Chapter X
Game Quantification

by PH. G. KOLAITIS

Game quantification interacts with the model theory of infinitary logics, abstract model theory, generalized recursion theory, and descriptive set theory. The aim of this chapter is to examine these connections and give some applications of the game quantifiers to the above areas of mathematical logic.

The chapter is divided into four sections. The first presents the basic notions and the interpretation of infinite strings of quantifiers via two-person infinite games. Section 2 deals with the interaction between game quantification and global definability theory, the main theme being that certain second-order statements can be reduced to formulas involving the game quantifiers which can, in turn, be approximated by formulas of $L_{\infty\omega}$. This section also includes a proof of Vaught's covering theorem, as well as applications of game quantification to the model theory of $L_{\omega_1\omega}$ and admissible fragments. In Section 3, we show that the game logics are absolute and unbounded, and most of the model-theoretic properties of these logics will then follow from this fact. Section 4, the final section, discusses the interaction with local definability theory. Here we consider the basic relation of the game quantifiers to inductive definability and higher recursion theory, and give some of their uses in descriptive set theory.

1. Infinite Strings of Quantifiers

This section presents the main definitions and basic results about infinite strings of quantifiers $(Q_0x_0Q_1x_1Q_2x_2\ldots)$ where, for each $i = 0, 1, 2, \ldots, Q_i$ is the *existential quantifier* $\exists$ or the *universal quantifier* $\forall$ on a set A. The interpretation of such strings is via two-person infinite games of perfect information. We first describe the interpretation in an informal way and indicate the expressive power of certain infinite strings. The precise definitions involve the notions of a winning strategy and a winning quasistrategy. The Gale–Stewart theorem is then proven and used to push negation through infinite strings in certain cases.

Throughout this section, A is a non-empty infinite set, $A^{<\omega} = \bigcup_{n\in\omega} A^n$ is the set of all finite sequences from A, and A^ω is the collection of all infinite sequences of elements of A. We use variables $x, y, z, \ldots$ to denote elements of A, variables

$s, t, u, \ldots$ to represent elements of $A^{<\omega}$, and variables $\alpha, \beta, \ldots$ to denote the members of A^ω. The empty sequence is denoted by $(\ )$, while $s^\frown t$ denotes the *concatenation* of two elements s, t of $A^{<\omega}$. Finally, if $\alpha \in A^\omega$ and $n \in \omega$, then $\alpha \upharpoonright n$ is the restriction of α to n, that is, $\alpha \upharpoonright n = (\alpha(0), \alpha(1), \ldots, \alpha(n-1)) \in A^n$.

1.1. Iterating the Existential and the Universal Quantifier Infinitely Often

1.1.1. The most natural infinite strings of quantifiers are obtained by iterating the existential quantifier or the universal quantifier—or, alternatively, the existential and the universal quantifier. If $R \subseteq A^\omega$ is a non-empty set of infinite sequences from A, then three infinite strings that result in this way are:

(1) $$(\exists x_0 \, \exists x_1 \, \exists x_2 \cdots) R(x_0, x_1, x_2, \ldots),$$

(2) $$(\forall x_0 \, \forall x_1 \, \forall x_2 \cdots) R(x_0, x_1, x_2, \ldots),$$

(3) $$(\exists x_0 \, \forall y_0 \, \exists x_1 \, \forall y_1 \, \exists x_2 \, \forall y_2 \cdots) R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots).$$

The first two strings, (1) and (2), respectively express existential and universal quantification over the set A^ω of infinite sequences from A. In order to interpret the infinite string given in (3), we associate it with the following two-person game $G(\exists\forall, R)$ of perfect information:

> A round of the game $G(\exists\forall, R)$ is played by players I and II alternatively choosing elements from A:
>
> $$\begin{array}{c|ccccccc} \text{I} & x_0 & & x_1 & & x_2 & & \cdots \\ \hline \text{II} & & y_0 & & y_1 & & y_2 & \cdots \end{array}.$$
>
> *Player* I *wins the above round* if $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots) \in R$, otherwise *Player* II *wins the round.*

We say that *Player* I *wins the game* $G(\exists\forall, R)$ if I has a systematic way to win every round of the game. Similarly, we say that *Player* II *wins the game* $G(\exists\forall, R)$ if II has a systematic way to win every round of the game. Finally, we put

$$(\exists x_0 \, \forall y_0 \, \exists x_1 \, \forall y_1 \, \exists x_2 \, \forall y_2 \cdots) R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$$
$$\text{iff Player I wins the game } G(\exists\forall, R).$$

In general, if $\bar{Q} = (Q_0, Q_1, Q_2, \ldots, Q_i, \ldots)$ is an arbitrary infinite string such that each Q_i is the existential or the universal quantifier, then the interpretation of the statement

(4) $$(Q_0 x_0 Q_1 x_1 Q_2 x_2 \cdots Q_i x_i \cdots) R(x_0, x_1, x_2, \ldots, x_i, \ldots)$$

is entirely analogous to the preceding one for (3). More specifically, we associate with $\bar{Q}$ and R a two-person infinite game $G(\bar{Q}, R)$ in a round of which, for each $i = 0, 1, 2, \ldots$, an element x_i in A is picked by Player I if $Q_i = \exists$ and by Player II if $Q_i = \forall$. At the end of the round, *Player* I *wins the round* if the infinite sequence $(x_0, x_1, x_2, \ldots, x_i, \ldots)$ is an element of R. Otherwise, *Player* II *wins the round.* We say that *Player* I *wins the game* $G(\bar{Q}, R)$ if I has a systematic way to win every round of it. Similarly, we say that *Player* II *wins the game* $G(\bar{Q}, R)$ if II has a systematic way to win every round of it. As before, we put

$$(Q_0 x_0 Q_1 x_1 Q_2 x_2 \cdots Q_i x_i \cdots) R(x_0, x_1, x_2, \ldots, x_i, \ldots)$$
$$\text{iff} \quad \text{Player I wins the game } G(\bar{Q}, R).$$

1.1.2 Remark. Often the infinite strings given in (1), (2), (3), and (4) are not applied to arbitrary relations $R \subseteq A^\omega$, but rather to relations which are either *open* or *closed.*

A relation $R \subseteq A^\omega$ is *open,* if it can be written as the infinitary disjunction of finitary relations; that is, if there are relations $R_n \subseteq A^n$, $n \in \omega$, such that

$$R(x_0, x_1, \ldots, x_{n-1}, x_n, \ldots) \Leftrightarrow \bigvee_{n \in \omega} R_n(x_0, x_1, \ldots, x_{n-1}).$$

Similarly, we say that a relation $R \subseteq A^\omega$ is *closed* if it can be written as the infinitary conjunction of finitary relations; that is, if there are relations $R_n \subseteq A^n$, for each $n \in \omega$, such that

$$R(x_0, x_1, \ldots, x_{n-1}, x_n, \ldots) \Leftrightarrow \bigwedge_{n \in \omega} R_n(x_0, x_1, \ldots, x_{n-1}).$$

This terminology is justified by the fact that a relation R is open (or closed) if it is an open set (or, respectively, a closed set) in the product topology on A^ω, where A is equipped with the discrete topology.

If the infinite strings in (1), (2), and (3) are applied to relations on A^ω which are open or closed, they can then be identified with certain monotone quantifiers on the set $A^{<\omega}$ of finite sequences from A. In order to make this idea precise, we introduce the following notions, which will be also used in Section 4 of this chapter.

1.1.3 Definitions. A *monotone quantifier* Q *on a set* A is a collection Q of subsets of A such that:

(i) Q is non-trivial; that is, $\varnothing \subsetneq Q \subsetneq \mathscr{P}(A)$;
(ii) Q has the *monotonicity property*, that is, if $X \in Q$ and $X \subseteq Y$, then $Y \in Q$.

Interchangeably, we write

$$QxR(x) \quad \text{iff} \quad R \in Q \quad \text{iff} \quad \{x \in A : R(x)\} \in Q.$$

The *dual* of a monotone quantifier Q is the collection $\breve{Q}$, where

$$X \in \breve{Q} \quad \text{iff} \quad (A - X) \notin Q.$$

It is quite clear that $\breve{Q}$ is also a monotone quantifier and that $(\breve{Q})^{\smile} = Q$.

Under these definitions, the *existential quantifier* $\exists$ *on* A is identified with the collection of non-empty subsets of A, and we write

$$\exists = \{X \subseteq A : X \neq \emptyset\},$$

while the *universal quantifier* $\forall$ *on* A is the singleton given by

$$\forall = \{A\}.$$

We obviously have that

$$\breve{\exists} = \forall \quad \text{and} \quad \breve{\forall} = \exists.$$

By iterating the existential and the universal quantifier on A infinitely often, we obtain the following interesting quantifiers on the set $A^{<\omega}$ of finite sequences from A:

(5) *The Suslin quantifier* $\mathscr{S}$

$$\mathscr{S} = \left\{X \subseteq A^{<\omega} : (\forall x_0\, \forall x_1\, \forall x_2 \cdots) \bigvee_n ((x_0, x_1, x_2, \ldots, x_{n-1}) \in X)\right\}.$$

(6) *The classical quantifier* $\mathscr{A}$

$$\mathscr{A} = \left\{X \subseteq A^{<\omega} : (\exists x_0\, \exists x_1\, \exists x_2 \cdots) \bigwedge_n ((x_0, x_1, x_2, \ldots, x_{n-1}) \in X)\right\}.$$

Here it is obvious that $\mathscr{A}$ is the dual of the Suslin quantifier.

(7) *The open game quantifier* $\mathscr{G}$,

$$\mathscr{G} = \left\{X \subseteq A^{<\omega} : (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \bigvee_n ((x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in X)\right\}.$$

(8) *The closed game quantifier* $\breve{\mathscr{G}}$

$$\breve{\mathscr{G}} = \left\{X \subseteq A^{<\omega} : (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_n ((x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in X)\right\}.$$

It will follow from results in Section 1.2 that the closed game quantifier is the dual of the open game quantifier.

1.1.4 Remark. The Suslin quantifier $\mathscr{S}$, the classical quantifier $\mathscr{A}$, and the two-game quantifiers can capture properties which are not, in general, expressible using the infinitary logic $L_{\omega_1\omega}$ or even the logic $L_{\infty\infty}$. The following examples indicate the expressive power of these quantifiers.

(i) The notion of well-foundedness can be expressed using the Suslin quantifier $\mathscr{S}$. Indeed, if R is a binary relation on a set A, then

$$R \text{ is } \textit{well-founded} \text{ iff } \quad (\forall x_0\, \forall x_1\, \forall x_2 \cdots) \bigvee_n (\neg(x_{n+1} R x_n)).$$

It is well known, of course, that this property is not expressible in the infinitary logic $L_{\omega_1\omega}$.

(ii) If $\mathfrak{A}$ is a structure which possesses a first-order definable coding machinery of finite sequences, then the Suslin quantifier and the classical quantifier $\mathscr{A}$ can be identified with monotone quantifiers on the universe A of the structure $\mathfrak{A}$. For example, this is the case with the structure $\mathbb{N} = \langle \omega, +, \cdot \rangle$ of natural numbers. On this structure, the Suslin quantifier and the classical quantifier $\mathscr{A}$ can capture second-order statements. This follows from the fact that on $\mathbb{N}$ every Π_1^1 relation $R(\bar{z})$ can be written in the form

$$R(\bar{z}) \Leftrightarrow (\forall x_0\, \forall x_1\, \forall x_2 \cdots)\left(\bigvee_n \psi(\langle x_0, x_1, \ldots, x_{n-1}\rangle, \bar{z})\right),$$

where ψ is a first-order formula and $\langle x_0, x_1, \ldots, x_{n-1}\rangle$ is an element of ω coding the sequence $(x_0, x_1, \ldots, x_{n-1})$.

The above is a rather special property of the structure $\mathbb{N}$ of natural numbers. At the other extreme, if $\mathbb{R} = \langle \omega^\omega \cup \omega, \omega, +, \cdot, Ap\rangle$, where $Ap(\alpha, n) = \alpha(n)$, is the structure of real numbers, then the Suslin quantifier and the classical quantifier $\mathscr{A}$ coincide respectively with the universal and the existential quantifier on the reals. This is a consequence of the fact that we can code infinitely many reals by a real in a first-order definable way.

(iii) The open game and the closed game quantifier have, in general, higher expressive power than the Suslin and the classical quantifier $\mathscr{A}$. If a structure $\mathfrak{A}$ possesses a first-order coding machinery of finite sequences, then the relation of satisfaction "$\mathfrak{A} \models \varphi$", where φ is a sentence of the first-order logic of the vocabulary of $\mathfrak{A}$, can be shown to be expressible in terms of the open game or the closed game quantifier, while this relation is not first-order definable on such structures. In particular, on the structure $\mathbb{R}$ of the real numbers the game quantifiers properly transcend the Suslin and the classical $\mathscr{A}$ quantifier.

The connections between local definability theory and game quantification will be investigated in Section 4 of this chapter.

(iv) Consider a vocabulary τ consisting of two binary relation symbols $<_1$, $<_2$ and the equality symbol $=$. Using the infinite string $(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots)$ and countable disjunctions and conjunctions, we can write a statement $\varphi(u, v, <_1, <_2)$ expressing that:

"$<_1$ and $<_2$ are well-orderings

and

u is in the field of $<_1$, v is in the field of $<_2$

and

the order type $|u|_1$ of u in $<_1$ is less than or equal to the order type $|v|_2$ of v in $<_2$."

The crucial property $|u|_1 \leq |v|_2$ is then expressed as follows:

$$(\forall x_0 \exists y_0 \forall x_1 \exists y_1 \cdots)\left[\left(\bigwedge_n (x_n <_1 u) \leftrightarrow \bigwedge_n (y_n <_2 v)\right)\right.$$
$$\left. \wedge \bigwedge_{m,n} (x_m <_1 x_n \leftrightarrow y_m <_2 y_n) \wedge \bigwedge_{m,n} (x_m = x_n \leftrightarrow y_m = y_n)\right].$$

The proof that this statement works can be obtained by induction on $|u|_1$.

From the above, it easily follows that using the infinite string $(\forall x_0 \exists y_0 \forall x_1 \exists y_1 \cdots)$ and countable disjunctions and conjunctions, we can write a statement $\psi(<)$ asserting that

"$<$ is a well-ordering of order type $\gamma + \gamma$ for some ordinal γ".

Malitz [1966] has shown, however, that this statement is not expressible by any formula of the infinitary logic $L_{\infty\infty}$. Thus, game quantification can give rise to infinitary logics which are different from the usual infinitary logics $L_{\kappa\lambda}$. These new infinitary logics will be introduced and studied in Section 3 of this chapter, while in Section 2 we will pursue the relationship between game quantification and global definability theory.

1.2. Winning Strategies and Winning Quasistrategies

Assume that $\bar{Q} = (Q_0, Q_1, Q_2, \ldots, Q_i, \ldots)$ is an infinite string such that for each $i = 0, 1, 2, \ldots$ Q_i is the existential or the universal quantifier on a set A. In the preceding section the interpretation of the statement

$$(Q_0 x_0 Q_1 x_1 Q_2 x_2 \cdots Q_i x_i \cdots) R(x_0, x_1, x_2, \ldots, x_i, \ldots)$$

was given in a rather informal way, since we defined the concept "Player I wins the game $G(\bar{Q}, R)$" by saying simply that "Player I has a systematic way to win every round of the game $G(\bar{Q}, R)$." This definition is intuitive, but not very precise. We will now give precise definitions of these concepts in a set-theoretic framework. It actually turns out that we can give at least two interpretations for infinite strings of quantifiers which are equivalent in the presence of the full axiom of choice, but which may nevertheless be different if only weaker choice principles are available. For the sake of clarity, we give the definitions and then state the results only for the infinite string $(\exists, \forall, \exists, \forall, \ldots, \exists, \forall, \ldots)$. However, these notions will generalize to arbitrary strings $\bar{Q} = (Q_0, Q_1, Q_2, \ldots, Q_i, \ldots)$ with only notational changes in the definitions or the proofs.

1.2.1. Let $R \subseteq A^\omega$ be a relation on the set of infinite sequences from A, and let $G(\exists\forall, R)$ be the two-person infinite game associated with the statement

$$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots)R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots). \tag{9}$$

A *strategy σ for Player* I *in the game* $G(\exists\forall, R)$ *is a function* $\sigma\colon \bigcup_{n\in\omega} A^{2n} \to A$ from the set of finite sequences of even length into A.

Intuitively, a strategy σ for I provides him with a next move. We say that I *follows the strategy σ in a round* $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ of the game $G(\exists\forall, R)$ if $x_0 = \sigma((\,))$ and $x_n = \sigma((x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}))$, for all $n = 1, 2, 3, \ldots$. We call σ *a winning strategy for* I *in the game* $G(\exists\forall, R)$ if I wins every round of the game in which he follows σ.

In an analogous way, we define a *strategy τ for Player* II *in* $G(\exists\forall, R)$ to be a function $\tau\colon \bigcup_{n\in\omega} A^{2n+1} \to A$. *Player* II *follows τ in a round* $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ of the game if $y_n = \tau((x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x_n))$ for all $n = 0, 1, 2, \ldots$. We say that τ *is a winning strategy for* II *in* $G(\exists\forall, R)$ if II wins every round of the game in which he follows τ.

Using the above notions, we rigorously interpret the statement given in (9) as follows:

$$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall x_2 \cdots)R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots) \tag{10}$$
$$\text{iff Player I has a winning strategy for the game } G(\exists\forall, R).$$

In practice, when we prove theorems about infinite strings of quantifiers, we must often use the axiom of choice to exhibit a winning strategy for one of the players in the game associated with the infinite string. There are situations, however, in which one is working in a set theory where the full axiom of choice is not available. In such cases, we can still prove the results about the infinite strings of quantifiers by reformulating the interpretation of the infinite string given in (9). The idea here is to replace the notion of a strategy by that of a quasistrategy, a quasistrategy being essentially a multiple-valued strategy that provides the player with a non-empty set of possible next moves instead of a single move.

A *quasistrategy Σ for Player* I *in the game* $G(\exists\forall, R)$ is a set $\Sigma \subseteq A^{<\omega}$ of finite sequences from A such that:

(i) there is some $x_0 \in A$ for which $(x_0) \in \Sigma$;
(ii) if $(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) \in \Sigma$, then there is some $x \in A$ for which $(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x) \in \Sigma$;
(iii) if $(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x) \in \Sigma$, then for every $y \in A$

$$(x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x, y) \in \Sigma.$$

Player I *follows the quasistrategy* Σ *in a round* $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ of $G(\exists\forall, R)$ if every initial segment of the round is in Σ. Furthermore, we say that Σ *is a winning quasistrategy for* I *in the game* $G(\exists\forall, R)$ if I wins every round of the game in which he follows Σ.

We define also the notions of *quasistrategy for* II *and winning quasistrategy for* II *in the game* $G(\exists\forall, R)$ in an analogous dual way.

We can now interpret the statement in (9) in an alternative way as follows:

(11) $$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots)R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$$
$$\text{iff}\quad \text{Player I has a winning quasistrategy in the game } G(\exists\forall, R).$$

It is quite obvious that if Player I has a winning strategy in the game $G(\exists\forall, R)$, then I also has a winning quasistrategy in this game. If, in addition, the set A can be well-ordered, then every winning quasistrategy for I in $G(\exists\forall, R)$ gives rise to a winning strategy for I in this game. We therefore see that, in the presence of the axiom of choice, the two interpretations given by (10) and (11) of the statement in (9) are equivalent. This equivalence, however, depends on the axiom of choice in an essential way.

If we interpret the infinite string $(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1\, \exists x_2\, \forall y_2 \cdots)$ via quasistrategies, then most theorems about this string can be proved using the axiom of dependent choices. A weaker principle than the full axiom of choice, *the axiom of dependent choices* states that, for every non-empty set B and for every binary relation $P \subseteq B \times B$ on B,

$$(\forall x \in B)(\exists y \in B)P(x, y) \Rightarrow (\exists f\colon \omega \to B)(\forall n)P(f(n), f(n+1)).$$

Observe that we used the axiom of dependent choices implicitly, when we asserted in Section 1.1.4 that the Suslin quantifier can express the notion of well-foundedness. Indeed, this axiom is precisely the choice principle needed to show that a relation is well-founded if and only if it has no infinite descending chains.

We will now investigate some simple properties of strategies and quasistrategies, beginning with

1.2.2 Lemma. *Let* $R \subseteq A^\omega$ *be a relation on the set of infinite sequences from* A. *Then,*

(i) *It is not possible that both Players* I *and* II *have winning strategies in the game* $G(\exists\forall, R)$.
(ii) (*Assuming the axiom of dependent choices*). *It is not possible that both Players* I *and* II *have winning quasistrategies in the game* $G(\exists\forall, R)$.

