \subset A_2, \quad B_1 \supset L'' \to L'' \subset B_2, \\ \psi &: A_1 \supset L' \to L'' \subset B_2, \quad B_1 \supset L'' \to L' \subset A_2. \end{aligned}$$

Then

$$N_1 \cup_\phi N_2 = \mathbf{R}P^2 + [a,c] \times S^1 \text{ and } N_1 \cup_\psi N_2 = D^2 + [-1,1] \times_{\mathbf{Z}_2} S^1.$$

Therefore, we have the equality since $[S^1] = 0$ and $[[-1,1] \times_{\mathbf{Z}_2} S^1]] = [[-1,1]][\mathbf{R}P^1] = 0$ in $SK_*(pt,pt)$, cf. [**4**, Lemma 1.5 (i) and (ii)].

Let $\rho = [H; V]$ be a G-slice type, then a G-vector bundle E over a G-manifold N is said to be of type ρ if the set of points in E having slice type ρ is precisely N. For example, the bundles $\eta : G \times_H V \to G/H$ and

$\widehat{\eta} : G \times_H V \times D^1 \to G/H \times D^1$ are of type ρ. We denote by $SK_*^G[\rho, \rho]$ the SK group resulting from equivariant cuttings and pastings of G-vector bundles of type ρ. Then we have that $SK_*^G[\rho, \rho] \cong SK_*(B\Gamma(\rho), B\Gamma(\rho))$ for some space $B\Gamma(\rho)$, cf. [**7**, Corollaries 1.5.3 and 2.2]. Further, let $(\mathcal{F}, \mathcal{F}_0)$ be a pair of families of G-slice types such that $\mathcal{F} - \mathcal{F}_0 = \{\rho\}$, $\rho = [H; V]$. If M is a G-manifold of type $\mathcal{F}$, then the set M_ρ of all points in $x \in M$ having slice type ρ is a G-submanifold of M, cf. [**4**, p. 37]. We see that a normal bundle $\nu(M_\rho)$ of M_ρ in M is of type ρ. With these notations, we have the following proposition from Lemma 1.3 and [**3**, Theorem 1.12].

Proposition 1.5. *$SK_*^G[\rho, \rho]$ is a free SK_*-module with basis $[\eta]$ and $[\widehat{\eta}]$. Further, the sequence*

$$0 \longrightarrow SK_*^G[\mathcal{F}_0, \mathcal{F}_0] \xrightarrow{i_*} SK_*^G[\mathcal{F}, \mathcal{F}] \xrightarrow{\nu} SK_*^G[\rho, \rho] \longrightarrow 0 \tag{1.5.1}$$

is a split exact sequence, where i_* is induced by the inclusion $\mathcal{F}_0 \subset \mathcal{F}$ and $\nu([M]) = [\nu(M_\rho)]$. A splitting map s to ν is defined by $s([\eta]) = [G \times_H D(V)]$ and $s([\widehat{\eta}]) = [G \times_H D(V \times \mathbf{R})]$.

Let $G = \mathbf{Z}_n$ be the cyclic group of order n for the remainder of this paper. Here $\mathbf{Z}_1 = \{1\}$. The nontrivial irreducible $\mathbf{Z}_{2s}$-modules are $V_0, V_1, \cdots, V_{s-1}$ where $V_0 = \mathbf{R}$ with a generator of $\mathbf{Z}_{2s}$ acting by multiplication by -1, while V_j, $j \geq 1$, is the set $\mathcal{C}$ of complex numbers with a generator of $\mathbf{Z}_{2s}$ acting by multiplication by $\exp(2\pi i j/2s)$, $i = \sqrt{-1}$. On the other hand, the nontrivial irreducible $\mathbf{Z}_{2t+1}$-modules are $V_1, V_2, \cdots, V_t$ where $V_j = \mathcal{C}$ with a generator of $\mathbf{Z}_{2t+1}$ acting by multiplication by $\exp(2\pi i j/2t+1)$. Then the $\mathbf{Z}_n$-slice types are of the forms:

$$\begin{aligned}
\sigma^{2s}(A) &= \sigma(a(0), a(1), \cdots, a(s-1)) \\
&= \Big[\mathbf{Z}_{2s}; \prod_j V_j^{a(j)}\Big], \\
\sigma^{2t+1}(B) &= \sigma(b(1), b(2), \cdots, b(t)) \\
&= \Big[\mathbf{Z}_{2t+1}; \prod_j V_j^{b(j)}\Big],
\end{aligned}$$

where $s \geq 1$ and $t \geq 0$ with $2s|n$, $2t+1|n$, i.e., $2s$, $2t+1$ are divisors of n, and $A = (a(0), \cdots, a(s-1))$, $B = (b(1), \cdots, b(t))$ are sequences of nonnegative integers. If $t = 0$ then by $\sigma^1(B)$ we mean $[1; 0]$ which is also denoted by σ_{-1}, cf. [**7; 4.8.2**].

Let $|\sigma^{2s}(A)| = a(0) + 2\sum_{j\geq 1} a(j)$ or $|\sigma^{2t+1}(B)| = 2\sum_{j\geq 1} b(j)$ be the dimension of $\sigma^{2s}(A)$ or $\sigma^{2t+1}(B)$, respectively. We give a total ordering on the family $St(\mathbf{Z}_n)$ as follows:

(1) $\sigma_{-1} < \sigma^k(A)$ for all $\sigma^k(A)$ with $k \geq 2$.

(2) $\sigma^k(A) < \sigma^l(B)$ if $|\sigma^k(A)| < |\sigma^l(B)|$.

(3) Suppose that $|\sigma^k(A)| = |\sigma^l(B)|$, then $\sigma^k(A) < \sigma^l(B)$ if $k < l$.

(4) Suppose that $|\sigma^k(A)| = |\sigma^l(B)|$ and $k = l$, then $\sigma^k(A) < \sigma^k(B)$ if $V^A = \prod_j V_j^{a(j)} < V^B = \prod_j V_j^{b(j)}$ in the ordering of $\mathbf{Z}_k$-modules induced lexicographically from an ordering in the $\mathbf{Z}_{2s}$-modules: $V_0 < V_1 < \cdots < V_{s-1}$ or the $\mathbf{Z}_{2t+1}$-modules: $V_1 < \cdots < V_t$, cf. [**7**, p. 29].

Definition 1.6. Let V be a $\mathbf{Z}_t$-module and s (≥ 1) be an integer with $s|t$. Then $\overline{V}$ denotes the $\mathbf{Z}_s$-module induced by the natural inclusion $\mathbf{Z}_s \subset \mathbf{Z}_t$. For any slice type $\sigma^t = [\mathbf{Z}_t; \prod_j V_j^{a(j)}]$, we define a slice type $\overline{\sigma}^s = [\mathbf{Z}_s; NT(\prod_j \overline{V}_j^{a(j)})]$ where $NT(-)$ is the nontrivial part of a $\mathbf{Z}_s$-module. Note that $\overline{\sigma}^1 = \sigma_{-1}$.

(1.7) If $\sigma^t = \sigma(a(0), a(1), \cdots, a([(t-1)/2]))$ then $\overline{\sigma}^s = \sigma(b(0), b(1), \cdots, b([(s-1)/2]))$, $s \geq 1$, is expressed as follows (here $a(0)$ or $b(0)$ is excluded if t or s is odd, respectively).

(1.7.1) If $t = s(2q)$, then

$$b(0) = 2\sum_{1\leq k\leq q} a\Big((2k-1)\frac{s}{2}\Big),$$

$$b(i) = a(i) + \sum_{1\leq k\leq q-1} (a(sk-i) + a(sk+i)) + a(sq-i),$$

$$1 \leq i \leq \Big[\frac{s-1}{2}\Big].$$

(1.7.2) If $t = s(2q+1)$, then

$$b(0) = a(0) + 2 \sum_{1 \le k \le q} a\Big((2k-1)\frac{s}{2}\Big),$$
$$b(i) = a(i) + \sum_{1 \le k \le q} (a(sk-i) + a(sk+i)), \quad 1 \le i \le \Big[\frac{s-1}{2}\Big].$$

(1.8) For each case, we define a subset $P(s|t;i)$ of $P(t) = \{0, 1, \cdots, [\frac{t-1}{2}]\}$ if t is even, or $\{1, \cdots, [\frac{t-1}{2}]\}$ if t is odd as follows:

(1.8.1) If $t = s(2q)$, then

$$P(s|t;0) = \Big\{(2k-1)\frac{s}{2}\Big|\ 1 \le k \le q\Big\} \text{ if } s \text{ is even, or } \varnothing \text{ if } s \text{ is odd,}$$
$$P(s|t;i) = \{i\} \cup \{sk-i, sk+i|\ 1 \le k \le q-1\} \cup \{sq-i\},$$
$$1 \le i \le \Big[\frac{s-1}{2}\Big].$$

(1.8.2) If $t = s(2q+1)$, then

$$P(s|t;0) = \{0\} \cup \Big\{(2k-1)\,\frac{s}{2}\Big|\ 1 \le k \le q\Big\} \text{ if } s \text{ is even, or } \varnothing \text{ if } s \text{ is odd,}$$
$$P(s|t;i) = \{i\} \cup \{sk-i, sk+i|\ 1 \le k \le q\}, \quad 1 \le i \le \Big[\frac{s-1}{2}\Big].$$

Further set $P(s|t;-1) = P(t) \setminus \cup_{i \ge 0} P(s|t;i)$ for each case. Then $P(t)$ is a disjoint union of these $P(s|t;i)$, $i \ge -1$. Note that $0 \in P(s|t;0)$ only if (s,t) satisfies (1.7.2), s even.

