On an ad hoc computability structure in a Hilbert space

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Abstract: Pour-El & Richards [3] discussed an ad hoc computability structure in an effectively separable Hilbert space taking as an effective generating set a slightly modified one from the original orthonormal basis. We show that an application of the Poincaré-Wigner orthogonalizing procedure to Pour-El & Richards' modified system gives an orthonormal effective generating set which yields a third computability structure.

Key words: Computability structure; effectively separable Hilbert space.

1. Introduction. Pour-El and Richards discussed an *ad hoc* computability structure in an effectively separable Hilbert space $\mathbf{X}$ (over the complex number field) with a computability structure $\mathcal{S}$, i.e., $\langle \mathbf{X}, \mathcal{S} \rangle$ ([3], Chapter 4, §§5, 6). Recall that a computability structure is the set of all the computable sequences and that computable sequences in $\mathbf{X}$ are specified by a set of three axioms ([3], Chapter 2, §1). Effective separability of $\mathbf{X}$ means that $\mathbf{X}$ admits a computable sequence, say $\mathcal{E}$, called effective generating set, whose linear combinations are dense in $\mathbf{X}$. Thus, when a countable basis is designated as a computable sequence, a computability structure is determined (Effective Densnity Lemma. [3], p. 86). In fact, Pour-El & Richards actually worked out the case of $\mathbf{X} = L^2[0,1]$, taking the standard complete orthonormal basis $\{ e^{2\pi i m x}, m = 0, \pm 1, \pm 2, \ldots\}$ as an effectively generating set, which determines the standard computability structure of $L^2[0,1]$. Since their crucial arguments were done in the space ℓ^2, we may replace $L^2[0,1]$ by a separable Hilbert space $\mathbf{X}$ and $\{e^{2\pi i m x}\}$ by any of its complete orthonormal bases $\mathcal{E} = \{\mathbf{e}_n;\, n = 0, 1, 2, \ldots\}$, and may then consider $\mathcal{E}$ as the *standard* basis of $\mathbf{X}$ and the computability structure $\mathcal{S}$ generated by it as the *standard* one.

Following [3] faithfully, we then have another computability structure, an *ad hoc* computability structure $\mathcal{T}$ in $\mathbf{X}$, effectively generated by a sequence $\mathcal{F} = \{\mathbf{f}, \mathbf{e}_1, \mathbf{e}_2, \ldots\}$ with $\mathbf{f} \in \mathbf{X}$, non-computable with respect to $\mathcal{S}$. To specify $\mathbf{f}$, Pour-El and Richards took a recursive function $a : \mathbf{N} \to \mathbf{N}$ which enumerates a recursively enumerable non-recursive set A in a one-to-one manner, supposing $0 \notin A$. Then they let

$$\alpha_n = 2^{-a(n)}, \quad n \geq 1, \tag{1}$$

and

$$\gamma^2 = 1 - \sum_{n=1}^{\infty} \alpha_n^2, \quad \gamma > 0, \tag{2}$$

whence finally

$$\mathbf{f} = \gamma\, \mathbf{e}_0 + \sum_{n=1}^{\infty} \alpha_n\, \mathbf{e}_n. \tag{3}$$

Notice that $\sqrt{2/3} < \gamma < 1$ and γ is not computable since the convergence (2) is not effective (cf. [3], pp. 16–17). Thus, $\mathbf{f}$ is not computable in $\langle \mathbf{X}, \mathcal{S} \rangle$. They subsequently applied the Gramm-Schmidt orthogonalization procedure to the system $\mathcal{F}$ to get an orthonormal basis $\{\mathbf{u}_n;\, n = 0, 1, 2, \ldots\}$ of $\mathbf{X}$.

Although their observations occupied only a part of the proof of the Eigenvector Theorem, they thus showed existence of a unitary operator $U : \mathbf{X} \to \mathbf{X}$, which maps $\mathcal{S}$ onto $\mathcal{T}$. However, they wondered how this operator U could be grasped more explicitly ([3], pp. 139–141). Actually, it is evident that the image $\mathcal{E}_V$ of $\mathcal{E}$ by any unitary operator V in $\mathbf{X}$ defines a computability structure $\mathcal{S}_V$ in $\mathbf{X}$. If $\mathcal{E}$ is an orthonormal basis, then so is $\mathcal{E}_V$. Thus, by means of Fourier coefficients, the question, as mentioned earlier, is reduced to a discussion of unitary matrices acting in the space ℓ^2 of square summable series. It is certainly interesting to obtain detailed knowledge about such matrices.