Proof. Part (i) is obvious and requires no choice principles. To prove part (ii) we will assume, towards a contradiction, that Player I has a winning quasistrategy Σ in $G(\exists\forall, R)$ and that II also has a winning quasistrategy T in this same game. Using dependent choices, we can then produce a round $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ of the game $G(\exists\forall, R)$ every initial segment of which is in both Σ and T. But then the round $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ is in both R and $\neg R$. This is a contradiction. $\square$

If $R \subseteq A^\omega$ is a relation on the set of infinite sequences from A, and if $G(\exists\forall, R)$ is the game associated with the statement

(9) $$(\exists x_0\ \forall y_0\ \exists x_1\ \forall y_1\ \exists x_2\ \forall y_2 \cdots)R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots),$$

then $G(\forall\exists, \neg R)$ is the game associated with the statement

(12) $$(\forall x_0\ \exists y_0\ \forall x_1\ \exists y_1\ \forall x_2\ \exists y_2 \cdots) \neg R(x_0, y_0, x_1, y_1, x_2, y_2, \ldots).$$

It is clear from the definitions that a winning strategy (respectively, quasistrategy) for II in the game $G(\exists\forall, R)$ is a winning strategy (respectively quasistrategy) for I in the game $G(\forall\exists, \neg R)$. We therefore have the following

1.2.3 Lemma. *Let $R \subseteq A^\omega$ be a relation on the set of infinite sequences from A. Then,*

(i) *Player* II *has a winning strategy (respectively quasistrategy) in $G(\exists\forall, R)$ if and only if Player* I *has a winning strategy (respectively quasistrategy) in $G(\forall\exists, \neg R)$.*
(ii) *Player* I *has a winning strategy (respectively quasistrategy) in $G(\exists\forall, R)$ if and only if Player* II *has a winning strategy (respectively quasistrategy) in $G(\forall\exists, \neg R)$.* $\square$

Assume now that $R \subseteq A^\omega$ is a relation such that Player I or Player II has a winning strategy (respectively a winning quasistrategy) in the game $G(\exists\forall, R)$. Combining this with Lemmas 1.2.2 and 1.2.3, we obtain the equivalence:

(13) $$\neg(\exists x_0\ \forall y_0\ \exists x_1\ \forall y_1 \cdots)R(x_0, y_0, x_1, y_1, \ldots) \Leftrightarrow (\forall x_0\ \exists y_0\ \forall x_1\ \exists y_1 \cdots) \neg R(x_0, y_0, x_1, y_1, \ldots),$$

where the interpretation of the statements given in (9) and (12) is via winning strategies as in (10) (respectively via winning quasistrategies as in (11)).

We say that the game $G(\exists\forall, R)$ is *determined* if Player I or Player II has a winning strategy in this game. We also say that $G(\exists\forall, R)$ is *weakly determined* if Player I or Player II has a winning quasistrategy in the game. The preceding facts show that if the game $G(\exists\forall, R)$ is determined or weakly determined, then to negate the statement given in (9), we can push the negation through the infinite string $(\exists x_0\ \forall y_0\ \exists x_1\ \forall y_1 \cdots)$ and apply it to the relation R. Although this manipulation is always true for finite strings and all relations R, it is not true for infinite strings and arbitrary relations $R \subseteq A^\omega$. Indeed using the axiom of choice, Gale and Stewart [1953] showed that there is a relation $R \subseteq 2^\omega$ such that the game $G(\exists\forall, R)$ is not determined. It turns out, however, that if the relation R is open or closed, then the associated game $G(\exists\forall, R)$ is determined.

1.2.4 Theorem (Gale–Stewart [1953]). *Let $R \subseteq A^\omega$ be a relation on the set of infinite sequences from A which is either open or closed. Then,*

(i) *(Assuming the axiom of choice). Player I or Player* II *has a winning strategy in the game $G(\exists\forall, R)$;*
(ii) *Player* I *or Player* II *has a winning quasistrategy in the game $G(\exists\forall, R)$.*

Proof. The first part of the theorem follows from the second by well-ordering the set A. Moreover, in view of Lemma 1.2.3, it is enough to establish the result for the case of a closed relation $R \subseteq A^\omega$. Therefore, assume that there are finitary relations $R_n \subseteq A^{2n+2}$, for each $n = 0, 1, 2, \ldots$, such that

$$R(x_0, y_0, x_1, y_1, \ldots, x_n, y_n, \ldots) \Leftrightarrow \bigwedge_{n \in \omega} R_n(x_0, y_0, x_1, y_1, \ldots, x_n, y_n).$$

We will show that Player I or Player II has a winning quasistrategy in the game $G(\exists\forall, R)$. The winning quasistrategy will be obtained by using an inductive analysis for the set of "winning positions" for Player I in the open game $G(\forall\exists, \neg R)$. More precisely, consider the following monotone operator $\varphi(u, S)$, where u ranges over the elements of $A^{<\omega}$ and S over the subsets of $A^{<\omega}$:

$$\varphi(u, S) \Leftrightarrow (u \text{ has even length}) \,\&\, \Bigl(\text{if } u = (x_0, y_0, \ldots, x_n, y_n),$$

$$\text{then } \bigvee_{m \leq n} \neg R_m(x_0, y_0, \ldots, x_m, y_m)\Bigr) \vee (\forall x\, \exists y)(u^\frown(x, y) \in S).$$

By induction on the ordinals define a sequence $\{\varphi^\xi\}_\xi$ of subsets of $A^{<\omega}$, where

$$u \in \varphi^0 \Leftrightarrow \varphi(u, \varnothing),$$
$$u \in \varphi^\xi \Leftrightarrow \varphi(u, \bigcup_{\eta < \xi} \varphi^\eta),$$

and let $\varphi^\infty = \bigcup_\xi \varphi^\xi$. Intuitively, the set φ^∞ consists of all "winning positions" for Player I in the game $G(\forall\exists, \neg R)$, since (using the axiom of dependent choices) we can show that

$$(14) \qquad (x_0, y_0, \ldots, x_n, y_n) \in \varphi^\infty$$
$$\Leftrightarrow (\forall x_{n+1}\, \exists y_{n+1}\, \forall x_{n+2}\, \exists y_{n+2} \cdots) \bigvee_{m \in \omega} \neg R_m(x_0, y_0, \ldots, x_m, y_m).$$

In completing the proof of the theorem, we will not use the above equivalence, but have included it in order to make the role of φ^∞ transparent.

We claim now that if the empty sequence ( ) is not in φ^∞, then Player I has a winning quasistrategy in the game $G(\exists\forall, R)$, while if $(\ ) \in \varphi^\infty$, then Player II has a winning quasistrategy in $G(\exists\forall, R)$. Indeed, if $(\ ) \notin \varphi^\infty$, then it can be easily verified that the set

$$\Sigma = \{u \in A^{<\omega}: (u \text{ has even length and } u \notin \varphi^\infty)$$
$$\vee\ (u \text{ has odd length and } (\forall y)(u^\frown(y) \notin \varphi^\infty)\}$$

is a winning quasistrategy for I in $G(\exists\forall, R)$. On the other hand, if $(\ ) \in \varphi^\infty$, then for $u \in \varphi^\infty$, we first put $|u|_\varphi =$ least ordinal ξ such that $u \in \varphi^\xi$, and then let

$$T = \{u \in A^{<\omega}\text{: for every } v \in A^{<\omega} \text{ if } v = (x_0, y_0, \ldots, x_i, y_i, x_{i+1}, y_{i+1}) \text{ is an initial segment of } u \text{ of even length, then } v \in \varphi^\infty \text{ and } |(x_0, y_0, \ldots, x_i, y_i)|_\varphi = 0 \text{ or } |(x_0, y_0, \ldots, x_i, y_i)|_\varphi > |(x_0, y_0, \ldots, x_i, y_i, x_{i+1}, y_{i+1})|_\varphi\}.$$

It is now quite easy to show that T is a winning quasistrategy for II in $G(\exists\forall, R)$. ▯

Combining the Gale–Stewart theorem with Lemmas 1.2.2 and 1.2.3 we have the following:

1.2.5 Corollary. *Let $R \subseteq A^\omega$ be a relation which is open or closed. Then,*

(i) *(Assuming the axiom of choice). Player* I *does not have a winning strategy in $G(\exists\forall, R)$ if and only if Player* II *has a winning strategy in $G(\exists\forall, R)$.*
(ii) *(Assuming the axiom of dependent choices). Player* I *does not have a winning quasistrategy in $G(\exists\forall, R)$ if and only if Player* II *has a winning quasistrategy in $G(\exists\forall, R)$.* ▯

The above corollary allows us to push the negation through the infinite string Thus, if $R \subseteq A^\omega$ is open or closed, then

$$(13) \qquad \neg(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots)R(x_0, y_0, x_1, y_1, \ldots) \Leftrightarrow (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \neg R(x_0, y_0, x_1, y_1, \ldots).$$

▯

1.2.6 Corollary. *The closed game quantifier $\breve{\mathcal{G}}$ is the dual of the open game quantifier $\mathcal{G}$.* ▯

As was mentioned in the introduction to this section, all the preceding results extend to arbitrary infinite strings. In general, if $\overline{Q} = (Q_0, Q_1, \ldots, Q_i, \ldots)$ is an infinite string such that for each $i = 0, 1, 2, \ldots$ Q_i is the existential or the universal quantifier on A, then the *dual string* $\overline{Q^\cup}$ is defined by

$$\overline{Q^\cup} = (\breve{Q}_0, \breve{Q}_1, \ldots, \breve{Q}_i, \ldots).$$

If a relation $R \subseteq A^\omega$ is open or closed, then we have the equivalence

$$(15) \qquad \neg(Q_0 x_0 Q_1 x_1 \cdots Q_i x_i \cdots)R(x_0, x_1, \ldots, x_i, \ldots) \Leftrightarrow (\breve{Q}_0 x_0 \breve{Q}_1 x_1 \cdots \breve{Q}_i x_i \cdots) \neg R(x_0, x_1, \ldots, x_i, \ldots).$$

Proof of the above equivalence requires the full axiom of choice if the interpretation is via winning strategies and the axiom of dependent choices if the interpretation is via winning quasistrategies.

1.2.7. In view of the preceding results for the open and the closed games, it is natural to ask whether or not there are other relations $R \subseteq A^\omega$ for which the game $G(\exists\forall, R)$ is determined. We say that the game $G(\exists\forall, R)$ is *Borel* if the relation R is a Borel set in the product topology on A^ω, where A discrete. Martin [1975] proved that in ZFC every Borel game is determined. His proof actually established that in ZF + axiom of dependent choices (DC) every Borel game is weakly determined; that is, that, one of the two players has a winning quasistrategy in such a game. The question of determinacy for games $G(\exists\forall, R)$, where R has higher complexity, is independent of ZF and leads into strong set-theoretic hypotheses.

1.2.8 Remarks. We have two reasons in mind for making explicit the distinction between winning quasistrategies and winning strategies. The first, is that it is often important to know the weakest possible metatheory in which we can formulate and prove results about infinite strings of quantifiers. This will be useful, in Section 3 of this chapter; for there we discuss the set-theoretic definability of the infinitary logics built by using the game quantifiers. The second reason is the connection between game quantification and descriptive set theory, a connection which will be briefly pursued in Section 4. Much of the current research in descriptive set theory is carried in ZF together with the axiom of dependent choices (DC) and the hypothesis that certain infinite games are weakly determined.

From now on, we will distinguish explicitly between strategies and quasistrategies in only a very few cases. Instead, we will use the statement "*Player* I *wins the game* $G(\exists\forall, R)$" for both interpretations, i.e., depending on the context or on the metatheory available, this means that Player I has a winning strategy or a winning quasistrategy in the game $G(\exists\forall, R)$.

1.2.9. We should point out that finite strings of quantifiers at the beginning can always be absorbed inside an infinite string. More precisely, for any relation $R \subseteq A^\omega$, we have the equivalence

$$
\begin{aligned}
(16) \qquad & (Q_0 x_0)(Q_1 x_1) \cdots (Q_n x_n)\{(Q_{n+1} x_{n+1})(Q_{n+2} x_{n+2}) \cdots\} \\
& \qquad R(x_0, x_1, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots) \\
& \Leftrightarrow (Q_0 x_0 Q_1 x_1 \cdots Q_n x_n Q_{n+1} x_{n+1} \cdots) R(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots),
\end{aligned}
$$

where $Q_i = \exists$ or $Q_i = \forall$, for each $i = 0, 1, 2, \ldots$.

In general, if the relation R is arbitrary, the proof of the above equivalence requires the axiom of choice, even though the interpretation may be via winning quasistrategies. However, in the case where R is open or closed, no choice principles are required in the proof, since there are canonical quasistrategies for such games.

We end this section with two simple propositions. These will provide a first insight into the relationship between game quantification and second-order logic.

If $R \subseteq A^{<\omega}$ is a relation on the set of finite sequences from A, then R gives rise to an *open* relation $\bigvee R$ and a *closed* relation $\bigwedge R$ on the set A^ω of infinite sequences from A, where

$$\bigvee R = \{\alpha \in A^\omega\colon \text{there is some } n \in \omega \text{ such that } (\alpha \restriction n) \in R\}$$

and

$$\bigwedge R = \{\alpha \in A^\omega\colon (\alpha \restriction n) \in R \text{ for all } n \in \omega\}.$$

1.2.10 Proposition. *Let $R \subseteq A^{<\omega}$ be a relation on the set of finite sequences from A. Then,*

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots \forall x_n\, \exists y_n \cdots) \bigwedge_n R(x_0, y_0, x_1, y_1, \ldots, x_n, y_n)$$

$$\text{iff} \quad (\exists T)(T \text{ is a winning quasistrategy for I in } G(\forall\exists, \textstyle\bigwedge R) \text{ and } T \subseteq R).$$

Proof. The result follows immediately from the observation that if T is a winning quasistrategy for Player I in $G(\forall\exists, \bigwedge R)$, then, using dependent choices, we see that any sequence $u = (x_0, y_0, \ldots, x_n, y_n)$ in T can be extended to a round $(x_0, y_0, \ldots, x_n, y_n, x_{n+1}, y_{n+1}, \ldots)$ of $G(\forall\exists, \bigwedge R)$ in which I follows T. ∎

The closed game quantifier can be expressed using second-order existential quantification. This is the content of the next proposition, a result that we will use repeatedly in the sequel.

1.2.11 Proposition. *Let $R \subseteq A^{<\omega}$ be a relation on the set of finite sequences from A. Then,*

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots \forall x_n\, \exists y_n \cdots) \bigwedge_n R(x_0, y_0, x_1, y_1, \ldots, x_n, y_n)$$

$$\begin{aligned}\text{iff} \quad (\exists T_1\, \exists T_2 \cdots \exists T_n \cdots)\Big\{\bigwedge_n (&T_n \subseteq A^{2n} \,\&\, T_1 \subseteq R \\ &\& (\forall x_0\, \exists y_0)((x_0, y_0) \in T_1) \\ &\& (\forall x_0\, \forall y_0 \cdots \forall x_{n-1}\, \forall y_{n-1})[(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in T_n \\ &\to (R(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \\ &\& (\forall x_n\, \exists y_n)(T_{n+1}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}, x_n, y_n))]\Big\}.\end{aligned}$$

Proof. In view of Proposition 1.2.10, it is enough to consider a winning quasistrategy T for I in the game $G(\forall\exists, \bigwedge R)$ and to put

$$T_n = \{(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in A^{2n}\colon (x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in T\}. \quad ∎$$

2. Projective Classes and the Approximations of the Game Formulas

In this section we will study the interactions between game quantification and global definability theory. The first basic result to be presented here is Svenonius' theorem which establishes that on countable structures the relations definable by the closed game quantifier coincide with the Σ_1^1 relations. Following this theorem, we will show that the game quantifier formulas can be approximated by formulas of the infinitary logic $L_{\omega_1\omega}$. These two results make it possible to analyze certain second-order statements, such as Σ_1^1 and Π_1^1 formulas, by the use of methods and techniques from the model theory of $L_{\omega_1\omega}$. As an illustration of these ideas, we will here outline a proof of Vaught's covering theorem. The section will end with applications of the approximations of the game formulas to descriptive set theory and to the model theory of $L_{\omega_1\omega}$ and admissible fragments.

2.1. Game Quantification and Projective Classes

Throughout this section we will be working with vocabularies which contain only relation and constant symbols. If τ is such a vocabulary, then $L_{\omega\omega}[\tau]$ is the set of all first-order formulas of vocabulary τ. As usual, $L_{\omega_1\omega}$ is the infinitary logic which allows for countable disjunctions and conjunctions, while $L_{\omega_1\omega}[\tau]$ is the set of all formulas of $L_{\omega_1\omega}$ of vocabulary τ. If the vocabulary is either fixed or understood from the context, then we will often write $L_{\omega\omega}$ and $L_{\omega_1\omega}$ instead of $L_{\omega\omega}[\tau]$ and $L_{\omega_1\omega}[\tau]$.

In what follows *countable* means of cardinality less than or equal to ω; that is, the cardinality is either finite or denumerably infinite. Moreover, we write HF for the set of hereditarily finite sets and HC for the set of hereditarily countable sets, so that

$$\mathrm{HF} = \{x : |Tc(x)| < \omega\} \quad \text{and} \quad \mathrm{HC} = \{x : |Tc(x)| < \omega_1\}.$$

All the vocabularies to be considered here are countable. If τ is such a countable vocabulary, then we can identify the formulas of $L_{\omega_1\omega}[\tau]$ with set-theoretic objects, so that if φ is in $L_{\omega_1\omega}[\tau]$, then $Tc(\{\varphi\}) \subseteq \mathrm{HC}$. In particular, we have that

$$L_{\omega\omega}[\tau] = L_{\omega_1\omega}[\tau] \cap \mathrm{HF} \quad \text{and} \quad L_{\omega_1\omega}[\tau] = L_{\omega_1\omega}[\tau] \cap \mathrm{HC}.$$

If A is an admissible set (possibly with urelements) and $\tau \in A$, then

$$L_A[\tau] = L_{\infty\omega}[\tau] \cap A$$

denotes the *admissible fragment of* $L_{\infty\omega}[\tau]$ *associated with* A, *where* $L_{\infty\omega}$ *is the* infinitary logic which allows for arbitrary disjunctions and conjunctions, but which only allows for finite strings of quantifiers.

2.1.1 Definitions. Let τ be a countable vocabulary containing only relation and constant symbols.

(i) We say that a *second-order formula φ is* $\mathrm{PC}_\Delta[\tau]$ (or simply PC_Δ) if it is of the form

$$\exists \bar{R} \bigwedge_{n \in \omega} \psi_n(\bar{R}),$$

where $\bar{R}$ is a countable set of relation symbols $\bar{R} = (R_1, R_2, \ldots)$ not in the vocabulary τ and where, for each $n \in \omega$, we have that $\psi_n(\bar{R})$ is a formula of $L_{\omega\omega}[\tau']$, with $\tau' = \tau \cup \bar{R}$.

(ii) We say that a second-order formula φ is Σ_1^1 *over* $L_{\omega_1\omega}[\tau]$, and we write φ is $\Sigma_1^1(L_{\omega_1\omega}[\tau])$ or simply $\Sigma_1^1(L_{\omega_1\omega})$ if it is of the form

$$\exists \bar{R} \psi(\bar{R}),$$

where $\bar{R}$ is a countable set of relation symbols not in τ and $\psi(\bar{R})$ is a formula of $L_{\omega_1\omega}[\tau']$, with $\tau' = \tau \cup \bar{R}$.

(iii) If A is an admissible set and $\tau \in A$, then we say that a formula φ is Σ_1^1 *over* $L_A[\tau]$, and we write φ is $\Sigma_1^1(L_A[\tau])$ or simply $\Sigma_1^1(L_A)$, in case φ is of the form

$$\exists \bar{R} \psi(\bar{R}),$$

where $\bar{R}$ is a countable set of relation symbols not in τ such that $\bar{R} \in A$ and $\psi(\bar{R})$ is a formula of the admissible fragment $L_A[\tau']$, with $\tau' = \tau \cup \bar{R}$.

We now introduce the notions of a *closed game formula* and an *open game formula*, which are obtained by applying the closed and the open game quantifier to formulas of the first-order logic $L_{\omega\omega}$.

2.1.2 Definitions. Let τ be a vocabulary which is countable and contains only relation and constant symbols.

(i) We say that $\Phi(\bar{z})$ is a *closed game formula* if it is of the form

$$(\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots) \bigwedge_{n < \omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}), \tag{1}$$

where φ_n is a formula of $L_{\omega\omega}[\tau]$ in the displayed free variables, for each $n \in \omega$.

(ii) We say that $\Phi(\bar{z})$ is an *open game formula* if it is of the form

$$(\exists x_0 \, \forall y_0 \, \exists x_1 \, \forall y_1 \cdots) \bigvee_{n < \omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}), \tag{2}$$

where φ_n is a formula of $L_{\omega\omega}[\tau]$ in the displayed free variables, for each $n \in \omega$.

The Gale–Stewart theorem (1.2.4) implies that the negation of a closed game formula is always logically equivalent to an open game formula, and vice-versa. It actually turns out that there is a strong connection between PC_Δ formulas and closed game formulas. However, in order to analyze $\Sigma_1^1(L_{\omega_1\omega})$ formulas we must consider the following generalization of the game formulas, a generalization introduced by Vaught [1973b].

(iii) A *closed Vaught formula* $\Phi(\bar{z})$ is one of the form

(3)
$$\left(\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} \cdots\right) \bigwedge_{n<\omega} \varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

where I is a countable set and, for each $(i_0, j_0, \ldots, i_{n-1}, j_{n-1}) \in I^{2n}$, we have that $\varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is a formula of $L_{\omega_1\omega}[\tau]$ in the displayed free variables.

(iv) An *open Vaught formula* $\Phi(\bar{z})$ is one of the form

(4)
$$\left(\exists x_0 \bigvee_{i_0 \in I} \forall y_0 \bigwedge_{j_0 \in I} \exists x_1 \bigvee_{i_1 \in I} \forall y_1 \bigwedge_{j_1 \in I} \cdots\right) \bigvee_{n<\omega} \varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

where I is a countable set and each $\varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is a formula of $L_{\omega_1\omega}[\tau]$ in the displayed free variables.

To simplify the already cumbersome notation, we will henceforth write

$$\bar{\imath}, \bar{\jmath} \text{ for the sequence } (i_0, j_0, \ldots, i_{n-1}, j_{n-1}) \text{ in } I^{2n}$$

and

$$\bar{x}, \bar{y} \text{ for the sequence of variables } (x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

so that

$$\varphi^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}) \text{ denotes the formula}$$

$$\varphi^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}).$$

(v) We say that $\Phi(\bar{z})$ is a *game formula* if it is either an open or a closed game formula. Similarly, a *Vaught formula* is one which is either an open or a closed Vaught formula.