Then we see that

$$(1.9) \qquad |\sigma^t| - |\overline{\sigma}^s| = \begin{cases} a(0) + 2 \displaystyle\sum_{j \in P(s|t;-1)\setminus\{0\}} a(j) & \text{in case (1.7.1)} \\ 2 \displaystyle\sum_{j \in P(s|t;-1)} a(j) & \text{in case (1.7.2).} \end{cases}$$

Let us consider a $\mathbf{Z}_t$-disk $D(\sigma^t) = D(V)$ for $\sigma^t = [\mathbf{Z}_t; V]$. Since $D(\sigma^t)$ has slice types $\{\overline{\sigma}^s|\ 1 \le s \le t,\ s|t\}$ from Definition 1.6, it is clear that each point $x \in D(\sigma^t)^{\mathbf{Z}_s}$ has a slice type $\overline{\sigma}^u$ for some u with $s \le u \le t$,

$s|u|t$. Further $D(\sigma^t)^{\mathbf{Z}_s} = D(\sigma(a(0)', \cdots, a([\frac{t-1}{2}])'))$ where $a(j)' = a(j)$ when $j \in P(s|t;-1)$ or 0 when $j \notin P(s|t;-1)$ from (1.7) and (1.8). Hence we have the following lemma from (1.9).

Lemma 1.10. *$D(\sigma^t)^{\mathbf{Z}_s}$ is a $\mathbf{Z}_t$-invariant disk $D^{|\sigma^t|-|\overline{\sigma}^s|}$ in $D(\sigma^t)$ with slice types $\{\overline{\sigma}^u; s|u|t\}$.*

We see that $\overline{\sigma^s} < \sigma^t$ since $s < t$ and $|\overline{\sigma}^s| \leq |\sigma^t|$. From now on, we denote the ordering $\overline{\sigma}^s < \sigma^t$ by $\overline{\sigma}^s \prec \sigma^t$ instead in this particular case.

Rename the $\mathbf{Z}_n$-slice types: $\sigma_{-1} = \rho_0 < \rho_1 < \cdots < \rho_m < \rho_{m+1} < \cdots$ by using the ordering on the family $St(\mathbf{Z}_n)$ and let $\mathcal{F}_m$ be defined by $\mathcal{F}_m = \{\rho_m | 0 \leq i \leq m\}$ for each $m \geq 0$. Then the ordering $\prec$ ensures that each $\mathcal{F}_m$ is a family of $\mathbf{Z}_n$-slice types. If m is sufficiently large compared with k, $SK_k^{\mathbf{Z}_n}[\mathcal{F}_m, \mathcal{F}_m] = SK_k^{\mathbf{Z}_n}(pt, pt)$. Hence we have $SK_*^{\mathbf{Z}_n}(pt, pt) = \oplus_{m \geq 0} SK_*^{\mathbf{Z}_n}[\rho_m, \rho_m]$ from the exact sequences (1.5.1) when $(\mathcal{F}, \mathcal{F}_0) = (\mathcal{F}_m, \mathcal{F}_{m-1})$, and obtain the following proposition.

Proposition 1.11 (cf. [**3**, Proposition 1.13]). *$SK_*^{\mathbf{Z}_n}(pt, pt)$ is a free SK_*-module with basis $\mathcal{B} = \{x_{\sigma^s}, \widehat{x}_{\sigma^s} | \sigma^s \in St(\mathbf{Z}_n)\}$ where $x_{\sigma^s} = [\mathbf{Z}_n \times_{\mathbf{Z}_s} D(\sigma^s)]$ and $\widehat{x}_{\sigma^s} = [\mathbf{Z}_n \times_{\mathbf{Z}_s} D(\sigma^s \times \mathbf{R})]$.*

2. $\mathbf{Z}_n$-SK invariants.

Definition 2.1. For any $\mathbf{Z}_n$-manifold M and $\sigma = \sigma^s \in St(\mathbf{Z}_n)$, we define M_σ to be the set consisting of $x \in M$ whose slice type $\sigma_x = \sigma$ or $\sigma_x \succ \sigma$, i.e., $\overline{\sigma}_x^s = \sigma$.

Remark 2.2. Note that if $\sigma^s \neq \sigma'^s$ then $M_{\sigma^s} \cap M_{\sigma'^s} = \varnothing$ and $M^{\mathbf{Z}_s} = \cup_{\sigma^s} M_{\sigma^s}$. We see that each M_{σ^s} is an invariant submanifold of M as follows. For any $x \in M_{\sigma^s}$, let $(\mathbf{Z}_n)_x = \mathbf{Z}_t$ for some $s \leq t \leq n$ with $s|t|n$, and let $U_x = \mathbf{R}^p \oplus V_x$ denote a $\mathbf{Z}_t$-invariant neighborhood of x in M where $\sigma_x = [\mathbf{Z}_t; V_x]$ is the slice type of x. Note that $U_x^{\mathbf{Z}_s} = \mathbf{R}^p \oplus V_x^{\mathbf{Z}_s}$ is a $\mathbf{Z}_s$-neighborhood (chart) of x in $M^{\mathbf{Z}_s}$, cf. [**4**, p. 37] and [**5**, Theorem 4.14]. Here we have $V_x^{\mathbf{Z}_s} = \mathbf{R}^q \times \{0\}$ in V_x where $q = |\sigma_x| - |\sigma^s|$, and every point $y \in V_x^{\mathbf{Z}_s}$ has a slice type σ_y such that $\sigma_y \succeq \sigma^s$ from Lemma 1.10. This implies that $U_x^{\mathbf{Z}_s} \subset M_{\sigma^s}$ and it is also a chart of M_{σ^s} as an invariant submanifold of M. We note that $\dim(M_{\sigma^s}) = p+q = \dim(M) - |\sigma^s|$. It is clear that $\partial(M_{\sigma^s}) = (\partial M)_{\sigma^s}$ by using a $\mathbf{Z}_n$-collar. Further, it follows from the definition that an equivariant SK process on M induces the one on M_{σ^s}.

Remark 2.3.

(i) If $\sigma = \sigma_{-1}$, then $M_{\sigma_{-1}} = M$ since $\sigma_{-1} \preceq \sigma^s$ for all σ^s.

(ii) If $\sigma^s(\mathbf{0}) = \sigma^s(0, \cdots, 0)$, then $M_{\sigma^s(\mathbf{0})}$ is the components of $M^{\mathbf{Z}_s}$ with $\dim M_{\sigma^s(\mathbf{0})} = \dim M - |\sigma^s(\mathbf{0})| = \dim M$.

Example 2.4. Let $M = \mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma)$ for $\sigma = \sigma^t$. Then $M_\tau = \mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma)^{\mathbf{Z}_s}$ if $s|t$, $\tau = \overline{\sigma}^s \preceq \sigma$, or $M_\tau = \varnothing$ otherwise. Therefore, we have that

$$\begin{aligned}
[M_{\overline{\sigma}}] &= \frac{n}{t}\,[D^{|\sigma|-|\overline{\sigma}|}] \\
&= \begin{cases} \dfrac{n}{t}\alpha^{(|\sigma|-|\overline{\sigma}|)/2}[D^0] & \text{if } |\sigma| \equiv |\overline{\sigma}| \pmod 2 \\ \dfrac{n}{t}\alpha^{(|\sigma|-|\overline{\sigma}|-1)/2}[D^1] & \text{if } |\sigma| \equiv |\overline{\sigma}|+1 \pmod 2 \end{cases}
\end{aligned}$$

in $SK_*(pt, pt)$ from Example 1.4 and Lemma 1.10.

We now define a $\mathbf{Z}_n$-SK invariant.

Definition 2.5. Let T be a map for k-dimensional $\mathbf{Z}_n$-manifolds, which is assumed to take values in $\mathbf{Z}$ and to be additive with respect to disjoint union $+$, that is, if $M = M_1 + M_2$ then $T(M) = T(M_1) + T(M_2)$. We call T a $\mathbf{Z}_n$-SK invariant if $T(N_1 \cup_\phi N_2) = T(N_1 \cup_\psi N_2)$ for any $\mathbf{Z}_n$-diffeomorphisms $\phi, \psi : L \to L$ in Definition 1.1. Thus T induces an additive homomorphism $T : SK_k^{\mathbf{Z}_n}(pt, pt) \to \mathbf{Z}$. The set $\mathcal{I}_k^{\mathbf{Z}_n}$ of all these invariants T is a $\mathbf{Z}$-module under the natural addition.

Example 2.6. For each $\sigma \in St(\mathbf{Z}_n)$, the map $\overline{\chi}_\sigma$ defined by $\overline{\chi}_\sigma(M) = \overline{\chi}(M_\sigma)$ is invariant, cf. Remark 2.2. On the other hand, $\overline{\chi}^{\mathbf{Z}_s}$ mentioned in the introduction is also invariant for each s with $s|n$. We note that $\overline{\chi}_{\sigma_{-1}} = \overline{\chi}^{\mathbf{Z}_1} = \overline{\chi}$, cf. Remark 2.3 (i). By considering χ instead of $\overline{\chi}$, we have other invariants χ_σ and $\chi^{\mathbf{Z}_s}$.

Let $\mathcal{S}^s$ be the set consisting of all $\sigma^s \in St(\mathbf{Z}_n)$.

Theorem 2.7. *For each $k \geq 0$, the class $\{\theta_\sigma\}$:*

$$\theta_\sigma = \frac{s}{n}\Big(\overline{\chi}_\sigma + \sum_{\tau \in \mathcal{S}^t, \sigma \prec \tau} \varphi\Big(\frac{t}{s}\Big)(-1)^{|\tau|-|\sigma|}\overline{\chi}_\tau\Big)$$

is a basis of $\mathcal{I}_k^{\mathbf{Z}_n}$ as a free $\mathbf{Z}$-module, where $\sigma \in \mathcal{S}^s$, $s \geq 1$, $s|n$, with $|\sigma| \leq k$ and φ is the Euler phi-function.

We have obtained the result when $n = 2$ or 4, cf. [**3**, Propositions 2.3 and 2.4].

