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ON AN EXAMPLE OF A FUNCTION WITH A DERIVATIVE WHICH DOES NOT HAVE A THIRD ORDER SYMMETRIC RIEMANN DERIVATIVE ANYWHERE

Abstract

In this paper we construct a differentiable function $F : \mathbb{R} \to \mathbb{R}$ that does not have a third order symmetric Riemann derivative at any point. In fact,

$$\underline{SRD}^3 F(x) = \liminf_{h\to 0} \frac{F(x+3h)-3F(x+h)+3F(x-h)-F(x-3h)}{(2h)^3} = -\infty$$

and

$$\overline{SRD}^3 F(x) = \limsup_{h\to 0} \frac{F(x+3h)-3F(x+h)+3F(x-h)-F(x-3h)}{(2h)^3} = +\infty$$

for every $x \in \mathbb{R}$.

1 Introduction

The three well-known classical theorems concerning convexity of a function, of a derivative and of a second derivative using second, third and fourth order Riemann derivates (see [4], [5], [6], [7]), require $\limsup, \liminf$ and $\liminf$ respectively, in their statements. Also, the non-classical but natural generalization (non-Riemann for orders greater or equal to five) using divided differences also uses $\liminf$ (see [3]). The present work, besides other consequences, states that we can not replace $\liminf$ by $\limsup$ for the third Riemann derivate.

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2 Construction of a Periodic Function and its Properties

Let $a \in \mathbb{R}^+$. Define $f_a : \mathbb{R} \to \mathbb{R}$ as follows:

i) f_a is periodic of period 13a;

ii)

$$f_a(x) = \begin{cases} -x^3, & \text{if } x \in [0, a); \\ -x^3 + 4(x-a)^3, & \text{if } x \in [a, 2a); \\ -x^3 + 4(x-a)^3 - 6(x-2a)^3, & \text{if } x \in [2a, 3a); \\ (x-4a)^3, & \text{if } x \in [3a, 4a); \\ 0, & \text{if } x \in [4a, 6.5a]. \end{cases}$$

iii) $f_a(6.5a + x) = -f_a(6.5a - x)$ if $x \in [0, 6.5a]$.
Let $b \in \mathbb{R}^+$ and define $G = G_{a,b} = \dfrac{b}{a^2} f_a$. It is easy to see that

$$G(10a) = G(12a) = -G(a) = -G(3a) = ab \tag{1}$$

$$G(11a) = G(-2a) = -G(15a) = \max_{[0,13a]} G = -\min_{[0,13a]} G = 4ab \tag{2}$$

$$0 \leq G(y) \leq 4ab \qquad \text{if } y \in [4a, 13a] \tag{3}$$

$$-4ab \leq G(y) \leq 0 \qquad \text{if } y \in [0, 9a] \tag{4}$$

$$-ab \leq G(y) \leq 0 \qquad \text{if } y \in [-10a, -4a] \tag{5}$$

$$0 \leq G(y) \leq ab \qquad \text{if } y \in [-9a, -3a] \tag{6}$$

$$G'' \quad \text{exist on } \mathbb{R} \quad \text{and} \quad |G'| \leq 4b, \quad |G''| \leq 12\frac{b}{a} \quad \text{on } \mathbb{R} \tag{7}$$

3 Main Auxiliary Inequalities

Let a, b, G as in 2. Then for every $x \in \mathbb{R}$ there are $h, k \in [a, 12a]$ such that

$$G(x+3h) - 3G(x+h) + 3G(x-h) - G(x-3h) \geq 8ab \tag{8}$$

$$G(x+3k) - 3G(x+k) + 3G(x-k) - G(x-3k) \leq -8ab \tag{9}$$

Proof. *i*) Let $x \in [-a, 14a]$. We consider the following cases:

(a) α. If $x \in [-a, 3a]$ take $h = x + 2a$. Then $x + h \in [0, 8a]$, $x - 3h \in [-12a, -4a], h \in [a, 5a]$. Thus by (2),(4),(5), $G(x - h) = G(-2a) = 4ab, G(x + h) \leq 0, G(x - 3h) \leq 0$. This proves (8).

