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A STOCHASTIC APPROACH TO THE RIEMANN-ROCH THEOREM

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1. Introduction

The index theorems for classical elliptic complexes, i.e., de Rham, signature, spin and Dolbeault complexes, are typical and substantial examples of the Atiyah-Singer index theorem. Restricting to these classical complexes, the *heat equation method*, which was first proposed by McKean-Singer [12] for the de Rham complex and was accomplished by Patodi [13], is nowadays well-known. This approach is based on the identity between the index and the alternating sum of traces, sometimes called supertrace, of heat kernels. So the problem is reduced to obtain the asymptotic expansion of the heat kernel. For this, Patodi [13], [14] used the parametrix and then, Gilkey [6] (cf. also, Atiyah-Bott-Patodi [2]) used the invariance theory and many researches followed.

Recently, J.-M. Bismut discussed this problem by using the stochastic analysis, especially the *Malliavin calculus*. He computed the index theorem for the twisted spin complex and his method is based on the splitting of the Wiener space and the pinned Wiener process. Then S. Watanabe computed the index theorem for the de Rham and the signature complexes ([9]) by a method somewhat different from Bismut's: He expressed the fundamental solution explicitly by using the composition of the Dirac delta function and the Brownian motion on a manifold, which is a typical generalized Wiener functional. Then he applied a theory of asymptotic expansion for generalized Wiener functionals, as developped in [9], [16], to obtain an asymptotic expansion of the fundamental solution. This method has an advantage that a formal Taylor expansion is applicable. In this paper, following Watanabe's method, we give a probabilistic proof of the *Riemann-Roch theorem*, i.e., the computation of the *index of Dolbeault complex* with coefficients in a holomorphic vector bundle V:

$$\bar{\partial}_V\colon \Gamma^\infty(\Lambda^{0,+}(M)\otimes V) \to \Gamma^\infty(\Lambda^{0,-}(M)\otimes V)$$

where M is a compact Kähler manifold, $\Lambda^{0,+}(M)$ and $\Lambda^{0,-}(M)$ are spaces of complex differential forms of degree (0, even) and (0, odd) respectively, V is a holomorphic vector bundle and $\bar{\partial}$ is the usual $\bar{\partial}$-operator obtained by a decomposition of the exterior differential d: $d=\partial+\bar{\partial}$ (see the section 2 for precise definitions). Although the Riemann-Roch theorem is valid for any compact almost complex manifold, we restrict ourselves to the compact Kähler manifold because the local theorem is true only for Kähler manifold (cf. Gilkey [6], Remark 3.6.1). This theorem was first proved in heat equation method by Patodi [14].

Here is an outline of our approach. First we construct the fundamental solution of $\frac{\partial}{\partial t}-\Delta_V^c$, where $\Delta_V^c=-(\bar{\partial}_V+\bar{\partial}_V^*)^2$. For this, we apply the Feynman-Kac formula to treat the term of multiplication operator appearing in a Weitzenböck type formula. Then we can obtain the fundamental solution explicitly by a generalized Wiener functional expectation. Secondly in order to calculate the index, we must obtain a cancellation of the supertrace of this fundamental solution and the Berezin formula is a main algebraic tool. This combined with the asymptotic expansion of generalized Wiener functional, enable us to obtain a cancellation at the level of functional before taking expectation, thereby compute the index.

Finally we explain the construction of this paper. In the section 2, we give a Weitzenböck type formula which is essential in constructing the fundamental solution. In the section 3, we express the fundamental solution by a generalized Wiener functional integration. In the section 4, we obtain the Berezin formula for the supertrace of Dolbeault complex. In the section 5, we give the proof of the index theorem.

2. Weitzenböck type formula

To fix notations, let us review the Kähler geometry. Let (M, g) be a compact Kähler manifold of complex dimension n and (V, h) be a holomorphic Hermitian vector bundle over M of fiber dimension k. We denote by $\Gamma^\infty(V)$ the set of all C^∞ sections of V. We use this notation for any vector bundle. Let $(z^1, \cdots, z^n)$ be a local holomorphic coordinate system for M and $T^cM=TM\otimes\boldsymbol{C}$ be the complexification of the real tangent bundle TM. Writing $z^j=x^j+iy^j$, set

$$\frac{\partial}{\partial z^j}=\frac{1}{2}\left(\frac{\partial}{\partial x^j}-i\frac{\partial}{\partial y^j}\right),\ \frac{\partial}{\partial \bar{z}^j}=\frac{1}{2}\left(\frac{\partial}{\partial x^j}+i\frac{\partial}{\partial y^j}\right).$$

Then T^cM can be decomposed as follows;

$$T^cM=T'M\oplus T''M$$

where

$$T'M=\operatorname{span}_{\boldsymbol{C}}\left\{\frac{\partial}{\partial z^j}\right\}_{j=1}^n,\quad T''M=\operatorname{span}_{\boldsymbol{C}}\left\{\frac{\partial}{\partial \bar{z}^j}\right\}_{j=1}^n.$$

Moreover, set

$$dz^j = dx^j + idy^j\,, \quad d\bar{z}^j = dx^j - idy^j\,,$$
$$\Lambda^{1,0}(M) = \mathrm{span}_{\boldsymbol{C}}\{dz^j\}_{j=1}^n\,, \quad \Lambda^{0,1}(M) = \mathrm{span}_{\boldsymbol{C}}\{d\bar{z}^j\}_{j=1}^n\,.$$

Then also it holds that

$$\Lambda^{1,0}(M) = (T'M)^*\,, \quad \Lambda^{0,1}(M) = (T''M)^*\,.$$

We set,

$$\Lambda^{p,q}(M) = \Lambda^p(\Lambda^{1,0}(M)) \otimes \Lambda^q(\Lambda^{0,1}(M))\,,$$
$$\Lambda^{0,*}(M) = \bigoplus_q \Lambda^{0,q}(M)\,,$$
$$\Lambda^{0,+}(M) = \bigoplus_{q\,:\,\mathrm{even}} \Lambda^{0,q}(M)\,, \quad \Lambda^{0,-}(M) = \bigoplus_{q\,:\,\mathrm{odd}} \Lambda^{0,q}(M)\,.$$

We decompose the exterior derivative d as $d = \partial + \bar{\partial}$ where

$$\partial\colon \Gamma^\infty(\Lambda^{p,q}(M)) \to \Gamma^\infty(\Lambda^{p+1,q}(M))\,,$$
$$\bar{\partial}\colon \Gamma^\infty(\Lambda^{p,q}(M)) \to \Gamma^\infty(\Lambda^{p,q+1}(M))\,.$$

We extend $\bar{\partial}$ to complex differential forms with coefficient V as follows;

$$\bar{\partial}_V\colon \Gamma^\infty(\Lambda^{0,q}(M)\otimes V) \to \Gamma^\infty(\Lambda^{0,q+1}(M)\otimes V)$$

by

$$\bar{\partial}_V(\omega\otimes s) = \bar{\partial}\omega\otimes s \tag{2.1}$$

where s is a local holomorphic section of V. This forms an elliptic complex called twisted Dolbeault complex.

Take a local holomorphic section $\{s_1, s_2, \cdots, s_k\}$ of V. We assume that the Riemannian metric g is extended to T^cM as a *Hermitian* inner product, i.e., complex linear in the first variable and conjugate linear in the second variable (be careful that in some literature, e.g., Kobayashi-Nomizu [10], g is to be complex bilinear). We set

$$g_{j\bar{h}} = g\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^h}\right), \quad g_{\bar{j}h} = g\left(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^h}\right).$$

Further we can introduce a metric on the cotangent bundle which is naturally defined from g. We also denote it by g and set

$$g^{j\bar{h}} = g(dz^j, dz^h)\,, \quad g^{\bar{j}h} = g(d\bar{z}^j, d\bar{z}^h)\,.$$

Then $(g^{j\bar{h}})$ is an inverse matrix of $(g_{j\bar{h}})$ in the sense that $g_{j\bar{h}}\, g^{k\bar{h}} = \delta_j^k$ where δ_j^k is the Kronecker delta: $\delta_j^k = 1$ for $j = k$ and $\delta_j^k = 0$ for $j \neq k$. Here and after we abbreviate the summation sign for repeated indices. Similarly, taking a local holomorphic section $\{s_1, s_2, \cdots, s_k\}$ of V, we set

$$h_{\alpha\bar\beta} = h(s_\alpha, s_\beta)\,,$$

and let $(h^{\alpha\bar\beta})$ be an inverse matrix in the sense that

$$h_{\alpha\bar\beta}\, h^{\gamma\bar\beta} = \delta^\gamma_\alpha\,.$$

Let ∇^V be a connection on V such that

(unitary) $\qquad h(\nabla^V_X s_\alpha, s_\beta)+h(s_\alpha, \nabla^V_{\bar X} s_\beta) = Xh(s_\alpha, s_\beta)\,,$ for $\forall X\in\Gamma^\infty(T^c M)\,,$

(holomorphic) $\quad \nabla^V_X s = 0$ for a holomorphic section s of V and $X\in\Gamma^\infty(T''M)$.

Such a connection exists uniquely and we call it the canonical connection on V. With respect to this connection, we have the following formulas for the covariant differentiation;

$$\nabla^V_j s_\alpha\,(= \nabla^V_{\partial/\partial z^j} s_\alpha) = l^\beta_{j\alpha} s_\beta\,, \quad \nabla^V_{\bar j} s_\alpha\,(= \nabla^V_{\partial/\partial \bar z^j} s_\alpha) = 0\,,$$

where

$$l^\beta_{j\alpha} = h_{\alpha\bar\gamma/j}\, h^{\beta\bar\gamma} \quad \left(h_{\alpha\bar\gamma/j} = \frac{\partial}{\partial z^j} h_{\alpha\bar\gamma}\right).$$

$T'M$ is a holomorphic vector bundle. So let ∇^M be the canonical connection on $T'M$. Then we have similarly

$$\nabla^M_j \frac{\partial}{\partial z^h} = \Gamma^l_{jh} \frac{\partial}{\partial z^l}\,, \quad \nabla^M_{\bar j} \frac{\partial}{\partial z^h} = 0\,,$$

where

$$\Gamma^l_{jh} = g_{h\bar p/j}\, g^{l\bar p}\,.$$

We extend ∇^M to $T^c M$ and $\Lambda^{0,*}(M)$ as usual. Then we have the following;

$$\nabla^M_j \frac{\partial}{\partial \bar z^h} = 0\,, \quad \nabla^M_{\bar j} \frac{\partial}{\partial \bar z^h} = \bar\Gamma^l_{jh} \frac{\partial}{\partial \bar z^l}\,, \quad \nabla^M_j dz^h = -\Gamma^h_{jl}\, dz^l\,,$$
$$\nabla^M_{\bar j} dz^h = \nabla^M_j d\bar z^h = 0\,, \quad \nabla^M_{\bar j} d\bar z^h = -\bar\Gamma^h_{lj}\, d\bar z^l\,.$$

(cf. Kobayashi-Nomizu [10], II, Chapter IX). We note that the above connection coincides with the Levi-Civita connection, since g is a Kähler metric.