Proof. For each $\sigma \in St(\mathbf{Z}_n)$, let $g_\sigma : SK_*^{\mathbf{Z}_n}(pt, pt) \to SK_{*-|\sigma|}(pt, pt)$ be a map given by $g_\sigma([M]) = [M_\sigma]$ and let us define a map f_σ by

$$f_\sigma = \frac{s}{n}\Big(g_\sigma + \sum_{\tau \in \mathcal{S}^t, \sigma \prec \tau} \varphi\Big(\frac{t}{s}\Big)[D^{|\tau|-|\sigma|}]g_\tau\Big). \tag{2.7.1}$$

We now recall the basis elements in Proposition 1.11. The values $f_{\sigma'}(x_\sigma)$ which do not vanish are as follows:

$$\begin{cases} f_{\sigma_{-1}} = [D^0] & \text{on } x_{\sigma_{-1}} = [\mathbf{Z}_n], \\ f_{\overline{\sigma}^i} = \alpha^{(|\sigma^s|-|\overline{\sigma}^i|)/2}[D^0], & \\ f_{\overline{\sigma}^j} = \alpha^{(|\sigma^s|-|\overline{\sigma}^j|-1)/2}[D^1], & \\ f_{\sigma^s} = [D^0] & \text{on } x_{\sigma^s}, \end{cases} \tag{2.7.2}$$

where $\overline{\sigma}^i \prec \sigma^s$, $\overline{\sigma}^j \prec \sigma^s$ with $1 \leq i, j \leq s$, $i|s$, $j|s$ and $|\overline{\sigma}^i| \equiv |\overline{\sigma}^j| + 1 \equiv |\sigma^s| \pmod 2$. Note that $\overline{\sigma}^1 = \sigma_{-1}$ and $|\sigma_{-1}| = 0$. For example,

$$\begin{aligned} f_{\overline{\sigma}^i}(x_{\sigma^s}) &= \frac{i}{n}\Big(g_{\overline{\sigma}^i}(x_{\sigma^s}) + \sum_{i|u|s, i<u} \varphi\Big(\frac{u}{i}\Big)[D^{|\overline{\sigma}^u|-|\overline{\sigma}^i|}]g_{\overline{\sigma}^u}(x_{\sigma^s})\Big) \\ &= \frac{i}{s}\Big(\sum_{i|u|s} \varphi\Big(\frac{u}{i}\Big)\Big)[D^{|\sigma^s|-|\overline{\sigma}^i|}] \\ &= \alpha^{(|\sigma^s|-|\overline{\sigma}^i|)/2}[D^0] \end{aligned}$$

from Example 2.4 and the equality $\sum_{i|u|s} \varphi(u/i) = s/i$. The values on the elements $\widehat{x}_\sigma$ are given by $f_{\sigma'}(\widehat{x}_\sigma) = [D^1] f_{\sigma'}(x_\sigma)$. Thus $\{f_\sigma\}$ are maps

$$f_\sigma : SK_*^{\mathbf{Z}_n}(pt, pt) \to SK_{*-|\sigma|}(pt, pt) \tag{2.7.3}$$

of degree $-|\sigma|$ from (2.7.2), which give an SK_*-homomorphism

$$f_* = \oplus_m f_{\rho_m} : SK_*^{\mathbf{Z}_n}(pt, pt) \to L = \oplus_m SK_{*-|\rho_m|}(pt, pt), \tag{2.7.4}$$

where $St(\mathbf{Z}_n) = \{\rho_m | m \geq 0\}$. Let $\mathcal{C}_m = \{[D^0], [D^1]\}$ be the basis of mth copy of $SK_*(pt, pt)$ in L. Then we can give a total ordering of the basis $\mathcal{B} = \{x_{\rho_m}, \widehat{x}_{\rho_m} | m \geq 0\}$ or $\mathcal{B}' = \cup_{m \geq 0} \mathcal{C}_m$ of L naturally by using that of $St(\mathbf{Z}_n)$, $x_{\rho_m} < \widehat{x}_{\rho_m}$ and $[D^0] < [D^1]$. From (2.7.2), we see that f_* is an isomorphism since the matrix relative to the ordered basis $\mathcal{B}$ and $\mathcal{B}'$ is triangular with components 1 on the diagonal. Therefore, from Lemma 1.3, any T is factorized as

$$T : SK_k^{\mathbf{Z}_n}(pt, pt) \overset{f_*}{\cong} \oplus_m SK_{k-|\rho_m|}(pt, pt) \overset{\oplus_m \overline{\chi}}{\cong} \oplus_m \mathbf{Z} \xrightarrow{T'} \mathbf{Z} \tag{2.7.5}$$

for some T', where the direct sum is taken over all $\rho_m \in St(\mathbf{Z}_n)$ such that $|\rho_m| \leq k$. Hence we have $T = \sum_m T'(1_m)\theta_{\rho_m}$ where $(\oplus_m \overline{\chi}) \circ f_* = \oplus \theta_{\rho_m}$ and $1_m = 1$ in the copy of $\mathbf{Z}$ in $\oplus_m \mathbf{Z}$ corresponding to ρ_m. This completes the proof. $\square$

Corollary 2.8. *Let M be a k-dimensional $\mathbf{Z}_n$-manifold such that $\overline{\chi}_\sigma(M) = 0$ (or $\chi_\sigma(M) = 0$) for all $\sigma \in St(\mathbf{Z}_n)$ with $|\sigma| \leq k$. Then $T(M) = 0$ for any $\mathbf{Z}_n$-SK invariant.*

By using the isomorphism $\oplus \theta_{\rho_m}$ in (2.7.5), we have the following propositions.

Proposition 2.9. *Let $[M_1]$ and $[M_2]$ be elements in $SK_k^{\mathbf{Z}_n}(pt, pt)$. Then $[M_1] = [M_2]$ if and only if $\overline{\chi}_\sigma(M_1) = \overline{\chi}_\sigma(M_2)$ (or $\chi_\sigma(M_1) = \chi_\sigma(M_2)$) for all $\sigma \in St(\mathbf{Z}_n)$ with $|\sigma| \leq k$.*

Proposition 2.10. *For each integer q (≥ 2), let $\mathcal{K}(q)$ be a submodule of $\mathcal{I}_*^{\mathbf{Z}_n} = \sum_k \mathcal{I}_k^{\mathbf{Z}_n}$ consisting of invariants T such that $T(M) \equiv 0$*

(mod q) *for all* $\mathbf{Z}_n$*-manifolds* M. *Then* $\mathcal{K}(q)$ *is generated by the class* $\{q\theta_\sigma | \sigma \in St(\mathbf{Z}_n)\}$.

Example 2.11. (i) Divide $\mathcal{S}^t$ as $\mathcal{S}^t = \mathcal{S}_1^t \cup \mathcal{S}_2^t$ where $\mathcal{S}_1^t$ consists of all $\sigma^t = \sigma(a(0), a(1), \cdots)$ with $a(0)$; even and $\mathcal{S}_2^t$ consists of $\sigma_*^t = \sigma(a(0)+1, a(1), \cdots)$ for any $\sigma^t \in \mathcal{S}_1^t$. Then, if $\sigma = \sigma^s \in \mathcal{S}_1^s$ then

$$\begin{aligned} \frac{n}{s}\,\theta_\sigma &= \overline{\chi}_\sigma + \sum_{s<t\le n, s|t|n} \varphi\Big(\frac{t}{s}\Big)\Big(\sum_{\tau\in\mathcal{S}_1^t, \sigma\prec\tau} \overline{\chi}_\tau - \sum_{\tau_*\in\mathcal{S}_2^t, \sigma\prec\tau_*} \overline{\chi}_{\tau_*}\Big) \\ &\equiv 0 \ \Big(\mathrm{mod}\ \frac{n}{s}\Big) \end{aligned} \tag{2.11.1}$$

from Theorem 2.7 since $|\sigma| \equiv |\tau| \equiv 0$, $|\tau_*| \equiv 1 \pmod 2$. Here we note that $\overline{\chi}(M_\tau) = (-1)^{\dim M - |\tau|}\chi(M_\tau)$ for any $\tau \in \mathcal{S}^t$ and $\mathbf{Z}_n$-manifolds M, cf. Lemma 1.3 and Remark 2.2. Hence $\overline{\chi}_\tau(M) = (-1)^{\dim M}\chi_\tau(M)$ if $\tau \in \mathcal{S}_1^t$ and $\overline{\chi}_{\tau_*}(M) = (-1)^{\dim M+1}\chi_{\tau_*}(M)$. Therefore, we have

$$\begin{aligned} \chi_\sigma + \sum_{s<t\le n, s|t|n} \varphi\Big(\frac{t}{s}\Big)\Big(\sum_{\tau\in\mathcal{S}_1^t, \sigma\prec\tau} \chi_\tau + \sum_{\tau_*\in\mathcal{S}_2^t, \sigma\prec\tau_*} \chi_{\tau_*}\Big) \\ \equiv 0 \ \Big(\mathrm{mod}\ \frac{n}{s}\Big) \end{aligned} \tag{2.11.2}$$

from 2.11.1. Now let $M_{\mathrm{ev}}^{\mathbf{Z}_s}$ (or $M_{\mathrm{od}}^{\mathbf{Z}_s}$) be a union of components of $M^{\mathbf{Z}_s}$ whose codimension are even (or odd) in M, respectively, and define $\chi_*^{\mathbf{Z}_s}(M) = \chi(M_*^{\mathbf{Z}_s})$ where $* = \mathrm{ev}$ or od. Since $M_{\mathrm{ev}}^{\mathbf{Z}_s} = \sum_{\sigma\in\mathcal{S}_1^s} M_\sigma$ and $M^{\mathbf{Z}_t} = \sum_{\tau\in\mathcal{S}^t} M_\tau$ in general, we have

$$\chi_{\mathrm{ev}}^{\mathbf{Z}_s} + \sum_{s<t\le n, s|t|n} \varphi\Big(\frac{t}{s}\Big)\chi^{\mathbf{Z}_t} \equiv 0 \ \Big(\mathrm{mod}\ \frac{n}{s}\Big). \tag{2.11.3}$$

from (2.11.2). In particular, if $s = 1$ then

$$\chi + \sum_{1<t\le n, t|n} \varphi(t)\chi^{\mathbf{Z}_t} \equiv 0 \pmod n. \tag{2.11.4}$$

(ii) Take $n = 2^r$ for example. Then $2^{r-t}\theta_{\tau_*} = \overline{\chi}_{\tau_*} \equiv 0 \pmod{2^{r-t}}$, that is, $\chi_{\tau_*} \equiv 0 \pmod{2^{r-t}}$ for each $\tau_* \in \mathcal{S}_2^{2^t}$ since there is no slice type ν such that $\tau_* \prec \nu$, cf. (1.7.1). Thus $\chi_{\mathrm{od}}^{\mathbf{Z}_{2^t}} \equiv 0 \pmod{2^{r-t}}$, and

$$\chi_{\mathrm{ev}}^{\mathbf{Z}_{2^s}} + \sum_{s<t\le r} 2^{t-s-1}\,\chi_{\mathrm{ev}}^{\mathbf{Z}_{2^t}} \equiv 0 \pmod{2^{r-s-1}} \tag{2.11.5}$$

from (2.11.3) since $\varphi(2^j) = 2^{j-1}$ in general and $2^{t-s-1}\chi_{\mathrm{od}}^{\mathbf{Z}_{2^t}} \equiv 0 \pmod{2^{r-s-1}}$.