The purpose of the present note is to apply the Poincaré-Wigner orthogonalization procedure to the

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above $\mathcal{F}$, which results producing an explit unitary matrix (See §3 below). It turns out that the orthonormal basis thus obtained defines a third computability structure in $\mathbf{X}$ which coincides neither with the standard one $\mathcal{S}$ nor with the structure $\mathcal{T}$ of Pour-El & Richards mentioned above (See §4).

2. The Poincaré-Wigner procedure. It is well-known that given an orthonormal basis $\mathcal{E} = \{\mathbf{e}_n;\ n = 0, 1, 2, \ldots\}$ of a Hilbert space $\mathbf{X}$, a unitary isomorphism $\Phi_{\mathcal{E}}$ from $\mathbf{X}$ to the Hilbert space ℓ^2 of square summable sequences of complex numbers is determined by the Fourier expansion

$$\begin{aligned} &\Phi_{\mathcal{E}} : \mathbf{X} \ni \mathbf{x} \mapsto x = (\xi_0, \xi_1, \xi_2, \ldots) \in \ell^2 \\ &\qquad \xi_n = (\mathbf{x}, \mathbf{e}_n), \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{4}$$

where (,) denotes the scalar product of the Hilbert space $\mathbf{X}$. The unitarity is nothing but the Parseval relation

$$(\mathbf{x}, \mathbf{x}) = \sum_{n=0}^{\infty} |\xi_n|^2. \tag{5}$$

The sequence $\mathcal{F}$ is not orthonormal, but serves as a basis of the space $\mathbf{X}$ (For Riesz bases and the related materials, see, e.g., Daubechies [1]).

Lemma 2.1. *The sequence $\mathcal{F}$ is a Riesz basis in the Hilbert space $\mathbf{X}$. In other words, $\mathcal{F}$ determines a linear isomorphism $R_{\mathcal{F}}$ from $\mathbf{X}$ onto the Hilbert space ℓ^2.*

Proof. First observe that any $\mathbf{x} \in \mathbf{X}$ is uniquely expressed as

$$\mathbf{x} = \eta_0\, \mathbf{f} + \sum_{n=1}^{\infty} \eta_n\, \mathbf{e}_n \tag{6}$$

the right-hand side converging in $\mathbf{X}$. In fact, in terms of the system $\{\mathbf{e}_n\}$,

$$\eta_0\, \gamma = \xi_0, \quad \eta_0\, \alpha_n + \eta_n = \xi_n, \quad n = 1, 2, \ldots . \tag{7}$$

Note then

$$(\mathbf{x}, \mathbf{x}) = |\eta_0|^2 + \sum_{n=1}^{\infty} \alpha_n(\eta_0 \overline{\eta_n} + \overline{\eta_0}\eta_n) + \sum_{n=1}^{\infty} |\eta_n|^2 \tag{8}$$

since $(\mathbf{f}, \mathbf{f}) = 1$. Since

$$\left| \sum_{n=1}^{\infty} \alpha_n\, (\eta_0 \overline{\eta_n} + \overline{\eta_0}\eta_n) \right| \leq \frac{1-\gamma^2}{\epsilon}\, |\eta_0|^2 + \epsilon \sum_{n=1}^{\infty} |\eta_n|^2$$

for any $\epsilon > 0$, we see

$$A \sum_{n=0}^{\infty} |\eta_n|^2 \leq (\mathbf{x}, \mathbf{x}) \leq B \sum_{n=0}^{\infty} |\eta_n|^2 \tag{9}$$

for some $A > 0$ and $B > 0$. In fact, taking $\epsilon = 1 - (1/2)\gamma^2$, we have

$$A = \min\left\{1 - \frac{1-\gamma^2}{\epsilon},\ 1 - \epsilon\right\} = \frac{1}{2}\gamma^2$$

while by taking $\epsilon = 1$

$$B = \max\left\{1 + \frac{1-\gamma^2}{\epsilon},\ 1 + \epsilon\right\} = 2.$$

Thus, (6) and (9) determine a linear isomorphism

$$\mathcal{R}_{\mathcal{F}} : \mathbf{x} \ \mapsto\ y = (\eta_0, \eta_1, \eta_2, \ldots) \tag{10}$$

from $\mathbf{X}$ to the Hilbert space ℓ^2 of square summable sequences. □

Now we rewrite the computability structure $\mathcal{T}$ in the following way (cf. [3], p. 135).