Let $\bar\partial^*_V\colon \Gamma^\infty(\Lambda^{0,q}(M)\otimes V\to\Gamma^\infty(\Lambda^{0,q-1}(M)\otimes V)$ be the formal adjoint of $\bar\partial_V$, i.e.,

$$\int_M (\bar\partial_V \omega, \eta)\,\mathrm{dvol} = \int_M (\omega, \bar\partial_V^* \eta)\,\mathrm{dvol} \qquad \text{for} \quad \omega, \eta \in \Gamma^\infty(\Lambda^{0,*}(M)\otimes V).$$

Let $\Delta_V^c = -(\bar\partial_V + \bar\partial_V^*)^2$ be the associated Laplacian of this complex. We shall get the Weitzenböck type formula for Δ_V^c to solve the heat equation for Δ_V^c. Let $\nabla = \nabla^M \otimes 1_V + 1_{\Lambda^{0,*}(M)} \otimes \nabla^V$ be a connection on $\Lambda^{0,*}(M)\otimes V$. Let $\mathrm{ext}(d\bar z^j)\colon \Lambda^{0,p}(M) \to \Lambda^{0,p+1}(M)$ be defined by exterior multiplication, i.e.,

$$\mathrm{ext}(d\bar z^j)\,\omega = d\bar z^j \wedge \omega \qquad \text{for} \quad \omega \in \Lambda^{0,p}(M).$$

Let $\mathrm{int}(dz^j)\colon \Lambda^{0,p}(M) \to \Lambda^{0,p-1}(M)$ be defined by interior multiplication, i.e.,

$$\mathrm{int}(dz^j)\,d\bar z^h = g(d\bar z^h, d\bar z^j).$$

In other words, $\mathrm{int}(dz^j)$ is the dual operator of $\mathrm{ext}(d\bar z^j)$. Using these notations, we put

$$(\bar\partial + \bar\partial^*)_V = (\mathrm{ext}(d\bar z^j)\,\nabla_{\bar j} - \mathrm{int}(dz^j)\,\nabla_j).$$

Lemma 2.1. *We have the following identity;*

$$(\bar\partial + \bar\partial^*)_V = \bar\partial_V + \bar\partial_V^*. \tag{2.2}$$

Proof. (cf. Gilkey [6], p. 149) Both $(\bar\partial + \bar\partial^*)_V$ and $\bar\partial_V + \bar\partial_V^*$ are invariantly defined first order differetial operators whose leading symbols coincide each other and whose 0-th order symbols are linear in the 1-jets of the metric. For each point, we can take a holomorphic coordinate and a holomorphic frame such that the 1-jets of the metric vanish at the point (cf. Gilkey [6], Lemma 3.7.1 and Lemma 3.7.2). Therefore $(\bar\partial + \bar\partial^*)_V = \bar\partial_V + \bar\partial_V^*$. □

Next, let us define the curvature transformation as follows;

$$R^V(X, Y) = [\nabla_X^V, \nabla_Y^V] - \nabla_{[X,Y]}^V \text{ on } \Gamma^\infty(V) \qquad \text{for} \quad X, Y \in \Gamma^\infty(T^c M),$$

where $[X, Y] = XY - YX$. This is valid for any vector bundle. Further we define the curvature tensors as follows;

$$R^{T'M}\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar z^d}\right)\frac{\partial}{\partial z^b} = R^a{}_{bc\bar d}\frac{\partial}{\partial z^a},$$

$$R^{T''M}\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar z^d}\right)\frac{\partial}{\partial \bar z_b} = R^{\bar a}{}_{\bar b c\bar d}\frac{\partial}{\partial \bar z^a},$$

$$R^V\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar z^d}\right) s_\beta = L^\alpha{}_{\beta c\bar d}\, s_\alpha.$$

Then, it holds that

$$R^a{}_{bc\bar d} = -\Gamma^a_{cb/\bar d}, \quad R^{\bar a}{}_{\bar b c\bar d} = \bar\Gamma^a_{db/c}, \quad L^\alpha{}_{\beta c\bar d} = -l^\alpha_{c\beta/\bar d},$$

and

$$R^{\Lambda^{0,1}(M)}\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar{z}^d}\right) d\bar{z}^a = -R^{\bar{a}}{}_{\bar{b}c\bar{d}}\, d\bar{z}^b\,.$$

For an $n\times n$ complex matrix $K=(K^{\bar{a}}{}_{\bar{b}})$, we define the derivation extension $D[K]$ $\in \mathrm{End}(\Lambda^{0,*}(M))$ as follows;

$$D[K]\,(d\bar{z}^{j_1}\wedge\cdots\wedge d\bar{z}^{j_p}) = \sum_{r=1}^{p} d\bar{z}^{j_1}\wedge\cdots\wedge K^{\bar{j}_r}{}_{\bar{c}} d\bar{z}^c\wedge\cdots\wedge d\bar{z}^{j_p}\,.$$

Then

$$R^{\Lambda^{0,*}(M)}\left(\frac{\partial}{\partial z^c}, \frac{\partial}{\partial \bar{z}^d}\right) = -D\,[R^{\bar{\cdot}}{}_{\bar{\cdot}c\bar{d}}]\,.$$

Let $\Omega^{T'M}$ be a curvature form on $T'M$ and Ω^V be a curvature form on V with respect to the canonical connection. Then it holds that

$$\Omega^{T'M}{}^a{}_b = R^a{}_{bc\bar{d}}\, dz^c\wedge d\bar{z}^d\,, \quad \Omega^V{}^\alpha{}_\beta = L^\alpha{}_{\beta c\bar{d}}\, dz^c\wedge d\bar{z}^d\,.$$

Now we can state the Weitzenböck type formula as follows;

Theorem 2.2. *It holds that*

$$\begin{aligned}\Delta_V^c = {}& \frac{1}{2} g^{j\bar{h}}\nabla_j\nabla_{\bar{h}} + \frac{1}{2} g^{\bar{j}h}\nabla_{\bar{j}}\nabla_h + \frac{1}{2} D[g^{j\bar{h}}R^{\bar{\cdot}}{}_{\bar{\cdot}j\bar{h}}]\otimes 1_V \\ & + \frac{1}{2}[\mathrm{int}(dz^j), \mathrm{ext}(d\bar{z}^h)]\otimes L^{\cdot}{}_{\cdot j\bar{h}}\,. \end{aligned} \tag{2.3}$$

Proof. Take any $z_0\in M$ and fix it. Since g is a Kähler metric, we can take a holomorphic coordinate and a holomorphic frame near z_0 so that

$$g_{j\bar{h}}(z_0) = \delta_{jh}\,, \quad g_{j\bar{h}/l}(z_0) = g_{j\bar{h}/\bar{l}}(z_0) = 0\,, \tag{2.4}$$

$$h_{\alpha\bar{\beta}}(z_0) = \delta_{\alpha\beta}\,, \quad h_{\alpha\bar{\beta}/l}(z_0) = h_{\alpha\bar{\beta}/\bar{l}}(z_0) = 0\,, \tag{2.5}$$

(cf. Gilkey [6], Lemma 3.7.1. and Lemma 3.7.2.). Then we have

$$\mathrm{ext}(d\bar{z}^j)\,\mathrm{int}(dz^h) + \mathrm{int}(dz^h)\,\mathrm{ext}(d\bar{z}^j) = \delta_{jh}\,, \tag{2.6}$$

$$\mathrm{ext}(d\bar{z}^j)\,\mathrm{ext}(d\bar{z}^h) + \mathrm{ext}(d\bar{z}^h)\,\mathrm{ext}(d\bar{z}^j) = 0\,, \tag{2.7}$$

$$\mathrm{int}(dz^j)\,\mathrm{int}(dz^h) + \mathrm{int}(dz^h)\,\mathrm{int}(dz^j) = 0\,. \tag{2.8}$$

Here every term is evaluated at z_0. This is valid throughout the proof. Hence,

$$\begin{aligned}(\bar{\partial}+\bar{\partial}^*)_V^2 = {}& \mathrm{ext}(d\bar{z}^j)\,\nabla_{\bar{j}}\,\mathrm{ext}(d\bar{z}^h)\,\nabla_{\bar{h}} - \mathrm{ext}(d\bar{z}^j)\,\nabla_{\bar{j}}\,\mathrm{int}(dz^h)\,\nabla_h \\ & - \mathrm{int}(dz^j)\,\nabla_j\,\mathrm{ext}(d\bar{z}^h)\,\nabla_{\bar{h}} + \mathrm{int}(dz^j)\,\nabla_j\,\mathrm{int}(dz^h)\,\nabla_h\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{ext}(d\bar z^j)\,\{\operatorname{ext}(\nabla_{\bar j}\, d\bar z^h)+\operatorname{ext}(d\bar z^h)\,\nabla_{\bar j}\}\,\nabla_{\bar h} \\
&\quad -\operatorname{ext}(d\bar z^j)\,\{\operatorname{int}(\nabla_{\bar j}\, dz^h)+\operatorname{int}(dz^h)\,\nabla_{\bar j}\}\,\nabla_h \\
&\quad -\operatorname{int}(dz^j)\,\{\operatorname{ext}(\nabla_j\, d\bar z^h)+\operatorname{ext}(d\bar z^h)\,\nabla_j\}\,\nabla_{\bar h} \\
&\quad +\operatorname{int}(dz^j)\,\{\operatorname{int}(\nabla_j\, dz^h)+\operatorname{int}(dz^h)\,\nabla_j\}\,\nabla_h \\
&= \operatorname{ext}(d\bar z^j)\operatorname{ext}(d\bar z^h)\,\nabla_{\bar j}\nabla_{\bar h}-\operatorname{ext}(d\bar z^j)\operatorname{int}(dz^h)\,\nabla_{\bar j}\nabla_h \\
&\quad -\operatorname{int}(dz^j)\operatorname{ext}(d\bar z^h)\,\nabla_j\nabla_{\bar h}+\operatorname{int}(dz^j)\operatorname{int}(dz^h)\,\nabla_j\nabla_h \\
&\quad (\text{since } \Gamma^h_{jl}(z_0)=0) \\
&= \frac{1}{2}\operatorname{ext}(d\bar z^j)\operatorname{ext}(d\bar z^h)\,(\nabla_{\bar j}\nabla_{\bar h}-\nabla_{\bar h}\nabla_{\bar j}) \\
&\quad -\{\operatorname{int}(dz^j)\operatorname{ext}(d\bar z^h)\,\nabla_j\nabla_{\bar h}+\operatorname{ext}(d\bar z^h)\operatorname{int}(dz^j)\,\nabla_{\bar h}\nabla_j\} \\
&\quad +\frac{1}{2}\operatorname{int}(dz^j)\operatorname{int}(dz^h)\,(\nabla_j\nabla_h-\nabla_h\nabla_j)\,.
\end{aligned}
$$

Since the connection is canonical, it holds that

$$\nabla_j\nabla_h-\nabla_h\nabla_j=0\,,\quad \nabla_{\bar j}\nabla_{\bar h}-\nabla_{\bar h}\nabla_{\bar j}=0\,.$$

Hence we have

$$(\bar\partial+\bar\partial^*)^2_V = -\{\operatorname{int}(dz^j)\operatorname{ext}(d\bar z^h)\,\nabla_j\nabla_{\bar h}+\operatorname{ext}(d\bar z^h)\operatorname{int}(dz^j)\,\nabla_{\bar h}\nabla_j\}\,.$$

Therefore, using (2.6), we get

$$(\bar\partial+\bar\partial^*)^2_V = -\nabla_j\nabla_{\bar j}+\operatorname{ext}(d\bar z^h)\operatorname{int}(dz^j)\,(\nabla_j\nabla_{\bar h}-\nabla_{\bar h}\nabla_j)\,,$$

and

$$(\bar\partial+\bar\partial^*)^2_V = -\nabla_{\bar j}\nabla_j-\operatorname{int}(dz^j)\operatorname{ext}(d\bar z^h)\,(\nabla_j\nabla_{\bar h}-\nabla_{\bar h}\nabla_j)\,.$$

Averaging these,

$$
\begin{aligned}
(\bar\partial+\bar\partial^*)^2_V &= -\frac{1}{2}\nabla_j\nabla_{\bar j}-\frac{1}{2}\nabla_{\bar j}\nabla_j \\
&\quad -\frac{1}{2}[\operatorname{int}(dz^j),\operatorname{ext}(d\bar z^h)]\,R^{\Lambda^{0,*}(M)\otimes V}\left(\frac{\partial}{\partial z^j},\frac{\partial}{\partial \bar z^h}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&R^{\Lambda^{0,*}(M)\otimes V}\left(\frac{\partial}{\partial z^j},\frac{\partial}{\partial \bar z^h}\right) \\
&\quad = R^{\Lambda^{0,*}(M)}\left(\frac{\partial}{\partial z^j},\frac{\partial}{\partial \bar z^h}\right)\otimes 1_V+1_{\Lambda^{0,*}(M)}\otimes R^V\left(\frac{\partial}{\partial z^j},\frac{\partial}{\partial \bar z^h}\right) \\
&\quad = -D[R^{\bar\cdot}{}_{\bar\cdot j\bar h}]\otimes 1_V+1_{\Lambda^{0,*}(M)}\otimes L^{\cdot}{}_{\cdot j\bar h}\,.
\end{aligned}
$$

Hence by Lemma 2.1,

$$
\begin{aligned}
\Delta_V^c &= -(\bar\partial_V+\bar\partial_V^*)^2 = -(\bar\partial+\bar\partial^*)_V^2 \\
&= \frac{1}{2}\nabla_j\nabla_{\bar j}+\frac{1}{2}\nabla_{\bar j}\nabla_j \\
&\quad -\frac{1}{2}[\mathrm{int}(dz^j), \mathrm{ext}(d\bar z^h)]\, D[R^{\bar\cdot}{}_{\bar\cdot j\bar h}]\otimes 1_V \\
&\quad +\frac{1}{2}[\mathrm{int}(dz^j), \mathrm{ext}(d\bar z^h)]\otimes L^{\cdot}{}_{\bar\cdot j\bar h}\,.
\end{aligned}
$$

This is valid even for non Kähler complex manifold.