Let $M \times N$ be the cartesian product of $\mathbf{Z}_n$-manifolds M and N straightening the angle, then it gives a multiplication on $SK_*^{\mathbf{Z}_n}(pt, pt)$ naturally.

Proposition 2.12. *The multiplicative relations on the basis elements in $\mathcal{B}$ are given by the following* (i) *and* (ii):

$$\text{(i)}\quad [\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t)] \cdot [\mathbf{Z}_n \times_{\mathbf{Z}_u} D(\tau^u)] = \frac{nv}{tu}\,[D^{(|\sigma^t|-|\overline{\sigma}^v|)+(|\tau^u|-|\overline{\tau}^v|)}][\mathbf{Z}_n \times_{\mathbf{Z}_v} D(\overline{\sigma}^v \times \overline{\tau}^v)],$$

where $v = (t, u)$ is the greatest common divisor of t and u, and $\overline{\sigma}^v \times \overline{\tau}^v$ is a product of the induced $\mathbf{Z}_v$-modules $\overline{\sigma}^v$ and $\overline{\tau}^v$, i.e., if $\overline{\sigma}^v = \sigma(\cdots, a(i), \cdots)$ and $\overline{\tau}^v = \sigma(\cdots, b(i), \cdots)$, then $\overline{\sigma}^v \times \overline{\tau}^v = \sigma(\cdots, a(i) + b(i), \cdots)$, and

(ii) $\widehat{y} \cdot z = y \cdot \widehat{z} = \widehat{y \cdot z}$ and $\widehat{(\widehat{y})} = \alpha y$ for any y, z,

where $\widehat{y} = [D^1]y$ in general.

Proof. To show (i), let $\{\overline{\sigma}^i\}$ or $\{\overline{\tau}^j\}$ be the class in Definition 1.6 for σ^t or τ^u, respectively. Then $\overline{\sigma}^v \times \overline{\tau}^v$ gives the class $\{\overline{\sigma}^i \times \overline{\tau}^i; i|v\}$, and each side of (i) has

$$\begin{aligned}(2.12.1)\quad \overline{\chi}_{\overline{\sigma}^i \times \overline{\tau}^i} &= \frac{n}{t}\,(-1)^{|\sigma^t|-|\overline{\sigma}^i|} \cdot \frac{n}{u}\,(-1)^{|\tau^u|-|\overline{\tau}^i|} \\ &= \frac{nv}{tu}\,(-1)^{(|\sigma^t|-|\overline{\sigma}^v|)+(|\tau^u|-|\overline{\tau}^v|)} \cdot \frac{n}{v}\,(-1)^{|\overline{\sigma}^v \times \overline{\tau}^v|-|\overline{\sigma}^i \times \overline{\tau}^i|}\end{aligned}$$

and $\overline{\chi}_\nu = 0$ if $\nu \notin \{\overline{\sigma}^i \times \overline{\tau}^i; i|v\}$ by Example 2.4. This implies (i). The equalities (ii) are clear. □

3. Multiplicative invariants.

Definition 3.1. Let T be a $\mathbf{Z}_n$-SK invariant for all $\mathbf{Z}_n$-manifolds. Then T is said to be multiplicative if $T(M \times N) = T(M)T(N)$ for any $\mathbf{Z}_n$-manifolds M and N.

We see that $T(D^0)=0$ or 1 since $T(D^0)^2=T(D^0)$, and T is trivial, i.e., $T\equiv 0$ if $T(D^0)=0$. From now on, we treat the case $T\not\equiv 0$, hence any T has the value $T(D^0)=1$.

Definition 3.2. Let T be an invariant which is not necessarily multiplicative. We define an invariant $T_{(k)}$ by $T_{(k)}(M)=T(M)$ if $k=\dim(M)$ and $T_{(k)}(M)=0$ if $k\neq\dim(M)$.

We first consider the case $\mathbf{Z}_1=\{1\}$.

Proposition 3.3. *Any multiplicative invariant* $T_0: SK_*(pt,pt)\to\mathbf{Z}$ *has a form* $T_0=\sum_{k\geq 0}(-a)^k\overline{\chi}_{(k)}$ *where* $a=T_0(D^1)$. *Here, if* $a=0$, *then we regard* a^0 *as* 1.

Proof. It is easy to check that these T_0 are multiplicative. Let T_0 be any multiplicative invariant, then we can write $T_0=\sum_{k\geq 0}r_k\overline{\chi}_{(k)}$ for some $r_k\in\mathbf{Z}$ from Lemma 1.3. Regarding r_1^0 as 1 if $r_1=0$, we have that $r_k=r_1^k$, $k\geq 0$, since $T_0(D^k)=T_0(D^1)^k$ and $r_0=T_0(D^0)=1$. Taking $a=-r_1$, we have the desired form. □

Remark 3.4. If $a=0$, then $T_0=(-0)^0\overline{\chi}_{(0)}=\overline{\chi}_{(0)}(=\chi_{(0)})$. If $a=1$, then $T_0=\sum_{k\geq 0}(-1)^k\overline{\chi}_{(k)}=\sum_{k\geq 0}\chi_{(k)}=\chi$ from Lemma 1.3. On the other hand, if $a=-1$, then $T_0=\sum_{k\geq 0}\overline{\chi}_{(k)}=\overline{\chi}$.

We now study the invariants on $\mathbf{Z}_n$-manifolds with $n\geq 2$.

Definition 3.5. Let s be an integer with $1\leq s\leq n$ and $s|n$. We say that a multiplicative invariant T is of type (s) if $T(\mathbf{Z}_n\times_{\mathbf{Z}_t}D(\sigma^t(\mathbf{0})))=0$ for each t with $1\leq t<s$, $t|n$ and $T(\mathbf{Z}_n\times_{\mathbf{Z}_s}D(\sigma^s(\mathbf{0})))=\beta\neq 0$, where $\sigma^1(\mathbf{0})=\sigma_{-1}$.

Lemma 3.6. *If* T *is of type* (s), *then*

(i) $T(\mathbf{Z}_n\times_{\mathbf{Z}_t}D(\sigma^t))=0$ *for any* σ^t *with* $t<s$ *or* $s<t$, $s\nmid t$,

(ii) $T(\mathbf{Z}_n\times_{\mathbf{Z}_t}D(\sigma^t(\mathbf{0})))=\frac{n}{t}$ *for* $s\leq t\leq n$ *with* $s|t|n$. *In particular,* $\beta=\frac{n}{s}$.

Proof. To prove (i), put $p_t = T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t))$. If $t < s$, applying T to Proposition 2.12 (i) when $\sigma = \sigma^t$, $\tau = \sigma^t(\mathbf{0})$, we have $p_t = \frac{n}{t}^{-1} p_t T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{0}))) = 0$ from Definition 3.5. Next suppose that $s < t < n$ and $s \nmid t$. Then we also have $p_t = 0$ by using the same equality when $\sigma = \sigma^t$, $\tau = \sigma^s(\mathbf{0})$ since $T(\mathbf{Z}_n \times_{\mathbf{Z}_v} D(\overline{\sigma}^v)) = 0$, $v = (s,t) < s$ from the above and $\beta \neq 0$. Thus (i) follows. Similarly we have $T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{0})))\beta = \frac{n}{t}\beta$ for $s \leq t \leq n$ with $s|t|n$, which gives (ii). $\square$

Proposition 3.7. *If T is of type* (1)*, then $T(M) = T_0(M_0)$ for any $\mathbf{Z}_n$-manifold M where $M_0 = M$ with ignoring group action.*

Proof. By Proposition 2.12, we see that $[M] \cdot [\mathbf{Z}_n] = [M_0] \cdot [\mathbf{Z}_n]$ in $SK_*^{\mathbf{Z}_n}(pt, pt)$. In fact, there is an equivariant diffeomorphism $f : M \times \mathbf{Z}_n \to M_0 \times \mathbf{Z}_n$ given by $f(x,g) = (g^{-1}x, g)$ for $(x,g) \in M \times \mathbf{Z}_n$. From this, we have $T(M)T(\mathbf{Z}_n) = T(M_0)T(\mathbf{Z}_n)$ and $T(M) = T_0(M_0)$ since $T(\mathbf{Z}_n) \neq 0$. $\square$

We next consider an invariant T of type (s) with $s > 1$.

Lemma 3.8. *For each t with $s|t|n$, let $P(t) = \cup_i P(s|t;i)$, $-1 \leq i \leq [\frac{s-1}{2}]$ be the partition in* (1.8)*, and let $\overline{\sigma}^s$ be the slice type for $\sigma^t = \sigma^t(\mathbf{e}_j)$ where $\mathbf{e}_j = \sigma(a(0), a(1), \cdots)$ such that $a(j) = 1$ or zero otherwise. Then we have that $\overline{\sigma}^s = \sigma^s(\mathbf{0})$ if $j \in P(s|t;-1)$, $\sigma^s(\mathbf{e}_0)$ if $j = 0 \in P(s|t;0)$ in case* (1.7.2)*, s; even, $\sigma^s(2,0,\cdots,0)$ if $j \in P(s|t;0)$, $j \geq 1$ in general, and $\sigma^s(\mathbf{e}_i)$ if $j \in P(s|t;i)$ with $i \geq 1$. Further, if T is of type (s), then*

(i-1) *in case* (1.7.1) *or* (1.7.2)*, s; odd,*

$$T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{e}_j))) = \begin{cases} \dfrac{n}{t}\, a & \text{if } j = 0 \in P(s|t;-1), \\ \dfrac{n}{t}\, a^2 & \text{if } j \in P(s|t;-1) \backslash \{0\}, \\ \dfrac{s^2}{tn}\, \xi_0^2 & \text{if } j \in P(s|t;0) \ (\text{in case } (1.7.1),\ s;\ \text{even}), \\ \dfrac{s}{t}\, \xi_i & \text{if } j \in P(s|t;i),\ i \geq 1, \end{cases}$$

and

(i-2) *in case* (1.7.2), *s; even,*

$$T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{e}_j))) = \begin{cases} \dfrac{n}{t}\, a^2 & \text{if } j \in P(s|t;-1), \\ \dfrac{s}{t}\, \xi_0 & \text{if } j = 0 \in P(s|t;0), \\ \dfrac{s^2}{tn}\, \xi_0^2 & \text{if } j \in P(s|t;0)\backslash\{0\}, \\ \dfrac{s}{t}\, \xi_i & \text{if } j \in P(s|t;i),\ i \geq 1 \end{cases}$$

where $a = T(D^1)$ *and* $\xi_i = T(\mathbf{Z}_n \times_{\mathbf{Z}_s} D(\sigma^s(\mathbf{e}_i)))$ *for each* i.