Proposition 2.1. *Let $\mathcal{S}_2$ be the standard computability structure of the Hilbert space ℓ^2. Then the linear isomorphism $R_{\mathcal{F}}$ maps the computability structure $\mathcal{T}$ of $\mathbf{X}$ onto $\mathcal{S}_2$.*

Remark 2.1. By means of the Fourier expansion $\Phi_{\mathcal{E}}$, we have the linear isomorphism

$$\mathcal{R} = R_{\mathcal{F}}\, \Phi_{\mathcal{E}}^{-1} :\ \ell^2 \to \ell^2.$$

In fact, write the Fourier expansion

$$\Phi_{\mathcal{E}}\left(\sum_{n=0}^{\infty} c_n\, \mathbf{e}_n\right) = \sum_{n=0}^{\infty} c_n\, e_{(n)}.$$

Here each $e_{(j)} \in \ell^2$ has only one non-vanishing component, the (j+1)-st, which is 1. Then $\Phi_{\mathcal{E}}(\mathbf{f}) = \gamma\, e_{(0)} + \sum_{n=1}^{\infty} \alpha_n\, e_{(n)}$ but $R_{\mathcal{F}}(\mathbf{f}) = e_{(0)}$ and $R_{\mathcal{F}}(\mathbf{e}_n) = e_{(n)}$ for $n \geq 1$. $\mathcal{R}$ is given by an infinite square matrix

$$\mathcal{R} = I - \frac{1}{\gamma}\, P_0, \tag{11}$$

where I is the identity matrix in the space ℓ^2 and

$$P_0 = \begin{pmatrix} \gamma - 1 & 0 & 0 & \cdots & \cdots \\ \alpha_1 & 0 & 0 & \ddots & \ddots \\ \alpha_2 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ \alpha_n & & & & \\ \vdots & & & & \end{pmatrix}. \tag{12}$$

Note that the operator P_0 is nilpotent since the matrix equation

$$P_0^2 + (1-\gamma)P_0 = 0 \tag{13}$$

is valid. The inverse of $\mathcal{R}$ is then a slightly simpler matrix

$$\mathcal{R}^{-1} = I + P_0 \tag{14}$$

because of (7). The operator $\mathcal{R}$ in fact induces an ad hoc computability structure $\mathcal{T}_2$ in the space ℓ^2 from the standard computability structure $\mathcal{S}_2$ in ℓ^2 in the sense $\mathcal{T}_2 = \mathcal{R}^{-1}(\mathcal{S}_2)$.

The following is an ℓ^2 version of Proposition 2.1.

Proposition 2.2. *Consider the sequence $\{x_{(n)}\}$, given by $x_{(n)} = \sum_{k=0}^{\infty} a_{nk}\, e_{(k)}$. $\{x_{(n)}\}$ is a computable sequence in $\mathcal{T}_2$ if and only if the following conditions hold*:

a. the sequences $\left\{\frac{1}{\gamma}\, a_{n0}\right\}$ and $\left\{a_{nk} - \frac{\alpha_k}{\gamma}\, a_{n0}\right\}$ are computable;

b. $\sum_{k=1}^{\infty} \left|a_{nk} - \frac{\alpha_k}{\gamma}\, a_{n0}\right|^2$ converge effectively in n and k.

Proof. Recall that $\{e_{(n)}\}$ is an effective generating set of $\mathcal{S}_2$. Thus, just write down the computability criterion of the sequence $\{\mathcal{R}(x_{(n)})\}$ in $\mathcal{S}_2$ (see [3], p. 136, Lemma 1). □

Now recall (8). By (10), the right-hand side of (8) is a positive definite quadratic form of $y \in \ell^2$

$$(\mathbf{x}, \mathbf{x}) = (y, Gy). \tag{15}$$

Here $G : \ell^2 \to \ell^2$ is a self-adjoint operator given by

$$G = (R_{\mathcal{F}}^{-1})^* \, R_{\mathcal{F}}^{-1}$$

with $Q^* : \mathbf{X} \to \ell^2$ being the adjoint of a linear operator $Q : \ell^2 \to \mathbf{X}$. We obviously have $G = (\mathcal{R}^{-1})^* \, \mathcal{R}^{-1}$. Therefore, G is represented as an infinite square matrix $G = I + P_0^* + P_0 + P_0^* \, P_0$.