Further by uisng (2.6), we have

$$
\begin{aligned}
&[\mathrm{int}(dz^j), \mathrm{ext}(d\bar z^h)]\, D[R^{\bar\cdot}{}_{\bar\cdot j\bar h}] \\
&\quad = -R^{\bar a}{}_{\bar b j\bar h}\,\{\mathrm{int}(dz^j)\,\mathrm{ext}(d\bar z^h)-\mathrm{ext}(d\bar z^h)\,\mathrm{int}(dz^j)\}\,\mathrm{ext}(d\bar z^b)\,\mathrm{int}(dz^a) \\
&\quad = R^{\bar a}{}_{\bar b j\bar j}\,\mathrm{ext}(d\bar z^b)\,\mathrm{int}(dz^a) \\
&\qquad -2\sum_{j\neq h} R^{\bar a}{}_{\bar b j\bar h}\,\mathrm{int}(dz^j)\,\mathrm{ext}(d\bar z^h)\,\mathrm{ext}(d\bar z^b)\,\mathrm{int}(dz^a)\,.
\end{aligned}
$$

By Kähler condition, ∇^M is torsion free, i.e., $\Gamma^a_{hb}=\Gamma^a_{bh}$, and hence $R^{\bar a}{}_{\bar b j\bar h}=R^{\bar a}{}_{\bar h j\bar b}$. Moreover by noting (2.7), we get,

$$
\begin{aligned}
&[\mathrm{int}(dz^j), \mathrm{ext}(d\bar z^h)]\, D[R^{\bar\cdot}{}_{\bar\cdot j\bar h}] \\
&\quad = -R^{a}{}_{\bar b j\bar j}\,\mathrm{ext}(d\bar z^b)\,\mathrm{int}(dz^a)\,.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\Delta_V^c &= \frac{1}{2}\nabla_j\nabla_{\bar j}+\frac{1}{2}\nabla_{\bar j}\nabla_j+\frac{1}{2}D[R^{\bar\cdot}{}_{\bar\cdot j\bar h}]\otimes 1_V \\
&\quad +\frac{1}{2}[\mathrm{int}(dz^j), \mathrm{ext}(d\bar z^h)]\otimes L^{\cdot}{}_{\cdot j\bar h}\,, \\
&= \frac{1}{2}g^{j\bar h}\nabla_j\nabla_{\bar h}+\frac{1}{2}g^{\bar j h}\nabla_{\bar j}\nabla_h+\frac{1}{2}D[g^{j\bar h}R^{\bar\cdot}{}_{\bar\cdot j\bar h}]\otimes 1_V \\
&\quad +[\mathrm{int}(dz^j), \mathrm{ext}(d\bar z^h)]\otimes L^{\cdot}{}_{\cdot j\bar h}\,,
\end{aligned}
$$

which completes the proof. □

3. A heat equation for Δ_V^c

In this section, we shall obtain the fundamental solution of the following heat equation on $\Lambda^{0,*}(M)\otimes V$:

$$
(3.1)\qquad
\begin{cases}
\dfrac{\partial u}{\partial t}(t, z) = \Delta_V^c\, u(t, z)\,, \quad t>0\,, \\
\lim_{\substack{t\downarrow 0\\ w\to z}} u(t, w) = \xi(z)\in\Gamma^\infty(\Lambda^{0,*}(M)\otimes V)\,.
\end{cases}
$$

Let $U(M)$ be the unitary frame bundle of $T'M$. Let $(\boldsymbol{C}^n)'$, $(\boldsymbol{C}^n)''$ and

$\Lambda^{p,q}(\boldsymbol{C}^n)$ be the canonical fiber of $T'M$, $T''M$ and $\Lambda^{p,q}(M)$, respectively. Let $\{\delta_j\}_{j=1}^n$ be a canonical basis of $(\boldsymbol{C}^n)'$ and $\{\bar{\delta}_j\}_{j=1}^n$ be the conjugate basis. Let $\{\delta^j\}$ and $\{\bar{\delta}^j\}$ be the dual and conjugate dual basis, respectively. We introduce the following representations ρ: $U(n)\to U((\boldsymbol{C}^n)')$ and $\hat{\rho}$: $U(n)\to U(\Lambda^{0,*}(\boldsymbol{C}^n))$ by

$$\begin{aligned}&\rho(u)\,\delta_j = u_j^h\,\delta_h\,,\\ &\hat{\rho}(u)\,\bar{\delta}^{j_1}\wedge\cdots\wedge\bar{\delta}^{j_q} = u_{j_1}^{h_1}\,\bar{\delta}^{h_1}\wedge\cdots\wedge u_{j_q}^{h_q}\,\bar{\delta}^{h_q}\,,\quad u\in U(n)\,,\end{aligned}$$

where $U(n)$ is the unitary group of degree n. Then

$$T'M = U(M)\times_\rho(\boldsymbol{C}^n)'\,,\quad \Lambda^{0,*}(M) = U(M)\times_{\hat{\rho}}\Lambda^{0,*}(\boldsymbol{C}^n)\,.$$

Similarly, let $U(V)$ be the unitary frame bundle of V, $\{b_\alpha\}_{\alpha=1}^k$ be the canonical basis of $\boldsymbol{C}^k$, $\boldsymbol{C}^k$ being the canonical fiber of V, and introduce the following representation σ: $U(k)\to U(\boldsymbol{C}^k)$ by

$$\sigma(u)\,b_\alpha = u_\alpha^\beta\,b_\beta\,.$$

Then V can be represented as an associated vector bundle by this representation:

$$V = U(V)\times_\sigma\boldsymbol{C}^k\,.$$

Let $U(M)+U(V)$ be the $U(n)\times U(k)$ principal fiber bundle whose base is M and fiber at $z\in M$ is $U_z(M)\times U_z(V)$. Let ω^M be the connection form on $U(M)$ for ∇^M and ω^V be the connection form on $U(V)$ for ∇^V. Then $\omega=\omega^M\oplus\omega^V\in\Gamma^\infty(T^*(U(M)+U(V))\otimes(\mathfrak{U}(n)\oplus\mathfrak{U}(k)))$ is the connection form on $U(M)+U(V)$ for ∇ where $\mathfrak{U}(n)$ is the Lie algebla of $U(n)$. We extend ω and the differential of projection π_* to be complex linear:

$$\begin{aligned}&\omega^c\colon T^c(U(M)+U(V)) \to (\mathfrak{U}(n)\oplus\mathfrak{U}(k))\otimes\boldsymbol{C}\,,\\ &\pi_*^c\colon T^c(U(M)+U(V)) \to T^cM\,.\end{aligned}$$

We define the complex canonical horizontal vector fields $L_1, \cdots, L_n\in\Gamma^\infty(T^c(U(M)+U(V)))$ so that

$$(\omega^c, L_j)(r) = 0\,,\quad \pi_*^c\,L_j(r) = e_j\,,\quad \text{for}\quad r = (z, e, v)\in U(M)+U(V)\,. \tag{3.2}$$

Here $z\in M$, $e=[e_1, \cdots, e_n]$ is a unitary frame at $T_z'M$ and $v=[v_1, \cdots, v_k]$ is a unitary frame at V_z. For $\xi\in\Gamma^\infty(\Lambda^{0,*}(M)\otimes V)$, we define the scalarization F_ξ: $U(M)+U(V)\to\Lambda^{0,*}(\boldsymbol{C}^n)\otimes\boldsymbol{C}^k$ by $F_\xi(r)=r^{-1}\,\xi(\pi(r))$. Here we regard $r\in U(M)+U(V)$ as a vector space isomorphism r: $\Lambda^{0,*}(\boldsymbol{C}^n)\otimes\boldsymbol{C}^k\to\Lambda_z^{0,*}(M)\otimes V_z$ (cf. Kobayashi-Nomizu [10], Proposition 5.4).

Lemma 3.1. *For any* $\xi\in\Gamma^\infty(\Lambda^{0,*}(M)\otimes V)$, *it holds that*

$$F_{g^{j\bar{h}}\nabla_j\nabla_{\bar{h}}\xi}(r) = L_j\bar{L}_jF_\xi(r)\,, \tag{3.3}$$

$$F_{g^{\bar{j}h}\nabla_{\bar{j}}\nabla_h\xi}(r) = \bar{L}_j L_j F_\xi(r) . \tag{3.4}$$

Proof. We decompose L_j to the real and the imaginary parts and do the same proof as in the real case. □

We introduce a local coordinate to $U(M)+U(V)$. For $r=(z, e, v)$, we denote the components of $e=[e_1, \cdots, e_n]$ and $v=[v_1, \cdots, v_k]$, as follows;

$$e_h = e_h^j \frac{\partial}{\partial z^j} , \quad v_\beta = v_\beta^\alpha s_\alpha .$$

Denoting by $\hat{R}^{\bar{a}}{}_{\bar{b}c\bar{d}}(r)$ the scalarization of $R^{\bar{a}}{}_{\bar{b}c\bar{d}}$, we have

$$R^{T''M}(e_c, \bar{e}_d)\, \bar{e}_b = \hat{R}^{\bar{a}}{}_{\bar{b}c\bar{d}}(r)\, \bar{e}_a ,$$
$$\hat{R}^{\bar{a}}{}_{\bar{b}cd}(r) = (\bar{e}^{-1})^a_p\, \bar{e}^q_b\, e^r_c\, \bar{e}^s_d\, R^{\bar{p}}{}_{\bar{q}r\bar{s}}(z) .$$

Similarly denoting by $\hat{L}^\alpha{}_{\beta c\bar{d}}(r)$ the scalarization of $L^\alpha{}_{\beta c\bar{d}}$:

$$R^V(e_c, \bar{e}_d)\, v_\beta = \hat{L}^\alpha{}_{\beta c\bar{d}}(r)\, v_\alpha ,$$
$$\hat{L}^\alpha{}_{\beta c\bar{d}}(r) = (v^{-1})^\alpha_\gamma\, v^\delta_\beta\, e^r_c\, \bar{e}^s_d\, L^\gamma{}_{\delta r\bar{s}}(z) .$$

The following lemma is straightforward from the definition.

Lemma 3.2. *For any $\xi\in\Gamma^\infty(\Lambda^{0,*}(M)\otimes V)$, it holds that*

$$F_{(D[g^{j\bar{h}}\, R^{\cdot}{}_{\cdot j\bar{h}}]\otimes 1_V)\xi}(r) = (D[\hat{R}^{\bar{\cdot}}{}_{\bar{\cdot}j\bar{j}}(r)]\otimes 1_{C^k})\, F_\xi(r) , \tag{3.5}$$

$$F_{([\mathrm{int}(dz^j),\, \mathrm{ext}(d\bar{z}^h)]\otimes L^{\cdot}{}_{\cdot j\bar{h}})\xi}(r) = ([\mathrm{int}(\delta^j), \mathrm{ext}(\bar{\delta}^h)]\otimes \hat{L}^{\cdot}{}_{\cdot j\bar{h}}(r))\, F_\xi(r) , \tag{3.6}$$

where the definition of D is extended to the basis $\{\delta^j\}$.