β. If $x \in [3a, 6a]$ take $h = 15a - x$.Then $x + h = 15a$, $x - h \in [-9a, -3a], x + 3h \in [33a, 39a], h \in [9a, 12a]$. Thus by (2),(6),(3), $G(x + h) = G(15a) = -4ab, G(x - h) \geq 0, G(x + 3h) \geq 0$. This proves (8).

γ. If $x \in [6a, 10a]$ take $h = x + 2a$. Then $x - h = -2a, x + h \in [14a, 22a], x - 3h \in [-26a, -18a], h \in [8a, 12a]$. Thus by (2),(4), $G(x - h) = 4ab, G(x + h) \leq 0, G(x - 3h) \leq 0$. This proves (8).

δ. If $x \in [10a, 14a]$ take $h = -x + 15a$. Then $x + h = 15a$, $x - h \in [5a, 13a], x + 3h \in [17a, 25a], h \in [a, 5a]$. Thus by (2),(3), $G(x + h) = -4ab, G(x - h) \geq 0, G(x + 3h) \geq 0$. This proves (8).

(b) α. If $x \in [-a, 3a]$ take $k = -x + 11a$. Then $x + k = 11a, x - k \in [-13a, -5a]$, $x + 3k \in [27a, 35a], k \in [8a, 12a]$. Thus by (2),(4), $G(x + k) = 4ab, G(x - k) \leq 0, G(x + 3k) \leq 0$. This proves (9).

β. If $x \in [3a, 6a]$ take $k = x - 2a$. Then $x - k = 2a, x + k \in [4a, 10a]$, $x - 3k \in [-6a, 0], k \in [a, 4a]$. Thus by (2),(6),(3),G(x - k) = - 4ab, G(x + k) $\geq$ 0, G(x - 3k) $\geq$ 0. This proves (9).

γ. If $x \in [6a, 10a]$ take $k = -x + 11a$. Then $x + k = 11a, x - k \in [a, 9a]$, $x + 3k \in [13a, 21a], k \in [a, 5a]$. Thus by (2),(4), $G(x + k) = 4ab, G(x - k) \leq 0, G(x + 3k) \leq 0$. This proves (9).

δ. If $x \in [10a, 14a]$ take $k = x - 2a$. Then $x - k = 2a, x + k \in [18a, 26a]$, $x - 3k \in [-22a, -14a], k \in [8a, 12a]$. Thus by (2),(3), $G(x - k) = -4ab, G(x + k) \geq 0, \quad G(x - 3k) \geq 0$. This proves (9).

ii) Let $x \in \mathbb{R}$. There exists a $n_0 \in \mathbb{Z}$ such that $n_0 \leq \dfrac{x}{13a} < n_0 + 1$. Then $x \in \big[13an_0, 13a(n_0+1)\big)$. So $x - 13an_0 \in [0, 13a)$. Putting $x_0 = x - 13an_0, x_0 \in [-a, 14a]$ and so (8) and (9) are true for $x = x_0$ by i). Since G has period $13a$, (8) and (9) are also true for all x. $\square$

4 A Mean Value Theorem for Divided Differences

Let n $\in \mathbb{N}$ and let f be continuous on $[c, d]$ such that $f_{(n)}$ exists on $[c, d]$. Let $x_1 < x_2 < \cdots < x_{n+1}; x_i \in [c, d], i = 1, 2, \ldots, n + 1$.

Then there is a $c \in (x_1, x_{n+1})$ such that

$$n! V_n(x_1, x_2, \ldots, x_{n+1}) = f_{(n)}(c).$$

(A proof may be found in [1] pp.193 th.III).