(vi) If $\Phi(\bar{z})$ is either a game formula or a Vaught formula and if A is an admissible set, then we say that $\Phi(\bar{z})$ *is in* A just in case the family of formulas $\{\varphi^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y})$: $(\bar{\imath}, \bar{\jmath}) \in I^{2n}, n < \omega\}$ is an element of A.

2.1.3. If $\mathfrak{A}$ is a structure of vocabulary τ, then the interpretation of a Vaught formula on $\mathfrak{A}$ is via a two-person infinite game in a round of which Player I and Player II take turns and each chooses an element from the universe A of the structure $\mathfrak{A}$ and an index from the set I. The definition of a winning strategy and a winning quasistrategy in this game is analogous to that given in Section 1.2. The Gale–Stewart theorem extends to Vaught formulas by essentially the same proof, so that the negation of a closed Vaught formula is logically equivalent to an open Vaught formula, and vice-versa.

In general, game formulas cannot capture statements expressible by formulas of the weak second-order logic L_{wII}. On the other hand, the infinitary logic $L_{\omega_1\omega}$ is stronger than L_{wII}, so that if we hope to study $\Sigma_1^1(L_{\omega_1\omega})$ formulas using some infinitary logic, then we must consider a logic which is at least as strong as L_{wII}. These comments provide a first justification for introducing the Vaught formulas. We should also point out here that if $I = \omega$ and $\mathfrak{A} = \langle A, \ldots \rangle$ is a structure of vocabulary τ such that $\omega \subseteq A$ and $\mathfrak{A}$ possesses a first-order coding machinery of finite sequences, then the open and the closed Vaught formulas have no more expressive power than the formulas obtained by applying the open and the closed game quantifier to formulas of $L_{\omega_1\omega}$. Of course, over such structures the weak second-order logic L_{wII} is subsumed by the first-order logic $L_{\omega\omega}$.

We now proceed to investigate the connections between PC_Δ and $\Sigma_1^1(L_{\omega_1\omega})$ formulas on the one hand and closed game and Vaught formulas on the other. All the results refer to a fixed vocabulary τ which is countable and contains only relation and constant symbols.

2.1.4 Proposition. (i) *Any closed game formula is logically equivalent to a* PC_Δ *formula.*

(ii) *Any closed Vaught formula* $\Phi(\bar{z})$ *is logically equivalent to a* $\Sigma_1^1(L_{\omega_1\omega})$ *formula. Moreover, if* A *is an admissible set and* $\Phi(\bar{z})$ *is in* A, *then* $\Phi(\bar{z})$ *is logically equivalent to a* $\Sigma_1^1(L_A)$ *formula.*

Proof. The first part of this proposition follows immediately from Proposition 1.2.11. On the other hand, the extension of this proposition to closed Vaught formulas gives easily the second part. ☐

Svenonius [1965] established a partial converse to Proposition 2.1.4. More specifically, he showed that over countable models the closed game formulas have the same expressive power as the PC_Δ formulas. Vaught [1973b] obtained a generalization of this result by introducing the class of formulas which here we call closed Vaught formulas and by showing that over countable structures they are equivalent to the $\Sigma_1^1(L_{\omega_1\omega})$ formulas. Before presenting these results, we will introduce the following notation:

$\models' \varphi$ means that the sentence φ is true in all countable structures.

Notice that if φ is a sentence of $L_{\omega_1\omega}[\tau]$, then

$$\models' \varphi \quad \text{if} \quad \models \varphi,$$

because the Skolem–Löwenheim theorem holds for the infinitary logic $L_{\omega_1\omega}$.

2.1.5 Theorem. (i) (Svenonius [1965]). *For any* PC_Δ *formula* $\exists\bar{R}\bigwedge_{n<\omega}\psi_n(\bar{z},\bar{R})$, *there is a sequence of quantifier-free formulas* $\varphi_n(\bar{z},\bar{x},\bar{y})$ *of* $L_{\omega\omega}[\tau]$ *such that if* $\Phi(\bar{z})$ *is the closed game formula* $(\forall x_0\,\exists y_0\,\forall x_1\,\exists y_1\cdots)\bigwedge_n\varphi_n(\bar{z},\bar{x},\bar{y})$, *then*

(a) $\models \exists\bar{R}\bigwedge_{n<\omega}\psi_n(\bar{z},\bar{R})\to\Phi(\bar{z})$;

(b) $\models'\Phi(\bar{z})\to\exists\bar{R}\bigwedge_{n<\omega}\psi_n(\bar{z},\bar{R})$; *and hence*

(c) $\models'\Phi(\bar{z})\leftrightarrow\exists\bar{R}\bigwedge_{n<\omega}\psi_n(\bar{z},\bar{R})$.

Moreover, the quantifier-free formulas $\varphi_n(\bar{z},\bar{x},\bar{y})$ *can be obtained recursively from* n, $\bar{R}$ *and the sequence* $\{\psi_n(\bar{z},\bar{R})\}$.

(ii) (Vaught [1973b]). *For any* $\Sigma_1^1(L_{\omega_1\omega})$ *formula* $\exists\bar{R}\psi(\bar{z},\bar{R})$, *there is a closed Vaught formula* $\Phi(\bar{z})$ *which does not contain symbols from* $\bar{R}$ *and such that*

(a) $\models\exists\bar{R}\psi(\bar{z},\bar{R})\to\Phi(\bar{z})$;

(b) $\models'\Phi(\bar{z})\to\exists\bar{R}\psi(\bar{z},\bar{R})$; *and hence*

(c) $\models'\Phi(\bar{z})\leftrightarrow\exists\bar{R}\psi(\bar{z},\bar{R})$.

Moreover, the formulas $\{\varphi^{\bar{\imath},\bar{\jmath}}(\bar{z},\bar{x},\bar{y})\colon(\bar{\imath},\bar{\jmath})\in I^{2n}, n<\omega\}$, *which determine* $\Phi(\bar{z})$, *can be chosen to be in* $L_{\omega\omega}[\tau]$ *and to depend on* $\exists\bar{R}\psi(\bar{z},\bar{R})$ *and* ω *in a primitive recursive way. In particular, if* A *is an admissible set,* $\omega\in A$ *and* $\exists\bar{R}\psi(\bar{z},\bar{R})$ *is* $\Sigma_1^1(L_A)$, *then the closed Vaught formula* $\Phi(\bar{z})$ *can be chosen in* A.

Sketch of Proof. In what follows we merely outline a proof of part (i) and give a hint of the proof of part (ii) of the theorem.

If we add new constant symbols, it will suffice to prove the result for a PC_Δ sentence $\exists\bar{R}\bigwedge_{n<\omega}\psi_n(\bar{R})$, where $\psi_n(\bar{R})$ is a sentence of $L_{\omega\omega}[\tau\cup\bar{R}]$, for each $n\in\omega$. Moreover, using the Skolem normal form, we may assume without loss of generality that the PC_Δ sentence $\exists\bar{R}\bigwedge_{n<\omega}\psi_n(\bar{R})$ is actually of the form

$$\exists\bar{R}\bigwedge_{n<\omega}(\forall x_1\cdots\forall x_{k_n})(\exists y_1\cdots\exists y_{l_n})\chi_n(x_1,\ldots,x_{k_n},y_1,\ldots,y_{l_n},\bar{R}),$$

where $\chi_n(x_1,\ldots,x_{k_n},y_1,\ldots,y_{l_n},\bar{R})$ is a quantifier-free formula of $L_{\omega\omega}[\tau\cup\bar{R}]$, for each $n\in\omega$.

To make the game-theoretic argument involved transparent, we will also assume that we have only one quantifier-free formula $\chi(x,y,\bar{R})$ in the variables x and y, so that the original PC_Δ sentence is

$$\exists\bar{R}(\forall x)(\exists y)\chi(x,y,\bar{R}).$$

It is easy to show that for any quantifier-free formula $\theta(\bar{w},\bar{R})$ in $L_{\omega\omega}[\tau\cup\bar{R}]$ one can find, recursively from θ, a quantifier-free formula $\theta^*(\bar{w})$ in $L_{\omega\omega}[\tau]$ such that

$$\models\exists\bar{R}\theta(\bar{w},\bar{R})\leftrightarrow\theta^*(\bar{w}).$$

Using the above fact, we let $\varphi_n(x_0, y_0, \ldots, x_n, y_n)$ be a quantifier free formula of $L_{\omega\omega}[\tau]$ which is logically equivalent to

$$\exists \bar{R} \bigwedge_{m \leq n} \chi(x_m, y_m, \bar{R})$$

and then consider the closed game sentence Φ:

$$(\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots) \bigwedge_n \varphi_n(x_0, y_0, x_1, y_1, \ldots, x_n, y_n).$$

We claim that this closed game sentence has the required properties, namely

(a) $\models \exists \bar{R}(\forall x)(\exists y)\chi(x, y, \bar{R}) \rightarrow \Phi$; and
(b) $\models' \Phi \leftrightarrow \exists \bar{R}(\forall x)(\exists y)\chi(x, y, \bar{R})$.

It is clear that if $\mathfrak{A}$ is a structure of vocabulary τ such that

$$\mathfrak{A} \models \exists \bar{R}(\forall x)(\exists y)\chi(x, y, \bar{R}),$$

then the set

$$\Sigma = \{u \in A^{<\omega}\text{: if } (x_0, y_0, \ldots, x_n, y_n) \subseteq u, \text{ then } (\mathfrak{A}, \bar{R}^{\mathfrak{A}}, x_n, y_n) \models \chi(x, y, \bar{R})\}$$

is a winning quasistrategy for Player I in the game associated with Φ.

Assume now that $\mathfrak{A}$ is a countable structure such that $\mathfrak{A} \models \Phi$. Consider a round of the game associated with Φ in which Player II enumerates the universe A of $\mathfrak{A}$ and Player I answers using his winning quasistrategy; that is, the round looks like:

II	a_0		a_1		a_2		$\cdots$	
I		b_0		b_1		b_2		$\cdots$

with $A = \{a_0, a_1, a_2, \ldots\}$.

Since I follows his winning quasistrategy in this round, we have that

$$\mathfrak{A} \models \exists \bar{R} \bigwedge_{m \leq n} \chi(a_m, b_m, \bar{R}), \quad \text{for all} \quad n \in \omega.$$

Let $\mathbf{a}_m$, $\mathbf{b}_m$, for $m < \omega$, be new constant symbols not in τ and consider the set of quantifier-free sentences T, where

$$T = \text{Diagram}(\mathfrak{A}) \cup \{\chi(\mathbf{a}_m, \mathbf{b}_m, \bar{R}): m < \omega\}.$$

T is finitely satisfiable; and, hence, by the compactness theorem it has a model. Since each sentence $\chi(\mathbf{a}_m, \mathbf{b}_m, \bar{R})$ is quantifier-free, this implies that there is a set $\bar{R}^{\mathfrak{A}}$ of relations on A such that

$$\mathfrak{A}, \bar{R}^{\mathfrak{A}} \models \chi(\mathbf{a}_m, \mathbf{b}_m, \bar{R}) \quad \text{for each} \quad m < \omega.$$

However, the sequence $\{a_0, a_1, a_2, \ldots\}$ exhausts the universe A of the structure $\mathfrak{A}$, and therefore we have

$$\mathfrak{A}, \bar{R}^{\mathfrak{A}} \models (\forall x)(\exists y)\chi(x, y, \bar{R}).$$

The main argument remains the same in the general case where we have infinitely many quantifier-free formulas $\chi_n(x_1, \ldots, x_{k_n}, y_1, \ldots, y_{l_n}, \bar{R})$ for $\eta < \omega$. There are only minor combinatorial complications which can be handled by enumerating the tuples $\bar{x}$, $\bar{y}$ of variables in such a way that the variables occurring at stage m of the enumeration have indices $\leq m$. This completes the proof of the first part of the theorem.

In order to establish part (ii) of our result we show first that a $\Sigma_1^1(L_{\omega_1\omega}[\tau])$ formula $\Psi(\bar{z})$ is equivalent to a PC_Δ formula $\Psi'(\bar{z})$ over an expanded vocabulary τ' which contains τ and subsumes weak second-order logic. By applying part (i) of the above, we can find a closed game formula $\Phi'(\bar{z})$ over τ' which is logically equivalent to $\Psi'(\bar{z})$ on countable structures. The closed game formula $\Phi'(\bar{z})$ over τ' can, in turn, be translated to a closed Vaught formula $\Phi(\bar{z})$ over τ. In such a translation the propositional part of the Vaught formula is used to capture the expanded vocabulary.

We should point out that Harnik [1974] and Makkai [1977a] gave direct proofs of part (ii) by associating an appropriate countable admissible fragment with the $\Sigma_1^1(L_{\omega_1\omega}[\tau])$ formula $\Psi(\bar{z})$. The proof is analogous to the one we gave for part (i) with the model existence theorem for fragments used in place of the compactness theorem. ☐

2.2. The Approximations of the Game and the Vaught Formulas

In Section 1 we pointed out that game formulas can be used to capture statements which are not expressible in $L_{\infty\infty}$. We will see here however that the Vaught formulas in general and the game formulas in particular can be approximated by formulas of $L_{\infty\omega}$. This result combined with Theorem 2.1.5 (the theorems of Svenonius and of Vaught) makes it possible to analyze $\Sigma_1^1(L_{\omega_1\omega})$ and $\Pi_1^1(L_{\omega_1\omega})$ formulas via $L_{\omega_1\omega}$ formulas.

2.2.1 Definition (Vaught [1973a]). Assume that $\Phi(\bar{z})$ is a closed Vaught formula of the form

$$\left(\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} \cdots\right) \bigwedge_n \varphi^{\bar{i}, \bar{j}}(\bar{z}, \bar{x}, \bar{y}).$$

Then, for any $n < \omega$, any $(\bar{\imath}, \bar{\jmath}) = (i_0, j_0, \ldots, i_{n-1}, j_{n-1}) \in I^{2n}$, and any ordinal α, by induction on α simultaneously define a formula

$$\delta_\alpha^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}) \equiv \delta_\alpha^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

as follows:

(1) $$\delta_0^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{is}$$

$$\bigwedge_{m \leq n} \varphi^{i_0 j_0 \cdots i_{m-1} j_{m-1}}(\bar{z}, x_0, y_0, \ldots, x_{m-1}, y_{m-1});$$

(2) $$\delta_{\alpha+1}^{i_0 j_0 \cdots i_{n-1} j_{n-1}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{is}$$

$$\forall x_n \bigwedge_{i_n \in I} \exists y_n \bigvee_{j_n \in I} \delta_\alpha^{i_0 j_0 \cdots i_n j_n}(\bar{z}, x_0, y_0, \ldots, x_n, y_n);$$

(3) $$\delta_\alpha^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}) \quad \text{is} \quad \bigwedge_{\beta < \alpha} \delta_\beta^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}), \quad \text{if } \alpha \text{ is a limit ordinal.}$$

We write $\delta_\alpha(\bar{z})$ for the formula $\delta_\alpha^{(\ )}(\bar{z})$, where ( ) is the empty sequence, and we call $\delta_\alpha(\bar{z})$ the *α-th approximation of* $\Phi(\bar{z})$. For each ordinal α, we let $\rho_\alpha(\bar{z})$ be the formula

(4) $$\bigwedge_{n<\omega} \Bigg[\Bigg(\forall x_0 \bigwedge_{i_0 \in I} \forall y_0 \bigwedge_{j_0 \in I} \cdots \forall x_{n-1} \bigwedge_{i_{n-1} \in I} \forall y_{n-1} \bigwedge_{j_{n-1} \in I} \Bigg)$$

$$(\delta_\alpha^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}) \rightarrow \delta_{\alpha+1}^{\bar{\imath}, \bar{\jmath}}(\bar{z}, \bar{x}, \bar{y})) \Bigg].$$

2.2.2. It is clear that for each ordinal α and each $(\bar{\imath}, \bar{\jmath}) \in I^{2n}$, where $n < \omega$, the formulas $\delta_\alpha^{\bar{\imath}, \bar{\jmath}}(\bar{z})$ and $\rho_\alpha(\bar{z})$ are formulas of $L_{\infty\omega}$. Moreover, if $\alpha < \omega_1$, then they are actually formulas of $L_{\omega_1\omega}$.

It is also quite easy to verify that the formulas $\delta_\alpha^{\bar{\imath}, \bar{\jmath}}(\bar{z})$ can be defined by a Σ-recursion as a function of the Vaught formula $\Phi(\bar{z})$, the sequence $\bar{\imath}, \bar{\jmath}$ and the ordinal α. Consequently, if A is an admissible set having ordinal $o(A)$ and if the Vaught formula $\Phi(\bar{z})$ is in A, then for every ordinal $\alpha < o(A)$, the formulas $\delta_\alpha(\bar{z})$ and $\rho_\alpha(\bar{z})$ are elements of A.

2.2.3. If $\Phi(\bar{z})$ is a closed game formula, then the approximations of $\Phi(\bar{z})$ are defined in an analogous way, although they are actually of a simpler form. More specifically, if $\Phi(\bar{z})$ is the closed game formula

$$(\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots) \bigwedge_{n<\omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

then

$$\text{(5)} \qquad \delta^n_0(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{is} \quad \bigwedge_{m \le n} \varphi_m(\bar{z}, x_0, y_0, \ldots, x_{m-1}, y_{m-1}),$$

$$\text{(6)} \qquad \delta^n_{\alpha+1}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{is} \quad \forall x_n \, \exists y_n \delta^{n+1}_\alpha(\bar{z}, x_0, y_0, \ldots, x_n, y_n),$$

$$\text{(7)} \qquad \delta^n_\alpha(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{is} \quad \bigwedge_{\beta < \alpha} \delta^n_\beta(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

for α limit.

We write $\delta_\alpha(\bar{z})$ for the formula $\delta^0_\alpha(\bar{z})$ and call it the α-th *approximation of the closed game formula* $\Phi(\bar{z})$.

Also, we put $\rho_\alpha(\bar{z})$ for the formula

$$\text{(8)} \qquad \bigwedge_{n<\omega} [(\forall x_0 \, \forall y_0 \cdots \forall x_{n-1} \, \forall y_{n-1})(\delta^n_\alpha(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \\ \rightarrow \delta^n_{\alpha+1}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}))].$$

If $\Phi(\bar{z})$ is an open Vaught formula (or an open game formula), then we define the approximations

$$\varepsilon^{i,j}_\alpha(\bar{z}, \bar{x}, \bar{y}) \quad (\text{respectively, } \varepsilon^n_\alpha(\bar{z}, \bar{x}, \bar{y}))$$

of $\Phi(\bar{z})$ in a dual way, so that if

$$\delta^{i,j}_\alpha(\bar{z}, \bar{x}, \bar{y}) \quad (\text{respectively, } \delta^n_\alpha(\bar{z}, \bar{x}, \bar{y}))$$

are the approximations of the closed Vaught formula (or the closed game formula) which is logically equivalent to $\neg\Phi(\bar{z})$, then

$$\varepsilon^{i,j}_\alpha(\bar{z}, \bar{x}, \bar{y}) \quad \text{is logically equivalent to} \quad \neg\delta^{i,j}_\alpha(\bar{z}, \bar{x}, \bar{y})$$

$$(\text{respectively, } \varepsilon^n_\alpha(\bar{z}, \bar{x}, \bar{y}) \text{ is logically equivalent to } \neg\delta^n_\alpha(\bar{z}, \bar{x}, \bar{y})).$$

2.2.4 Example. Let $<$ be a binary relation symbol in the vocabulary τ and let Φ be the open game sentence which asserts that $<$ is well-founded; that is to say, Φ is the sentence

$$(\forall x_0 \, \forall x_1 \, \forall x_2 \cdots)\left(\bigvee_{n \in \omega} \neg(x_{n-1} < x_{n-2})\right).$$

Below we compute the approximations ε_α of Φ and find their meaning:

(i) if $m < \omega$, then $\varepsilon_m = \varepsilon^0_m$ is the sentence

$$(\forall x_0 \, \forall x_1 \cdots \forall x_{m-1})\left(\bigvee_{k \le m} \neg(x_{k-1} < x_{k-2})\right).$$

(ii) $\varepsilon_\omega = \varepsilon^0_\omega$ is the sentence

$$\bigvee_{m<\omega} \varepsilon^0_m = \bigvee_{m<\omega} \left[(\forall x_0 \, \forall x_1 \cdots \forall x_{m-1}) \left(\bigvee_{k \le m} \neg (x_{k-1} < x_{k-2}) \right) \right].$$

Notice that ε_ω asserts that, for some $m < \omega$, there is no descending chain with m elements in $<$. Therefore, ε_ω states that $<$ is a well-founded relation of finite rank.

(iii) $\varepsilon_{\omega+1} = \varepsilon^0_{\omega+1}$ is the sentence

$$\forall x_0 \left(\bigvee_{m<\omega} \forall x_1 \, \forall x_2 \cdots \forall x_{m-1} \left(\bigvee_{k \le m} \neg (x_{k-1} < x_{k-2}) \right) \right).$$

This sentence asserts that, for every element x in the field of $<$, the set of predecessors of x has finite rank. Therefore, $\varepsilon_{\omega+1}$ is equivalent to the assertion that $<$ is a well-founded relation with rank $\le \omega < \omega + 1$.

The pattern revealed in (i), (ii), and (iii) holds in general. Indeed, by induction on α, we can show that for any ordinal α

ε_α asserts that "$<$ is a well-founded relation of rank less than α".