From Theorem 2.2, Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} F_{\Delta^c_V\xi}(r) &= \Big(\frac{1}{2} L_j\bar{L}_j + \frac{1}{2}\bar{L}_j L_j + \frac{1}{2} D[\hat{R}^{\bar{\cdot}}{}_{\bar{\cdot}j\bar{j}}(r)]\otimes 1_{C^k} \\ &\qquad + \frac{1}{2}[\mathrm{int}(\delta^j), \mathrm{ext}(\bar{\delta}^h)]\otimes \hat{L}^{\cdot}{}_{\cdot j\bar{h}}(r)\Big) F_\xi \\ &=: AF_\xi(r) . \end{aligned} \tag{3.7}$$

Now we consider the following initial value problem of a heat equation on $U(M)+U(V)$ taking values in $\Lambda^{0,*}(\boldsymbol{C}^n)\otimes\boldsymbol{C}^k$;

$$\begin{cases} \dfrac{\partial V}{\partial t}(t, r) = A\, V(t, r) , & t>0 , \\ \lim_{\substack{t\downarrow 0 \\ r'\to r}} V(t, r') = F_\xi(r) . \end{cases} \tag{3.8}$$

Let $(x^1(t), \cdots, x^n(t), y^1(t), \cdots, y^n(t))$ be an $\boldsymbol{R}^{2n}$-valued Brownian motion and set

$$z^j(t) = \frac{x^j(t)+iy^j(t)}{\sqrt{2}} \quad (j=1, \cdots, n).$$

$z(t)=(z^1(t), \cdots, z^n(t))$ is called an n-dimenitonal complex Brownian motion. We consider the following stochastic differential equation (SDE) in the form of the Stratonovich differentials;

$$\text{(3.9)} \qquad \begin{cases} d\boldsymbol{r}_t = L_j(\boldsymbol{r}_t)\circ dz^j(t)+\bar{L}_j(\boldsymbol{r}_t)\circ d\bar{z}^j(t)\,, \\ \boldsymbol{r}_0 = \boldsymbol{r}\,, \end{cases}$$

and we denote the solution of the SDE by

$$(\boldsymbol{r}(t, \boldsymbol{r}, z)) = (Z(t, \boldsymbol{r}, z), e(t, \boldsymbol{r}, z), v(t, \boldsymbol{r}, z)).$$

The meaning of (3.9) is as follows; we say that $\boldsymbol{r}_t$ is a solution of (3.9) if it is a $U(M)+U(V)$-valued continuous semimartingale in the sense that, for every $F\in C^\infty(U(M)+U(V))$, $F(\boldsymbol{r}_t)$ is a continuous semimartingale and satisfies

$$F(\boldsymbol{r}_t)-F(\boldsymbol{r}) = \int_0^t (L_jF)(\boldsymbol{r}_s)\circ dz^j(s)+\int_0^t (\bar{L}_jF)(\boldsymbol{r}_s)\circ d\bar{z}^j(s).$$

So we can rewrite the SDE (3.9) in the real form as follows;

$$\begin{cases} d\boldsymbol{r}_t = \sqrt{2}\ \mathrm{Re}L_j(\boldsymbol{r}_t)\circ dx^j_t-\sqrt{2}\ \mathrm{Im}L_j(\boldsymbol{r}_t)\circ dy^j_t\,. \\ \boldsymbol{r}_0 = \boldsymbol{r}\,. \end{cases}$$

Then $(\boldsymbol{r}(t, \boldsymbol{r}, z))$ is a diffusion process whose generator is

$$\sum_{j=1}^n \{(\mathrm{Re}L_j)^2+(\mathrm{Im}L_j)^2\} = \frac{1}{2}L_j\bar{L}_j+\frac{1}{2}\bar{L}_jL_j\,.$$

We define the $\mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n)\otimes\boldsymbol{C}^k)$-valued process $M(t, \boldsymbol{r}(\cdot, \boldsymbol{r}, z))$ by the solution of the following differential equation;

$$\text{(3.10)} \qquad \begin{cases} \dfrac{dM(t)}{dt} = M(t)\,\hat{J}(\boldsymbol{r}(t, \boldsymbol{r}, z))\,, \\ M(0) = I\,, \end{cases}$$

where

$$\hat{J}(\boldsymbol{r}) = \frac{1}{2}D[\hat{R}^{\bar{\cdot}}{}_{\cdot j\bar{j}}(\boldsymbol{r})]\otimes 1_{C^k}+\frac{1}{2}[\mathrm{int}(\delta^j), \mathrm{ext}(\bar{\delta}^h)]\otimes\hat{L}^{\cdot}{}_{\cdot j\bar{h}}(\boldsymbol{r}).$$

Lemma 3.3. *The unique solution of* (3.7) *is given by*

$$V(t, r) = E[M(t, r(\cdot, r, z))\, F_\xi(r(t, r, z))]\,. \tag{3.11}$$

Proof. By the Ito formula, for

$B(t, r) \in \Gamma^{1,2}([0, \infty) \times (U(M)+U(V)) \to \Lambda^{0,*}(C^n) \otimes C^k)$,

$$\begin{aligned} & M(t, r(\cdot, r, z))\, B(t, r(\cdot, r, z)) - B(0, r) \\ & = \int_0^t M(s) \frac{\partial}{\partial t} B(s, r(s))\, ds + \int_0^t M(s)\, L_j B(s, r(s))\, dz^j(s) \\ & + \int_0^t M(s)\, \bar{L}_j B(s, r(S))\, d\bar{z}^j(s) \\ & + \int_0^t M(s) \frac{1}{2} (L_j \bar{L}_j + \bar{L}_j L_j)\, B(s, r(s))\, ds \\ & + \int_0^t M(s)\, \hat{J}(r(s, r, z))\, B(s, r(s))\, ds\,. \end{aligned} \tag{3.12}$$

Using this formula we can complete the proof (cf. Ikeda-Watanabe [8] Chapter V, §3). □

For $V \in \Gamma^\infty(U(M)+U(V) \to \Lambda^{0,*}(C^n) \otimes C^k)$, V is called $U(n) \times U(k)$-equivariant if

$$\rho(u)^{-1} \otimes \sigma(v)^{-1} V(r) = V(R_{(u,v)}\, r) \quad \text{for} \quad u \in U(n) \quad \text{and} \quad v \in U(k)\,,$$

where $R_{(u,v)}$ is the right action by $(u, v) \in U(n) \times U(k)$ on $U(M)+U(V)$. Then there exists $\xi \in \Gamma^\infty(\Lambda^{0,*}(M) \otimes V)$ such that $F_\xi = V$ if and only if V is $U(n) \times U(k)$-equivariant.

Lemma 3.4. *$V(t, r)$ is $U(n) \times U(k)$-equivariant for each t where*

$$V(t, r) = E[M(t, r(\cdot, r, z))\, F_\xi(r(t, r, z))]\,.$$

Proof. We fix a $u \in U(n)$ and a $v \in U(k)$. It is easy to see that $R_{(u,v)} r(t, r, z)$ satisfies the following SDE;

$$\begin{cases} r(0) = R_{(u,v)}\, r\,, \\ dr(t) = (R_{(u,v)})_* L_j(r(t)) \circ dz^j(t) + (R_{(u,v)})_* \bar{L}_j(r(t)) \circ d\bar{z}^j(t) \\ \qquad = L_j(r(t)) \circ d(u^{-1}z)^j(t) + \bar{L}_j(r(t)) \circ d\overline{(u^{-1}z)}^j(t)\,. \end{cases}$$

By the uniqueness of the solution of the SDE, we have

$$r(t, R_{(u,v)}\, r, u^{-1}z) = R_{(u,v)}\, r(t, r, z)\,. \tag{3.13}$$

On the other hand, by the definition of the scalarization we have

$$\hat{J}(R_{(u,v)}\, r)\, \rho(u)^{-1} \otimes \sigma(v)^{-1} = \rho(u)^{-1} \otimes \sigma(v)\, \hat{J}(r)\,.$$

By the uniqueness of the solution of an initial value problem of an ordinary differential equation, we have

$$(3.14)\qquad \begin{aligned} &M(t, R_{(u,v)}\, r(\cdot, r, z))\, \rho(u)^{-1}\otimes\sigma(v)^{-1} \\ &\quad = \rho(u)^{-1}\otimes\sigma(v)^{-1}M(t, r(\cdot, r, z))\,. \end{aligned}$$

Then by using the $U(n)$-invariance of a complex Brownian motion,

$$(3.15)\qquad \begin{aligned} &V(t, R_{(u,v)}\, r) \\ &\quad = E[M(t, r(\cdot, R_{(u,v)}\, r, u^{-1}z))\, F_\xi(r(t, R_{(u,v)}\, r, u^{-1}z))] \\ &\quad = E[M(t, r(\cdot, R_{(u,v)}\, r, z))\, \rho(u)^{-1}\otimes\sigma(v)^{-1}F_\xi(r(t, r, z))] \\ &\quad = E[\rho(u)^{-1}\otimes\sigma(v)^{-1}M(t, r(\cdot, r, z))\, F_\xi(r(t, r, z))] \\ &\quad = \rho(u)^{-1}\otimes\sigma(v)^{-1}V(t, r)\,, \end{aligned}$$

which completes the proof. □

Thus the unique solution of the heat equation (3.1) is given by

$$(3.16)\qquad u(t, z) = E[rM(t, r(\cdot, r, z))\, r(t, r, z)^{-1}\, \xi(Z(t, r, z))]\,.$$

Then the fundamental solution of the heat equation (3.1) is expressed formally as follows;

$$(3.17)\qquad e(t, z, w) = E[rM(t, r(\cdot, r, z))\, r(t, r, z)^{-1}\tilde{\delta}_w\, (Z(t, r, z))]\,,$$

where $\pi(r)=z$, $\tilde{\delta}_w=\delta_w/|\det(g_{j\bar{k}})|$ and δ_w is the Dirac delta function at w. But $\tilde{\delta}_w(Z(t, r, z))$ is not a usual Wiener functional. It is a kind of distribution on the Wiener space W_0^{2n} as an element of a Sobolev class $\tilde{\boldsymbol{D}}^{-\infty}=\bigcup_{s>0}\bigcap_{p>1}\boldsymbol{D}_p^{-s}$ and the meaning of the expectation in (3.17) is a generalized expectation in the sence of the pairing;

$$_{\tilde{\boldsymbol{D}}^{\infty}(\mathrm{End}(\Lambda^{0,*}(M)\otimes V))}\langle 1, rM(t)\, r(t)^{-1}\, \tilde{\delta}_w(Z(t))\rangle_{\tilde{\boldsymbol{D}}^{-\infty}(\mathrm{End}(\Lambda^{0,*}(M)\otimes V))}\,.$$

For details of an analysis on the Wiener space, Sobolev spaces of generalized Wiener functionals and generalized expectations in particular, we refer to [9] and [16].

Next we will give a local expression of the SDE (3.9). Let $C(M)$ be the complex frame bundle for M and $C(V)$ be that of V. We extend ω^M to the connection form on $C(M)$ and ω^V to that on $C(V)$. As before, we extend these to be complex linear:

$$\begin{aligned} &\omega^M\colon T^cC(M) \to \mathfrak{gl}(n, \boldsymbol{C})\,, \\ &\omega^V\colon T^cC(V) \to \mathfrak{gl}(k, \boldsymbol{C})\,, \end{aligned}$$

where $\mathfrak{gl}(n, \boldsymbol{C})$ is the Lie algebra of the complex general group $GL(n, \boldsymbol{C})$. We define the restriction of ω^M to $T'C(M)$ by ω_1^M. Similarly we denote the restriction of ω^V to $T'C(V)$ by ω_1^V. Then we have the following lemma.