5 Bounds for the Numerator of Riemann Third Order Ratio

Let G as in 2 and $x, h \in \mathbb{R}$. Then

$$|G(x+3h) - 3G(x+h) + 3G(x-h) - G(x-3h)| \leq 144h^2 \frac{b}{a}.$$

PROOF. Let $h \neq 0$, then

$$|G(x+3h) - 3G(x+h) + 3G(x-h) - G(x-3h)| =$$

$$|G(x+3h) + 3G(x-h) - 4G(x) - (G(x-3h) + 3G(x+h) - 4G(x))| =$$

$$|12h^2 V_2(x+3h, x, x-h; G) - 12h^2 V_2(x-3h, x, x+h; G)| \leq$$

$$6h^2(|2! V_2(x+3h, x, x-h; G)| + |2! V_2(x-3h, x, x+h; G)|)$$

Now (7) and 4 complete the proof. □

6 Main Result

There exists a function $F : \mathbb{R} \to \mathbb{R}$ such that F' exist on $\mathbb{R}$
and $\overline{SRD}^3 F = \infty$, $\underline{SRD}^3 F = -\infty$ on $\mathbb{R}$.

PROOF. Let $b \in (0, 1)$. Define

$$g(y) = \frac{1}{12^3} - \frac{18y}{b-y} - \frac{4yb}{1-yb},\ y \in [0, b)$$

Then g is continuous on $[0, b)$ and $g(0) > 0$. Thus there is an $a \in (0, b)$ such that $g(a) > 0$. Let $G = G_{a,b}$ as in 2. Define $F_n = G_{a^n, b^n} \quad (n \in \mathbb{N})$,

$$F = \sum_{n=1}^{\infty} F_n$$

Then by (2),(7)

$$|F_n| \leq 4(ab)^n, |F'_n| \leq 4b^n (n \in \mathbb{N}).$$

Thus

$$\sum_{n=1}^{\infty} F_n \quad , \quad \sum_{n=1}^{\infty} F'_n$$

converge uniformly, thus F' exists on $\mathbb{R}$ and

$$F' = \sum_{n=1}^{\infty} F_n'.$$

Let $x \in \mathbb{R}$.Then by 3, for each $n \in \mathbb{N}$ there are $h_n = h_n(x), k_n = k_n(x) \in [a^n, 12a^n]$ such that

$$\frac{F_n(x+3h_n) - 3F_n(x+h_n) + 3F_n(x-h_n) - F_n(x-3h_n)}{(2h_n)^3} \geq \frac{1}{12^3}\left(\frac{b}{a^2}\right)^n$$

$$\frac{F_n(x+3k_n) - 3F_n(x+k_n) + 3F_n(x-k_n) - F_n(x-3k_n)}{(2k_n)^3} \leq -\frac{1}{12^3}\left(\frac{b}{a^2}\right)^n$$

Now fix an n.
Using the above estimates, 5 and (2) we get

$$\frac{F(x+3k_n) - 3F(x+k_n) + 3F(x-k_n) - F(x-3k_n)}{(2k_n)^3} =$$

$$\sum_{m=1}^{n-1} \frac{F_m(x+3k_n) - 3F_m(x+k_n) + 3F_m(x-k_n) - F_m(x-3k_n)}{(2k_n)^3} +$$

$$\frac{F_n(x+3k_n) - 3F_n(x+k_n) + 3F_n(x-k_n) - F_n(x-3k_n)}{(2k_n)^3} +$$

$$\sum_{m=n+1}^{\infty} \frac{F_m(x+3k_n) - 3F_m(x+k_n) + 3F_m(x-k_n) - F_m(x-3k_n)}{(2k_n)^3} \leq$$

$$\sum_{m=1}^{n-1} \frac{144k_n^2\left(\frac{b}{a}\right)^m}{(2k_n)^3} - \frac{1}{12^3}\left(\frac{b}{a^2}\right)^n +$$

$$\sum_{m=n+1}^{\infty} \frac{F_m(x+3k_n) - 3F_m(x+k_n) + 3F_m(x-k_n) - F_m(x-3k_n)}{(2k_n)^3} \leq$$

$$\frac{18}{a^n}\sum_{m=1}^{n-1}\left(\frac{b}{a}\right)^m - \frac{1}{(12)^3}\left(\frac{b}{a^2}\right)^n + \frac{4}{a^{3n}}\sum_{m=n+1}^{\infty}(ab)^m =$$

$$\frac{18}{a^n}\,\frac{\left(\frac{b}{a}\right)^n - \frac{b}{a}}{\frac{b}{a} - 1} - \frac{1}{12^3}\left(\frac{b}{a^2}\right)^n + \frac{4}{a^{3n}}\,\frac{(ab)^{n+1}}{1-ab} \leq$$