It follows, therefore, that if $\mathfrak{A}$ is a structure of cardinality $\le k$, then

$$\mathfrak{A} \models (\forall x_0 \, \forall x_1 \, \forall x_2 \cdots) \left(\bigvee_{n<\omega} \neg (x_{n-1} < x_{n-2}) \right) \quad \text{iff} \quad \mathfrak{A} \models \bigvee_{\alpha < \kappa^+} \varepsilon_\alpha .$$

Later on we will show that the above equivalence holds for arbitrary open games or for open Vaught formulas. Before developing the general theory of the approximations, we will present the main properties of the finite approximations of game formulas on saturated structures. Consequently, we now consider

2.2.5 Theorem. *Let $\Phi(\bar{z})$ be the closed game formula*

$$(\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots) \bigwedge_{n<\omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

and let $\mathfrak{A}$ be a structure of vocabulary τ.

(i) *If $\mathfrak{A}$ is ω-saturated, then*

$$\mathfrak{A} \models \forall \bar{z} \left(\Phi(\bar{z}) \leftrightarrow \bigwedge_{m<\omega} \delta_m(\bar{z}) \right);$$

(ii) *If $\mathfrak{A}$ is recursively saturated and the sequence $\{\varphi_n(\bar{z}, \bar{x}, \bar{y}) \colon n < \omega\}$ is recursive, then again*

$$\mathfrak{A} \models \forall \bar{z} \left(\Phi(\bar{z}) \leftrightarrow \bigwedge_{m<\omega} \delta_m(\bar{z}) \right).$$

Proof. We outline the argument for part (ii) since that part is the effective version of part (i).

Let $\mathfrak{A}$ be a recursively saturated structure and assume that the sequence $\{\varphi_n(\bar{z}, \bar{x}, \bar{y}) : n < \omega\}$ is recursive. It is clear from the definition of the finite approximations that for any structure $\mathfrak{A}$

$$\mathfrak{A} \models \forall \bar{z}(\Phi(\bar{z}) \to \delta_m(\bar{z})) \quad \text{for all} \quad m < \omega.$$

Thus, it remains to show that, under the above hypotheses,

$$\mathfrak{A} \models \forall \bar{z}\left(\bigwedge_{m<\omega} \delta_m(\bar{z}) \to \Phi(\bar{z})\right).$$

The main idea comes from the proof of the Gale–Stewart theorem in Section 1. More specifically, as in Theorem 1.2.4, we consider the monotone operator $\varphi(\bar{z}, u, S)$, where

$$\begin{aligned} \varphi(\bar{z}, u, S) \Leftrightarrow (u \in A^{<\omega} &\text{ and } u \text{ has even length}) \\ &\& \text{ (if } u = (x_0, y_0, \ldots, x_{n-1}, y_{n-1}), \\ &\text{then } (\bigvee_{k \le n} \neg\, \varphi_k(\bar{z}, x_0, y_0, \ldots, x_{k-1}, y_{k-1}) \\ &\vee (\exists x\, \forall y)((\bar{z}, u^\frown(x, y)) \in S)). \end{aligned}$$

Let φ^α be the stages of the inductive definition generated by φ. That is,

$$\varphi^0 = \{(\bar{z}, u) : \varphi(\bar{z}, u, \varnothing)\}, \quad \text{and} \quad \varphi^\alpha = \left\{(\bar{z}, u) : \varphi\left(\bar{z}, u, \bigcup_{\beta<\alpha} \varphi^\beta\right)\right\}.$$

From this, it is easy to show that, for any $m < \omega$ and any $n < \omega$, we have

(1) $$(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \in \varphi^m \quad \text{iff} \quad (\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \notin \delta_m^n.$$

Since the sequence $\{\varphi_n(\bar{z}, \bar{x}, \bar{y}) : n < \omega\}$ is recursive, we can view $\varphi(\bar{z}, u, S)$ as a Σ_1 monotone inductive definition on $\mathrm{HYP}_{\mathfrak{A}}$. But $\mathfrak{A}$ is recursively saturated and so $o(\mathrm{HYP}_{\mathfrak{A}}) = \omega$. Therefore, by Gandy's theorem, (see Barwise [1975]) the inductive definition must close off at ω steps, so that we then have

(2) $$\varphi^\infty = \bigcup_{m<\omega} \varphi^m.$$

Assume now that $\mathfrak{A}, \bar{z} \models \bigwedge_{m<\omega} \delta_m(\bar{z})$. Then $\bar{z} \notin \varphi^m$ for all $m < \omega$ by the equivalence given in (1). Hence, $\bar{z} \notin \varphi^\infty$ by (2). The proof of the Gale–Stewart theorem implies then that Player I has a winning quasistrategy in the closed game $G(\forall\exists, \bigwedge_{n<\omega}\varphi_n)$. Hence, $\mathfrak{A}, \bar{z} \models \Phi(\bar{z})$. □

2.2.6. In many respects, the idea behind the approximations has its origins in classical descriptive set theory and the approximations of the operator $\mathscr{A}$ (see, for example, Kuratowski [1966]). The finite approximations of closed game formulas were introduced by Keisler [1965c], who established, among other results, the first part of Theorem 2.2.5. Moschovakis [1969, 1971, 1974a] developed the theory of positive elementary inductive definability on arbitrary structures $\mathfrak{A}$ which possess a first-order coding machinery of finite sequences. He obtained the basic connection between inductive definability and game quantification; and, in essence, discovered the properties of the approximations δ_α of closed game formulas.

However, Moschovakis' results were of a local nature, since they dealt with an arbitrary but fixed structure. In the abstracts Chang–Moschovakis [1968], Chang [1968a], and the paper by Chang [1971b], the approximations of the game formulas are used implicitly in the study of global definability. The approximations of the Vaught formulas were introduced by Vaught [1973b] who established their main properties and used them in the study of $\Sigma_1^1(L_{\omega_1\omega})$ and $\Pi_1^1(L_{\omega_1\omega})$ formulas.

2.2.7 Theorem (Vaught [1973b]). *Let $\Phi(\bar{z})$ be a closed Vaught formula of the form*

$$\left(\forall x_0 \bigwedge_{i_0\in I} \exists y_0 \bigvee_{j_0\in I} \forall x_1 \bigwedge_{i_1\in I} \exists y_1 \bigvee_{j_1\in I} \cdots\right) \bigwedge_{n} \varphi^{\bar{\imath},\bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}).$$

Then, we have

(i) *for any ordinals α, β with $\alpha > \beta$ and for any $\bar{\imath}, \bar{\jmath}$,*

$$\models \delta_\alpha^{\bar{\imath},\bar{\jmath}}(\bar{z}, \bar{x}, \bar{y}) \to \delta_\beta^{\bar{\imath},\bar{\jmath}}(\bar{z}, \bar{x}, \bar{y});$$

(ii) *for any ordinal α,*

$$\models \Phi(\bar{z}) \to \delta_\alpha(\bar{z}) \text{ and } \models (\delta_\alpha(\bar{z}) \wedge \rho_\alpha(\bar{z})) \to \Phi(\bar{z});$$

(iii) *for any structure $\mathfrak{A}$ of cardinality $\leq \kappa$,*

(1) $$\mathfrak{A} \models \forall\bar{z}\left(\bigvee_{\alpha<\kappa^+} \delta_\alpha(\bar{z})\right);$$

(2) $$\mathfrak{A} \models \forall\bar{z}\left[\Phi(\bar{z}) \leftrightarrow \bigwedge_{\alpha<\kappa^+} \delta_\alpha(\bar{z})\right];$$

(3) $$\mathfrak{A} \models \forall\bar{z}\left[\Phi(\bar{z}) \leftrightarrow \bigvee_{\alpha<\kappa^+} (\rho_\alpha(\bar{z}) \wedge \delta_\alpha(\bar{z}))\right];$$

(iv) *Moreover, if M is an admissible set, $o(M) > \omega$, $\Phi(\bar{z})$ is in M and $\mathfrak{A} \in M$, then*

$$\mathfrak{A} \models \forall \bar{z} \rho_{o(M)}(\bar{z})$$

and hence

$$\mathfrak{A} \models \forall \bar{z}\left(\Phi(\bar{z}) \leftrightarrow \bigwedge_{\alpha < o(M)} \delta_\alpha(\bar{z})\right).$$

Hint of Proof. Part (i) is proven by induction on the ordinal α. Part (ii) follows easily from the definitions of the formulas δ_α and ρ_α. For example, if $\mathfrak{A}$ is a structure such that $\mathfrak{A}, \bar{z} \models \delta_\alpha(\bar{z}) \wedge \rho_\alpha(\bar{z})$, then the set

$$\Sigma = \{u \in (A \times I)^{<\omega} : (\forall v)((v = (x_0, i_0, y_0, j_0, \ldots, x_{n-1}, i_{n-1}, y_{n-1}, j_{n-1}) \\ \& \ (v \subseteq u)) \rightarrow \mathfrak{A}, \bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1} \models \delta_\alpha^{\bar{i},\bar{j}})\}$$

is a winning quasistrategy for Player I in the game associated with $\Phi(\bar{z})$. Hence, $\mathfrak{A}, \bar{z} \models \Phi(\bar{z})$.

The proof of parts (iii) and (iv) requires the inductive analysis of the dual open game and is similar to the proof of Theorem 2.2.5. In (iii), a cardinality argument shows that the corresponding monotone operator closes off at some ordinal $\alpha < \kappa^+$. In (iv) this is proved using Gandy's theorem or directly using a boundedness argument. $\square$

The following result is an immediate consequence of Theorem 2.2.7 in which we take $\kappa = \omega$ in part (iii). It has interesting applications in descriptive set theory.

2.2.8 Corollary. *Let $\Phi(\bar{z})$ be a closed Vaught formula. Then*

(i) $\models' (\forall \bar{z})\left(\Phi(\bar{z}) \leftrightarrow \bigwedge_{\alpha < \omega_1} \delta_\alpha(\bar{z})\right)$

(ii) $\models' (\forall \bar{z})\left(\Phi(\bar{z}) \leftrightarrow \bigvee_{\alpha < \omega_1} (\delta_\alpha(\bar{z}) \wedge \rho_\alpha(\bar{z}))\right)$ $\square$

Theorem 2.2.7 is the main result on the approximations of the closed Vaught and the closed game formulas. We can, of course, formulate and prove an analogous "dual" result on the approximations of the open Vaught and the open game formulas.

Burgess [1977] introduced a notion of approximations for formulas of abstract logics and showed that if $(L^*, \models^*)$ is an absolute logic, then the formulas of L^* can be approximated by formulas of $L_{\infty\omega}$. His proof makes use of Theorem 2.2.7, since he shows first that any formula of L^* can be approximated by formulas involving game quantification and arbitrary disjunctions and conjunctions. More about these results can be found in Chapter XVII of this volume.

In what follows we will combine the Svenonius–Vaught result which is given in Theorem 2.1.5 with the results on approximations in order to study properties of the $\Sigma_1^1(L_{\omega_1\omega})$ and the $\Pi_1^1(L_{\omega_1\omega})$ formulas. We begin by proving a strong version of the interpolation theorem for $L_{\omega\omega}$.

2.2.9 Theorem. *Let Φ, Ψ be $\Sigma_1^1(L_{\omega\omega})$ sentences and let $\delta_m^{\Psi^*}$, where $m < \omega$, be the finite approximations of the closed game sentence Ψ^* which is equivalent to Ψ on countable structures.*

If $\models \Phi \to \neg\Psi$, then there is some $m < \omega$ such that $\models \Phi \to \neg\delta_m^{\Psi^*}$.

Proof. In order to derive a contradiction, we assume that $\models \Phi \to \neg\Psi$, but for all $m < \omega$, the sentence $\Phi \wedge \delta_m^{\Psi^*}$ has a model. Consider then the closed game sentence Φ^* which is equivalent to Φ on countable structures and let $\delta_n^{\Phi^*}$, where $n < \omega$, be its finite approximations. Since $\models \Phi \to \Phi^*$, $\models \Phi^* \to \bigwedge_{n<\omega} \delta_n^{\Phi^*}$ and $\models \delta_m^{\Psi^*} \to \delta_{m'}^{\Psi^*}$, for $m > m'$, the set

$$T = \{\delta_n^{\Phi^*} \wedge \delta_m^{\Psi^*} : n, m < \omega\}$$

is finitely satisfiable. Let $\mathfrak{A}$ be a countable, recursively saturated model of T. Then $\mathfrak{A} \models (\bigwedge_{n<\omega} \delta_n^{\Phi^*}) \wedge (\bigwedge_{m<\omega} \delta_m^{\Psi^*})$. But by Theorem 2.2.5, we have

$$\mathfrak{A} \models \Phi^* \leftrightarrow \bigwedge_{n<\omega} \delta_n^{\Phi^*} \quad \text{and} \quad \mathfrak{A} \models \Psi^* \leftrightarrow \bigwedge_{m<\omega} \delta_m^{\Psi^*}$$

so that $\mathfrak{A} \models \Phi^* \wedge \Psi^*$. However, since $\mathfrak{A}$ is countable, $\mathfrak{A} \models (\Phi \leftrightarrow \Phi^*) \wedge (\Psi \leftrightarrow \Psi^*)$ and hence

$\mathfrak{A} \models \Phi \wedge \Psi$. But this is a condiction of the hypothesis that

$\models \Phi \to \neg\Psi$. □

The next result was established by Vaught [1973b] and has turned out to have many interesting consequences.

2.2.10 Vaught's Covering Theorem. *Let Φ, Ψ be $\Sigma_1^1(L_{\omega_1\omega})$ sentences and let $\delta_\alpha^{\Psi^*}$, for α an ordinal, be the approximations of the closed Vaught sentence Ψ^* which is equivalent to Ψ on countable structures.*

(i) *If $\models \Phi \to \neg\Psi$, then there is an ordinal $\beta < \omega_1$ such that $\models \Phi \to \neg\delta_\beta^{\Psi^*}$.*
(ii) *Moreover, if A is a countable admissible set, Φ and Ψ are $\Sigma_1^1(L_A)$ and $\models \Phi \to \neg\Psi$, then there is some ordinal $\beta < o(A)$ such that $\models \Phi \to \neg\delta_\beta^{\Psi^*}$.*

Proof. Here we give the proof for the case where Φ and Ψ are $\Sigma_1^1(L_{\omega_1\omega})$ sentences and, at the same time, point out the modifications that are needed if Φ and Ψ are $\Sigma_1^1(L_A)$.

Let Φ and Ψ be $\Sigma_1^1(L_{\omega_1\omega})$ sentences such that $\models \Phi \to \neg\Psi$ and let Φ^* and Ψ^* be the closed Vaught sentences which are respectively equivalent to Φ and Ψ on countable structures. The key idea is that if $\neg\Psi$ holds, then we can use the inductive analysis of the open Vaught formula which is equivalent to $\neg\Psi^*$ in order to extract a $\Sigma_1^1(L_{\omega_1\omega})$ sentence which pins down ordinals. But then the undefinability of well-order in $L_{\omega_1\omega}$ implies that all ordinals pinned down in this way are bounded by some ordinal $\beta < \omega_1$. From this, it will follow that $\models \Phi \to \neg\delta_\beta^{\Psi^*}$. We now provide some of the technical details there are necessary to make this idea precise.

The closed Vaught sentence Ψ^* is of the form

$$\left(\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} \cdots\right) \bigwedge_{n<\omega} \psi^{\bar{i},\bar{j}}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

where I is a countable set and the $\psi^{\bar{i},\bar{j}}(\bar{x}, \bar{y})$ are formulas of $L_{\omega_1\omega}$. It is easy to see that if $\delta_\alpha^{\Psi^*,\bar{i},\bar{j}}$ are the approximations of Ψ^* for α an ordinal and $(\bar{i}, \bar{j}) \in I^{2n}$, then

$$(1) \qquad \delta_\alpha^{\Psi^*,\bar{i},\bar{j}}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) \quad \text{iff}$$

$$\bigwedge_{\beta<\alpha} \left(\forall x_n \bigwedge_{i_n \in I} \exists y_n \bigvee_{j_n \in I}\right) \delta_\beta^{\Psi^*,\bar{i},i_n,\bar{j},j_n}(x_0, y_0, \ldots, x_n, y_n)$$

$$\wedge \bigwedge_{k\le n} \psi^{\bar{i}_k,\bar{j}_k}(x_0, y_0, \ldots, x_{k-1}, y_{k-1}).$$

It is clear from the above equivalence that the approximations of Ψ^* would have the same meaning if, instead by induction on the ordinals, they were defined by induction on the rank of an arbitrary well-ordering $<$. We will now consider new relation symbols $<$, $P^{\bar{i},\bar{j}}$ for $(\bar{i}, \bar{j}) \in I^{2n}$, $n < \omega$, and a new constant symbol c.

We claim that in the expanded vocabulary $\tau' = \tau \cup \{<, c\} \cup \{P^{\bar{i},\bar{j}}: (\bar{i}, \bar{j}) \in I^{2n}, n < \omega\}$ we can find a sentence χ of $L_{\omega_1\omega}[\tau']$ which asserts that $<$ is a linear ordering and that the relations $P^{\bar{i},\bar{j}}$ satisfy the equivalence given in (1) above along $<$. More precisely, we let χ be the conjunction of the following sentences of $L_{\omega_1\omega}[\tau']$:

(i) "$<$ is a linear ordering with greatest element c";
(ii) $P^{(\ )}(c)$;
(iii) the universal closure of the formula,

$$P^{\bar{i},\bar{j}}(u, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

$$\leftrightarrow (\forall v < u)\left(\forall x_n \bigwedge_{i_n \in I} \exists y_n \bigvee_{j_n \in I}\right) P^{\bar{i},i_n,\bar{j},j_n}(v, x_0, y_0, \ldots, x_n, y_n)$$

$$\wedge \bigwedge_{k\le n} \psi^{\bar{i}_k,\bar{j}_k}(x_0, y_0, \ldots, x_{k-1}, y_{k-1}),$$

for $(\bar{i}, \bar{j}) \in I^{2n}$, $n \in \omega$.

It follows from the preceding comments that if a structure $\mathfrak{A}$ is a model of χ and u is an element of $<^{\mathfrak{A}}$ of rank α, then for any $\bar{i}, \bar{j}$, we have

$$\{(\bar{x}, \bar{y}): P^{\mathfrak{A},\bar{i},\bar{j}}(u, \bar{x}, \bar{y})\} = \{(\bar{x}, \bar{y}): \mathfrak{A}, \bar{x}, \bar{y} \models \delta_\alpha^{\Psi^*,\bar{i},\bar{j}}\}.$$

We will show now that the sentence $(\neg\Psi^*) \wedge \chi$ pins down ordinals. Indeed, we claim that:

(2) if $\mathfrak{A}$ is a structure of vocabulary τ' such that $\mathfrak{A} \models (\neg\Psi^*) \wedge \chi$, then $<^{\mathfrak{A}}$ is a well-ordering of its field.

Otherwise, let $\mathfrak{A} \models (\neg\Psi^*) \wedge \chi$ and let $c >^{\mathfrak{A}} v_1 >^{\mathfrak{A}} v_2 >^{\mathfrak{A}} \cdots >^{\mathfrak{A}} v_n >^{\mathfrak{A}} v_{n+1} >^{\mathfrak{A}} \cdots$ be an infinite descending chain in the field of $<^{\mathfrak{A}}$. Since $\mathfrak{A} \models \chi$, we can use then the conjucts given in (ii) and (iii) of χ and the infinite descending chain above to define a winning quasistrategy for Player I in the game associated with Ψ^*. Hence we have that $\mathfrak{A} \models \Psi^*$. But this is a contradiction.

In order to complete the proof of the theorem, we observe that since $\models \Phi \to \neg\Psi$ and $\models' \Psi \leftrightarrow \Psi^*$, we must have that $\models \Phi \to \neg\Psi^*$. It thus follows from (2) above that we have

(3) if $\mathfrak{A}$ is a structure of vocabulary τ' such that $\mathfrak{A} \models \Phi \wedge \chi$, then $<^{\mathfrak{A}}$ is a well-ordering of its field.

The undefinability of well-order in $L_{\omega_1\omega}$ now implies that there is an ordinal $\beta < \omega_1$ such that if $\mathfrak{A} \models \Phi \wedge \chi$, then $<^{\mathfrak{A}}$ has rank less than β. As a consequence, the sentence $\Phi \wedge \delta_\beta^{\Psi^*}$ has no model and therefore $\models \Phi \to \neg\delta_\beta^{\Psi^*}$.

If Φ and Ψ are $\Sigma_1^1(L_A)$, where A is a countable admissible set, then the result can be proved by an entirely analogous argument using the effective versions of Theorems 2.1.5 and 2.2.7, and the theorem for pinning down ordinals in admissible fragments (for the latter result, see Barwise [1975] or Chapter VIII of this volume). Notice also that if A = HF, then the result was proved in Theorem 2.2.9. □

Although Vaught's covering theorem is a generalization of Theorem 2.2.9, its proof appears to be quite different from the one given for Theorem 2.2.9. Therefore, it is natural to ask if Vaught's covering theorem can be proved by combining compactness results with recursive saturation. Harnik [1974] gave such a proof (his proof can be found also in Makkai [1977a]) using the Barwise compactness theorem for a countable admissible fragment A and the existence of Σ_A-saturated models. For the definition and related results about Σ_A-saturation, the reader should also see Section 7, Chapter VIII of this volume.

2.3. Some Applications of Game Quantification

The results in Sections 2.1 and 2.2 have many interesting applications to the model theory of $L_{\omega_1\omega}$ and admissible fragments L_A. It actually turns out that we can derive the main theorems about compactness, abstract completeness, and interpolation in $L_{\omega_1\omega}$ or in L_A from the Svenonius–Vaught theorem, the approximations and the covering theorem. Since these results are well known and are discussed in Chapter VIII of the present volume, we will here restrict ourselves to merely listing some of the applications and making occasional brief comments on the proofs.