Lemma 3.5. *We can express ω_1^M and ω_1^V locally as follows:*

$$\omega_1^{M\,a}{}_b = (e^{-1})^a_c\,(de^c_b + \Gamma^c_{jd}\, e^d_b\, dz^j)\,, \tag{3.18}$$

$$\omega_1^{V\,\alpha}{}_\beta = (v^{-1})^\alpha_\gamma\,(dv^\gamma_\beta + l^\gamma_{j\delta}\, v^\delta_\beta\, dz^j)\,. \tag{3.19}$$

Proof. The proof is similar to that in the real case (cf. Kobayashi-Nomizu [10], p. 142, Proposition 7.3), so we omit it. □

Then for the connection form on $C(M)+C(V)$, we have

$$\begin{aligned}\omega &= \begin{bmatrix} (\omega_1^{M\,j}{}_h) & 0 \\ 0 & (\omega_1^{V\,\alpha}{}_\beta) \end{bmatrix} \\ &= \begin{bmatrix} ((e^{-1})^j_l\,(de^l_h + \Gamma^l_{qp}\, e^p_h\, dz^q)) & 0 \\ 0 & ((v^{-1})^\alpha_\gamma\,(dv^\gamma_\beta + l^\gamma_{q\delta}\, v^\delta_\beta\, dz^q)) \end{bmatrix}.\end{aligned} \tag{3.20}$$

By considering the condition (3.2), we see the following expression for L_j:

$$L_j = e^p_j \frac{\partial}{\partial z^p} - \Gamma^q_{rs}\, e^s_t\, e^r_j \frac{\partial}{\partial e^q_t} - l^\delta_{u\varepsilon}\, v^\varepsilon_\zeta\, e^u_j \frac{\partial}{\partial v^\delta_\zeta}\,. \tag{3.21}$$

By taking conjugate,

$$\bar{L}_j = \bar{e}^p_j \frac{\partial}{\partial \bar{z}^p} - \bar{\Gamma}^q_{rs}\, \bar{e}^s_t\, \bar{e}^r_j \frac{\partial}{\partial \bar{e}^q_t} - \bar{l}^\delta_{u\varepsilon}\, \bar{v}^\varepsilon_\zeta\, \bar{e}^u_j \frac{\partial}{\partial \bar{v}^\delta_\zeta}\,. \tag{3.22}$$

Now we can express the SDE (3.9) locally. Since $z^j \in C^\infty(C(M)+C(V) \to \boldsymbol{C})$ is holomorphic, we have

$$z^j(\boldsymbol{r}(t)) - z^j(\boldsymbol{r}) = \int_0^t e^j_h(s) \circ dz^h(s)\,.$$

Similary, since e^p_q and v^α_β are also holomorphic,

$$\begin{aligned} e^p_q(\boldsymbol{r}(t)) - e^p_q(\boldsymbol{r}) &= \int_0^t \Gamma^p_{rs}(Z(s))\, e^s_q(s)\, e^r_j(s) \circ dz^j(s)\,, \\ v^\alpha_\beta(\boldsymbol{r}(t)) - v^\alpha_\beta(\boldsymbol{r}) &= \int_0^t \Gamma^\alpha_{r\varepsilon}(Z(s))\, v^\varepsilon_\beta(s)\, e^r_j(s) \circ dz^j(s)\,. \end{aligned}$$

Thus we have the following SDE:

$$\begin{cases} dZ^j(t) = e^j_h(t) \circ dz^h(t)\,, \\ de^p_q(t) = -\Gamma^p_{rs}(Z(t))\, e^s_q(t)\, e^r_j(t) \circ dz^j(t) \\ \qquad\quad\; = -\Gamma^p_{rs}(Z(t))\, e^s_q(t) \circ dZ^r(t)\,, \\ dv^\alpha_\beta(t) = -l^\alpha_{r\varepsilon}(Z(t))\, v^\varepsilon_\beta(t)\, e^r_j(t) \circ dz^j(t) \\ \qquad\quad\; = -l^\alpha_{r\varepsilon}(Z(t))\, v^\varepsilon_\beta(t) \circ dZ^r(t)\,, \\ Z^j(0) = z^j,\ e^p_q(0) = e^p_q,\ v^\alpha_\beta(0) = v^\alpha_\beta\,. \end{cases} \tag{3.23}$$

This form is exactly the same as in a real case (cf. Ikeda-Watanabe [8]).

4. Berezin formulas

To prove the index theorem we must study a supertrace. Berezin formulas provide us a very powerfull algebraic methods to discuss it, cf. Cycon et al. [5] §12.2.

We consider on $\Lambda^{0,*}(\boldsymbol{C}^n)$. Setting,

$$(a^j)^*: = \mathrm{ext}(\bar{\delta}^j)\,, \quad a^j: = \mathrm{int}(\delta^j) \qquad (j = 1, \cdots, n)\,. \tag{4.1}$$

then it holds that

$$\begin{cases} (a^j)^*(a^h)^* + (a^h)^*(a^j)^* = a^j a^h + a^h a^j = 0\,, \\ a^j(a^h)^* + (a^h)^* a^j = \delta^{jh}\,. \end{cases} \tag{4.2}$$

Moreover, setting,

$$\gamma^j: = (a^j)^* - a^j\,, \quad \hat{\gamma}^j: = i((a^j)^* + a^j)\,, \tag{4.3}$$

it holds that

$$\gamma^j \gamma^h + \gamma^j \gamma^h = -2\delta^{jh} \quad (j, h = 1, \cdots, 2n,\ \gamma^{j+n} = \hat{\gamma}^j)\,. \tag{4.4}$$

Thus $\mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n))$ is a Clifford algebla generated by $\gamma^1, \cdots, \gamma^n, \hat{\gamma}^1, \cdots, \hat{\gamma}^n$. For $K = \{(1 \leq) k_1 < \cdots < k_p (\leq n)\}$ and $L = \{(1 \leq) l_1 < \cdots < l_q (\leq n)\}$, we set

$$\gamma^K = \gamma^{k_1} \cdots \gamma^{k_p}\,, \quad \hat{\gamma}^L = \hat{\gamma}^{l_1} \cdots \hat{\gamma}^{l_q}\,.$$

Then we have

$$\mathrm{tr}_{\Lambda^{0,*}(\boldsymbol{C}^n)}(\gamma^K \hat{\gamma}^L) = \begin{cases} \dim \Lambda^{0,*}(\boldsymbol{C}^n) = 2^n & \text{if } \ K = L = \phi\,, \\ 0 & \text{if } \ K \neq \phi \text{ or } L \neq \phi\,, \end{cases} \tag{4.5}$$

(cf. Cycon et al. [5] (12.18) and also Atiyah-Bott [1], Proposition 8.28). From this we have

$$\mathrm{tr}_{\Lambda^{0,*}(\boldsymbol{C}^n)}((\gamma^K \hat{\gamma}^L)^*(\gamma^{K'} \hat{\gamma}^{L'})) = \begin{cases} 2^n & \text{if } \ (K, L) = (K', L')\,, \\ 0 & \text{if } \ (K, L) \neq (K', L')\,. \end{cases} \tag{4.6}$$

By noting that $\#\{\gamma^K \hat{\gamma}^L\} = (2^n)^2 = \dim \mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n))$, $\{\gamma^K \hat{\gamma}^L\}$ is an orthogonal basis of $\mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n))$ with respect to the Hilbert-Schmidt inner product. So for any $A \in \mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n))$, we can express it uniquely as

$$A = \sum_{K, L} C_{K\hat{L}}(A)\, \gamma^K \hat{\gamma}^L \quad (C_{K\hat{L}}(A) \in \boldsymbol{C})\,. \tag{4.7}$$

Then the Berezin formula is as follows:

$$\mathrm{tr}_{\Lambda^{0,*}(\boldsymbol{C}^n)}(A) = 2^n C_{\phi\hat{\phi}}(A) . \tag{4.8}$$

We define $(-1)^F \in \mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n))$ by

$$(-1)^F \omega_p = (-1)^p \omega_p \qquad \text{for} \quad \omega_p \in \Lambda^{0,p}(\boldsymbol{C}^n) .$$

Then we have the following:

Lemma 4.1. *$(-1)^F$ is expressed by $\{\gamma^K \hat{\gamma}^L\}$ as follows;*

$$(-1)^F = i^n \gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n . \tag{4.9}$$

Proof. By (4.3), we get

$$\gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n = i^n((a^1)^* a^1 - a^1(a^1)^*) \cdots ((a^n)^* a^n - a^n(a^n)^*) .$$

Then for $1 \in \Lambda^{0,0}(\boldsymbol{C}^n)$,

$$\gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n 1 = i^n(-a^1(a^1)^*) \cdots ((-a^n(a^n)^*) 1 = (-i)^n 1 .$$

So (4.9) holds on $\Lambda^{0,0}(\boldsymbol{C}^n)$. Futhermore by (4.4),

$$\gamma^j \gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n = -\gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n \gamma^j ,$$
$$\hat{\gamma}^j \gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n = -\gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n \hat{\gamma}^j ,$$

and hence by noting $(a^j)^* = (\gamma^j - i\hat{\gamma}^j)/2$,

$$(a^j)^* \gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n = -\gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n (a^j)^* .$$

Therefore we have

$$\begin{aligned} \gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n \bar{\delta}^{j_1} \wedge \cdots \wedge \bar{\delta}^{j_p} &= \gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n (a^{j_1})^* \cdots (a^{j_p})^* 1 \\ &= (-1)^p (a^{j_1})^* \cdots (a^{j_p})^* \gamma^1 \hat{\gamma}^1 \cdots \gamma^n \hat{\gamma}^n 1 \\ &= (-i)^n (-1)^p \bar{\delta}^{j_1} \wedge \cdots \wedge \bar{\delta}^{j_p} , \end{aligned}$$

which completes the proof. $\square$

Thus for $A \in \mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n))$ such that $A(\Lambda^{0,\pm}(\boldsymbol{C}^n)) \subset \Lambda^{0,\pm}(\boldsymbol{C}^n)$,

$$\begin{aligned} \mathrm{tr}_{\Lambda^{0,+}(\boldsymbol{C}^n)} A - \mathrm{tr}_{\Lambda^{0,-}(\boldsymbol{C}^n)} A &= \mathrm{tr}_{\Lambda^{0,*}(\boldsymbol{C}^n)} ((-1)^F A) \\ &= (-2i)^n C_{\{1\hat{1} \dots n\hat{n}\}}(A) . \end{aligned} \tag{4.10}$$

The supertrace that we must consider to prove the index theorem is the following:

$$\begin{aligned} &T[C] = \mathrm{tr}_{\Lambda^{0,+}(\boldsymbol{C}^n)\otimes \boldsymbol{C}^k}(C) - \mathrm{tr}_{\Lambda^{0,-}(\boldsymbol{C}^n)\otimes \boldsymbol{C}^k}(C) , \\ &\quad \text{for} \quad C \in \mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n) \otimes \boldsymbol{C}^k) \quad \text{such that} \\ &\quad C(\Lambda^{0,\pm}(\boldsymbol{C}^n) \otimes \boldsymbol{C}^k) \subset \Lambda^{0,\pm}(\boldsymbol{C}^n) \otimes \boldsymbol{C}^k . \end{aligned} \tag{4.11}$$

With respect to this supertrace we have the following;

$$\begin{aligned} &T[A\otimes B]=(-2i)^n C_{\{1\hat{1}\cdots n\hat{n}\}}(A)\,\mathrm{tr}_{C^k}(B)\,, \\ (4.12)\qquad &\text{for}\quad A\in\mathrm{End}(\Lambda^{0,*}(C^n))\ \text{such that}\ A(\Lambda^{0,\pm}(C^n))\subset\Lambda^{0,\pm}(C^n) \\ &\text{and}\quad B\in\mathrm{End}(C^k)\,. \end{aligned}$$

On the other hand, it holds that

$$(4.13)\qquad D[M]=-\frac{1}{4}M^{j}{}_{\bar{k}}(\gamma^h-i\hat{\gamma}^h)(\hat{\gamma}^j+i\hat{\gamma}^j)\,,$$

$$(4.14)\qquad [\mathrm{int}(\delta^j),\mathrm{ext}(\bar{\delta}^h)]=-\frac{1}{2}(\gamma^j+i\hat{\gamma}^j)(\gamma^h-i\hat{\gamma}^h)-\delta^{jh}\,.$$

They are order 2 with respect to γ and $\hat{\gamma}$. Then we have the following two lemmas for calculating the supertrace T.