(ii) $\xi_i = \frac{n}{s}\,\gamma_i$ *for some* $\gamma_i \in \mathbf{Z}$.

Proof. We write $\overline{\sigma}^s$ as $\overline{\sigma}^s = \sigma(b(0), b(1), \cdots)$ for $\sigma^t = \sigma^t(\mathbf{e}_j)$. Suppose that $j \in P(s|t;i)$ with $i \geq 1$. Then $b(i) = 1$ and $b(p) = 0$, $p \neq i$, from (1.7) and (1.8), which means $\overline{\sigma}^s = \sigma^s(\mathbf{e}_i)$. In the same way, if $j \in P(s|t;0)$ then $\overline{\sigma}^s = \sigma^s(2, 0, \cdots, 0)$ when $j \geq 1$, and $\sigma^s(\mathbf{e}_0)$ when $j = 0 \in P(s|t;0)$ in case (1.7.2), s; even. Finally we see that $\overline{\sigma}^s = \sigma^s(\mathbf{0})$ if and only if $j \in P(s|t;-1)$. We next prove the parts (i) and (ii). Put $\lambda_j = T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{e}_j)))$. In case $j \in P(s|t;i)$ with $i \geq 0$, apply T to Proposition 2.12 (i) when $\sigma = \sigma^t(\mathbf{e}_j)$ and $\tau = \sigma^s(\mathbf{0})$. Then we have $\lambda_j = \frac{s}{t}\, T(\mathbf{Z}_n \times_{\mathbf{Z}_s} D(\overline{\sigma}^s))$ since $|\sigma^t(\mathbf{e}_j)| = |\overline{\sigma}^s|$ ($= 1$ or 2) from the above result and the facts that $T(D^0) = 1$ and $\beta = \frac{n}{s}$, cf. Lemma 3.6 (ii). This implies the result when $i \geq 1$ in each case or $(i, j) = (0, 0)$ in case (1.7.2), s; even. If $i = 0$ and $j \geq 1$, substituting the equality $T(\mathbf{Z}_n \times_{\mathbf{Z}_s} D(\sigma^s(2, 0, \cdots, 0))) = \frac{s}{n}\,\xi_0^2$ into λ_j, we have $\lambda_j = \frac{s^2}{tn}\,\xi_0^2$. On the other hand, suppose that $j \in P(s|t;-1)$. Then we have $\lambda_j = \frac{n}{t}\, T(D^{|\sigma^t(\mathbf{e}_j)| - |\overline{\sigma}^s|}) = \frac{n}{t}\, a^{|\sigma^t(\mathbf{e}_j)|}$ similarly since $\overline{\sigma}^s = \sigma^s(\mathbf{0})$, from which the result follows. Thus we obtain (i). Consider the case $t = n$ in particular. Set $\gamma_i = T(D(\sigma^n(\mathbf{e}_j)))$ with $j \in P(s|n;i)$ if $i \geq 1$. When $i = 0$, we also set $\gamma_0 = T(D(\sigma^n(\mathbf{e}_0)))$ in case (1.7.2), s; even, while in case (1.7.1), s; even we set $\gamma_0 = \frac{s}{n}\,\xi_0$. Then γ_0 is an integer since so is $T(D(\sigma^n(\mathbf{e}_j))) = \gamma_0^2$ with $j \in P(s|n;0)$. These imply (ii). □

Lemma 3.9. $T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t)) = \frac{n}{t}\, a^{|\sigma^t| - |\overline{\sigma}^s|}\gamma_{\overline{\sigma}^s}$ *for any slice type* σ^t *with* $s|t|n$, *where* $\gamma_{\overline{\sigma}^s} = \prod_i \gamma_i^{b(i)}$ *if* $\overline{\sigma}^s = \sigma(b(0), b(1), \cdots)$. *If* a *or* $\gamma_i = 0$ *for some* i, *then we regard* a^0 *or* γ_i^0 *as* 1, *respectively.*

Proof. Write σ^t as $\sigma^t = \sigma(a(0), a(1), \cdots, a([\frac{t-1}{2}]))$ and set $\lambda_j = T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{e}_j)))$ again. First we have that

$$T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(0, \cdots, 0, a(j), 0, \cdots, 0))) = \left(\frac{t}{n}\right)^{a(j)-1} \lambda_j^{a(j)}, \quad a(j) \geq 0 \tag{3.9.1}$$

by using Proposition 2.12 (i) when $\sigma = \sigma^t(\mathbf{e}_j)$ and $\tau = \sigma^t(0, \cdots, 0, a(j), 0, \cdots, 0)$ inductively. Then

$$T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t)) = \left(\frac{t}{n}\right)^{l(\sigma)-1} \prod_{j \geq -1} \prod_{j \in P_i} \lambda_j^{a(j)} \tag{3.9.2}$$

where $l(\sigma) = \sum_i a(i)$ and $P_i = P(s|t; i)$ are the subsets of $P(t)$ in Lemma 3.8. Set $L_i = \prod_{j \in P_i} \lambda_j^{a(j)}$ and $L_* = \prod_{i \geq 1} L_i$. Then, in case (i-1) in Lemma 3.8,

$$\begin{aligned} L_{-1} &= \left(\frac{n}{t}\, a\right)^{a(0)} \prod_{j \in P_{-1} \setminus \{0\}} \left(\frac{n}{t}\, a^2\right)^{a(j)} = \left(\frac{n}{t}\right)^{l_{-1}} a^{|\sigma| - |\overline{\sigma}|}, \\ L_0 &= \left(\frac{s^2}{tn}\, \xi_0^2\right)^{l_0} = \left(\frac{n}{t}\right)^{l_0} \gamma_0^{2l_0} \quad \text{(in case (1.7.1), } s\text{; even)}, \\ L_* &= \left(\frac{s}{t}\right)^{l_*} \prod_{i \geq 1} \xi_i^{l_i} = \left(\frac{n}{t}\right)^{l_*} \prod_{i \geq 1} \gamma_i^{l_i} \end{aligned} \tag{3.9.3}$$

from (1.9) and Lemma 3.8 (ii), where $l_i = \sum_{j \in P_i} a(j)$ and $l_* = \sum_{i \geq 1} l_i$. Substitute these into (3.9.2). Then we obtain the desired formula by using the equalities $l(\sigma) = l_{-1} + l_0 + l_*$, $b(0) = 2l_0$ and $b(i) = l_i$, $i \geq 1$, from (1.7) and (1.8). (Remark: If s is odd, then $b(0)$ does not exist. In this case, we regard l_0 as 0, i.e., $L_0 = 1$.) Note that λ_j may be zero. Hence we must regard λ_j^0 as 1 if $\lambda_j = 0$ in (3.9.1) since $T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{0}))) = \frac{n}{t}$, cf. Lemma 3.6 (ii); in other words, a^0 or γ_j^0 is regarded as 1 if a or $\gamma_j = 0$, respectively. We can prove the lemma in case (i-2) similarly, so we omit it here. $\square$

Theorem 3.10. *Let T be any multiplicative invariant of type (s) with $s > 1$, and let $\{a, \gamma_i\}$ be the class of integers in Lemma 3.8. Then T has the form*

$$T = \sum_{k,\sigma} (-a)^k \gamma_\sigma \overline{X}_{\sigma,(k+|\sigma|)},$$

where the sum is taken over all slice types $\sigma = \sigma^s \in \mathcal{S}^s$ and $k \geq 0$, and $\overline{\chi}_{\sigma,(j)} = (\overline{\chi}_\sigma)_{(j)}$ is the invariant defined in Definition 3.2. *Further, if a or $\gamma_i = 0$ for some i, then we regard a^0 or γ_i^0 as* 1, *respectively.*

Remark. We may consider that $\overline{\chi}_{\sigma,(j)}$ is defined for $j \geq |\sigma|$ since $\overline{\chi}_\sigma(M) = 0$ if $\dim(M) < |\sigma|$. Hence we write $j = k + |\sigma|$ with $k \geq 0$.