2.3.1 Applications of the Svenonius–Vaught Theorem. Vaught [1973b] obtained a proof of the Barwise compactness theorem using tools from the theory of game quantification. His argument consists of the following two independent parts:

(i) Let A be an arbitrary admissible set such that $\omega \in A$ and consider the class of bounded open game formulas. These are game formulas for which the associated game is bounded for Player II in the sense that his next move must belong to the union of the moves played thus far. More precisely, a *bounded open game formula* $\Phi(z)$ is of the form

$$[(\exists x_0)(\forall y_0 \in z \cup x_0)(\exists x_1)(\forall y_1 \in z \cup x_0 \cup y_0 \cup x_1)\cdots]$$
$$\bigvee_{n<\omega} \varphi^n(z, x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

where each $\varphi^n(z, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is a Δ_0 formula.

Vaught [1973b] showed that every admissible set A with $\omega \in A$ *reflects* bounded open game formulas. That is, if $\Phi(z)$ is such a formula and $A, z \models \Phi(z)$, then there is a transitive set w such that $z \in w \in A$ and $\langle w, \in\rangle, z \models \Phi(z)$.

(ii) The proof of the Svenonius–Vaught theorem (2.1.5) can be easily adapted to show that if A is in addition countable, then every strict-Π_1^1 formula is equivalent on A to a bounded open game formula. It then follows from part (i) that if A is a countable admissible set with $\omega \in A$, then A satisfies strict Π_1^1-reflection, and hence A is Σ_1-compact.

2.3.2 Applications of the Approximations. (i) Every $\Sigma_1^1(L_{\omega_1\omega})$ class of countable models is the intersection of $\aleph_1$ $L_{\omega_1\omega}$-elementary classes.

(ii) Every $\Sigma_2^1(L_{\omega_1\omega})$ class of countable models is the union of $\aleph_1$ $L_{\omega_1\omega}$-elementary classes.

These two results are rather direct consequences of Corollary 2.2.8. The first result, in turn, implies that every analytic set of reals is the intersection of $\aleph_1$ Borel sets. On the other hand, the second result yields Scott's isomorphism theorem for countable structures, since if $\mathfrak{A}$ is countable, then the collection $\{\mathfrak{B}: \mathfrak{B} \approx \mathfrak{A}\}$ is a $\Sigma_1^1(L_{\omega_1\omega})$ class of countable models.

Other applications of the approximation theorem given in Section 2.2.7 include:

(iii) *The Reduction Principle for $\Pi_1^1(L_{\omega_1\omega})$ Classes of Countable Models.* This principle asserts that if $\mathcal{K}_1$, $\mathcal{K}_2$ are two $\Pi_1^1(L_{\omega_1\omega})$ classes of countable models, then we can find two other $\Pi_1^1(L_{\omega_1\omega})$ classes $\mathcal{K}'_1$, $\mathcal{K}'_2$ such that $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{K}'_1 \cup \mathcal{K}'_2$ and $\mathcal{K}'_1 \cap \mathcal{K}'_2 = \varnothing$.

(iv) *The Abstract Completeness Theorem.* This result states that if A is a countable admissible set, then the set of valid sentences in L_A is Σ_1 on A uniformly.

2.3.3 Applications of the Covering Theorem. In this discussion, we will examine:

(i) The interpolation theorem for $L_{\omega_1\omega}$ and countable admissible fragments.
(ii) The undefinability of well-order in $L_{\omega_1\omega}$ and the theorem on pinning down ordinals in countable admissible fragments.

The interpolation theorem follows immediately from the covering theorem. Actually, in addition we obtain some information about the interpolant. For the undefinability of well-order, we will assume that $\varphi(<)$ is a $\Sigma_1^1(L_{\omega_1\omega})$ sentence such that if $\mathfrak{A} \models \varphi(<)$, then $<^{\mathfrak{A}}$ is a well-ordering. Then $\models \varphi(<) \to \neg(\exists x_0 \exists x_1 \cdots) \bigwedge_{n<\omega} (x_{n+1} < x_n)$, hence there is an ordinal $\beta < \omega_1$ such that $\models \varphi(<) \to \neg\delta_\beta$, where δ_α are the approximations of $(\exists x_0 \exists x_1 \cdots) \bigwedge_{n<\omega} (x_{n+1} < x_n)$. It follows now immediately from Sections 2.2.3 and 2.2.4 that $\neg\delta_\beta$ asserts that the rank of $<$ is less than β.

The proof of the covering theorem we gave here makes use of the undefinability of well-order. However, Harnik's [1974] proof of this result does not depend on it, so that we can first prove the covering theorem and then establish the undefinability of well-order. This is, for example, the approach taken by Makkai [1977a].

Further applications of this material can be found in Makkai [1973b, 1974b], Vaught [1974], Harnik [1976] and Harnik–Makkai [1976].

2.4. On the Connection with Invariant Descriptive Set Theory

We have here tried to develop the theory of game quantification in a more or less self-contained way by using methods from the model theory of $L_{\omega_1\omega}$ and admissible fragments.

At this point we should mention that there is also a very interesting connection between game quantification and invariant descriptive set theory. It is part of the general interaction between infinitary logic and descriptive set theory, which arises by identifying countable structures with elements of a product of topological spaces of the form 2^{ω^n}, ω^{ω^n}, or ω^n. If φ is a sentence of some infinitary logic, then the collection of all countable models of φ can be viewed as a subset of such a product which is invariant under a certain action of the group $\omega!$ of the permutations on ω, or under a natural equivalence relation. Topological methods and results from invariant descriptive set theory can then be used to derive theorems of infinitary logic. In particular, some of the results we have presented here can be studied by these methods. This direction has been pursued with much success by Vaught [1974], Burgess–Miller [1975], Miller [1978] and others.

3. Model Theory for Game Logics

The aim of this section is to present an overview of the model theory for the infinitary logics $L_{\infty G}$ and $L_{\infty V}$ associated with game quantification. The main result is that the logics $L_{\infty G}$ and $L_{\infty V}$ are absolute in the sense of Barwise [1972a]. Many model-theoretic properties of $L_{\infty G}$ and $L_{\infty V}$ then follow from this result and from the fact that both of these logics can express the notion of well-foundedness.

3.1. The Infinitary Logics $L_{\infty G}$ and $L_{\infty V}$

We will begin our discussion with

3.1.1 Definition. The infinitary logic $(L_{\infty G}, \models_{L_{\infty G}})$ is determined by the class $L_{\infty G}[\tau]$ of *$L_{\infty G}$-formulas of vocabulary* τ and the *relation of satisfaction* $\models_{L_{\infty G}}$ between sentences of $L_{\infty G}[\tau]$ and structures of vocabulary τ. If τ is a vocabulary, then $L_{\infty G}[\tau]$ is the smallest class which:

(i) contains all atomic formulas over the vocabulary τ;
(ii) is closed under negation $\neg$;
(iii) is closed under single existential $\exists$ and single universal $\forall$ quantification;
(iv) if Φ is a set of formulas of $L_{\infty G}[\tau]$ with only finitely many free variables in Φ, then the conjunction $\bigwedge \Phi$ and the disjunction $\bigvee \Phi$ are also formulas of $L_{\infty G}[\tau]$;
(v) if $\{\varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1}) : n < \omega\}$ are formulas of $L_{\infty G}[\tau]$ in the displayed free variables, then the expressions

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_{n<\omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

and

$$(\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \bigvee_{n<\omega} \varphi_n(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

are also formulas of $L_{\infty G}[\tau]$ with $\bar{z}$ as free variables.

The *relation of satisfaction* "$\mathfrak{A} \models_{L_{\infty G}} \psi$" between sentences of $L_{\infty G}[\tau]$ and structures of vocabulary τ is defined inductively, using the game theoretic interpretation from Section 1 for the clause given in (v). It is understood that if the full axiom of choice is available in the metatheory, then the interpretation is via winning strategies. If one is working only with the axiom of dependent choices, then the interpretation of the clause in (v) is given using winning quasistrategies.

If τ is a vocabulary and HC is the set of hereditarily countable sets, then we put

$$L_{\omega_1 G}[\tau] = L_{\infty G}[\tau] \cap \mathrm{HC}.$$

Notice that the open game and closed game formulas that we considered in Section 2 are actually elements of $L_{\omega_1 G}[\tau]$.

3.1.2 Definition. The infinitary logic $(L_{\infty V}, \models_{L_{\infty V}})$ is defined as follows:
If τ is a vocabulary, then the collection $L_{\infty V}[\tau]$ is the smallest class of formulas which satisfies the closure properties (i), (ii), (iii), and (iv) in the previous definition and in addition is such that:

(v′) if I is a non-empty set and for every $n \in \omega$ and every $(\bar{\imath}, \bar{\jmath}) \in I^{2n}$ $\varphi^{\bar{\imath},\bar{\jmath}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ is a formula of $L_{\infty V}[\tau]$ in the displayed free variables, then the expressions

$$\Big(\forall x_0 \bigwedge_{i_0 \in I} \exists y_0 \bigvee_{j_0 \in I} \forall x_1 \bigwedge_{i_1 \in I} \exists y_1 \bigvee_{j_1 \in I} \cdots\Big) \bigwedge_{n < \omega} \varphi^{\bar{i}, \bar{j}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

and

$$\Big(\exists x_0 \bigvee_{i_0 \in I} \forall y_0 \bigwedge_{j_0 \in I} \exists x_1 \bigvee_{i_1 \in I} \forall y_1 \bigwedge_{j_1 \in I} \cdots\Big) \bigvee_{n < \omega} \varphi^{\bar{i}, \bar{j}}(\bar{z}, x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

are also formulas of $L_{\infty V}[\tau]$ with $\bar{z}$ as free variables.

The relation of satisfaction "$\mathfrak{A} \models_{L_{\infty V}} \psi$" between sentences of $L_{\infty V}[\tau]$ and structures of vocabulary τ is defined inductively, again associating a game with the formulas in (v′).

We put

$$L_{\omega_1 V}[\tau] = L_{\infty V}[\tau] \cap \mathrm{HC}$$

and observe that the open Vaught and closed Vaught formulas of Section 2 are elements of $L_{\omega_1 V}[\tau]$.

It is not hard to verify that the logic $L_{\omega_1 V}$ is stronger than the logic $L_{\omega_1 G}$. Indeed, $L_{\omega_1 V}$—and, of course, $L_{\infty V}$—can express infinitary connectives which cannot be captured by $L_{\omega_1 G}$ (nor by $L_{\infty G}$ for that matter).

Vaught [1974] pointed out that the weak second-order version of $L_{\omega_1 V}$ coincides with $L_{\omega_1 V}$, so that $L_{\omega_1 V}$ is invariant under passage to weak second-order logic, while $L_{\omega_1 G}$ is not. However, as we have mentioned before, over countable models possessing a first-order coding machinery of finite sequences, the infinitary logics $L_{\omega_1 G}$ and $L_{\omega_1 V}$ have the same expressive power.

3.1.3. We now recall the definition of an absolute logic from Chapter XVII, a definition which was originally given in Barwise [1972a].

Let T be a set theory at least as strong as the admissible set theory KP and let $(L, \models_L)$ be an abstract logic. We say that the logic $(L, \models_L)$ is *absolute relative to T* if:

(i) The relation "φ is a sentence of $L[\tau]$" is a Σ_1^T predicate of φ and the vocabulary τ; and
(ii) if φ is a sentence of $L[\tau]$ and $\mathfrak{A}$ is a structure of vocabulary τ, then the predicate "$\mathfrak{A} \models_L \varphi$" is a Δ_1^T predicate of $\mathfrak{A}$, φ and τ.

A logic $(L, \models_L)$ is *strictly absolute* if it is absolute relative to the admissible set theory KP.

One of the main results of Barwise [1972a] (see also Chapter XVII of the present volume) asserts that if $(L, \models_L)$ is a strictly absolute logic, then $L \leq L_{\infty\omega}$. However, we showed in Section 1.1.4 that there is a formula of $L_{\omega_1 G}$ which asserts that:

"$<$ is a well-ordering of order type $\gamma + \gamma$ for some ordinal γ."

Since the above statement is not expressible in $L_{\infty\infty}$, we obtain the following

3.1.4 Theorem. *The infinitary logics $L_{\omega_1 G}$, $L_{\omega_1 V}$, $L_{\infty G}$, $L_{\infty V}$ are not strictly absolute.* ☐

It is now natural to ask whether or not the game logics are absolute relative to some true set theory. The answer to this question is provided by the following result of Barwise [1972a].

3.1.5 Theorem. *The infinitary logics $L_{\omega_1 G}$, $L_{\omega_1 V}$, $L_{\infty G}$ and $L_{\infty V}$ are all absolute relative to the theory* KP + Σ_1*-separation* + *Axiom of Dependent Choices.*

Sketch of Proof. Once more the main idea comes from the inductive analysis of the open games, which was given in the proof of the Gale–Stewart theorem. An inspection of the proof given there reveals, first of all, that the Gale–Stewart theorem is itself provable in KP + Σ_1-separation + axiom of dependent choices. To establish that satisfaction is absolute for, say, the infinitary logic $L_{\omega_1 G}$, we define by induction on the construction of the $L_{\omega_1 G}[\tau]$-formulas a Σ_1 predicate $P(\tau, \mathfrak{A}, \psi, i)$ such that if $\mathfrak{A}$ is a structure of vocabulary τ, then

$$P(\tau, \mathfrak{A}, \psi, i) \quad \text{iff} \quad (i = 0 \,\&\, \mathfrak{A} \models \psi) \vee (i = 1 \,\&\, \mathfrak{A} \not\models \psi).$$

This automatically takes care of the negations, while for the crucial clause given in (v) of Definition 3.1.1 we use the Gale–Stewart theorem and Σ_1-separation. More precisely, if ψ is the sentence

$$(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_{n<\omega} \psi_n(x_0, y_0, \ldots, x_{n-1}, y_{n-1}),$$

then

$$P(\tau, \mathfrak{A}, \psi, 0) \Leftrightarrow \text{Player I has a winning quasistrategy in } G\Big(\forall\exists, \bigwedge_{n<\omega} \psi_n\Big)$$

$$\Leftrightarrow (\ ) \notin \varphi^\infty,$$

and

$$P(\tau, \mathfrak{A}, \psi, 1) \Leftrightarrow \text{Player I has a winning quasistrategy in } G\Big(\exists\forall, \bigvee_{n<\omega} \neg\psi_n\Big)$$

$$\Leftrightarrow (\ ) \in \varphi^\infty,$$

where φ^∞ is the smallest fixed point of the monotone operator $\varphi(u, S)$ associated with the open game $G(\exists\forall, \bigvee_{n<\omega} \neg\psi_n)$, just as in the proof of the Gale–Stewart theorem given in Section 1.2.4. □

3.2. Model-theoretic Properties of the Logics $L_{\infty G}$ and $L_{\infty V}$

The following model-theoretic properties of the infinitary logic $L_{\infty V}$ follow from its absoluteness and the results in Chapter XVII of this volume.

3.2.1 Theorem. (i) *The logic $L_{\infty V}$ has the downward Skolem–Löwenheim property to ω. That is, if a sentence φ of $L_{\infty V}[\tau]$ has a model, then it has a countable model.*

(ii) *The logic $L_{\infty V}$ has the Karp property. That is to say, if $\mathfrak{A}$, $\mathfrak{B}$ are structures of vocabulary τ which satisfy the same sentences of $L_{\infty\omega}[\tau]$, then they satisfy the same sentences of $L_{\infty V}[\tau]$.* □

Barwise [1972a] showed that these properties are shared by any abstract logic which is absolute. Moreover, Barwise [1972a] and Burgess [1977] established certain negative results about logics which are absolute and *unbounded*. That is, the collection of well-founded structures is a PC class. Since the infinitary logics $L_{\omega_1 G}$, $L_{\omega_1 V}$, $L_{\infty G}$ and $L_{\infty V}$ can all express the notion of well-foundedness, we have

3.2.2 Theorem. (i) (Failure of the Abstract Completeness Theorem). *The set of valid sentences of the infinitary logic $L_{\omega_1 G}$ is a complete Π_1 set on* HC. *The same is true for the validities of the infinitary logic $L_{\omega_1 V}$.*

(ii) *The infinitary logics $L_{\omega_1 G}$ and $L_{\omega_1 V}$ do not satisfy: the Craig interpolation theorem, the Δ-interpolation theorem, the Beth definability theorem, and the weak Beth definability theorem.* □

The reader is referred to Chapter II for the definitions of these notions and to Chapter XVII for the proof of the above theorem.

3.2.3. The approximation theory for Vaught formulas, which was developed in Section 2, can be easily extended to arbitrary formulas of $L_{\infty V}$, the main result being that with any sentence ψ of $L_{\infty V}$ we can associate sentences δ_α^ψ of $L_{\infty\omega}$, for α an ordinal, such that

$$\models \psi \leftrightarrow \bigwedge_\alpha \delta_\alpha^\psi.$$

Green [1979] used these approximations to introduce consistency properties for $L_{\infty V}$ and obtained a model existence theorem for game logics. As we mentioned in

Corollary 2.2.8, Burgess [1977] extended the approximation theory to any absolute logic. Finally, Harnik [1976], using the approximations and model theoretic forcing, established certain strong preservation theorems for $L_{\infty V}$ which partially compensate for the failure of interpolation.

We conclude this section by pointing out that certain sublogics and extensions of the game logics $L_{\infty G}$ and $L_{\infty V}$ have also been studied. For example, Ellentuck [1975], Burgess [1978b] and Green [1978] have investigated the *Suslin logics* which can be described intuitively as the propositional part of $L_{\infty V}$, since they allow for infinite alternations of the connectives $\bigwedge$ and $\bigvee$, but not of the quantifiers $\forall$ and $\exists$. Burgess [1977] introduced the *Borel-game logic* $L_{\infty B}$, an extension of $L_{\infty V}$. In this logic, the infinite strings of quantifiers and connectives are applied not only to matrices which are open or closed, but also to matrices which can be coded by a Borel set. Of course, it takes Martin's [1975] theorem on Borel determinacy to show that negations can be pushed inside. The Borel-game logic is absolute relative to ZF + axiom of dependent choices.

4. *Game Quantification and Local Definability Theory*

This section contains the connections between game quantification, generalized recursion theory, and descriptive set theory. The first basic result asserts that on structures with a first-order coding machinery, the (positive elementary) inductive relations coincide with the ones that are explicitly definable using the open game quantifier. This result is due to Moschovakis [1972] and constitutes an absolute version of Svenonius' theorem (see Theorem 2.1.5). Aczel [1975] generalized this result and showed that the Q-inductive relations on a structure can be characterized using infinite strings $(Qx_0 Qx_1 Qx_2 \cdots)$, where Q is an arbitrary monotone quantifier. To present these theorems, we introduce infinite strings $(Qx_0 Qx_1 Qx_2 \cdots)$ and interpret them via two-person infinite games. We will pursue the study of the Q-inductive relations and state their characterizations in terms of functional recursion, representability in stronger logics, and admissible sets with quantifiers. We will also briefly indicate some of the tools of inductive definability which are used to derive local versions of the global results given in Section 2. That done, we will discuss the connections with non-monotone inductive definitions and the recursion-theoretic difference between the open game and the closed game quantifier. The chapter will end with some results and comments concerning the interactions of game quantification with descriptive set theory.

Because of the limitations of space, most of the results in this section will be given without proofs. However, we have included the definitions of the basic notions as well as all the relevant references to the literature.

4.1. Iterating a Monotone Quantifier Infinitely Often

4.1.1. Assume that Q is a monotone quantifier on a set A; that is, suppose that Q is a non-empty, proper subset of $\mathscr{P}(A)$ which is closed under supersets. In order to iterate the quantifier Q infinitely often, we must give meaning to the string

$$(Qx_0 Qx_1 Qx_2 \cdots).$$

The following interpretation is due to Aczel [1975] and is motivated by the observation that, since Q has the monotonicity property,

$$QxP(x) \quad \text{iff} \quad (\exists X \in Q)(\forall x \in X)P(x),$$

so that intuitively we should have the equivalence

$$\begin{aligned}&(Qx_0 Qx_1 Qx_2 \cdots)R(x_0, x_1, x_2, \ldots)\\ &\quad \text{iff} \quad (\exists X_0 \in Q)(\forall x_0 \in X_0)(\exists X_1 \in Q)(\forall x_1 \in X_1) \cdots R(x_0, x_1, \ldots).\end{aligned}$$

This suggests associating with Q as well as with a relation $R \subseteq A^\omega$ the following two-person infinite game $G(Q, R)$ of perfect information:

A *round of the game* $G(Q, R)$ is played by Players I and II who make alternate moves in such a way that I picks a set $X_i \in Q$ and II responds by picking an element $x_i \in X_i$, $i = 0, 1, 2, \ldots$

$$\begin{array}{c|ccccccccc} \text{I} & X_0 & & X_1 & & X_2 & & \cdots & & (X_i \in Q, \text{ all } i \in I) \\ \hline \text{II} & & x_0 & & x_1 & & x_2 & \cdots & & (x_i \in X_i, \text{ all } i \in I) \end{array}.$$

Player I wins the above round if $(x_0, x_1, x_2, \ldots) \in R$; otherwise, Player II wins We say that *Player* I *wins the game* $G(Q, R)$ if I has a systematic way to win every round of the game. This can be made precise by requiring that Player I have a *winning strategy for* $G(Q, R)$; that is, that there be a function $\sigma\colon \bigcup_{n<\omega} (Q \times A)^n \to Q$ with the property that $(x_0, x_1, x_2, \ldots) \in R$ for any round $(X_0, x_0, X_1, x_1, X_2, x_2, \ldots)$ of $G(Q, R)$ in which $X_0 = \sigma((\ ))$ and $X_{i+1} = \sigma(X_0, x_0, \ldots, X_i, x_i)$, for every $i \in \omega$. Similarly, we say that *Player* II *wins the game* $G(Q, R)$ if II has a winning strategy $\tau\colon \bigcup_{n<\omega} (Q \times A)^n \times Q \to A$ with which he can win every round of $G(Q, R)$. Finally, we put

$$\begin{aligned}&(Qx_0 Qx_1 Qx_2 \cdots)R(x_0, x_1, x_2, \ldots)\\ &\quad \text{iff} \quad \text{Player I wins the game } G(Q, R).\end{aligned}$$

The following proposition is a simple, but useful tool in manipulating infinite strings of quantifiers. Its proof follows easily from the definitions and the axiom of choice.