Lemma 4.2. (Cancellation lemma)

Let $M^{(1)},\cdots,M^{(p)},N^{(1)},\cdots,N^{(q)}$, be $n\times n$ complex matrices and $\alpha^{(1)},\cdots,\alpha^{(p)}$, $\beta^{(1)},\cdots,\beta^{(r)}$, be $k\times k$ complex matrices. Let $C_p\in\mathrm{End}(\Lambda^{0,*}(C^n)\otimes C^k)$, $A_q\in\mathrm{End}(\Lambda^{0,*}(C^n))$, and $B_r\in\mathrm{End}(C^k)$ as

$$\begin{aligned} C_p&=(D[M^{(1)}]\otimes 1_{C^k}+[\mathrm{int}(\delta^j),\mathrm{ext}(\bar{\delta}^h)]\otimes\alpha^{(1)}) \\ &\quad\cdots(D[M^{(p)}]\otimes 1_{C^k}+[\mathrm{int}(\delta^j),\mathrm{ext}(\bar{\delta}^h)]\otimes\alpha^{(p)})\,, \\ A_q&=D[N^{(1)}]\cdots D[N^{(q)}]\,, \\ B_r&=\beta^{(1)}\cdots\beta^{(r)}\,. \end{aligned}$$

Then it holds that

$$(4.15)\qquad T[C_p(A_q\otimes B_r)]=0\qquad \text{if}\quad p+q<n\,.$$

Proof. $C_p(A_q\otimes B_r)$ is order $2p+2q$ with respect to γ and $\hat{\gamma}$. So by (4.12), (4.15) holds immediately. □

Lemma 4.3. *Assume $p+q=n$. Let $M, N^{(1)},\cdots,N^{(q)}$, be $n\times n$ complex matrices and $\alpha^{\bullet}{}_{\cdot jh}$ be $k\times k$ complex matrices and we define $A\in\mathrm{End}(\Lambda^{0,*}(C^n)\otimes C^k)$ by*

$$\begin{aligned} A&=\left(\frac{1}{2}D[M]\otimes 1_{C^k}+\frac{1}{2}[\mathrm{int}(\delta^j),\mathrm{ext}(\bar{\delta}^h)]\otimes\alpha^{\bullet}{}_{\cdot jh}\right)^p \\ &\quad\times(D[N^{(1)}]\cdots D[N^{(q)}]\otimes 1_{C^k})\,. \end{aligned}$$

Then the following identity holds;

$$\begin{aligned} &T[A]\,dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n \\ (4.16)\qquad &=\left(\frac{i}{2}\right)^n\sum_{c=0}^{p}\frac{p!}{c!(p-c)!}\,\mathrm{tr}((\alpha^{\bullet}{}_{\cdot jh}\,dz^j\wedge d\bar{z}^h)^{\wedge(p-c)}) \end{aligned}$$

$$\wedge\Big(\frac{1}{2}M^{\bar{r}}{}_{\bar{s}}\,d\bar{z}^{s}\wedge dz^{r}\Big)^{\wedge c}$$

$$\wedge N^{(1)\bar{a}_1}{}_{\bar{b}_1}\,d\bar{z}^{b_1}\wedge dz^{a_1}\wedge\cdots\wedge N^{(q)a_q}{}_{\bar{b}_q}\,d\bar{z}^{b_q}\wedge dz^{a_q}\,.$$

Proof. We note that A can be expressed by γ^j and $\hat{\gamma}^j$ as follows,

$$A=\Big(-\frac{1}{8}M^{\bar{r}}{}_{\bar{s}}(\gamma^{s}-i\hat{\gamma}^{s})(\gamma^{r}+i\hat{\gamma}^{r})\otimes 1_{C^k}-\frac{1}{4}\{(\gamma^{j}+i\hat{\gamma}^{j})(\gamma^{h}-i\hat{\gamma}^{h})-2\delta_{jh}\}$$

$$\otimes\alpha^{\bullet}{}_{\bullet jh}\Big)^{p}\Big(\{-\frac{1}{4}N^{(1)\bar{a}_1}{}_{\bar{b}_1}(\gamma^{b_1}-i\hat{\gamma}^{b_1})(\gamma^{a_1}+i\hat{\gamma}^{a_1})\}\cdots$$

$$\{-\frac{1}{4}N^{(q)\bar{a}_q}{}_{\bar{b}_q}(\gamma^{b_q}-i\hat{\gamma}^{b_q})(\gamma^{a_q}+i\hat{\gamma}^{a_q})\}\otimes 1_{C^k}\Big)\,.$$

Then by using (4.12) we have

$$T[A]=\Big(\frac{i}{2}\Big)^{n}\sum_{c=0}^{p}\frac{p!}{c!(p-c)!}\,\mathrm{tr}(\alpha^{\bullet}{}_{\bullet j_1h_1}\cdots\alpha^{\bullet}{}_{\bullet j_{p-c}h_{p-c}})$$

$$C_{\{1\hat{1}\cdots n\hat{n}\}}\,[(\gamma^{j_1}+i\hat{\gamma}^{j_1})(\gamma^{h_1}-i\hat{\gamma}^{h_1})\cdots(\gamma^{j_{p-c}}+i\hat{\gamma}^{j_{p-c}})$$

$$(\gamma^{h_{p-c}}-i\hat{\gamma}^{h_{p-c}})\Big(\frac{1}{2}M^{\bar{r}}{}_{\bar{s}}(\gamma^{s}-i\hat{\gamma}^{s})(\gamma^{r}+i\hat{\gamma}^{r})\Big)^{c}$$

$$(N^{(1)\bar{a}_1}{}_{\bar{b}_1}(\gamma^{b_1}-i\hat{\gamma}^{b_1})(\gamma^{a_1}+i\hat{\gamma}^{a_1})\cdots$$

$$N^{(q)a_q}{}_{\bar{b}_q}(\gamma^{b_q}-i\hat{\gamma}^{b_q})(\gamma^{a_q}i+\hat{\gamma}^{a_q}))]\,.$$

Since the $2n$-th order part of the Clifford algebra and the $2n$-th order part of the exterior algebla are isomorphic, we have

$$T[A]\,dx^1\wedge dy^1\wedge\cdots\wedge dx^n\wedge dy^n$$

$$=\Big(\frac{i}{2}\Big)^{n}\sum_{c=0}^{p}\frac{p!}{c!(p-c)!}\,\mathrm{tr}((\alpha^{\bullet}{}_{\bullet jh}(dx^j+idy^j)\wedge(dx^h-idy^h))^{\wedge(p-c)})$$

$$\wedge\Big(\frac{1}{2}M^{\bar{r}}{}_{\bar{s}}(dx^{s}-idy^{s})\wedge(dx^{r}+idy^{r})\Big)^{\wedge c}$$

$$\wedge N^{(1)\bar{a}_1}{}_{\bar{b}_1}(dx^{b_1}-idy^{b_1})\wedge(dx^{a_1}+idy^{a_1})$$

$$\wedge\cdots N^{\bar{a}_q}{}_{\bar{b}_q}(dx^{b_q}-idy^{b_q})\wedge(dx^{a_q}+idy^{a_q})\,,$$

which completes the proof. □

5. Riemann-Roch theorem

In this section we will prove the Riemann-Roch theorem. For the complex (2.1), we define the cohomology $H^q(V)$ by

$$H^q(V)=\mathrm{Ker}\,\bar{\partial}_V/\mathrm{Im}\,\bar{\partial}_V\quad\text{on}\quad\Gamma^\infty(\Lambda^{0,q}(M)\otimes V)\,. \tag{5.1}$$

Since the complex (2.1) is elliptic, dim $H^q(V)$ is finite. So we define the index of the complex (2.1) by

$$\operatorname{Ind}(\bar\partial_V) = \sum_{q=0}^{n} (-1)^q \dim H^q(V) . \tag{5.2}$$

Then the Riemann-Roch theorem can be stated as follows;

Theorem 5.1. *The index of the twisted Dolbeault complex, denoted by* $\operatorname{Ind}(\bar\partial_V)$, *can be expressed in terms of* $ch(V)$ *and* $Td(T'M)$ *as follows;*

$$\operatorname{Ind}(\bar\partial_V) = \int_M ch(V)\wedge Td(T'M) . \tag{5.3}$$

To prove this theorem, we use the following well-known fact. Let $e(t, z, w)$ be a fundamental solution for Δ_V^c and T be a supertrace defined by (4.11). Then it holds that

$$\operatorname{Ind}(\bar\partial_V) = \int_M T[e(t, z, z)] \operatorname{dvol}(z) , \quad \forall t>0 \tag{5.4}$$

and

$$T[e(t, z, z)] \sim \sum_{h=0}^{\infty} t^{(h-2n)/2} a_h(z) \quad \text{as} \quad t \downarrow 0 . \tag{5.5}$$

So we have

$$\int_M a_h(z) \operatorname{dvol}(z) = \begin{cases} \operatorname{Ind}(\bar\partial_V) & \text{if} \quad h = 2n \\ 0 & \text{if} \quad h \neq 2n \end{cases} \tag{5.6}$$

(cf. Gilkey [6] p. 58, Theorem 1.7.6). Hence, we only have to show

$$a_{2n}(z) \operatorname{dvol}(z) = \{ch(V)\wedge Td(T'M)\}_{2n}$$

where $\{\ \}_{2n}$ is the $2n$-form part. This is called a heat equation method.

We will study the short time asymptotics of the fundamental solution. For this, it is convenient to introduce the parameter $\varepsilon>0$ as follows. Let $\boldsymbol{r}^\varepsilon(t)=(Z^\varepsilon(t), e^\varepsilon(t), v^\varepsilon(t))$ be the solution of the following SDE;

$$\begin{cases} d\boldsymbol{r}^\varepsilon(t) = \varepsilon L_j(\boldsymbol{r}^\varepsilon(t))\circ dz^j(t) + \varepsilon \bar L_j(\boldsymbol{r}^\varepsilon(t))\circ d\bar z^j(t) \\ \boldsymbol{r}^\varepsilon(0) = \boldsymbol{r} . \end{cases} \tag{5.7}$$

Let $M^\varepsilon(t, \boldsymbol{r}^\varepsilon(\cdot))$ be $\operatorname{End}(\Lambda^{0,*}(\boldsymbol{C}^n)\otimes \boldsymbol{C}^k)$-valued process defined as the solution of the following differential equation;

$$\begin{cases} dM^\varepsilon(t)/dt = \varepsilon^2 M^\varepsilon(t)\hat J(\boldsymbol{r}^\varepsilon(t)) \\ M^\varepsilon(0) = I . \end{cases} \tag{5.8}$$

Then by the scaling property of the complex Brownian motion, it holds that

$$(5.9)\qquad e(\varepsilon^2, z, w) = E[\boldsymbol{r}M^\varepsilon(1, \boldsymbol{r}^\varepsilon(\cdot, \boldsymbol{r}))\,\boldsymbol{r}^\varepsilon(1, \boldsymbol{r})^{-1}\,\tilde{\delta}_w(Z^\varepsilon(1, \boldsymbol{r}))]$$

where $\pi(\boldsymbol{r})=z$. We take an arbitrary point $z_0\in M$ and fix it. Further we take coordinate neighborhoods (U_1, φ_1) and (U_2, φ_2) such that $\bar{U}_1\subset U_2$, $\varphi_2|_{U_1}=\varphi_1$, $V|_{U_2}$ is trivial and U_2 is relatively compact. We identify U_2 and $\varphi_2(U_2)\subset \boldsymbol{C}^n$ by φ_2. If $\tilde{g}$ is a metric on $\boldsymbol{C}^n$ which coincides with g on U_2 and $\tilde{h}$ is a fibre metric on $\boldsymbol{C}^n\times\boldsymbol{C}^k$ which coincide with h on $V|_{U_2}$ and $\tilde{e}(t, z, w)$ is the corresponding fundamental solution on $\boldsymbol{C}^n\times\boldsymbol{C}^k$, then there exists a constant $c>0$ such that

$$\sup_{z\in\bar{U}_1}\|e(t, z, z)-\tilde{e}(t, z, z)\|=O(e^{-c/t})\quad\text{as}\quad t\downarrow 0$$