Proof of the theorem. First we see that the above T is multiplicative since so it is for the basis elements of $SK_*^{\mathbf{Z}_n}(pt, pt)$, cf. Proposition 1.11. To show this, set $M = \mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t)$ and $N = \mathbf{Z}_n \times_{\mathbf{Z}_u} D(\tau^u)$ for example. If $s|t$ and $s|u$ then $\overline{\sigma}^s \times \overline{\tau}^s$ is a slice type of $M \times N$, and

$$\overline{\chi}_{\overline{\sigma}\times\overline{\tau},(|\sigma|+|\tau|)}(M \times N) = \overline{\chi}_{\overline{\sigma},(|\sigma|)}(M) \cdot \overline{\chi}_{\overline{\tau},(|\tau|)}(N) \tag{3.10.1}$$

from the first equality in (2.12.1) when $i = s$, where $\overline{\sigma} = \overline{\sigma}^s$ and $\overline{\tau} = \overline{\tau}^s$. Hence

$$\begin{aligned} T(M \times N) &= (-a)^k \gamma_{\overline{\sigma}\times\overline{\tau}} \overline{\chi}_{\overline{\sigma}\times\overline{\tau},(|\sigma|+|\tau|)}(M \times N) \\ &= (-a)^{k'} \gamma_{\overline{\sigma}} \overline{\chi}_{\overline{\sigma},(|\sigma|)}(M) \cdot (-a)^{k''} \gamma_{\overline{\tau}} \overline{\chi}_{\overline{\tau},(|\tau|)}(N) \\ &= T(M)T(N) \end{aligned} \tag{3.10.2}$$

by the equality $\gamma_{\overline{\sigma}}\gamma_{\overline{\tau}} = \gamma_{\overline{\sigma}\times\overline{\tau}}$, where $k = |\sigma|+|\tau|-|\overline{\sigma}\times\overline{\tau}|$ and $k = k'+k''$, $k' = |\sigma| - |\overline{\sigma}|$, $k'' = |\tau| - |\overline{\tau}|$. If $s \nmid t$ or $s \nmid u$, then each side of (3.10.1) is zero. It is easy to see that the above T takes values a and $\frac{n}{s}\gamma_i$ as follows: $T(D^n) = (-a)^n \gamma_{\sigma^s(\mathbf{0})} \overline{\chi}_{\sigma^s(\mathbf{0}),(n)}(D^n) = a^n$ since $\overline{\chi}_{\sigma^s(\mathbf{0})}(D^n) = \overline{\chi}(D^n) = (-1)^n$ from Remark 2.3 (ii) and $\gamma_{\sigma^0(\mathbf{0})} = \prod_i \gamma_i^0 = 1$. In the same way, we have $T(\mathbf{Z}_n \times_{\mathbf{Z}_s} D(\sigma^s(\mathbf{e}_i))) = \frac{n}{s}\gamma_i$. Further we also see that $T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{0}))) = \frac{n}{s}$ if $t = s$ or zero, if $t < s$, $t|n$. Thus, T is of type (s). Now let T be any multiplicative invariant of type (s) with $s > 1$. From Theorem 2.7, we can write T as

$$T = \sum_{k,\sigma} a_{\sigma,(k+|\sigma|)} \theta_{\sigma,(k+|\sigma|)}, \tag{3.10.3}$$

where $\sigma \in St(\mathbf{Z}_n)$, $k \geq 0$ and $a_{\sigma,(k+|\sigma|)} \in \mathbf{Z}$. By assumption $s > 1$, we have $T(\mathbf{Z}_n) = 0$ and

$$\begin{aligned} 0 &= T(D^k)T(\mathbf{Z}_n) = T(D^k \times \mathbf{Z}_n) \\ &= a_{\sigma_{-1},(k)} \theta_{\sigma_{-1},(k)}(D^k \times \mathbf{Z}_n) = a_{\sigma_{-1},(k)} \cdot (-1)^k \end{aligned} \tag{3.10.4}$$

since $\theta_{\sigma_{-1}}(\mathbf{Z}_n) = 1$. Hence $a_{\sigma_{-1},(k)} = 0$ for any $k \geq 0$. Next we suppose that $a_{\tau,(k+|\tau|)} = 0$ for any $\tau = \tau^u$ with $1 \leq u < t < s$ and $k \geq 0$. Let σ^t be a slice type and set $M = D^k \times \mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t)$. Then

$$\begin{aligned} 0 = T(M) &= \sum_{\overline{\sigma}^u \preceq \sigma^t} a_{\overline{\sigma},(k+|\sigma|)}\, \theta_{\overline{\sigma},(k+|\sigma|)}(M) \\ &= a_{\sigma^t,(k+|\sigma^t|)}\, \theta_{\sigma^t,(k+|\sigma^t|)}(M) \end{aligned} \tag{3.10.5}$$

by Lemma 3.6 (i), Example 2.4 and inductive assumption. Thus $a_{\sigma^t,(k+|\sigma^t|)} = 0$ since $\theta_{\sigma^t,(k+|\sigma^t|)}(M) = (-1)^k$. Therefore we may write T as

(3.10.6)

$$T = \sum_{k,\sigma^s} \alpha_{\sigma^s,(k+|\sigma^s|)} \overline{\chi}_{\sigma^s,(k+|\sigma^s|)} + \sum_{s<t,s|t|n} \sum_{k,\tau^t} \beta_{\tau^t,(k+|\tau^t|)} \overline{\chi}_{\tau^t,(k+|\tau^t|)}$$

by the definition of θ_σ. Consider now $M = D^k \times \mathbf{Z}_n \times_{\mathbf{Z}_s} D(\sigma^s)$. Then $T(M) = \alpha_{\sigma^s,(k+|\sigma^s|)}(-1)^k \frac{n}{s}$ from (3.10.6), while $T(M) = T(D^k)T(\mathbf{Z}_n \times_{\mathbf{Z}_s} D(\sigma^s)) = a^k \cdot \frac{n}{s}\gamma_{\sigma^s}$ from Lemma 3.9 when $t = s$. Therefore we have

$$\alpha_{\sigma^s,(k+|\sigma^s|)} = (-a)^k \gamma_{\sigma^s}. \tag{3.10.7}$$

To complete the proof, we must show that $\beta_{\tau^t,(k+|\tau^t|)} = 0$. Set $M = D^k \times \mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau)$ where $\tau = \tau^t$. Then $T(M) = x + y$ where

$$\begin{cases} x = \alpha_{\overline{\tau}^s,(k+|\tau|)}\, \overline{\chi}_{\overline{\tau}^s,(k+|\tau|)}(M) \\ y = \displaystyle\sum_{s<u\leq t,s|u|t,k} \beta_{\overline{\tau}^u,(k+|\tau|)}\, \overline{\chi}_{\overline{\tau}^u,(k+|\tau|)}(M) \end{cases} \tag{3.10.8}$$

from (3.10.6). Here

$$\begin{aligned} x &= (-a)^{k+|\tau|-|\overline{\tau}^s|}\gamma_{\overline{\tau}^s} \cdot (-1)^k \frac{n}{t}\,(-1)^{|\tau|-|\overline{\tau}^s|} \\ &= a^k \cdot \frac{n}{t}\, a^{|\tau|-|\overline{\tau}^s|}\gamma_{\overline{\tau}^s} \end{aligned}$$

from (3.10.7) and Example 2.4. Therefore, $x = T(D^k)T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau)) = T(M)$ from Lemma 3.9 when $\sigma = \tau$, which implies

$$y = \sum_{u,k} \beta_{\overline{\tau}^u,(k+|\tau|)} \frac{n}{t}\,(-1)^{k+|\tau|-|\overline{\tau}^u|} = 0. \tag{3.10.9}$$

From this, we have $\beta_{\tau^t,(k+|\tau^t|)} = 0$ by induction on t, whose proof is the same as in the case (3.10.5). This completes the proof. $\square$

Remark 3.11. If $n = 1$ then $St(\mathbf{Z}_1) = \{\sigma_{-1}\}$. Hence, regarding $\gamma_{\sigma_{-1}}$ as 1 we have $T = \sum_k (-a)^k \overline{\chi}_{\sigma_{-1},(k+|\sigma_{-1}|)} = \sum_k (-a)^k \overline{\chi}_{(k)} = T_0$ since $\overline{\chi}_{\sigma_{-1}} = \overline{\chi}$ from Remark 2.3 (i), cf. Proposition 3.3.

Example 3.12. We now give typical multiplicative invariants.

(1) If $a = 0$ then $T = \sum_{k,\sigma} (-0)^k \gamma_\sigma \overline{\chi}_{\sigma,(k+|\sigma|)} = \sum_\sigma \gamma_\sigma \overline{\chi}_{\sigma,(|\sigma|)}$ since $(-0)^0 = 1$. In this case, if $\gamma_i = 0$ for any i, then $T = \overline{\chi}_{\sigma^s(\mathbf{0}),(0)} = (\overline{\chi}^{\mathbf{Z}_s})_{(0)} = (\chi^{\mathbf{Z}_s})_{(0)}$ since $\gamma_{\sigma^s(\mathbf{0})} = \prod_i 0^0 = 1$ and $\gamma_\sigma = 0$ otherwise. On the other hand, if $\gamma_i = 1$ for any i, then $T(M) = \overline{\chi}(F^0) = \chi(F^0)$ for any $\mathbf{Z}_n$-manifold M where F^0 is the set of isolated points in $M^{\mathbf{Z}_s}$.

(2) If $a = 1$ then $T = \sum_{k,\sigma} (-1)^k \gamma_\sigma \overline{\chi}_{\sigma,(k+|\sigma|)}$. Hence, if $\gamma_i = 0$ for any i, then $T = \sum_k (-1)^k \overline{\chi}_{\sigma^s(\mathbf{0}),(k)}$. In other words, $T(M) = (-1)^k \overline{\chi}(M_{\sigma^s(\mathbf{0})}) = \chi(M_{\sigma^s(\mathbf{0})})$ since $\dim M_{\sigma^s(\mathbf{0})} = \dim M$ $(= k)$. On the other hand, if $\gamma_i = 1$ for any i, then $T(M) = \sum_{k+|\sigma|=m} (-1)^k \overline{\chi}_{\sigma,(m)}(M)$ $= \sum_{k+|\sigma|=m} (-1)^k \overline{\chi}(M_\sigma)$ where $m = \dim M$. Thus $T(M) = \sum_\sigma \chi(M_\sigma)$ $= \chi^{\mathbf{Z}_s}(M)$ since $\dim(M_\sigma) = m - |\sigma| = k$ and $M^{\mathbf{Z}_s} = \cup_\sigma M_\sigma$, cf. Lemma 1.3, Remarks 2.2 and 2.3 (ii).

(3) Suppose that $a = -1$. Then $T(M) = \overline{\chi}(M_{\sigma^s(\mathbf{0})}) = (-1)^{\dim M} \times \chi(M_{\sigma^s(\mathbf{0})})$ if $\gamma_i = 0$ for any i, and $T(M) = \overline{\chi}^{\mathbf{Z}_s}(M)$ if $\gamma_i = 1$ for any i.

We conclude this section with an application of the above theorem. Let p (≥ 2) be a prime integer. Here we consider mod p invariants, that is, $\mathbf{Z}_n$-SK invariants which take values in the field $\mathbf{Z}/p\mathbf{Z}$ of integers modulo p. To distinguish this field from the action group $\mathbf{Z}_n$, we denote it by $\mathbf{Z}/p\mathbf{Z}$ for the rest of this section.