4.1.2 Proposition. *Let Q be a monotone quantifier on A and let $R \subseteq A^\omega$. Then we have,*

$$\begin{aligned}&Qx\{Qx_0 Qx_1 Qx_2 \cdots\}R(x, x_0, x_1, x_2, \ldots)\\ &\quad \text{iff} \quad (QxQx_0 Qx_1 Qx_2 \cdots)R(x, x_0, x_1, x_2, \ldots). \quad \square\end{aligned}$$

The next theorem provides the basic connection between winning strategies for Player I in the game $G(Q, R)$ and winning strategies for Player II in the dual game $G(\breve{Q}, \neg R)$ associated with the statement

$$(\breve{Q}x_0 \breve{Q}x_1 \breve{Q}x_2 \cdots) \neg R(x_0, x_1, x_2, \ldots), \text{ where of course } \neg R = A^\omega - R.$$

4.1.3 Theorem. *Let Q be a monotone quantifier on a set A and let $R \subseteq A^\omega$. Then the following are equivalent:*

(i) $(Qx_0 Qx_1 Qx_2 \cdots)R(x_0, x_1, x_2, \ldots)$; *that is to say, Player* I *wins the game* $G(Q, R)$

(ii) *Player* II *wins the game* $G(\breve{Q}, \neg R)$.

Proof. Let σ be a winning strategy for Player I in the game $G(Q, R)$. We will informally describe a winning strategy for Player II in the dual game $G(\breve{Q}, \neg R)$. The argument uses the axiom of choice and the fact that if $X \in Q$ and $Y \in \breve{Q}$, then $X \cap Y \neq \emptyset$. Assume then that Player I starts a round of $G(\breve{Q}, \neg R)$ by playing a set $Y_0 \in \breve{Q}$. If $X_0 = \sigma((\ ))$, then $X_0 \in Q$, and hence $X_0 \cap Y_0 \neq \emptyset$. Now,

> Player II answers Player I in $G(\breve{Q}, \neg R)$ by picking an element $x_0 \in X_0 \cap Y_0$.

If I plays $Y_1 \in \breve{Q}$, then II responds by playing some element x_1 of the non-empty set $X_1 \cap Y_1$, where $X_1 = \sigma(X_0, x_0) \in Q$. If Player II continues in this way, then at the end of time he has produced a round $(Y_0, x_0, Y_1, x_1, \ldots)$ of the game $G(\breve{Q}, \neg R)$ for which there is a round $(X_0, x_0, X_1, x_1, \ldots)$ of $G(Q, R)$ played according to the winning strategy σ for Player I in that game, hence $(x_0, x_1, \ldots) \in R$.

As to the other direction, we will assume that Player II wins the game $G(\breve{Q}, \neg R)$. We will indicate how to define a winning strategy for I in the game $G(Q, R)$. The idea is similar to the one presented earlier; namely, I plays in such a way that he forces his opponent to produce a sequence $(x_0, x_1, x_2, \ldots)$ which corresponds to moves of II in $G(\breve{Q}, \neg R)$ played according to his winning strategy. More precisely, I starts by playing the set

$$\begin{aligned}X_0 = \{x\colon &\text{there is a round of } G(\breve{Q}, \neg R) \text{ of the form } (Y, x, \ldots)\\ &\text{in which Player II follows his winning strategy}\}.\end{aligned}$$

Notice that $X_0 \in Q$, since otherwise its complement $(A - X_0) \in \breve{Q}$ and it is thus a legitimate move for I in $G(\breve{Q}, \neg R)$. But then the winning strategy of II in this game produces an element of $X_0 \cap (A - X_0)$. This is a contradiction.

Suppose now that Player II responds with an element $x_0 \in X_0$. Then there is a round of $G(\breve{Q}, \neg R)$ of the form $(Y_0, x_0, \ldots)$ in which II follows his winning strategy. The next move of I in $G(Q, R)$ is the set

$$X_1 = \{x\colon \text{there is a round of } G(\breve{Q}, \neg R) \text{ of the form } (Y_0, x_0, Y, x, \ldots) \text{ in which Player II follows his winning strategy}\}.$$

It is easy to see that $X_1 \in Q$. Moreover, if II responds with an element $x_1 \in X_1$, then there is a round of $G(\breve{Q}, \neg R)$ of the form $(Y_0, x_0, Y_1, x_1, \ldots)$ in which II plays according to his winning strategy. In this way, at the end of time the two players in $G(Q, R)$ have produced a sequence $(X_0, x_0, X_1, x_1, X_2, x_2, \ldots)$ such that there is a round $(Y_0, x_0, Y_1, x_1, Y_2, x_2, \ldots)$ of $G(\breve{Q}, \neg R)$ in which II follows his winning strategy. $\square$

The proof of the Gale–Stewart theorem (1.2.4) can be easily modified to yield the determinacy of open or closed games associated with the infinite string $(Qx_0Qx_1Qx_2\cdots)$. Thus, if Q is a monotone quantifier and R is a relation which is either open or closed, then Player I or Player II wins the game $G(Q, R)$. By combining this fact with Theorem 4.1.3 we immediately obtain the following

4.1.4 Corollary. *Let Q be a monotone quantifier on A and let $R \subseteq A^\omega$ be a relation which is either open or closed. Then*

$$\text{Player I does not win } G(Q, R) \quad \text{iff} \quad \text{Player I wins } G(\breve{Q}, \neg R)$$

and hence

$$\neg(Qx_0Qx_1Qx_2\cdots)R(x_0, x_1, x_2, \ldots) \Leftrightarrow$$
$$(\breve{Q}x_0\breve{Q}x_1\breve{Q}x_2\cdots) \neg R(x_0, x_1, x_2, \ldots). \quad \square$$

4.1.5. Thus far we have considered infinite strings obtained by iterating only one monotone quantifier infinitely often. We might also consider a sequence $\bar{Q} = \{Q_n\}_{n\in\omega}$ of arbitrary monotone quantifiers Q_n, $n \in \omega$, on a set A and the corresponding infinite string $(Q_0x_0Q_1x_1\cdots Q_nx_n\cdots)$. If $R \subseteq A^\omega$ is a collection of infinite sequences from A, then the statement

$$(Q_0x_0Q_1x_1\cdots Q_nx_n\cdots)R(x_0, x_1, \ldots, x_n, \ldots)$$

is interpreted via a game $G(\bar{Q}, R)$ which is suggested by the intuitive equivalence

$$(Q_0 x_0 Q_1 x_1 \cdots Q_n x_n \cdots) R(x_0, x_1, \ldots, x_n, \ldots) \quad \text{iff}$$
$$(\exists X_0 \in Q_0)(\forall x_0 \in X_0)(\exists X_1 \in Q_1)(\forall x_1 \in X_1)$$
$$\cdots (\exists X_n \in Q_n)(\forall x_n \in X_n) \cdots R(x_0, x_1, \ldots, x_n).$$

The preceding results extend naturally to such arbitrary strings with only minor modifications in the definitions and the proofs. In particular, if $R \subseteq A^\omega$ is either open or closed, then we can push the negation inside, so that we have

$$\neg(Q_0 x_0 Q_1 x_1 \cdots Q_n x_n \cdots) R(x_0, x_1, \ldots, x_n, \ldots)$$
$$\Leftrightarrow (\breve{Q}_0 x_0 \breve{Q}_1 x_1 \cdots \breve{Q}_n x_n \cdots) \neg R(x_0, x_1, \ldots, x_n, \ldots)$$

We should point out here that for the infinite string $(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots)$, the interpretation of the statement $(\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) R(x_0, y_0, x_1, y_1, \ldots)$ given above is equivalent to the one given in Section 1 of this chapter. Notice, however, that a strategy for I in the sense of this section essentially coincides with a quasi-strategy for I in the sense of Section 1, rather than with a strategy. This is because we have identified the existential quantifier $\exists$ on A with the collection $\{X \subseteq A : X \neq \varnothing\}$.

4.1.6. The infinite string $(Qx_0 Qx_1 Qx_2 \cdots)$ can be viewed as defining a new monotone quantifier Q^* on the set A^ω of infinite sequences from A. More specifically, the *quantifier Q^* on A^ω* is the collection

$$Q^* = \{X \subseteq A^\omega : (Qx_0 Qx_1 Qx_2 \cdots) X(x_0, x_1, x_2, \ldots)\}.$$

If the infinite string $(Qx_0 Qx_1 Qx_2 \cdots)$ is applied to relations R on A^ω which are open or closed, then it gives rise to two monotone quantifiers $Q^\vee$ and $Q^\wedge$ on the set $A^{<\omega}$ of finite sequences from A.

The *quantifier $Q^\vee$ on $A^{<\omega}$* is the collection

$$Q^\vee = \left\{X \subseteq A^{<\omega} : (Qx_0 Qx_1 Qx_2 \cdots) \bigvee_n X(x_0, x_1, \ldots, x_{n-1})\right\},$$

while *the quantifier $Q^\wedge$ on $A^{<\omega}$* is defined by

$$Q^\wedge = \left\{X \subseteq A^{<\omega} : (Qx_0 Qx_1 Qx_2 \cdots) \bigwedge_n X(x_0, x_1, \ldots, x_{n-1})\right\}.$$

The quantifiers $Q^\vee$ and $Q^\wedge$ can be expressed using the quantifier Q^* on A^ω and infinitary connectives. Indeed, if $R \subseteq A^{<\omega}$ is a relation on the set of finite sequences from A, then we first introduce the relations $\bigvee R$ and $\bigwedge R$ on the set of infinite sequences, where

$$\bigvee R = \left\{\alpha \in A^\omega : \bigvee_n R(\alpha \upharpoonright n)\right\} \quad \text{and} \quad \bigwedge R = \left\{\alpha \in A^\omega : \bigwedge_n R(\alpha \upharpoonright n)\right\}.$$

It is now clear that

$$Q^{\vee} sR(s) \Leftrightarrow \text{Player I wins } G(Q, \bigvee R) \Leftrightarrow Q^*\alpha \bigvee R(\alpha)$$

and

$$Q^{\wedge} sR(s) \Leftrightarrow \text{Player I wins } G(Q, \bigwedge R) \Leftrightarrow Q^*\alpha \bigwedge R(\alpha).$$

Since the quantifiers $Q^{\vee}$ and $Q^{\wedge}$ give rise to games which are open or closed, we can use Corollary 4.1.4 to find their dual quantifiers.

4.1.7 Corollary. *Let Q be a monotone quantifier on A. Then*:

(i) *the dual of the quantifier $Q^{\vee}$ is the quantifier $\breve{Q}^{\wedge}$; that is, $(Q^{\vee})^{\cup} = \breve{Q}^{\wedge}$;*
(ii) *the dual of the quantifier $Q^{\wedge}$ is the quantifier $\breve{Q}^{\vee}$; that is $(Q^{\wedge})^{\cup} = \breve{Q}^{\vee}$.* □

4.1.8. The Suslin and the classical $\mathscr{A}$ quantifier are special cases of the quantifiers $Q^{\vee}$ and $Q^{\wedge}$. Indeed, it is obvious that $\forall^{\vee}$ is the Suslin quantifier on the set $A^{<\omega}$, while $\exists^{\wedge}$ is the classical quantifier $\mathscr{A}$ on $A^{<\omega}$. Notice also that $\forall^{\wedge}$ and $\exists^{\vee}$ are respectively the universal and the existential quantifier on the set $A^{<\omega}$ of finite sequences from A.

We now consider the quantifiers $\exists\forall$ and $\forall\exists$ on the set $A^2 = A \times A$, where

$$\exists\forall = \{X \subseteq A^2 \colon (\exists x\, \forall y)((x, y) \in X)\}$$

and

$$\forall\exists = \{X \subseteq A^2 \colon (\forall x\, \exists y)((x, y) \in X)\}.$$

Of course, the quantifier $\forall\exists$ is the dual of $\exists\forall$. Moreover,

$$(\exists\forall)^{\vee} \text{ is the open game quantifier } \mathscr{G} \text{ on } A^{<\omega},$$

and

$$(\forall\exists)^{\wedge} \text{ is the closed game quantifier } \breve{\mathscr{G}} \text{ on } A^{<\omega}.$$

Observe that here we have tacitly identified the sequence $((x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots)$ in $(A \times A)^{\omega}$ with the sequence $(x_0, y_0, x_1, y_1, x_2, y_2, \ldots)$ in A^{ω}.

If Q is a monotone quantifier on A, *then the next quantifier Q^+ of Q* is the quantifier

$$Q^+ = (Q\breve{Q}\exists\forall)^{\vee},$$

where $Q\breve{Q}\,\exists\forall = \{X \subseteq A^4 \colon (Qx\breve{Q}y\, \exists z\, \forall w)((x, y, z, w) \in X)\}$. Therefore, if $R \subseteq A^{<\omega}$, then we have

$$Q^+ sR(s) \Leftrightarrow (Qx_0 \breve{Q}y_0\, \exists z_0\, \forall w_0 Qx_1 \breve{Q}y_1\, \exists z_1\, \forall w_1 \cdots) \bigvee_n R(x_0, y_0, z_0, w_0, \ldots, x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}).$$

It follows from the above that the dual quantifier of $Q^+ = (Q\breve{Q}\,\exists\forall)^\vee$ is the quantifier $(\breve{Q}Q\,\forall\exists)^\wedge$. Notice that the open game quantifier $\mathscr{G}$ is the next quantifier of $(\exists\forall)$. As we will see in the sequel, the next quantifier plays an important role in the theory of inductive definability.

4.2. Game Quantification and Positive Elementary Induction in a Quantifier

4.2.1. Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k\rangle$ be a structure and let Q be a monotone quantifier on the universe A of the structure. The *first-order logic* $\mathscr{L}^{\mathfrak{A}}(Q)$ of the structure $\mathfrak{A}$ has both first-order variables $x, y, z, \ldots$ and second-order variables $S, T, U, \ldots$, but the quantifiers $\forall, \exists, Q, \breve{Q}$ range only over the first-order variables. The "*boldface*" *first-order logic* $\boldsymbol{\mathscr{L}}^{\mathfrak{A}}(Q)$ of the structure $\mathfrak{A}$ is obtained from $\mathscr{L}^{\mathfrak{A}}(Q)$ by adding to the vocabulary a new constant symbol **a** for each element $a \in A$. If we do not consider an additional quantifier Q, then we have the logics $\mathscr{L}^{\mathfrak{A}}$ and $\boldsymbol{\mathscr{L}}^{\mathfrak{A}}$ respectively.

If $\varphi(x_1, \ldots, x_n, S)$ is a formula of $\mathscr{L}^{\mathfrak{A}}(Q)$ in which S is a n-ary relation symbol with only positive occurrences, then $\varphi(\bar{x}, S)$ gives rise to a transfinite sequence $\{I_\varphi^\xi\}_{\xi\in\mathbf{Ord}}$ of n-ary relations on A, where

$$I_\varphi^\xi = \left\{\bar{x} \in A^n\colon \varphi\left(\bar{x}, \bigcup_{\eta<\xi} \varphi^\eta\right)\right\}.$$

We put

$$I_\varphi = \bigcup_{\xi\in\mathbf{Ord}} I_\varphi^\xi$$

and call I_φ the set *inductively defined by* φ. It is easy to see that

$$\bar{x} \in I_\varphi \Leftrightarrow \varphi(\bar{x}, I_\varphi)$$

and

$$I_\varphi = \bigcap \{S\colon (\forall\bar{x})(\varphi(\bar{x}, S) \leftrightarrow \bar{x} \in S)\},$$

so that I_φ is the smallest fixed point of φ.

If R is an m-ary relation on A, we say that R is Q-(*positive*) *inductive* in case there is a formula $\varphi(\bar{u}, \bar{v}, S)$ of $\mathscr{L}^{\mathfrak{A}}(Q)$ with S occurring positively and a finite sequence $\bar{a}$ of elements of A such that

$$R(\bar{y}) \Leftrightarrow (\bar{a}, \bar{y}) \in I_\varphi.$$

We say that a relation $R \subseteq A^m$ is *Q-(positive) hyperelementary* if both R and $A^m - R$ are Q-inductive relations. We write

$$\mathbf{IND}(\mathfrak{A}, Q) = \text{the collection of all } Q\text{-(positive) inductive relations on } \mathfrak{A},$$

and

$$\mathbf{HYP}(\mathfrak{A}, Q) = \text{the collection of all } Q\text{-(positive) hyperelementary relations on } \mathfrak{A}.$$

If we do not consider an additional quantifier Q on A, then we have the notions of the (positive) inductive and the (positive) hyperelementary relations on $\mathfrak{A}$. In this case we put

$$\mathbf{IND}(\mathfrak{A}) = \text{all (positive) inductive relations on } \mathfrak{A},$$

and

$$\mathbf{HYP}(\mathfrak{A}) = \text{all (positive) hyperelementary relations on } \mathfrak{A}.$$

The theory of the inductive and the hyperelementary relations has been developed in the monograph Moschovakis [1974a]. Here we will purposely restrict ourselves to stating the results which are directly related to game quantification.

4.2.2. Henceforth, we will confine our attention to structures possessing a first-order coding machinery of finite sequences. We say that a structure $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k\rangle$ is *acceptable* if ω, $\leq_\omega$ are first-order on $\mathfrak{A}$ and there is a total, one-to-one coding function $\langle\ \rangle : A^{<\omega} \to A$ such that the relation seq and the functions lh and q are first-order on $\mathfrak{A}$, where

$$\mathrm{seq}(x) \Leftrightarrow \text{there are } x_1, x_2, \ldots, x_n \text{ such that } x = \langle x_1, x_2, \ldots, x_n\rangle;$$

$$lh(x) = \begin{cases} 0, & \text{if } \neg\ \mathrm{seq}(x) \\ n, & \text{if } \mathrm{seq}(x) \text{ and } x = \langle x_1, x_2, \ldots, x_n\rangle; \end{cases}$$

and

$$q(x, i) = (x)_i = \begin{cases} x_i, & \text{if } x = \langle x_1, x_2, \ldots, x_n\rangle \text{ and } 1 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Typical examples of acceptable structures are the structure of arithmetic $\mathbb{N} = \langle \omega, +, \cdot\rangle$, the rationals $\mathbb{Q} = \langle Q, +, \cdot\rangle$, the structure of analysis $\mathbb{R} = \langle \omega \cup \omega^\omega, \omega, +, \cdot, Ap\rangle$ (where $Ap(\alpha, n) = \alpha(n)$, with $\alpha \in \omega^\omega$ and $n \in \omega$), and the structures $\mathbb{V}_\lambda = \langle V_\lambda, \varepsilon \rangle$, for each ordinal $\lambda \geq \omega$, where V_λ is the collection of sets of rank less than λ.

Many of the results in this section are true under a much weaker hypothesis, namely that the structure $\mathfrak{A}$ under consideration has an *inductive pairing function*. Such a function is, of course, a total, one-to-one function $\langle\ \rangle: A \times A \to A$ with an inductive graph. Examples of such structures include the structures $\boldsymbol{\lambda} = \langle \lambda, < \rangle$ and $\mathbb{L}_\lambda = \langle L_\lambda, < \rangle$ for any infinite ordinal λ, all models of Peano arithmetic, and any structure of the form $\mathfrak{A} = \langle A, \in \rangle$ where A is a transitive set closed under pairs.

Every acceptable structure has the property that the weak second-order logic $\mathscr{L}_{\mathrm{wII}}$ on $\mathfrak{A}$ can be subsumed by the first-order logic $\mathscr{L}^{\mathfrak{A}}$ of the structure $\mathfrak{A}$.

If we want to avoid the assumption of acceptability, then we must consider a larger class of inductive definitions, namely the inductive* and the Q-inductive* relations of Barwise [1975, 1978b], or pass from an arbitrary structure $\mathfrak{A} = \langle A, R_1, \ldots, R_m, c_1, \ldots, c_k \rangle$ to the expanded structure $\mathfrak{A}^* = \langle A \cup A^{<\omega} \cup \omega, A, \omega, R_1, \ldots, R_m, \leq_\omega, Ap, c_1, \ldots, c_k \rangle$, where $Ap((a_1, \ldots, a_n), i) = a_i$.

If $\mathfrak{A}$ is an acceptable structure and T is a quantifier on the set $A^{<\omega}$ of finite sequences from A, then T can be identified with a quantifier on A, which we also denote by T and which is defined as follows:

$$T = \{X \subseteq A : \{(x_1, \ldots, x_n) \in A^{<\omega} : \langle x_1, \ldots, x_n \rangle \in X\} \in T\},$$

with $\langle\ \rangle: A^{<\omega} \to A$ a fixed coding function as in the definition of acceptability.