Lemma 2.2. *Let Γ_0 be an infinte square matrix*:

$$\Gamma_0 = P_0^* + P_0 + P_0^* \, P_0$$

$$= \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \cdots & \cdots \\ \alpha_1 & 0 & 0 & \cdots & \cdots \\ \alpha_2 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ \alpha_n & 0 & & & \\ \vdots & \vdots & & & \end{pmatrix}. \tag{16}$$

Then we have

$$G = I + \Gamma_0, \quad G^{-1} = I - \frac{1}{\gamma^2}\, \Gamma_0 + \frac{1}{\gamma^2}\, \Gamma_0^2. \tag{17}$$

Proof. Note

$$\Gamma_0^2 = \begin{pmatrix} 1-\gamma^2 & 0 & 0 & \cdots & \cdots \\ 0 & \alpha_1^2 & \alpha_1\alpha_2 & \alpha_1\alpha_3 & \cdots \\ 0 & \alpha_2\alpha_1 & \alpha_2^2 & \alpha_2\alpha_3 & \cdots \\ \vdots & \vdots & & \ddots & \\ \vdots & & & & \end{pmatrix}.$$

To compute G^{-1}, observe

$$\Gamma_0^3 = (1-\gamma^2)\, \Gamma_0 \tag{18}$$

by a simple computation. □

Remark 2.2. We also have

$$(P_0 - P_0^*)^2 + \Gamma_0^2 = 0,$$

and

$$\begin{aligned} (P_0 - P_0^*)\, \Gamma_0^2 &= \Gamma_0^2\, (P_0 - P_0^*) \\ &= (1-\gamma^2)\, (P_0 - P_0^*). \end{aligned}$$

Note G^{-1} is a bounded, positive definite self-adjoint operator. Hence, we may talk of its square root $G^{-1/2}$ which we will compute shortly (See §3).

Now the Poincaré-Wigner orthogonalization procedure reads as follows:

Proposition 2.3. *Let $e_{(0)} = R_{\mathcal{F}}(\mathbf{f})$ and $e_{(n)} = R_{\mathcal{F}}(\mathbf{e}_n)$, $n = 1, 2, \ldots$. Let*

$$\mathbf{v}_j = R_{\mathcal{F}}^{-1}\big(G^{-1/2}\, e_{(j)}\big), \quad j = 0, 1, 2, \ldots. \tag{19}$$

Then the system $\mathcal{V} = \{\mathbf{v}_j;\ j = 0, 1, 2, \ldots\}$ is an orthonormal basis.

Proof. Recall that each $e_{(j)}$ has only one non-vanishing component 1 at the $j+1$-st place. Let $j, k = 0, 1, 2, \ldots$. Then

$$\begin{aligned} (\mathbf{v}_j, \mathbf{v}_k) &= \big(R_{\mathcal{F}}^{-1}(G^{-1/2}\, e_{(j)}), R_{\mathcal{F}}^{-1}(G^{-1/2}\, e_{(k)})\big) \\ &= (G^{-1/2}\, e_{(j)}, G\, G^{-1/2}\, e_{(k)}). \end{aligned}$$

Here the third term is due to (15). But

$$\begin{aligned} (G^{-1/2}\, e_{(j)}, G\, G^{-1/2}\, e_{(k)}) &= (e_{(j)}, e_{(k)}) \\ &= \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}. \end{aligned}$$

Thus, $\mathcal{V}$ is orthonormal. Completeness is obvious from Lemma 2.1. □

3. The inverse square root $G^{-1/2}$.

Recall the following formula.

Lemma 3.1. *Let H be a bounded self-adjoint positive definite linear operator in a Hilbert space. Then its inverse square root $H^{-1/2}$ is given by*

$$H^{-1/2} = \frac{2}{\pi}\int_0^{+\infty} \sqrt{t}\,(t\,I + H)^{-2}\,dt. \tag{20}$$

Proof. Here we sketch its derivation using the spectral decomposition of H:

$$H = \int_a^b s\,dE(s), \quad 0 < a < b,$$

with the spectral projection operators $\{E(s)\}$. Then the right-hand side of (20) turns out

$$\int_a^b \frac{2}{\pi}\int_0^{+\infty} \frac{\sqrt{t}}{(t+s)^2}\,dt\,dE(s) = \int_a^b \frac{1}{\sqrt{s}}\,dE(s)$$

which is nothing but the left-hand side of (20). Note (20) is actually valid for general non-negative closed linear operators (See Komatsu [2]). □

Now we compute the inverse square root $G^{-1/2}$.