Let $\mathcal{I}(p)_*^{\mathbf{Z}_n} = \mathrm{Hom}\,(SK_*^{\mathbf{Z}_n}(pt, pt), \mathbf{Z}/p\mathbf{Z})$ consisting of all these invariants T. Then a natural map $i_* : \mathcal{I}_*^{\mathbf{Z}_n} \to \mathcal{I}(p)_*^{\mathbf{Z}_n}$ induced by $i : \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ is epic since $SK_*^{\mathbf{Z}_n}(pt, pt)$ is a direct sum of copies of the integers $\mathbf{Z}$, cf. Lemma 1.3 and (2.7.5).

Definition 3.13. Such T is said to be multiplicative if $T(M \times N) = T(M)T(N)$ in $\mathbf{Z}/p\mathbf{Z}$ for any $\mathbf{Z}_n$-manifolds M and N.

We also study an invariant T which is nontrivial, hence $T(D^0) \equiv 1$.

Now we have the results in our case, which are similar to the previous ones, as follows.

Proposition 3.14. *In case* $\mathbf{Z}_1 = \{1\}$, *any* mod p *multiplicative invariant* T *has a form* $T \equiv \sum_{k\geq 0}(-a)^k\overline{\chi}_{(k)} \pmod p$ *where* $a = T(D^1)$ *is an integer modulo* p. *Here, if* $a \equiv 0$, *then we regard* a^0 *as* 1. *In particular,* $\overline{\chi}_{(0)} = \chi_{(0)}$ (*when* $a \equiv 0$) *and* $\overline{\chi} = \chi$ (*when* $a \equiv 1$) *are possible* mod 2 *multiplicative invariants.*

Proof. We may write T as $T \equiv \sum_{k\geq 0} r_k\overline{\chi}_{(k)} \pmod p$ for some $r_k \in \mathbf{Z}$ by using the surjection $i_* : \mathcal{I}_*^{\mathbf{Z}_1} \to \mathcal{I}(p)_*^{\mathbf{Z}_1}$. The multiplicative property of T in $\mathbf{Z}/p\mathbf{Z}$ implies the result in the same way as the original case in Proposition 3.3. When $p = 2$, see Remark 3.4. $\square$

We next consider an invariant for $\mathbf{Z}_n$-manifolds with $n \geq 2$.

Definition 3.15. Let s be an integer with $1 \leq s \leq n$ and $s|n$. Then a mod p multiplicative invariant T is said to be of type (s) if $T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{0}))) \equiv 0$ for each t with $1 \leq t < s$ and $t|n$, and $T(\mathbf{Z}_n \times_{\mathbf{Z}_s} D(\sigma^s(\mathbf{0}))) = \beta \not\equiv 0$.

Lemma 3.16. *If* T *is of type* (s) *then*

(i) $T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t)) \equiv 0$ *for any* σ^t *with* $t < s$ *or* $s < t$, $s \nmid t$,

(ii) $T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{0}))) = \frac{n}{t} \not\equiv 0$ *for* $s \leq t \leq n$ *with* $s|t|n$. *In particular,* $\beta \equiv \frac{n}{s}$.

Proof. We will obtain (i) in the proof of the following Proposition 3.18, while we have (ii) since β is a unit in $\mathbf{Z}/p\mathbf{Z}$, see the proof of Lemma 3.6 (ii). Note that $\frac{n}{t} \not\equiv 0$ since $\frac{n}{t} \cdot \frac{t}{s} \equiv \frac{n}{s} \not\equiv 0$. $\square$

Proposition 3.17. *If* T *is of type* (1), *then* $T(M) \equiv T_0(M_0)$ *for any* $\mathbf{Z}_n$*-manifold* M *where* $M_0 = M$ *with ignoring group action.*

The proof is similar to that of Proposition 3.7 since $T(\mathbf{Z}_n)$ is a unit in $\mathbf{Z}/p\mathbf{Z}$.

When $s > 1$, we then have two lemmas (Lemmas 3.8 and 3.9) in our modulo p case. Each proof is exactly the same as the original case. Here we only use the fact that both $\frac{n}{t}$ and $\frac{t}{s}$ are units in $\mathbf{Z}/p\mathbf{Z}$ for each t with $s|t|n$. Using these results, we have a proposition.

Proposition 3.18. *Let T be any* $\bmod p$ *multiplicative invariant of type (s) with $s > 1$ and $\{a, \gamma_i\}$ be the class of integers modulo p in Lemma* 3.8. *Then T has the form*

$$T \equiv \sum_{k,\sigma} (-a)^k \gamma_\sigma \overline{X}_{\sigma,(k+|\sigma|)} \pmod p$$

where the sum is taken over all slice types $\sigma = \sigma^s \in \mathcal{S}^s$ and $k \geq 0$. If a or $\gamma_i \equiv 0$ for some i, then we regard a^0 or γ_i^0 as 1, respectively.

Proof. We use the notations in the proof of Theorem 3.10. Write T as

$$T \equiv \sum_{k,\sigma} a_{\sigma,(k+|\sigma|)} \theta_{\sigma,(k+|\sigma|)} \pmod p \tag{3.18.1}$$

by using the surjection $i_* : \mathcal{I}_*^{\mathbf{Z}_n} \to \mathcal{I}(p)_*^{\mathbf{Z}_n}$. By assumption $s > 1$, we have $a_{\sigma_{-1},(k)} \equiv 0$ in $\mathbf{Z}/p\mathbf{Z}$ for any $k \geq 0$, cf. (3.10.4). Suppose that $a_{\tau,(k+|\tau|)} \not\equiv 0$ for some $\tau = \tau^t$, $1 < t < s$, and some k, and $a_{\nu,(l+|\nu|)} \equiv 0$ for any $\nu = \nu^u$ with $1 \leq u < t$ and any l. Then, putting $M = D^k \times \mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau^t)$, we have

$$\begin{aligned} T(D^k)T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau^t)) &\equiv T(M) \\ &\equiv \sum_{\overline{\tau}^u \preceq \tau^t} a_{\overline{\tau}^u,(k+|\tau^t|)}\, \theta_{\overline{\tau}^u,(k+|\tau^t|)}(M) \\ &\equiv (-1)^k a_{\tau^t,(k+|\tau^t|)} \not\equiv 0 \end{aligned} \tag{3.18.2}$$

by inductive assumption, cf. (3.10.5). This implies that $T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau^t)) \not\equiv 0$ in particular. Then, from Definition 3.15 and Proposition 2.12 (i) we have

$$0 \equiv T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\sigma^t(\mathbf{0})))T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau^t)) \equiv \frac{n}{t}\, T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau^t)),$$

which implies that $\frac{n}{t} \equiv 0$. On the other hand,

$$0 \not\equiv T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau^t))^2 \equiv \frac{n}{t} T(\mathbf{Z}_n \times_{\mathbf{Z}_t} D(\tau^t \times \tau^t))$$

and $\frac{n}{t} \not\equiv 0$, which contradicts the above. Therefore, we have $a_{\tau,(k+|\tau|)} \equiv 0$ for any τ^t with $1 \leq t < s$ and $k \geq 0$. Thus, T is written as (3.10.6). (Equivalently we have the first part of Lemma 3.16 (i). The second part is obtained from the first one.) Then there is no trouble proving (3.10.7) and (3.10.9) to show $\beta_{\tau^t,(k+|\tau^t|)} \equiv 0$ for any τ^t with $s < t \leq n$ and $s|t|n$, paying attention to the fact that $\frac{n}{t} \not\equiv 0$ in $\mathbf{Z}/p\mathbf{Z}$. This completes the proof. $\square$

As an application, we consider possible mod p multiplicative invariants in case $G = \mathbf{Z}_n$ ($n = 2$ or 3). By the above proposition, such T is determined by the values $a = T(D^1)$ and $\gamma = T(D(V))$ in $\mathbf{Z}/p\mathbf{Z}$, where $V = \mathbf{R}$ with a natural $\mathbf{Z}_2$ action if $n = 2$ or $\mathcal{C}$ with a natural $\mathbf{Z}_3$ action if $n = 3$. We see that $St(\mathbf{Z}_n) = \{\sigma_{-1}\} \cup \{\sigma_i = [\mathbf{Z}_n; V^i] | i \geq 0\}$ where $\sigma_0 = \sigma(\mathbf{0})$. Note that each T is of type (1) or (n), and if T is of type (1) then $T \equiv T_0$, cf. Proposition 3.17. We now treat the case T is of type (n).

Proposition 3.19. *If* $p=2$ *or* 3, *then* mod p *multiplicative invariant of type* (n) *is as follows.*

n = 3 and p = 2

T is:

(1) $(a, \gamma) \equiv (0,0)$: $(\overline{\chi}^{\mathbf{Z}_3})_{(0)} = (\chi^{\mathbf{Z}_3})_{(0)}$,

(2) $(a, \gamma) \equiv (0,1)$: $\overline{\chi}(F^0) = \chi(F^0)$,

(3) $(a, \gamma) \equiv (1,0)$: $\overline{\chi}_{\sigma(\mathbf{0})} = \chi_{\sigma(\mathbf{0})}$,

(4) $(a, \gamma) \equiv (1,1)$: $\overline{\chi}^{\mathbf{Z}_3} = \chi^{\mathbf{Z}_3}$.

n = 3 and p = 3

T is:

(1) $(a, \gamma) \equiv (0,0)$: $(\overline{\chi}^{\mathbf{Z}_3})_{(0)} = (\chi^{\mathbf{Z}_3})_{(0)}$,

(2) $(a, \gamma) \equiv (0,1)$: $\overline{\chi}(F^0) = \chi(F^0)$,

(3) $(a,\gamma)\equiv(0,2)$: $(-1)^{\dim M/2}\overline{\chi}(F^0)=(-1)^{\dim M/2}\chi(F^0)$, $T(M)\equiv 0$ if $\dim M$ is odd,

(4) $(a,\gamma)\equiv(1,0)$: $\chi_{\sigma(\mathbf{0})}=(-1)^{\dim M}\overline{\chi}_{\sigma(\mathbf{0})}$,

(5) $(a,\gamma)\equiv(1,1)$: $\chi^{\mathbf{Z}_3}=(-1)^{\dim M}\overline{\chi}^{\mathbf{Z}_3}$,

(6) $(a,\gamma)\equiv(1,2)$: $\sum_i(-1)^i\chi(F_{2i})=(-1)^{\dim M}\sum_i(-1)^i\overline{\chi}(F_{2i})$,

(7) $(a,\gamma)\equiv(2,0)$: $\overline{\chi}_{\sigma(\mathbf{0})}=(-1)^{\dim M}\chi_{\sigma(\mathbf{0})}$,

(8) $(a,\gamma)\equiv(2,1)$: $\overline{\chi}^{\mathbf{Z}_3}=(-1)^{\dim M}\chi^{\mathbf{Z}_3}$,

(9) $(a,\gamma)\equiv(2,2)$: $\sum_i(-1)^i\overline{\chi}(F_{2i})=(-1)^{\dim M}\sum_i(-1)^i\chi(F_{2i})$.