$$\frac{18}{a^n}\frac{\left(\frac{b}{a}\right)^n}{\frac{b}{a}-1}-\frac{1}{12^3}\left(\frac{b}{a^2}\right)^n+\frac{4}{a^{3n}}\frac{(ab)^{n+1}}{1-ab}=$$

$$\frac{18a}{b-a}\left(\frac{b}{a^2}\right)^n-\frac{1}{12^3}\left(\frac{b}{a^2}\right)^n+\frac{4ab}{1-ab}\left(\frac{b}{a^2}\right)^n=-\left(\frac{b}{a^2}\right)^n g(a).$$

Similarly,

$$\frac{F(x+3h_n)-3F(x+h_n)+3F(x-h_n)-F(x-3h_n)}{(2h_n)^3}\geq\left(\frac{b}{a^2}\right)^n g(a).$$

Now since n was arbitrary fixed point of $\mathbb{N}$ and since $\lim\limits_{n\to\infty}k_n=0$, $\lim\limits_{n\to\infty}h_n=0$ and $\frac{b}{a^2}>1, g(a)>0$ we get

$$\lim_{n\to\infty}\frac{F(x+3h_n)-3F(x+h_n)+3F(x-h_n)-F(x-3h_n)}{(2h_n)^3}=+\infty$$

and

$$\lim_{n\to\infty}\frac{F(x+3k_n)-3F(x+k_n)+3F(x-k_n)-F(x-3k_n)}{(2k_n)^3}=-\infty$$

These show that

$$\limsup_{h\searrow 0}\frac{F(x+3h)-3F(x+h)+3F(x-h)-F(x-3h)}{(2h)^3}=+\infty$$

and

$$\liminf_{k\searrow 0}\frac{F(x+3k)-3F(x+k)+3F(x-k)-F(x-3k)}{(2k)^3}=-\infty.$$

These easily imply

$$\overline{SRD}^3F(x)=\infty,\ \underline{SRD}^3F(x)=-\infty$$

respectively,which complete the proof. □

7 Some First Category Subsets of $C[0,1]$

On C[0, 1] with $d(f,g)=\max\limits_{x\in[0,1]}|f(x)-g(x)|$

i) There is a $F \in C[0,1]$ such that $\overline{SRD}^3 F = \infty$ and $\underline{SRD}^3 F = -\infty$ on the open interval $(0,1)$.

ii) Let H_1 be the set of all functions f in $C[0,1]$ such that there is a x in $(0,1)$ with $\overline{SRD}^3 f(x) < \infty$ and H_2 be the set of all functions f in $C[0,1]$ such that there is a x in $(0,1)$ with $\underline{SRD}^3 f(x) > -\infty$. Then H_1, H_2 are of first category in $C[0,1]$.

iii) Let Θ be the set of all functions f in $C[0,1]$ such that $\overline{SRD}^3 f = \infty$ and $\underline{SRD}^3 f = -\infty$ on $(0,1)$. Then $H = C[0,1] - \Theta$ is of first category in $C[0,1]$.

PROOF. i) Follows easily from 6.

ii) For H_1, we will prove that the complement of H_1 is dense in $C[0,1]$ and that the set H_1 is of type F_σ. Take $\epsilon > 0$ and let $U(p,\epsilon)$ be the set of all functions f in $C[0,1]$ such that $d(f,p) < \epsilon$ where p is a polynomial. To show $U(p,\epsilon) \cap (C[0,1] - H_1) \neq \emptyset$. Each function of the form $p + \eta F$ $(\eta > 0)$ where F is the function of 7 i) belongs to $C[0,1] - H_1$. Indeed, if the polynomial p satisfies $|p^{(3)}| < L$ on $[0,1]$ then, by 4