Lemma 3.2. *We have*

$$G^{-1/2} = I + \beta_1\,\Gamma_0 + \beta_2\,\Gamma_0^2, \tag{21}$$

where

$$\beta_1 = -\frac{1}{2}\frac{1}{\gamma}\sqrt{\frac{2}{1+\gamma}},$$

$$\beta_2 = \frac{1}{2}\frac{1}{1-\gamma}\frac{1}{\gamma}\sqrt{\frac{2}{1+\gamma}} - \frac{1}{1-\gamma^2}.$$

Proof. We apply the formula (20) to the operator G. Note

$$\begin{aligned}(t\,I + G)^{-1} = {} & \frac{1}{t+1}\,I - \frac{1}{t^2+2t+\gamma^2}\,\Gamma_0 \\ & + \frac{1}{(t+1)(t^2+2t+\gamma^2)}\,\Gamma_0^2,\end{aligned}$$

whence

$$\begin{aligned}(t\,I + G)^{-2} = {} & \frac{1}{(t+1)^2}\,I - \frac{2(t+1)}{(t^2+2t+\gamma^2)^2}\,\Gamma_0 \\ & + \left\{ -\frac{1}{1-\gamma^2}\frac{1}{(t+1)^2} \right. \\ & \quad + \frac{1}{1-\gamma^2}\frac{1}{t^2+2t+\gamma^2} \\ & \quad \left. + \frac{2}{(t^2+2t+\gamma^2)^2} \right\} \Gamma_0^2\end{aligned} \tag{22}$$

for $t > 0$. Note

$$\int_0^{+\infty} \frac{\sqrt{t}}{(t+1)^2}\,dt = \frac{\pi}{2},$$

$$\int_0^{+\infty} \frac{\sqrt{t}}{t^2+2t+\gamma^2}\,dt = \frac{\pi}{2}\sqrt{\frac{2}{1+\gamma}},$$

$$\int_0^{+\infty} \frac{\sqrt{t}}{(t^2+2t+\gamma^2)^2}\,dt = \frac{\pi}{8}\frac{1}{\gamma(1+\gamma)}\sqrt{\frac{2}{1+\gamma}},$$

$$\int_0^{+\infty} \frac{(t+1)\sqrt{t}}{(t^2+2t+\gamma^2)^2}\,dt = \frac{\pi}{8}\frac{1}{\gamma}\sqrt{\frac{2}{1+\gamma}}.$$

Hence, computing the right-hand side of (20) for $H = G$, we get (21). □

Corollary 3.1. *The square root $G^{1/2}$ is given by the formula*

$$G^{1/2} = I - \beta_1\,\gamma\,\Gamma_0 + (\beta_1 + \beta_2)\,\Gamma_0^2.$$

Proof. Employ (17) and (18). □

To get some idea about the system $\mathcal{V} = \{\mathbf{v}_j\}$ (Proposition 2.3), we state the following.

Proposition 3.1. *Let $v_{(j)} = \mathcal{R}^{-1}G^{-1/2}(e_{(j)})$, $j = 0, 1, 2, \ldots$. We have*

$$v_{(0)} = -\frac{1}{2\beta_1\gamma}\,e_{(0)} + \beta_1\gamma\;(0, \alpha_1, \alpha_2, \ldots)$$

and

$$v_{(n)} = e_{(n)} + \alpha_n\beta_1\gamma\,e_{(0)} + \alpha_n(\beta_1+\beta_2)\,(0, \alpha_1, \alpha_2, \ldots)$$

for $n \geq 1$. The system $\{v_{(n)};\ n = 0, 1, 2, \ldots\}$ is complete and orthonormal in the Hilbert space ℓ^2.