(see e.g. [9] for this reduction). So our problem is reduced to the simpler case that $M=\boldsymbol{C}^n$, the Kähler metric $(g_{j\bar{h}}(z))$ coincides with the identity matrix I_n outside of a compact set, $V=M\times\boldsymbol{C}^k$ and $(h_{\alpha\bar{\beta}}(z))$ coincides with I_k outside of a compact set. Furthermore we will take a nice coordinate and a nice frame. Let $(z^1, \cdots, z^n)$ be a holomorphic coordinate around z_0 satisfying (2.4) and $(s_1, \cdots, s_k)$ be a holomorphic frame of V around z_0 satisfying (2.5). Then we have the following properties near the origin;

$$(5.10)\qquad \begin{cases} g_{j\bar{h}}(z) = \delta_{jh}-R^h{}_{jp\bar{q}}(0)z^p\,\bar{z}^q+O(|z|^3) \\ \Gamma^l_{jh}(z) = R^l{}_{\bar{h}p\bar{j}}(0)z^p+O(|z|^2) \\ l^\beta_{j\alpha}(z) = -L^\beta{}_{\alpha j\bar{p}}(0)\bar{z}^p+O(|z|^2) \\ R^{\bar{a}}{}_{\bar{b}c\bar{d}}(0) = g_{a\bar{b}/c\bar{d}}(0) = g_{c\bar{d}/a\bar{b}}(0) = -R^d{}_{ca\bar{b}}(0)\,. \end{cases}$$

By this coordinate we can take the coordinate for $U(M)+U(V)$ as follows;

$$\begin{aligned} &\boldsymbol{r}^\varepsilon(t) = (Z^\varepsilon(t), e^\varepsilon_1(t), \cdots, e^\varepsilon_n(t), v^\varepsilon_1(t), \cdots, v^\varepsilon_k(t))\,, \\ &Z^\varepsilon(t) = (Z^{\varepsilon 1}(t), \cdots, Z^{\varepsilon n}(t))\,, \\ &e^\varepsilon_h(t) = e^{\varepsilon j}_h(t)\frac{\partial}{\partial z^j}\,,\quad v^\varepsilon_\beta(t) = v^{\varepsilon\alpha}_\beta(t)s_\alpha\,. \end{aligned}$$

Then we can rewrite the SDE (3.23) as follows;

$$(5.11)\qquad \begin{cases} dZ^{\varepsilon j}(t) = \varepsilon e^{\varepsilon j}_h(t)\circ dz^h(t)\,, \\ de^{\varepsilon j}_h(t) = -\Gamma^j_{lp}(Z^\varepsilon(t))\,e^{\varepsilon p}_h(t)\circ dZ^{\varepsilon l}(t)\,, \\ dv^{\varepsilon\alpha}_\beta(t) = -l^\alpha_{j\delta}(Z^\varepsilon(t))\,v^{\varepsilon\delta}_\beta(t)\circ dZ^{\varepsilon j}(t)\,, \\ Z^{\varepsilon j}(0) = 0\,,\quad e^{\varepsilon j}_h(0) = e^j_h\,,\quad v^{\varepsilon\alpha}_\beta(0) = v^\alpha_\beta\,. \end{cases}$$

Furthermore we choose $\boldsymbol{r}\in U(M)+U(V)$ so that

$$e^j_h = \delta^j_h\,,\quad v^\alpha_\beta = \delta^\alpha_\beta\,.$$

This $\boldsymbol{r}$ defines an isomorphism from $\Lambda^{0,*}(\boldsymbol{C}^n)\otimes\boldsymbol{C}^k$ onto $\Lambda^{0,*}_{z_0}(M)\otimes V_{z_0}$, so we identify $\Lambda^{0,*}(\boldsymbol{C}^n)\otimes\boldsymbol{C}^k$ with $\Lambda^{0,*}_{z_0}(M)\otimes V_{z_0}$.

Next, we will get the expression of a local SDE for $r^\varepsilon(t)^{-1}=\Pi^\varepsilon(t)\otimes\Xi^\varepsilon(t)$, where

$$\Pi^\varepsilon(t)\colon \Lambda^{0,*}_{Z^\varepsilon(t)}(M) \to \Lambda^{0,*}(C^n)$$

and

$$\Xi^\varepsilon(t)\colon V_{Z^\varepsilon(t)} \to C^k .$$

Lemma 5.2. *$\Pi^\varepsilon(t)$ satisfies the following SDE;*

$$\left\{\begin{array}{l} d\Pi^\varepsilon(t) = \Pi^\varepsilon(t)\circ d\Theta^\varepsilon(t) \\ \Pi^\varepsilon(0) = I , \end{array}\right. \tag{5.12}$$

where $\Theta^\varepsilon(t)=D[\theta^\varepsilon(t)]$ and $\theta^\varepsilon(t)\in\mathfrak{gl}(\Lambda^{0,1}(C^n))$ is given by

$$\theta^{\varepsilon\bar{h}}{}_{\bar{j}}(t) = -\int_0^t \Gamma^h_{pj}(Z^\varepsilon(s))\circ d\bar{Z}^{\varepsilon p}(s) . \tag{5.13}$$

Moreover $\Xi^\varepsilon(t)$ satisfies the following SDE;

$$\left\{\begin{array}{l} d\Xi^\varepsilon(t) = \Xi^\varepsilon(t)\circ d\iota^\varepsilon(t) \\ \Xi^\varepsilon(0) = I , \end{array}\right. \tag{5.14}$$

where $\iota^\varepsilon(t)=(\iota^{\varepsilon\alpha}{}_\beta(t))\in\mathfrak{u}(k)$ is given by

$$\iota^{\varepsilon\alpha}{}_\beta(t) = \int_0^t l^\alpha_{p\beta}(Z^\varepsilon(s))\circ dZ^{\varepsilon p}(s) . \tag{5.15}$$

Proof. We note that the expression of the isomorphism

$$r^\varepsilon(t)^{-1}\colon \Lambda^{0,1}_{Z^\varepsilon(t)}(M) \to \Lambda^{0,1}(C^n)$$

in the matrix form with respect to a local frame $\{dz^j\}_{j=1}^n$ is $(\bar{e}^{\varepsilon h}_j(t))\in GL(\Lambda^{0,1}(C^n))$. Further, by (5.11) we have

$$d\bar{e}^{\varepsilon h}_j(t) = \bar{e}^{\varepsilon p}_j(t)\circ d\theta^{\varepsilon\bar{h}}{}_{\bar{p}}(t) ,$$

where $\theta^\varepsilon(t)$ is given by (5.13). $\Theta^\varepsilon(t)=D[\theta^\varepsilon(t)]\in\mathfrak{gl}(\Lambda^{0,*}(C^n))$ is the extension of $\theta^\varepsilon(t)$ to $\mathfrak{gl}(\Lambda^{0,*}(C^n))$ with the derivation property. So the extension of $(\bar{e}^{\varepsilon h}_j(t))$ to $GL(\Lambda^{0,*}(C^n))$ is determined by the solution $\Pi^\varepsilon(t)$ of the SDE (5.12).

Similarly we have the SDE (5.15) for $\Xi^\varepsilon(t)$. $\square$

By standard arguments in the Malliavin calculus, all of $Z^\varepsilon(t)\in D^\infty(C^n)$, $e^\varepsilon(t)\in D^\infty(\mathrm{End}(C^n)')$ and $v^\varepsilon(t)\in D^\infty(\mathrm{End}(C^k))$ have asymptotic expansions in the space $D^\infty(C^n)$, $D^\infty(\mathrm{End}(C^n)')$ and $D^\infty(\mathrm{End}(\Lambda^{0,*}(C^n)))$ respectively. More precisely, we have

$$Z^\varepsilon(t) = \varepsilon z(t)+O(\varepsilon^2) \quad \text{in} \quad D^\infty(C^n) \quad \text{as} \quad \varepsilon\downarrow 0 . \tag{5.16}$$

Furthermore $Z^\varepsilon(1)/\varepsilon$ is uniformly non-degenerate as $\varepsilon\downarrow 0$ in the sense of

Malliavin and hence we have

$$\delta_0(Z^\varepsilon(1)) = \varepsilon^{-2n}\,\delta_0(z(1)) + O(\varepsilon^{-2n+1}) \quad \text{in} \quad \tilde{\boldsymbol{D}}^{-\infty} \quad \text{as} \quad \varepsilon \downarrow 0\,. \tag{5.17}$$

(cf. Ikeda-Watanabe [9] or Watanabe [16]). By (5.12), we have

$$\Pi^\varepsilon(1) = I + A_1 + A_2 + \cdots$$

where

$$A_p = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{p-1}} \circ d\Theta^\varepsilon(t_p) \circ d\Theta^\varepsilon(t_{p-1}) \circ \cdots \circ d\Theta^\varepsilon(t_1)\,. \tag{5.18}$$

By (5.10), (5.13) and (5.16), we can expand $\theta^\varepsilon(t)$ as follows;

$$\theta^{\varepsilon\bar{k}}{}_{\bar{j}}(t) = \varepsilon^2\, c^{\bar{k}}{}_{\bar{j}}(t) + O(\varepsilon^3) \quad \text{in} \quad \boldsymbol{D}^\infty(\boldsymbol{C}^n) \quad \text{as} \quad \varepsilon \downarrow 0\,, \tag{5.19}$$

where

$$c^{\bar{k}}{}_{\bar{j}}(t) = -R^{\bar{k}}{}_{\bar{j}q\bar{p}}(0) \int_0^t z^q(s) \circ d\bar{z}^p(s)\,.$$

By using this, we have

$$\left\{\begin{aligned} &\Pi^\varepsilon(1) = I + A_1 + \cdots + A_n + O(\varepsilon^{2n+2}) \quad \text{in} \quad \boldsymbol{D}^\infty(\mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n)))\,, \\ &A_p = \varepsilon^{2p} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{p-1}} \circ dD[c(t_p)] \circ dD[c(t_{p-1})] \circ \cdots \\ &\qquad \circ dD[c(t_1)] + O(\varepsilon^{2p+1}) \quad \text{in} \quad \boldsymbol{D}^\infty(\mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n)))\,. \end{aligned}\right. \tag{5.20}$$

Similarly, setting

$$B_p = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{p-1}} \circ d\iota^\varepsilon(t_p) \circ d\iota^\varepsilon(t_{p-1}) \circ \cdots \circ d\iota^\varepsilon(t_1)\,, \tag{5.21}$$

$$b^\alpha{}_\beta(t) = L^\alpha{}_{\beta q\bar{r}}(0) \int_0^t \bar{z}^r(s) \circ dz^q(s)\,, \tag{5.22}$$

we have

$$\left\{\begin{aligned} &\Xi^\varepsilon(1) = I + B_1 + \cdots + B_n + O(\varepsilon^{2n+2}) \quad \text{in} \quad \boldsymbol{D}^\infty(\mathrm{End}(\boldsymbol{C}^k)) \\ &B_p = \varepsilon^{2p} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{p-1}} \circ db(t_p) \circ db(t_{p-1}) \circ \cdots \circ db(t_1) \\ &\qquad + O(\varepsilon^{2p+1}) \quad \text{in} \quad \boldsymbol{D}^\infty(\mathrm{End}((\boldsymbol{C}^k))\,. \end{aligned}\right. \tag{5.23}$$

On the other hand, by (5.8) we obtain

$$M^\varepsilon(1) = I + C_1 + \cdots + C_n + O(\varepsilon^{2n+2}) \quad \text{in} \quad \boldsymbol{D}^\infty(\mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n) \otimes \boldsymbol{C}^k))\,, \tag{5.24}$$

where

$$C_p = \varepsilon^{2p} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{p-1}} \mathcal{J}(r^\varepsilon(t_p)) \cdots \mathcal{J}(r^\varepsilon(t_1))\, dt_p\, dt_{p-1} \cdots dt_1\,.$$

Furthermore,

$$\hat{R}^{\bar{a}}{}_{\bar{b}j\bar{h}}(r^{\varepsilon}(t)) = R^{\bar{a}}{}_{\bar{b}j\bar{h}}(0)+O(\varepsilon) \quad \text{in} \quad \boldsymbol{D}^{\infty}(\boldsymbol{C}),$$
$$\hat{L}^{\alpha}{}_{\beta j\bar{h}}(r^{\varepsilon}(t)) = L^{\alpha}{}_{\beta j\bar{h}}(0)+O(\varepsilon) \quad \text{in} \quad \boldsymbol{D}^{\infty}(\boldsymbol{C}).$$

