5. Compact Complex Manifolds Containing "Global" Spherical Shells

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0. Introduction. Fix an integer $n\geq 2$. For ε, $0<\varepsilon<1$, we put

$$S_\varepsilon=\{z\in \boldsymbol{C}^n : 1-\varepsilon<\|z\|<1+\varepsilon\},$$
$$B_\varepsilon=\{z\in \boldsymbol{C}^n : \|z\|<1+\varepsilon\}, \text{ and}$$
$$\Sigma=\{z\in \boldsymbol{C}^n : \|z\|=1\},$$

where $\|z\|=(\sum_{j=1}^n |z_j|^2)^{1/2}$, $z=(z_j)$.

Let X be a compact complex manifold of dimension n. An open subset N of X is called a *spherical shell* if N is biholomorphic to S_ε for some ε.

Definition 1. A spherical shell N in X is said to be *global* if $X-N$ is connected. Otherwise, N is said to be *local.*

It is clear that, if N is local, then $X-N$ has two connected components. Any complex manifolds contain *local* spherical shells. But *global* spherical shells can be contained in only special types of manifolds.

Before stating the main results, we recall the definition of Hopf manifolds.

Definition 2. A compact complex manifold of dimension n (≥ 2) is called a *Hopf manifold* if its universal covering manifold is biholomorphic to $\boldsymbol{C}^n-\{0\}$. A Hopf manifold is said to be *primary* if its fundamental group is infinite cyclic.

1. Main results. Theorem 1. *Suppose that a compact complex manifold X of dimension n (≥ 2) contains a global spherical shell. Then we can construct a complex analytic family $\pi: \mathfrak{X}\rightarrow T=\{t\in \boldsymbol{C}: |t|<1\}$ of small deformations of X such that*

(i) $X=\pi^{-1}(0)$,

(ii) $X_t=\pi^{-1}(t)$ $(t\neq 0)$ *is biholomorphic to a compact complex manifold which is a modification of a primary Hopf manifold at finitely many points.*

Corollary 1. *The fundamental group of X is infinite cyclic. In particular, X is non-Kähler.*

We note that X itself is not always a modification of a Hopf manifold. In fact, if $n=2$, *all compact complex surfaces constructed by M. Inoue in* [2] *and* [3], *which are of Class* VII_0 *with positive second Betti numbers, contain global spherical shells, but none of them is a modi-*

fication of a Hopf surface. Consequently, by Theorem 1, we have

Theorem 2. *All compact complex surfaces constructed by Inoue in* [2] *and* [3] *are deformations of modifications of primary Hopf surfaces.*

2. All compact complex manifolds of dimension n containing global spherical shells are constructed as follows. (Here we shall make only a rough explanation.) Let B be the *open* unit ball in $\boldsymbol{C}^n$ and $\sigma: B_1 \to B$ a modification of B at finitely many points. Remove a small *closed* ball $\bar{B}_0$ from B_1, where B_0 is the interior of $\bar{B}_0$. Then the compact complex manifold X is obtained by identifying the two boundaries of $\bar{B}_1 - B_0$, where $\bar{B}_1 = B_1 \cup \Sigma$.

This identification induces a biholomorphic mapping $\zeta: B \to B_0 \subset B_1$. We denote by A the maximal compact analytic subset in B_1. If $\zeta \circ \sigma(A) \cap A = \emptyset$, then X is a modification of a primary Hopf manifold. If $\zeta \circ \sigma(A) \cap A \neq \emptyset$, then, by a slight translation of B_0 in B_1, we can make the set $\zeta \circ \sigma(A)$ lie outside of A. In other words, X can be deformed to a modification of a primary Hopf manifold.

By this method, we can determine all small deformations of the surface constructed in [2].

3. Griffiths ([1], p. 40) considered the following

Problem. Let M be a compact *Kähler* manifold of dimension n. Then is it true that *any* holomorphic mapping $\varphi: S_\varepsilon \to M$ extends meromorphically to B_ε?

He gave a partial answer to this problem and also gave a counter example in the case where M is *non-Kähler*. Namely he considered a primary Hopf manifold M and a restriction φ to S_ε ($\subset \boldsymbol{C}^n - \{0\}$) of the covering projection $\boldsymbol{C}^n - \{0\} \to M$. Note that in this case φ is biholomorphic near Σ. Concerning this problem, we obtain

Corollary 2. *Let M be a compact complex manifold of dimension n (≥ 2) and $\varphi: S_\varepsilon \to M$ a holomorphic mapping which is biholomorphic near Σ. Suppose that φ can not be extended meromorphically to the whole of B_ε. Then M is a deformation of a compact complex manifold which is a modification of a primary Hopf manifold at finitely many points.*

Details of the results will appear in the Proceedings of the International Symposium on Algebraic Geometry, Kyoto, Jan., 1977.

References

[1] Griffiths, P. A.: Two theorems on extensions of holomorphic mappings. Inv. Math., **14**, 27–62 (1971).

[2] Inoue, M.: New Surfaces with No Meromorphic Functions. Proc. Int. Congress of Math. Vancouver (1974).

[3] ——: New Surfaces with No Meromorphic Functions. II (to appear). Complex Analysis and Algebraic Geometry. Iwanami Shoten, Pub. Tokyo.