Remark 3.1. $\beta_1\gamma$ and $\beta_1 + \beta_2$ are not computable. To see this, note that γ is expressible as algebraic functions either of $\beta_1\gamma$ from

$$\beta_1\gamma = -\frac{1}{2}\sqrt{\frac{2}{1+\gamma}}, \quad \text{i.e.,} \quad \gamma = \frac{1}{2(\beta_1\gamma)^2} - 1,$$

or of $\beta_1 + \beta_2$ from

$$\beta_1 + \beta_2 = \frac{1}{1-\gamma^2}\left(\sqrt{\frac{1+\gamma}{2}} - 1\right),$$

i.e., γ now is a somewhat involved algebraic function of $\beta_1 + \beta_2$. Thus, if $\beta_1\gamma$ or $\beta_1 + \beta_2$ were computable, then so would be γ, contradicting its non-computability. Similarly, β_1 and β_2 are shown to be not computable.

Here is another interpretation of Lemma 3.2 and Proposition 3.1.

Proposition 3.2. *$\mathcal{R}^{-1}G^{-1/2}$ is a unitary operator from the Hilbert space ℓ^2 onto itself. $\mathcal{R}^{-1}G^{-1/2}$ is explicitly given as*

$$\mathcal{R}^{-1}G^{-1/2} = I + \frac{1}{2}\sqrt{\frac{2}{1+\gamma}}\,(\,P_0 - P_0^*\,) + (\beta_1 + \beta_2)\,\Gamma_0^2.$$

Proof. Obvious from the meaning. Note also that

$$G^{1/2}\mathcal{R} = I - \frac{1}{2}\sqrt{\frac{2}{1+\gamma}}\,(P_0 - P_0^*) + (\beta_1+\beta_2)\,\Gamma_0^2$$

by an explicit computation. □

4. The computability structure generated by $\mathcal{V}$. Let $\mathcal{S}_\mathcal{V}$ be the computability structure in $\mathbf{X}$ effectively generated by the orthonormal basis $\mathcal{V}$. Let $\{\mathbf{x}_m\}$ be a sequence in $\mathbf{X}$, given by

$$\mathbf{x}_m = \sum_{k=0}^{\infty} c_{mk}\,\mathbf{v}_k.$$

The sequence $\{\mathbf{x}_m\}$ is computable with respect to $\mathcal{S}_\mathcal{V}$ if and only if

(i) the double sequence $\{c_{mk}\}$ is computable;

(ii) the series $\sum_{k=0}^{\infty} |c_{mk}|^2$ converges effectively in k and m.

(See [3], p. 136).

In passing, we have the following observation.

Lemma 4.1. *Let c_{mn} be a computable double sequence as in the above. Then the sequence $\{\sum_{n=1}^{\infty} \alpha_n\, c_{mn}\}$ is computable.*

Proof. We show that $\{\sum_{n=1}^{k} \alpha_n\, c_{mn}\}$ effectively converges in m and k as $k\to\infty$. We have a recursive function $e(m,N)$ such that $\sum_{n\ge k} |c_{mn}|^2 \le 2^{-2N}$ for $k \ge e(m,N)$. Then

$$\left|\sum_{n\ge k} \alpha_n\, c_{mn}\right| \le \sqrt{\sum_{n\ge k} |\alpha_n|^2}\sqrt{\sum_{n\ge k} |c_{mn}|^2} \le \sqrt{\sum_{n\ge k} |c_{mn}|^2} \le 2^{-N}$$

for $k \ge e(m,N)$. □

We show that the computability structure $\mathcal{S}_\mathcal{V}$ is different from the structures $\mathcal{S}$ and $\mathcal{T}$.

Proposition 4.1. $\mathbf{f}$ *is not computable in the structure $\mathcal{S}_\mathcal{V}$. Thus, $\mathcal{T}$ and $\mathcal{S}_\mathcal{V}$ are different.*

Proof. Note $\mathbf{f} = \sum_{n=0}^{\infty} \varphi_n\,\mathbf{v}_n$, where

$$\varphi_n = (\mathbf{f}, \mathbf{v}_n) = (e_{(0)}, G^{1/2}\, e_{(n)}) = \begin{cases} -\dfrac{1}{2\beta_1\gamma}, & n=0 \\ -\alpha_n\,\beta_1\,\gamma + (\beta_1+\beta_2)\alpha_n^2, & n\ge 1 \end{cases}$$

by virtue of Poposition 2.3 and Corollary 3.1. □

Proposition 4.2. *The computability structure $\mathcal{S}_\mathcal{V}$ in $\mathbf{X}$ is different from the standard computability structure $\mathcal{S}$.*