Using these, we have

$$\hat{J}(r^{\varepsilon}(t)) = J(0)+O(\varepsilon) \quad \text{in} \quad \boldsymbol{D}^{\infty}(\mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n)\otimes\boldsymbol{C}^k)), \tag{5.25}$$

where

$$J(0) = \frac{1}{2} D[R^{\bar{\cdot}}{}_{\bar{\cdot}j\bar{j}}(0)]\otimes 1_{\boldsymbol{C}^k}+\frac{1}{2}[\mathrm{int}(\delta^j),\ \mathrm{ext}(\bar{\delta}^h)]\otimes L^{\bullet}{}_{j\bar{h}}(0).$$

So we obtain

$$C_p = \varepsilon^{2p} J(0)^p/p!+O(\varepsilon^{2p+1}) \quad \text{in} \quad \boldsymbol{D}^{\infty}(\mathrm{End}(\Lambda^{0,*}(\boldsymbol{C}^n)\otimes\boldsymbol{C}^k)). \tag{5.26}$$

Note that A_p, B_p and C_p are of order$\geq 2p$ with respect to ε. Now we can apply Lemma 4.2 for A_p, B_p, C_p. Combining these with (5.20), (5.23) and (5.24), we have

$$T[M^{\varepsilon}(1)(\Pi^{\varepsilon}(1)\otimes\Xi^{\varepsilon}(1))] = \sum_{p+q=n} T[C_p(A_q\otimes I)]+O(\varepsilon^{2n+2}). \tag{5.27}$$

Thus the stochastic parallel displacement for V does not affect to the conclusion. Furthermore by (5.20) and (5.26),

$$\begin{aligned} &T[M^{\varepsilon}(1)(\Pi^{\varepsilon}(1)\otimes\Xi^{\varepsilon}(1))] \\ &= \varepsilon^{2n}\sum_{p+q=n} T\Big[\frac{1}{p!}J(0)^p\Big(\int_0^1\int_0^{t_1}\cdots\int_0^{t_{q-1}}\circ dD[c(t_q)]\circ\cdots\circ dD[c(t_1)]\otimes I\Big)\Big] \\ &\quad +O(\varepsilon^{2n+1}). \end{aligned} \tag{5.28}$$

Now by using Lemma 4.3 and the Itô formula, we have

$$\begin{aligned} &T[M^{\varepsilon}(1)(\Pi^{\varepsilon}(1)\otimes\Xi^{\varepsilon}(1))]\,\mathrm{dvol}(z_0) \\ &= \varepsilon^{2n}\Big(\frac{i}{2}\Big)^n \sum_{p+q=n}\frac{1}{p!}\sum_{c=0}^{p}\frac{p!}{c!(p-c)!} \\ &\quad \times \mathrm{tr}_{\boldsymbol{C}^k}[(L^{\bullet}{}_{\cdot j\bar{h}}(0)\,dz^j\wedge d\bar{z}^h)^{\wedge(p-c)}] \\ &\quad \wedge\Big(\frac{1}{2}R^{\bar{r}}{}_{\bar{s}t\bar{t}}(0)\,d\bar{z}^s\wedge dz^r\Big)^{\wedge c} \\ &\quad \wedge\int_0^1\int_0^{t_1}\cdots\int_0^{t_{q-1}}\circ dc^{\bar{a}_q}{}_{\bar{b}_q}(t_q)\,d\bar{z}^{b_q}\wedge dz^{a_q} \\ &\quad \wedge\cdots\wedge\circ dc^{\bar{a}_1}{}_{\bar{b}_1}(t_1)\,d\bar{z}^{b_1}\wedge dz^{a_1}+O(\varepsilon^{2n+1}) \\ &= \varepsilon^{2n}\Big(\frac{i}{2}\Big)^n\sum_{p+q+r=n}\frac{1}{r!}\mathrm{tr}_{\boldsymbol{C}^k}[(L^{\bullet}{}_{\cdot j\bar{h}})(0)\,dz^j\wedge d\bar{z}^h)^{\wedge r}] \end{aligned} \tag{5.29}$$

$$\wedge \frac{1}{p!}\Big(\frac{1}{2} R^{\bar{r}}{}_{\bar{s}l\bar{l}}(0)\, d\bar{z}^s \wedge dz^r\Big)^{\wedge p} \wedge \frac{1}{q!}(c^{\bar{a}}{}_{\bar{b}}(1)\, d\bar{z}^b \wedge dz^b)^{\wedge q} + O(\varepsilon^{2n+1}).$$

On the other hand, by (5.10) we have

$$\begin{aligned} c^{\bar{a}}{}_{\bar{b}}(1)\, d\bar{z}^b \wedge dz^a &= -R^{\bar{a}}{}_{\bar{b}q\bar{p}}(0)\, d\bar{z}^b \wedge dz^a \int_0^1 z^q(t) \circ d\bar{z}^p(t) \\ &= -R^p{}_{qa\bar{b}}(0)\, dz^a \wedge d\bar{z}^b \int_0^1 z^q(t) \circ d\bar{z}^p(t) \\ &= -\Omega^{T'M\,p}{}_q \int_0^1 z^q(t) \circ d\bar{z}^p(t) \end{aligned}$$

and

$$R^{\bar{r}}{}_{\bar{s}l\bar{l}}(0)\, d\bar{z}^s \wedge dz^r = R^l{}_{lr\bar{s}}(0)\, dz^r \wedge d\bar{z}^s = \Omega^{T'M\,l}{}_l\,.$$

Also by noting $L^{\bullet}{}_{\cdot j\bar{h}}(0)\, dz^j \wedge d\bar{z}^h = \Omega^V$, we have

$$\begin{aligned} &T[M^\varepsilon(1)\,(\Pi^\varepsilon(1) \otimes \Xi^\varepsilon(1))]\, \mathrm{dvol}(z_0) \\ &= \varepsilon^{2n}\Big(\frac{i}{2}\Big)^n \sum_{p+q+r=n} \frac{1}{r!} \mathrm{tr}((\Omega^V)^{\wedge r}) \wedge \frac{1}{p!}\Big(\frac{1}{2}\,\Omega^{T'M\,l}{}_l\Big)^{\wedge p} \\ &\quad \wedge \frac{1}{q!}\Big(-\Omega^{T'M\,j}{}_h \int_0^1 z^h(t) \circ d\bar{z}^j(t)\Big)^{\wedge q} + O(\varepsilon^{2n+1})\,. \end{aligned} \tag{5.30}$$

By (5.9), (5.17) and (5.30), we have

$$\begin{aligned} &T[e(\varepsilon^2, z_0, z_0)]\, \mathrm{dvol}(z_0) \\ &= E[T[M^\varepsilon(1)\,(\Pi^\varepsilon(1) \otimes \Xi^\varepsilon(1))]\, \mathrm{dvol}(z_0)\, \delta_0(Z^\varepsilon(1))] \\ &= \Big(\frac{i}{2}\Big)^n \sum_{p+q+r=n} \frac{1}{r!} \mathrm{tr}((\Omega^V)^{\wedge r}) \wedge \frac{1}{p!}\Big(\frac{1}{2}\,\Omega^{T'M\,l}{}_l\Big)^{\wedge p} \\ &\quad \wedge \frac{1}{q!} E[(-\Omega^{T'M\,j}{}_h \int_0^1 z^h(t) \circ d\bar{z}^j(t))^{\wedge q}\, \delta_0(z(1))] + O(\varepsilon)\,. \end{aligned} \tag{5.31}$$

On the other hand, the following identity for the conditional expectation is well-known;

$$E[\Phi(z)\, \delta_0(z(1))] = (1/\pi)^n\, E[\Phi(z) \,|\, z(1) = 0]\,.$$

So we get

$$\begin{aligned} &T[e(\varepsilon^2, z_0, z_0)]\, \mathrm{dvol}(z_0) \\ &= \sum_{p+q+r=n} \frac{1}{r!} \mathrm{tr}\Big(\Big(\frac{i\Omega^V}{2\pi}\Big)^{\wedge r}\Big) \wedge \frac{1}{p!}\Big(\frac{i}{4\pi}\,\Omega^{T'M\,l}{}_l\Big)^{\wedge p} \\ &\quad \wedge \frac{1}{q!} E\Big[\Big(-\frac{i}{2\pi}\,\Omega^{T'M\,j}{}_h \int_0^1 z^h(t) \circ d\bar{z}^j(t)\Big)^{\wedge q} \Big|\, z(1) = 0\Big] + O(\varepsilon) \\ &= \{\sum_{r=0}^\infty \frac{1}{r!} \mathrm{tr}\Big(\Big(\frac{i\Omega^V}{2\pi}\Big)^{\wedge r}\Big) \wedge \sum_{p=0}^\infty \frac{1}{p!}\Big(\frac{i}{4\pi}\,\Omega^{T'M\,l}{}_l\Big)^{\wedge p} \end{aligned} \tag{5.32}$$

$$\wedge \sum_{q=0}^{\infty} \frac{1}{q!} E\left[\left(-\frac{i}{2\pi} \Omega^{T'M\, j}{}_h \int_0^1 z^h(t)\circ d\bar{z}^j(t)\right)^{\wedge q} \middle| z(1)=0\right]\}_{2n}$$
$$+O(\varepsilon)$$
$$= \{ch(V)\wedge P(\Omega^{T'M})\}_{2n}+O(\varepsilon)\,,$$

where

$$P(X) = \exp\left(\frac{i}{4\pi} X^l{}_l\right) E\left[\exp\left(-\frac{i}{2\pi} X^j{}_h \int_0^1 z^h(t)\circ d\bar{z}^j(t)\right) \middle| z(1)=0\right]$$

for any $n\times n$ complex matrix $X=(X^j{}_h)$. By the $U(n)$-invariance of the complex Brownian motion, $P(X)$ is $U(n)$-invariant, i.e.,

$$P(U^*XU) = P(X) \qquad \text{for} \quad U\in U(n)\,.$$

So let us obtain the generating function of $P(X)$:

$$p(x_1, x_2, \cdots, x_n) := P\begin{bmatrix} -2\pi i x_1 & & & 0 \\ & -2\pi i x_2 & & \\ & & \ddots & \\ 0 & & & -2\pi i x_n \end{bmatrix}$$
$$= \exp\left(\sum_{j=1}^n \frac{x_j}{2}\right) E\left[\exp\left(-\sum_{j=1}^n x_j \int_0^1 z^j(t)\circ d\bar{z}^j(t)\right) \middle| z(1)=0\right]$$
$$= \prod_{j=1}^n \exp\left(\frac{x_j}{2}\right) E\left[\exp\left(-x_j \int_0^1 z^j(t)\circ d\bar{z}^j(t)\right) \middle| z^j(1)=0\right]$$
$$= \prod_{j=1}^n \exp\left(\frac{x_j}{2}\right) E\left[\exp\left(\frac{i x_j}{2} \{\int_0^1 x^j(t)\circ dy^j(t)\right.\right.$$
$$\left.\left. -\int_0^1 y^j(t)\circ dx^j(t)\}\right) \middle| x^j(1)=y^j(1)=0\right]$$
$$= \prod_{j=1}^n \exp\left(\frac{x_j}{2}\right) \frac{x_j}{\exp\left(\frac{x_j}{2}\right)-\exp\left(-\frac{x_j}{2}\right)}$$
$$= \prod_{j=1}^n \frac{x_j}{1-e^{-x_j}}\,.$$

Here we used the well-known formula for the stochastic area due to P. Lévy (cf. Ikeda-Watanabe [8], p. 388). Hence $p(x_1, x_2, \cdots, x_n)$ is the generating function for Todd polynomial (cf. Gilkey [6], p. 97). Thus we have $P(\Omega^{T'M})= Td(T'M)$. By (5.4) and (5.32), we conclude that

$$\mathrm{Ind}(\bar{\partial}_V) = \int_M \{ch(V)\wedge Td(T'M)\}_{2n}+O(\varepsilon)\,.$$

This completes the proof of Theorem 5.1.

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