$$\left| \frac{p(x+3h) - 3p(x+h) + 3p(x-h) - p(x-3h)}{(2h)^3} \right| \leq L$$

and for each x in $(0,1)$ and $h \neq 0$

$$\frac{1}{(2h)^3}\Big\{(p+\eta F)(x+3h) - 3(p+\eta F)(x+h) +$$

$$3(p+\eta F)(x-h) - (p+\eta F)(x-3h)\Big\} =$$

$$\frac{p(x+3h) - 3p(x+h) + 3p(x-h) - p(x-3h)}{(2h)^3} +$$

$$\eta \frac{F(x+3h) - 3F(x+h) + 3F(x-h) - F(x-3h)}{(2h)^3} \geq$$

$$-L + \eta \frac{F(x+3h) - 3F(x+h) + 3F(x-h) - F(x-3h)}{(2h)^3}.$$

Thus $\overline{SRD}^3 (p+\eta F)(x) = \infty$.This easily implies that $\overline{SRD}^3 (p+\eta F) = \infty$ on $(0,1)$, thus $p + \eta F \in C[0,1] - H_1$.

Set $\eta = \dfrac{\epsilon}{2\|F\|}$. Then $p + \eta F \in U(p,\epsilon)$. Thus $U(p,\epsilon) \cap (C[0,1] - H_1) \neq \emptyset$.

Let F_n be the set of all functions f in $C[0,1]$ with the property that there is a x in $\left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ such that if $0 < |h| < \frac{1}{3n}$ then

$$\frac{f(x+3h) - 3f(x+h) + 3f(x-h) - f(x-3h)}{(2h)^3} \leq n,$$

$n = 2, 3, \ldots$. Since

$$H_1 = \bigcup_{n=2}^{\infty} F_n$$

and $C[0,1]$ is a complete space, by Baire's theorem it is sufficient to show that F_n is closed in $C[0,1]$.

Let n be fixed. We prove that F_n is closed. Let $\{f_k\}$ be any sequence in F_n such that $f_k \to f$ in $C[0,1]$ as $k \to \infty$. Then the sequence of functions $\{f_k\}$ converges to f uniformly on $[0,1]$. Since $f_k \in F_n$, there is for each k a point $x_k \in [\frac{1}{n}, 1 - \frac{1}{n}]$ such that if $0 < |h| < \frac{1}{3n}$ then

$$\frac{f_k(x_k+3h) - 3f_k(x_k+h) + 3f_k(x_k-h) - f_k(x_k-3h)}{(2h)^3} \leq n.$$

Since $\{x_k\} \subset [\frac{1}{n}, 1 - \frac{1}{n}]$ there is a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ such that $\{x_{k_l}\}$ converges to a point $x_0 \in [\frac{1}{n}, 1 - \frac{1}{n}]$. Clearly the subsequence $\{f_{k_l}\}$ of $\{f_k\}$ converges uniformly to f on $[0,1]$. Also if $0 < |h| < \frac{1}{3n}$ then

$$\frac{f_{k_l}(x_{k_l}+3h) - 3f_{k_l}(x_{k_l}+h) + 3f_{k_l}(x_{k_l}-h) - f_{k_l}(x_{k_l}-3h)}{(2h)^3} \leq n,$$

$l = 1, 2, \ldots$. Since $\{f_{k_l}\}$ converges to f uniformly and $x_{k_l} \to x_0$ as $l \to \infty$, letting $l \to \infty$

$$\frac{f(x_0+3h) - 3f(x_0+h) + 3f(x_0-h) - f(x_0-3h)}{(2h)^3} \leq n.$$

This shows that $f \in F_n$ and so F_n is closed. Therefore H_1 is of the first category in $C[0,1]$. Similarly H_2 is also of the first category in $C[0,1]$.

iii) It follows easily, since $H = H_1 \cup H_2$ and 7 ii). □

8 A Specific Set of First Category

Let H_s be the set of all functions f in $C[0,1]$ for which a third order symmetric Riemann derivative exist in at least one point x of $(0,1)$. Then H_s is a set of first category in $C[0,1]$.

PROOF. Since H is of first category by 7 iii) in $C[0,1]$ and $H_s \subseteq H$, the set H_s is of first category in $C[0,1]$. $\square$

Remark. There exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $\overline{SRD}^1 F = \infty$, and $\underline{SRD}^1 F = -\infty$ on $\mathbb{R}$.

This is the work of [2].

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