In particular, the quantifiers $Q^\vee$, $Q^\wedge$, Q^+ and $(Q^+)^\cup$ can all be viewed, and indeed will so be viewed from here on, as quantifiers on the universe A of the structure $\mathfrak{A}$. Thus, for example, the open game quantifier on $A^{<\omega}$ is identified with the quantifier

$$\mathscr{G} = \left\{X \subseteq A : (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \bigvee_n (\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle \in X)\right\}$$

on A, while the closed game quantifier $\breve{\mathscr{G}}$ on $A^{<\omega}$ becomes the quantifier

$$\breve{\mathscr{G}} = \left\{X \subseteq A : (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots) \bigwedge_n (\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle \in X)\right\}$$

on A. For the remainder of this section, if $\mathfrak{A}$ is an acceptable structure, then $\langle\ \rangle: A^{<\omega} \to A$ will always denote a total, one-to-one function such that the associated coding and decoding relations and functions seq, lh, q are first-order on $\mathfrak{A}$.

The next theorem provides the basic connection between inductive definability and game quantification. We credit this result to Moschovakis [1974a], [1972] for the inductive relations and to Aczel [1975] for the Q-inductive relations.

4.2.3 Theorem. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_m, c_1, \ldots, c_k\rangle$ be an acceptable structure and let Q be a monotone quantifier on A. Then,*

(i) *a relation R on A is Q-(positive) inductive if and only if there is a formula $\varphi(u, \bar{z})$ of the "boldface" logic $\mathscr{L}^{\mathfrak{A}}(Q)$ of the structure $\mathfrak{A}$ such that $R(\bar{z}) \Leftrightarrow Q^+u\varphi(u, \bar{z})$; that is,*

$$R(\bar{z}) \Leftrightarrow (Qv_0 \breve{Q}w_0\ \exists x_0\ \forall y_0\ Qv_1\breve{Q}w_1\ \exists x_1\ \forall y_1 \cdots)$$
$$\bigvee_n \varphi(\langle v_0, w_0, x_0, y_0, \ldots, v_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}\rangle, \bar{z});$$

(ii) *in particular, a relation R on A is (positive) inductive if and only if there is a formula φ of the "boldface" logic $\mathscr{L}^{\mathfrak{A}}$ of the structure $\mathfrak{A}$ such that*

$$R(\bar{z}) \Leftrightarrow \mathscr{G}u\varphi(u, \bar{z}) \Leftrightarrow$$
$$(\exists x_0\ \forall y_0\ \exists x_1\ \forall y_1 \cdots) \bigvee_n \varphi(\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle, \bar{z}).$$

Hint of Proof. The inductive analysis of open games given in the proof of the Gale–Stewart theorem (1.2.4) can be used to show that if $(R(\bar{z}) \Leftrightarrow Q^+u\varphi(u, \bar{z}))$, then the relation R is Q-inductive. For the other direction, one has to show first that if $\psi(\bar{z}, S)$ is a formula of $\mathscr{L}^{\mathfrak{A}}(Q)$ in which S occurs positively, then there is a quantifier-free formula $\theta(\bar{v}, \bar{w}, \bar{x}, \bar{y}, \bar{z})$ such that

$$\psi(\bar{z}, S) \Leftrightarrow (Qv_0)(\breve{Q}w_0)(\exists x_0)(\forall y_0) \cdots (Qv_m)(\breve{Q}w_m)(\exists x_m)(\forall y_m)(\forall \bar{t})$$
$$[\theta(\bar{v}, \bar{w}, \bar{x}, \bar{y}, \bar{z}) \vee S(\bar{t})].$$

Using the equivalence above and the coding machinery on $\mathfrak{A}$, it is not hard to verify that the smallest fixed point I_ψ of the formula $\psi(\bar{z}, S)$ is explicitly definable by the next quantifier Q^+ applied to a formula $\varphi(u, \bar{z})$ of $\mathscr{L}^{\mathfrak{A}}(Q)$. □

4.2.4. The above identification of the inductive relations with the ones definable by open game formulas is an absolute version of Svenonius' theorem (2.1.5), and has many applications in either direction. In particular, results from inductive definability have implications for game quantification and vice-versa. For example, we can use the proof of Theorem 4.2.3 to discover the main properties of the approximations of the open game formulas. Indeed, if $\Phi(\bar{z})$ is an open game formula and $\varphi(\bar{z}, S)$ is a positive in S formula of $\mathscr{L}^{\mathfrak{A}}$ such that $\mathfrak{A} \models (\forall \bar{z})(\Phi(\bar{z}) \leftrightarrow I_\varphi(\bar{z}))$, then the approximations ε_α^Φ of Φ are equivalent on $\mathfrak{A}$ to the stages I_φ^α of φ. In the other direction, Moschovakis [1974a] used Theorem 4.2.3 to show the existence of universal inductive relations on acceptable structures. As a consequence, on every acceptable structure there are inductive relations which are not hyperelementary. Moreover, on such structures the relation of satisfaction "$\mathfrak{A} \models \varphi$", where φ is a sentence of $\mathscr{L}^{\mathfrak{A}}$, is hyperelementary; but it is not, of course, first-order.

The tools of inductive definability can be used to obtain local versions of such global results as Vaught's covering theorem (See Section 2.2.10), the separation and reduction principles and others. One of the main tools is the stage comparison theorem of Moschovakis [1974a] which asserts, intuitively, that we can compare the stages of an inductive definition in an inductive way. Its consequences include the following theorem, a theorem which is true for an arbitrary structure $\mathfrak{A}$.

4.2.5 Theorem. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be a structure and let Q be a monotone quantifier on A. Then the class* $\mathbf{IND}(\mathfrak{A}, Q)$ *of the Q-inductive relations has the pre-well-ordering property. That is, if $P \subseteq A^n$ is a Q-inductive relation, then there is a map $\sigma\colon P \xrightarrow{\text{onto}} \lambda$, where λ some ordinal, such that the relations $\leq_\sigma^*$ and $<_\sigma^*$ are Q-inductive, where*

$$\bar{x} \leq_\sigma^* \bar{y} \Leftrightarrow (\bar{x} \in P) \mathbin{\&} (\bar{y} \notin P \vee \sigma(\bar{x}) \leq \sigma(\bar{y}))$$

and

$$\bar{x} <_\sigma^* \bar{y} \Leftrightarrow (\bar{x} \in P) \mathbin{\&} (\bar{y} \notin P \vee \sigma(\bar{x}) < \sigma(\bar{y})). \quad \square$$

If P is a Q-inductive relation and $\sigma\colon P \xrightarrow{\text{onto}} \lambda$ is a map such that the relations $\leq_\sigma^*$ and $<_\sigma^*$ are Q-inductive, then we say that σ is a *Q-inductive norm* on P. The existence of Q-inductive norms easily implies the reduction principle for the Q-inductive relations and the separation principle for the complements of the Q-inductive relations.

With any structure $\mathfrak{A}$ we associate the ordinal $\kappa^{\mathfrak{A}}$, where

$$\kappa^{\mathfrak{A}} = \sup\{\operatorname{rank}(<) \colon < \text{ is a hyperelementary pre-well-ordering on } A\}.$$

If Q is a monotone quantifier on the universe A of the structure $\mathfrak{A}$, then we consider also the ordinal

$$\begin{aligned}\kappa^{\mathfrak{A}(Q)} = \sup\{&\operatorname{rank}(<)\colon \\ &< \text{ is a } Q\text{-hyperelementary pre-well-ordering on } A\}.\end{aligned}$$

The stage comparison theorem also yields the following useful boundedness principle.

4.2.6 Theorem. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be a structure and let Q be a monotone quantifier on A. Assume further that P is a Q-inductive relation and $\sigma\colon P \xrightarrow{\text{onto}} \lambda$ is a Q-inductive norm. Then*

(i) $\lambda \leq \kappa^{\mathfrak{A}(Q)}$;
(ii) *for each $\xi < \lambda$ the set $P^\xi = \{\bar{x} \in P \colon \sigma(\bar{x}) \leq \xi\}$ is Q-hyperelementary*;
(iii) *P is Q-hyperelementary if and only if $\lambda < \kappa^{\mathfrak{A}(Q)}$.* $\square$

The above result can be thought of as a local version of the approximation theorem (2.2.7) and the undefinability of well-order. Actually, Moschovakis [1974a] showed that it implies a covering theorem for the Q-inductive relations on any structure.

4.2.7 The Covering Theorem. *Let P be a Q-inductive relation on a structure $\mathfrak{A}$ and let $\sigma: P \xrightarrow{\text{onto}} \lambda$ be a Q-inductive norm. If R is the complement of a Q-inductive relation and $R \subseteq P$, then there is an ordinal $\xi < \kappa^{\mathfrak{A}(Q)}$ such that*

$$R \subseteq P^{\xi} = \{\bar{x} \in P: \sigma(\bar{x}) \leq \xi\}.$$

In particular, R is contained in a Q-hyperelementary subset of P. $\square$

In order to gain more insight into the relations definable by the game quantifiers on an acceptable structure, we next state various characterizations of the Q-inductive relations in terms of Spector classes, functional recursion, representability in stronger logics, and, finally, admissible sets with quantifiers.

4.2.8. Let Γ be a class of relations on A and let Q be a monotone quantifier on A. We say that Γ is *closed under Q* if, whenever a relation $P \subseteq A^{n+1}$ is in Γ, then the relation $R \subseteq A^n$ is also in Γ, where $R(\bar{x}) \Leftrightarrow (Qy)P(y, \bar{x})$.

The class Γ has the *pre-well-ordering property* if, for each relation P in Γ, there is a mapping $\sigma: P \xrightarrow{\text{onto}} \lambda$, where λ an ordinal, such that the relations $\leq_\sigma^*$ and $<_\sigma^*$ are in Γ.

Assume that $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ is an acceptable structure and Γ is a collection of relations on A. We call Γ a *Spector class on* $\mathfrak{A}$ if:

(i) Γ contains all first-order relations on A with parameters from A and is closed under $\cup, \cap, \forall, \exists$;
(ii) Γ has the pre-well-ordering property; and
(iii) Γ is *A-parametrized*. That is to say, for each $n \in \omega$, there is a $(n + 1)$-ary relation U^n in Γ with the property that a relation $R \subseteq A^n$ is in Γ if and only if there is some $a \in A$ such that $R = \{\bar{x} \in A^n: (a, \bar{x}) \in U^n\}$.

It actually turns out that the collections **IND**($\mathfrak{A}$) and **IND**($\mathfrak{A}$, Q) of the inductive and the Q-inductive relations are both Spector classes. The notion of a Spector class was introduced by Moschovakis [1974a] and provides a framework for developing abstract recursion theory. The following is a theorem of Moschovakis [1974a] and Aczel [1975]. On the one hand, it summarizes the main closure and structural properties of the inductive and the Q-inductive relations while, on the other, it yields a minimality characterization for these classes of relations.

4.2.9 Theorem. *Let $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be an acceptable structure and let Q be a monotone quantifier on A.*

(i) *The collection* **IND**($\mathfrak{A}$, Q) *of the Q-inductive relations on A is the smallest Spector class on $\mathfrak{A}$ closed under Q and $\breve{Q}$.*
(ii) *In particular, the collection* **IND**($\mathfrak{A}$) *of the inductive relations on A is the smallest Spector class on $\mathfrak{A}$.*

We state one further result about Spector classes, a result which shows that these classes possess interesting closure properties and are related to game quantification.

4.2.10 Theorem. *Let $\mathfrak{A}$ be an acceptable structure, Q a monotone quantifier on A and Γ a Spector class on $\mathfrak{A}$. Then*

(i) *Γ is closed under the quantifier Q if and only if Γ is closed under the quantifier $Q^\vee$. In particular,*

(ii) *Γ is closed under Q and $\breve{Q}$ if and only if Γ is closed under the next quantifier Q^+.*

(iii) *Every Spector class is closed under the open game quantifier $\mathcal{G}$.* ∎

4.2.11. Let A be a set such that $\omega \subseteq A$ and let $\mathcal{PF}_k$ be the collection of all k-ary partial functions from A to ω. A *functional on A* is a partial mapping

$$\Phi: A^l \times \mathcal{PF}_{k_1} \times \cdots \times \mathcal{PF}_{k_m} \to \omega,$$

which is *monotone*. That is, if $\Phi(\bar{x}, g_1, \ldots, g_m) = w$ and $g_1 \subseteq h_1, \ldots, g_m \subseteq h_m$, then $\Phi(\bar{x}, h_1, \ldots, h_m) = w$.

If $\bar{\Phi} = (\Phi_1, \ldots, \Phi_s)$ is a finite sequence of functionals on the universe of a structure $\mathfrak{A}$, then we can define the notion of a *recursive in* Φ *m*-ary partial function from A to ω. This is done by first associating with $\bar{\Phi}$ the smallest class of functionals having certain closure properties and containing $\bar{\Phi}$, and then iterating the operative functionals in that class. The detailed definitions of functional recursion can be found in Kechris–Moschovakis [1977].

A relation P on A is *semi-recursive in* $\bar{\Phi}$ if it is the domain of a recursive in $\bar{\Phi}$ partial function. We say that P is *recursive in* $\bar{\Phi}$ if its characteristic function χ_P is recursive in $\bar{\Phi}$. We put

$$\mathbf{ENV}[\bar{\Phi}] = \text{the collection of all semirecursive in } \bar{\Phi} \text{ relations}$$

and

$$\mathbf{SEC}[\bar{\Phi}] = \text{the collection of all recursive in } \bar{\Phi} \text{ relations.}$$

These classes of relations are called, respectively, the *envelope of* $\bar{\Phi}$ and the *section of* $\bar{\Phi}$.

Any monotone quantifier Q on A gives rise to a functional $\mathbf{F}_Q^\#$ which embodies existential quantification with respect to Q and $\breve{Q}$. This functional is defined by

$$\mathbf{F}_Q^\#(p) = \begin{cases} 0, & \text{if } (Qx)(p(x) = 0), \\ 1, & \text{if } (\breve{Q}x)(p(x)\downarrow \neq 0), \\ \uparrow, & \text{otherwise,} \end{cases}$$

where p varies over the partial functions from A to ω. Here $\downarrow$ abbreviates "is defined", while $\uparrow$ stands for "is undefined". If Q is the existential quantifier $\exists$, then we write $\mathbf{E}^{\#}$ for $\mathbf{F}_{\exists}^{\#}$ so that

$$\mathbf{E}^{\#}(p) = \begin{cases} 0, & \text{if } (\exists x)(p(x) = 0), \\ 1, & \text{if } (\forall x)(p(x)\downarrow \neq 0), \\ \uparrow, & \text{otherwise.} \end{cases}$$

It is not hard to show that positive elementary induction in the quantifier Q coincides with recursion in the functionals $\mathbf{E}^{\#}$, $\mathbf{F}_Q^{\#}$.

4.2.12 Theorem. *Let $\mathfrak{A}$ be an acceptable structure and Q a monotone quantifier on A, then,*

(i) *A relation is Q-inductive if and only if it is semirecursive in $\mathbf{E}^{\#}$, $\mathbf{F}_Q^{\#}$ and hence*

$$\mathbf{IND}(\mathfrak{A}, Q) = \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}_Q^{\#}].$$

(ii) *A relation is Q-hyperelementary if and only if it is recursive in $\mathbf{E}^{\#}$, $\mathbf{F}_Q^{\#}$ and hence*

$$\mathbf{HYP}(\mathfrak{A}, Q) = \mathbf{SEC}[\mathbf{E}^{\#}, \mathbf{F}_Q^{\#}].$$

In particular, we have

$$\mathbf{IND}(\mathfrak{A}) = \mathbf{ENV}[\mathbf{E}^{\#}] \quad \textit{and} \quad \mathbf{HYP}(\mathfrak{A}) = \mathbf{SEC}[\mathbf{E}^{\#}]. \quad \Box$$

4.2.13. Assume that $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ is a structure and T is a system of axioms and rules of inference in a logic $\mathscr{L}$ which has a constant $\mathbf{a}$ for each element $a \in A$. We say that a relation P on A is *weakly representable in* T if there is a formula φ of $\mathscr{L}$ such that

$$P(a_1, \ldots, a_n) \Leftrightarrow T \vdash \varphi(\mathbf{a}_1, \ldots, \mathbf{a}_n).$$

We say that P is *strongly representable* in T if both P and $\neg P$ are weakly representable. Aczel [1970, 1977] characterized the inductive and the Q-inductive relations on an acceptable structure in terms of representability in certain systems. If $\mathfrak{A}$ is a given structure, then the *infinitary system* $T^{\infty}(\mathfrak{A})$ consists of the following axioms and rules of inference:

(i) All standard first-order axioms and rules of inference for the "boldface" first-order logic $\mathscr{L}^{\mathfrak{A}}$.
(ii) All atomic and negated atomic sentences of $\mathscr{L}^{\mathfrak{A}}$ which are true in $\mathfrak{A}$.
(iii) *A-rule*: From $\varphi(\mathbf{a})$ for all $a \in A$, infer $(\forall x)\varphi(x)$.

If Q is a monotone quantifier on A, then the *infinitary system* $T^{\infty}(\mathfrak{A}, Q)$ *has,* in addition to (i), (ii), and (iii), the following rules:

(iv) *Q-rule*: From $\varphi(\mathbf{a})$ for all $a \in X$, with $X \in Q$, infer $(Qx)\varphi(x)$.
(v) $\breve{Q}$*-rule*: From $\varphi(\mathbf{a})$ for all $a \in X$, with $X \in \breve{Q}$, infer $(\breve{Q}x)\varphi(x)$.

Notice that the $\forall$-rule is the same as the A-rule, while the $\exists$-rule is an axiom of first-order logic, namely from $\varphi(\mathbf{a})$, for some a, infer $\exists x\varphi(x)$.

4.2.14 Theorem. *Let $\mathfrak{A}$ be an acceptable structure and Q a monotone quantifier on A.*

(i) *A relation P on A is weakly representable in $T^{\infty}(\mathfrak{A}, Q)$ if and only if it is Q-inductive.*
(ii) *A relation P on A is strongly representable in $T^{\infty}(\mathfrak{A}, Q)$ if and only if it is Q-hyperelementary.*

In particular, the inductive relations are exactly the weakly representable ones in $T^{\infty}(\mathfrak{A})$ and the hyperelementary relations are the strongly representable ones in $T^{\infty}(\mathfrak{A})$. $\square$

Notice that if $\mathfrak{A}$ is a countable, acceptable structure, then Svenonius theorem (2.1.5), when combined with Theorems 4.2.3 and 4.2.14, yields a completeness result about the infinitary system $T^{\infty}(\mathfrak{A})$, namely that if a formula $\varphi(X_1, \ldots, X_n)$ of $\mathscr{L}^{\mathfrak{A}}$ is universally valid, then $T^{\infty}(\mathfrak{A}) \vdash \varphi(X_1, \ldots, X_n)$. This completeness theorem also has a direct proof which uses the omitting types theorem. In this case, Theorems 4.2.3 and 4.2.14 can be used to give an alternative proof of Svenonius' theorem. On the structure of arithmetic $\mathbb{N} = \langle \omega, +, \cdot \rangle$ these results become the classical representability characterization of the Π_1^1 relations in ω-logic.

Finally, we mention the characterizations of the Q-inductive relations in terms of admissible sets with quantifiers. For simplicity, we restrict our attention to acceptable structures of the form $\mathfrak{A} = \langle A, \in \restriction A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ where A is a transitive set.

If A and M are transitive sets, $A \in M$, and Q is a quantifier on A, then we can define what it means for M to be a $Q^{\#}$, $\breve{Q}^{\#}$*-admissible set.* The crucial additional axioms are the schemata of Q and $\breve{Q}$-collection, where

$$Q\text{-}collection\text{: } (Qx \in A)(\exists y)\varphi \rightarrow (\exists w)(Qx \in A)(\exists y \in w)\varphi,$$

$$\breve{Q}\text{-}collection\text{: } (\breve{Q}x \in A)(\exists y)\varphi \rightarrow (\exists w)(\breve{Q}x \in A)(\exists y \in w)\varphi,$$

with φ a $\Delta_0(Q, \breve{Q})$ formula. The detailed definitions are given in Moschovakis [1974a] and Barwise [1978b], while the next theorem comes from Barwise–Gandy–Moschovakis [1971] and Moschovakis [1974a].

4.2.15 Theorem. *Let $\mathfrak{A} = \langle A, \in \restriction A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ be an acceptable structure such that A is a transitive set and let Q be a quantifier on A. Put*

$$\mathfrak{A}^{\#}(Q) = \bigcap \{M\colon \mathfrak{A} \in M \textit{ and } M \textit{ is a } Q^{\#}, \breve{Q}^{\#}\textit{-admissible set}\}.$$

Then $\mathfrak{A}^{\#}(Q)$ is a $Q^{\#}$, $\breve{Q}^{\#}$-admissible set, $o(\mathfrak{A}^{\#}(Q)) = \kappa^{\mathfrak{A}(Q)}$ and moreover, for any relation P on A

(i) *P is Q-inductive if and only if P is $\Sigma_1(Q, \breve{Q})$ on $\mathfrak{A}^{\#}(Q)$*
(ii) *P is Q-hyperelementary if and only if $P \in \mathfrak{A}^{\#}(Q)$.* ☐

At this point, we will collect all the characterizations of the Q-inductive relations into one result which we now present

4.2.16 Theorem. *Let $\mathfrak{A}$ be an acceptable structure and Q a monotone quantifier on A. If $P \subseteq A^n$ is a relation on A, then the following are equivalent:*

(i) *P is explicitly definable by the next quantifier Q^+; that is, there is a formula*

$$\varphi(u, \bar{z}) \text{ of } \mathscr{L}^{\mathfrak{A}}(Q) \quad \text{such that} \quad (\forall \bar{z})(P(\bar{z}) \Leftrightarrow Q^+ u\varphi(u, \bar{z})).$$

(ii) *P is Q-inductive.*
(iii) *P is in the smallest Spector class on $\mathfrak{A}$ closed under Q and $\breve{Q}$.*
(iv) *P is semi-recursive in $\mathbf{E}^{\#}$, $\mathbf{F}_Q^{\#}$.*
(v) *P is weakly representable in $T^{\infty}(\mathfrak{A}, Q)$.*
(vi) *P is $\Sigma_1(Q, \breve{Q})$ on the smallest $Q^{\#}$, $\breve{Q}^{\#}$-admissible set having $\mathfrak{A}$ as element, provided that the universe A of the structure $\mathfrak{A}$ is transitive and $\in \restriction A$ is among the relations of $\mathfrak{A}$.* ☐

The local results given above suggest certain generalizations of the global results in Section 2. The approximation theory extends to formulas involving the next quantifier; that is to say, it extends to expressions of the form $Q^+ u\varphi(u, \bar{z})$ and $(Q^+)^{\smile} u\varphi(u, \bar{z})$, where Q is an arbitrary monotone quantifier. However, in general, Svenonius' theorem does not hold for an arbitrary quantifier Q—in fact, it is actually false if Q is the open game quantifier $\mathscr{G}$. An interesting problem is to find natural monotone quantifiers Q for which Theorem 2.1.5 goes through. This, of course, is equivalent to the completeness theorem for the infinitary system $T^{\infty}(\mathfrak{A}, Q)$.