$\mathbf{n=2}$

If $p=2$ or $p=3$, $(a,\gamma)\equiv(0,0),(0,1),(1,0)$ or $(2,0)$, then such T is the same as the above, replacing $\mathbf{Z}_3$ with $\mathbf{Z}_2$. On the other hand, if $p=3$, $(a,\gamma)\equiv(0,2),(1,1),(1,2),(2,1)$ or $(2,2)$, then

T is:

(3)′ $(a,\gamma)\equiv(0,2)$: $(-1)^{\dim M}\overline{\chi}(F^0)=(-1)^{\dim M}\chi(F^0)$,

(5)′ $(a,\gamma)\equiv(1,1)$: $\chi^{\mathbf{Z}_2}$,

(6)′ $(a,\gamma)\equiv(1,2)$: $(-1)^{\dim M}\overline{\chi}^{\mathbf{Z}_2}$,

(8)′ $(a,\gamma)\equiv(2,1)$: $\overline{\chi}^{\mathbf{Z}_2}$,

(9)′ $(a,\gamma)\equiv(2,2)$: $(-1)^{\dim M}\chi^{\mathbf{Z}_2}$.

In the above, F_{2i} is a union of codimension $2i$ fixed point sets of $\mathbf{Z}_3$-manifold M. See Example 3.12 for the other notations.

Proof. Consider the case $n=3$, then each T has a form

$$T\equiv\sum_{k,i}(-a)^k\gamma^i\,\overline{\chi}_{i,(k+2i)}\pmod p \tag{3.19.1}$$

from Proposition 3.18 where $\overline{\chi}_i=\overline{\chi}_{\sigma_i}$, $i\geq 0$. Let $p=3$ in particular, then we have obtained the form in case $(a,\gamma)\equiv(0,0),(0,1),(1,0)$, $(1,1)$, $(2,0)$ or $(2,1)$ in Example 3.12. Note that, in case $(1,1)$ or $(2,1)$,

$$\begin{aligned}\chi^{\mathbf{Z}_3}(M)&=(-1)^{\dim M}(-1)^{\dim M^{\mathbf{Z}_3}}\chi^{\mathbf{Z}_3}(M)\\&=(-1)^{\dim M}\overline{\chi}^{\mathbf{Z}_3}(M)\end{aligned}\tag{3.19.2}$$

since $\operatorname{codim} M^{\mathbf{Z}_3}$ is even. The form in case $(1,2)$ or $(2,2)$ is given in a general manner later. Thus we give the proof in case $(0,2)$ here. From (3.19.1),

$$\begin{aligned} T &\equiv \sum_{k,i} (-0)^k 2^i \,\overline{\chi}_{i,(k+2i)} \\ &\equiv \sum_i 2^i \,\overline{\chi}_{i,(2i)} \quad (\text{by } (-0)^0 \equiv 1) \\ &\equiv \sum_i \overline{\chi}_{2i,(4i)} - \sum_i \overline{\chi}_{2i+1,(4i+2)} \quad (\text{by } 2^{2i} \equiv 1 \quad \text{and} \quad 2^{2i+1} \equiv -1), \end{aligned}$$

which implies the result since $T(M^{2m}) = (-1)^m \overline{\chi}_{m,(2m)}(M) = (-1)^m \overline{\chi}(F^0)$ for any $\mathbf{Z}_3$-manifold M of dimension $2m$ and $T(M) \equiv 0$ if $\dim M$ is odd. The proof in case $n = 2$ is similar, so we omit it here. $\square$

Finally, when $n = 3$ and $p \geq 3$ in general, we study $\operatorname{mod} p$ multiplicative invariants in some cases. Let $a \equiv 1 \pmod p$ and let γ be an integer with $1 \leq \gamma < p$. Then

$$\begin{aligned} T \equiv& \sum_{k,i} \gamma^{2i} \,\overline{\chi}_{2i,(2k+4i)} + \sum_{k,i} \gamma^{2i+1} \,\overline{\chi}_{2i+1,(2k+4i+2)} \\ &- \Big(\sum_{k,i} \gamma^{2i} \,\overline{\chi}_{2i,(2k+4i+1)} + \sum_{k,i} \gamma^{2i+1} \,\overline{\chi}_{2i+1,(2k+4i+3)} \Big) \pmod p \end{aligned}$$

from (3.19.1). Now let $e = e(\gamma^2)$ be the multiplicative order of γ^2 in $\mathbf{Z}/p\mathbf{Z}$, then

$$\begin{aligned} T(M^{2m}) &\equiv \sum_{k+2i=m} \gamma^{2i} \overline{\chi}_{2i,(2k+4i)}(M) + \sum_{k+2i+1=m} \gamma^{2i+1} \overline{\chi}_{2i+1,(2k+4i+2)}(M) \\ &\equiv \sum_{0 \leq i \leq [m/2]} \gamma^{2i} \overline{\chi}(F_{4i}) + \sum_{0 \leq i \leq [(m-1)/2]} \gamma^{2i+1} \overline{\chi}(F_{4i+2}) \\ &\equiv \sum_{o \leq l \leq e-1} \gamma^{2l} \Big(\sum_{i \equiv l} \overline{\chi}(F_{4i}) \Big) + \sum_{o \leq l \leq e-1} \gamma^{2l+1} \Big(\sum_{i \equiv l} \overline{\chi}(F_{4i+2}) \Big), \end{aligned}$$

where $i \equiv l$ means $i \equiv l \pmod e$. When $\dim M = 2m+1$, we have the same form with minus sign. Therefore,

$$\begin{aligned} T(M) &\equiv (-1)^{\dim M}\Big\{\sum_{o\le l\le e-1}\gamma^{2l}\Big(\sum_{i\equiv l}\overline{\chi}(F_{4i})\Big) \\ &\qquad\qquad + \sum_{0\le l\le e-1}\gamma^{2l+1}\Big(\sum_{i\equiv l}\overline{\chi}(F_{4i+2})\Big)\Big\} \\ &\equiv \sum_{o\le l\le e-1}\gamma^{2l}\Big(\sum_{i\equiv l}\chi(F_{4i})\Big) + \sum_{0\le l\le e-1}\gamma^{2l+1}\Big(\sum_{i\equiv l}\chi(F_{4i+2})\Big) \end{aligned}$$

from (3.19.2). For example, if $p=3$ and $(a,\gamma)\equiv(1,2)$, then $e(2^2)=1$ and

$$\begin{aligned} T(M) &\equiv \sum_i \chi(F_{4i}) - \sum_i \chi(F_{4i+2}) \\ &\equiv \sum_i (-1)^i \chi(F_{2i}), \end{aligned}$$

which gives (6) in Proposition 3.19. If $p=5$ and $(a,\gamma)\equiv(1,2)$, then $e(2^2)=2$ and

$$T(M) \equiv \sum_i \chi(F_{8i}) + 2\sum_i \chi(F_{8i+2}) + 4\sum_i \chi(F_{8i+4}) + 3\sum_i \chi(F_{8i+6}).$$

Next consider the case $a \equiv \gamma \pmod p$. Then

$$\begin{aligned} T &\equiv \sum_{k,i}\gamma^{2k+2i}\overline{\chi}_{2i,(2k+4i)} + \sum_{k,i}\gamma^{2k+2i+1}\overline{\chi}_{2i+1,(2k+4i+2)} \\ &\quad - \Big(\sum_{k,i}\gamma^{2k+2i+1}\overline{\chi}_{2i,(2k+4i+1)} + \sum_{k,i}\gamma^{2k+2i+2}\overline{\chi}_{2i+1,(2k+4i+3)}\Big) \end{aligned}$$

from (3.19.1). In the same way as the above, we have

$$T(M^{2m}) \equiv \sum_{0\le l\le e-1}\gamma^{2l}\Big(\sum_{m-i\equiv l}\overline{\chi}(F_{4i})\Big) + \sum_{1\le l\le e}\gamma^{2l-1}\Big(\sum_{m-i\equiv l}\overline{\chi}(F_{4i+2})\Big),$$

and

$$\begin{aligned} T(M^{2m+1}) \equiv -\Big(&\sum_{0\le l\le e-1}\gamma^{2l+1}\Big(\sum_{m-i\equiv l}\overline{\chi}(F_{4i})\Big) \\ &+ \sum_{0\le l\le e-1}\gamma^{2l}\Big(\sum_{m-i\equiv l}\overline{\chi}(F_{4i+2})\Big)\Big). \end{aligned}$$

For example, if $p = 3$ and $a \equiv \gamma \equiv 2$, then we have (9) in Proposition 3.19 similarly. Further, if $p = 5$ and $a \equiv \gamma \equiv 2$, then

$$\begin{aligned} T(M^{2m}) \equiv (-1)^m \Big(\sum_i \overline{\chi}(F_{8i}) + 3 \sum_i \overline{\chi}(F_{8i+2}) \\ + 4 \sum_i \overline{\chi}(F_{8i+4}) + 2 \sum_i \overline{\chi}(F_{8i+6}) \Big), \end{aligned}$$

and

$$\begin{aligned} T(M^{2m+1}) \equiv (-1)^m \Big(3 \sum_i \overline{\chi}(F_{8i}) + 4 \sum_i \overline{\chi}(F_{8i+2}) \\ + 2 \sum_i \overline{\chi}\Big(F_{8i+4}) + \sum_i \overline{\chi}(F_{8i+6}) \Big). \end{aligned}$$

In fact, we see that T is multiplicative by straightforward calculation.

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