Proof. Let us check how this computability criterion is related to the standard computability structure $\mathcal{S}$ of $\mathbf{X}$. To do so, we export the question to the space ℓ^2 and compare computability structures respectively induced by $\{v_{(j)}\}$ and by $\{e_{(j)}\}$. Thus, the above criterion ensures that the sequence $\{x_{(m)}\}$, given by

$$x_{(m)} = \sum_{k=0}^{\infty} c_{mk}\, v_{(k)}, \tag{23}$$

is computable with respect to the structure corresponding to $\{v_{(j)}\}$. Rewriting (23) in terms of $\{e_{(j)}\}$, we have

$$x_{(m)} = \sum_{n=0}^{\infty} \tilde{c}_{mn}\, e_{(n)}$$

where $\tilde{c}_{m0} = -1/(2\beta_1\gamma)\, c_{m0} + \beta_1\gamma \sum_{k=1}^{\infty} \alpha_k\, c_{mk}$ and $\tilde{c}_{mn} = c_{mn} + \alpha_n\beta_1\gamma\, c_{m0} + \alpha_n(\beta_1+\beta_2)\sum_{k=1}^{\infty} \alpha_k c_{mk}$ for $n\ge 1$. Thus, by virtue of Lemma 4.1 and Remark 3.1, $\tilde{c}_{m0}$ is not computable unless $c_{m0}=0$ and $\sum_{k=1}^{\infty} \alpha_k\, c_{mk} = 0$. When $c_{m0}=0$, $\tilde{c}_{mn}$, $n\ge 1$, are not computable unless $\sum_{k=1}^{\infty} \alpha_k\, c_{mk} = 0$. This shows that the sequence $\{x_{(n)}\} \notin \mathcal{S}_2$ at least when $\sum_{k=1}^{\infty} \alpha_k\, c_{mk} \ne 0$ (for some m). □

Remark 4.1. In a similar manner, it can be shown that $\{x_{(n)}\} \notin \mathcal{T}_2$ when $\sum_{k=1}^{\infty} \alpha_k\, c_{mk} \ne 0$ for some m while $c_{n0}=0$ for all n (See Proposition 2.2. See also Stability Lemma, [3], p. 79).

5. The Pour-El & Richards' operator T. Pour-El and Richards originally considered the following self-adjoint opertor T defined by

$$T\,\mathbf{e}_0 = 0, \quad T\,\mathbf{e}_n = 2^{-n}\,\mathbf{e}_n,\ n\ge 1. \tag{24}$$

Its matrix representation in ℓ^2 in the basis $\{e_{(n)}\}$ is given by

$$\Phi_\mathcal{E}\, T\, \Phi_\mathcal{E}^{-1} = \begin{pmatrix} 0 & 0 & \cdots & \\ 0 & 2^{-1} & 0 & \\ 0 & 0 & 2^{-2} & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

To obtain the matrix representation in the basis $\{v_{(n)}\}$, we have to compute

$$\tilde{T} = \mathcal{R}^{-1} G^{-1/2} \Phi_\mathcal{E}\, T\, \Phi_\mathcal{E}^{-1} G^{1/2}\mathcal{R}.$$

Let $\rho = \sum_{n=1}^{\infty} 2^{-n}\,\alpha_n^2$. Then $\tilde{T} = (t_{ij})$ $(i,j = 0,1,2,\ldots)$ is given by

$$t_{00} = \frac{1}{2}\frac{1}{1+\gamma}\rho$$

$$t_{0i} = t_{i0} = \beta_1\gamma\left(\frac{1}{2^i} + (\beta_1+\beta_2)\rho\right)\alpha_i,$$

$i \geq 1$, and, for $i, j \geq 1$,

$$\begin{aligned} t_{ij} &= t_{ji} \\ &= (\beta_1 + \beta_2)\left\{\frac{1}{2^i} + \frac{1}{2^j} + (\beta_1 + \beta_2)\rho\right\}\alpha_i\alpha_j \\ &\quad + \frac{1}{2^i}\,\delta_{ij} \end{aligned}$$

where δ_{ij} is Kronecker's delta. Thus, none of the components are computable.

However, any eigenvector corresponding to the eigenvalue 0 is a multiple of $v_{(0)}$. Recall $v_{(0)}$ is computable in the computability structure $\mathcal{S}_{\mathcal{V}}$.

Remark 5.1. ρ is computable. In fact,

$$\sum_{n=N}^{\infty} \frac{\alpha_n^2}{2^n} \leq 2^{-N}\,(1-\gamma^2) \leq 2^{-N}$$

for any positive integer N.

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