4.3. *Non-monotone Induction and Recursion in the Game Quantifiers*

4.3.1. A *second-order relation on a set A* is a relation $\varphi(x_1, \ldots, x_n, S)$ with arguments elements $x_1, \ldots, x_n$ of A and subsets S of a cartesian product A^m for some $m < \omega$. If $\varphi(x_1, \ldots, x_n, S)$ is a second-order relation on A and $S \subseteq A^n$, then we iterate φ and, by induction on the ordinals, define a sequence of n-ary relations $\{\varphi^{\xi}\}_{\xi}$ on A, where

$$\bar{x} \in \varphi^{\xi} \Leftrightarrow \left(\bar{x} \in \bigcup_{\eta < \xi} \varphi^{\eta}\right) \vee \varphi\left(\bar{x}, \bigcup_{\eta < \xi} \varphi^{\eta}\right).$$

We put

$$\varphi^{\infty} = \bigcup_{\xi} \varphi^{\xi}$$

and call φ^{∞} the set *inductively defined by* φ.

Notice that if φ is a monotone relation, then $(\bar{x} \in \varphi^{\xi} \Leftrightarrow \varphi(\bar{x}, \bigcup_{\eta<\xi} \varphi^{\eta}))$. This was indeed the case for the second-order relations determined by positive formulas in Section 4.1. Here we consider second-order relations which in general are non-monotone.

If $\mathfrak{A}$ is a structure and $\mathscr{F}$ is a collection of second-order relations on A, then we call a (first-order) relation P on A $\mathscr{F}$-*(non-monotone) inductive* in case there is a second-order relation $\varphi(\bar{u}, \bar{v}, S)$ in $\mathscr{F}$ and a sequence $\bar{a}$ of elements of A such that

$$P(\bar{y}) \Leftrightarrow (\bar{a}, \bar{y}) \in \varphi^{\infty}.$$

Let $\mathfrak{A}$ be an acceptable structure, let $\mathscr{G}$ be the open game quantifier on A

$$\mathscr{G} = \left\{ X \subseteq A : (\exists x_0\, \forall y_0\, \exists x_1\, \forall y_1 \cdots) \right.$$
$$\left. \bigvee_{n} (\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1} \rangle \in X) \right\},$$

and let $P(\bar{x}, S)$ be a second-order relation on A. We say that $P(\bar{x}, S)$ is $\mathscr{G}_1$ *on* $\mathfrak{A}$ if there is a formula $\varphi(u, \bar{x}, S)$ of $\mathscr{L}^{\mathfrak{A}}$ such that

$$P(\bar{x}, S) \Leftrightarrow \mathscr{G}u\varphi(u, \bar{x}, S).$$

We write

$$\mathscr{G}_1 = \text{the collection of all } \mathscr{G}_1 \text{ second-order relations on } \mathfrak{A}.$$

Theorem 4.2.3 has a relativized second-order version which shows that the $\mathscr{G}_1$ relations are exactly the second-order (positive) inductive relations on $\mathfrak{A}$. We will state now a characterization of the $\mathscr{G}_1$-(non-monotone) inductive relations on $\mathfrak{A}$. To do this, however, we need some notions from admissible set theory.

Let M and N be two admissible sets such that $M \subseteq N$. We say that M is N-*stable* if M is a Σ_1-elementary submodel of N, i.e. if for every Σ_1 formula $\varphi(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in M$

$$\langle M, \in \rangle \models \varphi(a_1, \ldots, a_n) \Leftrightarrow \langle N, \in \rangle \models \varphi(a_1, \ldots, a_n).$$

We say that an admissible set M is $\mathscr{G}_1$-*reflecting* if, for any formula $\varphi(u, \bar{z})$ of set theory and any sequence $\bar{a} = (a_1, \ldots, a_n)$ of parameters from M, we have

$$\langle M, \in \rangle \models \mathscr{G}u\varphi(u, \bar{a}) \Rightarrow \text{there is some admissible set } w \in M \text{ such that } \langle w, \in \rangle \models \mathscr{G}u\varphi(u, \bar{a}).$$

Observe that by Svenonius' theorem (2.1.5) we have that a countable admissible set is $\mathscr{G}_1$-reflecting if and only if it is Π_1^1-reflecting.

4.3.2 Theorem. *An admissible set M is $\mathscr{G}_1$-reflecting if and only if M is M^+ stable, where M^+ is the smallest admissible set having M as element.* ☐

This result is credited to Richter–Aczel [1974] for countable admissible sets. Richter–Aczel [1974] and Moschovakis [1974b] characterized the non-monotone inductions in the open game quantifier using $\mathscr{G}_1$-reflecting admissible sets.

4.3.3 Theorem. *Let $\mathfrak{A} = \langle A, \in \restriction A, R_1, \ldots, R_n, c_1, \ldots, c_k\rangle$ be an acceptable structure such that A is a transitive set. A relation P on A is $\mathscr{G}_1$-(non-monotone) inductive if and only if P is $\mathbf{\Sigma}_1$ on the smallest admissible set which is $\mathscr{G}_1$-reflecting and contains $\mathfrak{A}$ as an element.* ☐

This theorem is an absolute version of the following:

4.3.4 Corollary. *Let Π_1^1 be the class of Π_1^1 second-order relations on the structure of arithmetic $\mathbb{N} = \langle \omega, +, \cdot\rangle$. Then a relation P on ω is Π_1^1-(non-monotone) inductive if and only if P is $\mathbf{\Sigma}_1$ on the smallest Π_1^1-reflecting admissible set.* ☐

We next examine the non-monotone inductions in the closed game quantifier

$$\breve{\mathscr{G}} = \left\{X \subseteq A \colon (\forall x_0\, \exists y_0\, \forall x_1\, \exists y_1 \cdots)\right.$$

$$\left.\bigwedge_n (\langle x_0, y_0, \ldots, x_{n-1}, y_{n-1}\rangle \in X)\right\}$$

on an acceptable structure $\mathfrak{A}$.

We say that a second-order relation $P(\bar{x}, S)$ is $\breve{\mathscr{G}}_1$ *on* $\mathfrak{A}$ if there is a formula $\varphi(u, \bar{x}, S)$ of $\mathscr{L}^{\mathfrak{A}}$ such that

$$P(\bar{x}, S) \Leftrightarrow \breve{\mathscr{G}} u \varphi(u, \bar{x}, S).$$

We put

$$\breve{\mathscr{G}}_1 = \text{all } \breve{\mathscr{G}}_1 \text{ second-order relations on } \mathfrak{A}.$$

Harrington–Moschovakis [1974] obtained the following characterization of the non-monotone inductive relations in the quantifier $\breve{\mathscr{G}}$.

4.3.5 Theorem. *Let $\mathfrak{A}$ be an acceptable structure. Then a relation P on A is $\breve{\mathscr{G}}_1$-(non-monotone) inductive if and only if it is $\mathscr{G}$-(positive) inductive, and hence*

$$\breve{\mathscr{G}}_1\text{-IND} = \mathbf{IND}(\mathfrak{A}, \mathscr{G}) = \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\#}_{\mathscr{G}}]. \quad \square$$

4.3.6 Corollary. *Let* $\mathfrak{A} = \langle A, \in \restriction A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ *be an acceptable structure such that* A *is a transitive set. A relation* P *on* A *is* $\breve{\mathscr{G}}_1$*-inductive if and only if* P *is* Σ_1 *on the smallest* $\mathscr{G}^{\#}$, $\breve{\mathscr{G}}^{\#}$*-admissible set with* $\mathfrak{A}$ *as an element.* $\square$

4.3.7. In the light of the preceding theorems, it is natural to ask how do the classes $\mathscr{G}_1$-IND and $\breve{\mathscr{G}}_1$-IND compare. The main theorem of Aanderaa [1974] and the pre-well-ordering property for the second-order (positive) inductive relations (which is the relativized version of Theorem 4.2.5) immediately imply that

$$\mathscr{G}_1\text{-IND} \subsetneqq \breve{\mathscr{G}}_1\text{-IND}.$$

In other words, every $\mathscr{G}_1$-inductive relation is $\breve{\mathscr{G}}_1$-inductive, but the converse is not true. Moreover, the closure ordinals of the $\breve{\mathscr{G}}_1$-inductive relations is much bigger than the closure ordinal of the $\mathscr{G}_1$-inductive relations.

These results show that inductive definability provides ways to distinguish between the open game quantifier and the closed game quantifier. Such distinctions usually do not occur in model theory where a quantifier and its dual are treated on an equal basis, and the properties of the dual are obtained from the ones of the quantifier by involution.

Notice that the functionals $\mathbf{F}^{\#}_{\mathscr{G}}$ and $\mathbf{F}^{\#}_{\breve{\mathscr{G}}}$ do not differentiate the open game quantifier from the closed game quantifier, since it is easy to see that on any acceptable structure

$$\mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\#}_{\mathscr{G}}] = \mathbf{IND}(\mathfrak{A}, \mathscr{G}) = \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\#}_{\breve{\mathscr{G}}}].$$

The recursion-theoretic difference between the quantifiers $\mathscr{G}$ and $\breve{\mathscr{G}}$ is captured by the functional $\mathbf{F}^{\wedge}_{Q}$, which was introduced by Kolaitis [1980] and which, in general, distinguishes the quantifier Q from its dual $\breve{Q}$. The functional $\mathbf{F}^{\wedge}_{Q}$ is defined by

$$\mathbf{F}^{\wedge}_{Q}(p) = \begin{cases} 0, & \text{if } (Qx)(p(x) = 0) \\ 1, & \text{if } p \text{ is total } \& \ (\breve{Q}x)(p(x)\downarrow \neq 0), \\ \uparrow, & \text{otherwise} \end{cases}$$

where p varies over the partial functions from A to ω.

4.3.8 Theorem. *Let* $\mathfrak{A} = \langle A, R_1, \ldots, R_n, c_1, \ldots, c_k \rangle$ *be an acceptable structure. Then*

$$\mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\wedge}_{\mathscr{G}}] \subsetneqq \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\wedge}_{\breve{\mathscr{G}}}].$$

Moreover

$$\mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\wedge}_{\mathscr{G}}] \subsetneqq \mathscr{G}_1\text{-IND} \subsetneqq \breve{\mathscr{G}}_1\text{-IND} = \mathbf{ENV}[\mathbf{E}^{\#}, \mathbf{F}^{\wedge}_{\breve{\mathscr{G}}}]. \quad \square$$

4.4. Game Quantification and Descriptive Set Theory

4.4.1. As mentioned in Section 4.1.6, the infinite string $(\exists x_0 \, \forall y_0 \, \exists x_1 \, \forall y_1 \cdots)$ gives rise to a monotone quantifier $(\exists\forall)^*$ on the set A^ω of infinite sequences from A, where

$$(\exists\forall)^* = \{X \subseteq A^\omega : (\exists x_0 \, \forall y_0 \, \exists x_1 \, \forall y_1 \cdots) X(x_0, y_0, x_1, y_1, \ldots)\}.$$

If $A = \omega$, then the quantifier $(\exists\forall)^*$ is usually denoted by $\mathcal{D}^1$ or simply by $\mathcal{D}$ and is called the *game quantifier on* ω^ω, while if $A = R = \omega^\omega$, then $(\exists\forall)^*$ is the *game quantifier* $\mathcal{D}^2$ *on the set* R^ω of infinite sequences of reals. The properties of the quantifier $\mathcal{D}$ have been studied in depth by descriptive set theorists. We refer the reader to the book Moschovakis [1980] for a systematic treatment of $\mathcal{D}$ and its uses in definability theory. Here we will restrict ourselves to stating a sample of the results on the game quantifiers $\mathcal{D}$ and $\mathcal{D}^2$, results which are related to topics covered earlier in this chapter.

Assume that Γ is a collection of relations on integers and reals; that is, if $P \in \Gamma$, then P is a relation of the form $P(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)$, where $x_i \in \omega$ for $1 \le i \le n$ and $\alpha_j \in \omega^\omega$ for $1 \le j \le m$. If we quantify every relation in Γ by $\mathcal{D}$, we then obtain the class

$$\mathcal{D}\Gamma = \{\mathcal{D}\alpha P(\bar{x}, \alpha, \bar{\beta}) : P(x_1, \ldots, x_n, \alpha, \beta_1, \ldots, \beta_m) \text{ is a relation in } \Gamma\}.$$

In a similar way, we can define the class $\mathcal{D}^2\Gamma$ for a collection Γ of relations on integers, reals and infinite sequences of reals.

Some of the deeper results in descriptive set theory depend on *transfer theorems* which, in effect, assert that, *under certain assumptions*, properties of a class Γ transfer to the class $\mathcal{D}\Gamma$ or to the class $\mathcal{D}^2\Gamma$. In proving such transfer theorems, we usually need certain determinacy theorems or hypotheses about the class Γ.

We say that a relation P on A^ω is *determined* if Player I or Player II wins the game $G(\exists\forall, P)$ associated with P. Of course, for such relations P we have that

$$\neg(\exists x_0 \, \forall y_0 \, \exists x_1 \, \forall y_1 \cdots) P(x_0, y_0, x_1, y_1, \ldots)$$
$$\Leftrightarrow (\forall x_0 \, \exists y_0 \, \forall x_1 \, \exists y_1 \cdots) \neg P(x_0, y_0, x_1, y_1, \ldots).$$

We say that *determinacy holds for a class* Γ of relations on A^ω, and we write $\mathrm{Det}(\Gamma)$, if every relation in Γ is determined.

Martin [1975] established that every Borel set of reals is determined, or equivalently $\mathrm{Det}(\mathbf{\Delta}_1^1)$. This is an optimal result in ZFC, since it is well known that $\mathrm{Det}(\mathbf{\Sigma}_1^1)$ is not provable in ZFC. Much of the current research in descriptive set theory is carried on under the assumption that certain definable sets of reals are determined. The *hypothesis of projective determinacy* (PD) asserts that every projective set of reals is determined. The *projective sets* are the subsets of the reals which are definable by first-order formulas with parameters over the structure $\mathbb{R} = \langle \omega^\omega \cup \omega, \omega, +, \cdot, Ap \rangle$ of analysis. They are further classified as $\mathbf{\Sigma}_n^1$ or $\mathbf{\Pi}_n^1$ sets depending on the number of alternations of quantifiers in the defining formula starting respectively with an existential or a universal quantifier. If no parameters are allowed, then we have the "lightface" classes of Σ_n^1 and Π_n^1 sets of reals.

We next state a transfer theorem for the pre-well-ordering property, a result that is due to Moschovakis, and then discuss some of its applications in descriptive set theory.

4.4.2 Theorem. *Let Γ be a class of relations on integers and reals which contains all recursive relations and is closed under finite unions, finite intersections, and substitutions by recursive functions. If Γ has the pre-well-ordering property and* Det(Γ) *holds, then the class $\mathfrak{D}\Gamma$ also has the pre-well-ordering property.* ☐

In order to give concrete applications of this transfer theorem, we first need the following definition. We say that a relation $P(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m)$ on integers and reals is Σ_k^0 if there is a recursive relation R such that

$$P(\bar{x}, \alpha_1, \ldots, \alpha_m)$$
$$\Leftrightarrow (\exists l_1)(\forall l_2) \cdots (?l_k)R(\bar{x}, l_1, \ldots, l_k, \bar{\alpha}_1(l_k), \ldots, \bar{\alpha}_m(l_k)),$$

where all the quantifiers vary over the integers, and if $\alpha \in \omega^\omega$ and $k \in \omega$, then $\bar{\alpha}(k) = \langle\alpha(0), \ldots, \alpha(k-1)\rangle$.

It is quite easy to verify that for each $k \geq 1$ the class of all Σ_k^0 relations is closed under finite unions, finite intersections, recursive substitutions, and has the pre-well-ordering property. Martin's Borel determinacy and the transfer theorem of this section (4.4.2) now immediately imply the following:

4.4.3 Corollary. *The class $\mathfrak{D}\Sigma_k^0$ has the pre-well-ordering property, where $k \geq 1$. Moreover, each $\mathfrak{D}\Sigma_k^0$ is a Spector class.* ☐

The classical normal form for the Π_1^1 relations on the integers and Theorem 2.1.5, in effect, state that

$$\mathfrak{D}\Sigma_1^0 = \Pi_1^1.$$

Solovay has obtained the characterization of the class $\mathfrak{D}\Sigma_2^0$ in terms of non-monotone inductive definitions and this we present in

4.4.4 Theorem. *Let $\mathbb{N} = \langle\omega, +, \cdot\rangle$ be the structure of arithmetic and let Σ_1^1 be the collection of all Σ_1^1 second-order relations on ω. Then a relation P of integers and reals in $\mathfrak{D}\Sigma_2^0$ if and only if it is Σ_1^1-(non-monotone) inductive; that is to say,*

$$\mathfrak{D}\Sigma_2^0 = \Sigma_1^1\text{-IND}. \quad \square$$

In another direction, we first notice that

$$\mathfrak{D}\Pi_{2n+1}^1 = \Sigma_{2n+2}^1$$

for any $n = 0, 1, 2, \ldots$. Moreover, using the hypothesis of projective determinacy (PD), it is easy to see that

$$\Game\Sigma^1_{2n} = \Pi^1_{2n+1}$$

for any $n = 1, 2, \ldots$.

The computations given above when combined with the transfer theorem (4.4.2) give the next result, a result which was first proved directly by Martin and Moschovakis.

4.4.5 Theorem. *Assuming projective determinacy* (PD), *the classes* Π^1_{2n+1} *and* Σ^1_{2n+2} *have the pre-well-ordering property for all* $n = 0, 1, 2, \ldots$. *In fact*, Π^1_{2n+1} *and* Σ^1_{2n+2} *are Spector classes for all* $n = 0, 1, 2, \ldots$. □

This result is part of the periodicity picture for the projective sets, assuming projective determinacy. For more on the periodicity phenomena as well as on transfer theorems involving much stronger properties, we again refer the reader to Moschovakis [1980].

Recently work has been done on the game quantifier $\Game^2$ on the set R^ω of infinite sequences of reals. This includes transfer theorems of the type we have described here as well as a very useful characterization of the Σ^2_1 in $L(\mathbb{R})$ sets of reals.

The *inner model* $L(\mathbb{R})$ is the smallest model of ZF which contains the structure $\mathbb{R} = \langle \omega^\omega \cup \omega, \omega, +, \cdot, Ap\rangle$ of analysis and all the ordinals as elements. If P is a relation on integers and reals, we say that P is Σ^2_1 *in* $L(\mathbb{R})$ if there is a formula $\varphi(\bar{x}, \bar{a}, X)$ of the first-order language $\mathscr{L}^{\mathbb{R}}$ of the structure $\mathbb{R}$ such that

$$P(\bar{x}, \bar{a}) \Leftrightarrow (\text{in } L(\mathbb{R}) \text{ we have that } \mathbb{R} \models (\exists X)\varphi(\bar{x}, \bar{a}, X))$$

where, of course, the existential quantifier $(\exists X)$ ranges over subsets of reals.

In the terminology of Sections 1 and 2 of this chapter, the Σ^2_1 in $L(\mathbb{R})$ sets of reals are exactly the sets of reals definable in the sense of $L(\mathbb{R})$ by Σ^1_1 second-order formulas of the structure $\mathbb{R}$ of analysis.

We will end this chapter with a theorem of Martin and Steel. This result can be found in Martin–Moschovakis–Steel [1982].

4.4.6 Theorem. *A relation P on integers and reals is* Σ^2_1 *in* $L(\mathbb{R})$ *if and only if it is* $\Game^2\Pi^1_1$; *that is to say, if and only if there is a* Π^1_1 *relation S such that*

$$P(\bar{x}, \bar{\alpha}) \Leftrightarrow (\exists\beta_0\, \forall\gamma_0\, \exists\beta_1\, \forall\gamma_1 \cdots)S(\bar{x}, \bar{\alpha}, \langle\beta_0, \gamma_0, \beta_1, \gamma_1, \ldots\rangle),$$

where the quantifiers in the infinite string range over the reals. □

The above result provides a representation of the Σ^2_1 in $L(\mathbb{R})$ sets of reals in terms of the game quantifier $\Game^2$ applied to a very simple matrix. This representation, together with appropriate transfer theorems and determinacy hypotheses, makes it possible to obtain important structural properties for the class Σ^2_1 in $L(\mathbb{R})$.

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