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Nagoya Math. J.
Vol. 112 (1988), 81-115

GRADED LIE ALGEBRAS AND GENERALIZED JORDAN TRIPLE SYSTEMS

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Dedicated to Professor Akihiko Morimoto on his sixtieth birthday

Introduction

One frequently encounters (real) semisimple graded Lie algebras in various branches of differential geometry (e.g. [16], [9], [14], [18]). It is therefore desirable to study semisimple graded Lie algebras, including those which have been studied individually, in a unified way. One of our concerns is to classify (finite-dimensional) semisimple graded Lie algebras in a way that enables us to construct them. A graded Lie algebra $\mathfrak{g}$ of the form $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ is said to be of the ν-th kind. The classification of semisimple graded Lie algebras of the ν-th kind was done by Kobayashi-Nagano [4] for $\nu = 1$, and by J.H. Cheng [3] for $\nu = 2$ and $\dim \mathfrak{g}_{-2} = 1$. The first aim of this paper is to obtain a classification theorem (Theorem 1.7) for semisimple graded Lie algebras, which establishes a bijective correspondence between isomorphism classes of all gradations in a real semisimple Lie algebra $\mathfrak{g}$ and certain equivalence classes of partitions $(\Pi_0, \Pi_1, \cdots, \Pi_s)$ of a restricted fundamental root system Π of $\mathfrak{g}$. For the complex semisimple case, a similar but weaker assertion has been obtained by V.G. Kac [5]. Theorem 1.7 and its proof enable us to construct all gradations in a semisimple Lie algebra. A graded Lie algebra $\mathfrak{g} = \sum_{k=-\infty}^{\infty} \mathfrak{g}_k$ (not necessarily of finite dimension) is said to be of type α_0, if $\sum_{k\leqslant -1} \mathfrak{g}_k$ and $\sum_{k\geqslant 1} \mathfrak{g}_k$ are generated by $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ respectively. In Theorem 2.6 we give a necessary and sufficient condition for a gradation to be of type α_0. By using this, we will construct explicitly (up to isomorphisms) all gradations of the first and the second kind in each classical real simple Lie algebra (§§2.3 and 4.2).

Our second concern is the problem of classifying a wider class of triple systems, called generalized Jordan triple systems which contain all

Received April 22, 1987.

Jordan triple systems as a special case. To this problem we apply the classification and construction of semisimple graded Lie algebras given in §§ 1 and 2. To a Jordan triple system there associates a graded Lie algebra of the first kind [9]. I.L. Kantor [7] considered a similar situation in a more general setting to obtain: To a generalized Jordan triple system (U_{-1}, B) there corresponds a graded Lie algebra $\mathscr{L}(B)$ which is not necessarily of finite dimension. If the graded Lie algebra $\mathscr{L}(B)$ is of the ν-th kind (resp. classical), then we say that (U_{-1}, B) is of the ν-th kind (resp. classical). On the other hand, in our paper [1] we introduced the class of compact generalized Jordan triple systems, as a natural generalization of a similar concept for Jordan triple systems. Compact real simple Jordan triple systems were classified by Loos [10]. The second aim of this paper is to classify compact classical real simple generalized Jordan triple systems of the ν-th kind, where $\nu = 1, 2$ (Theorems 4.1 and 4.2). Our result covers the above-mentioned result of Loos for classical ones. There are twelve families of compact classical simple generalized Jordan triple systems of the second kind. It turns out that the classification we are concerned is equivalent to that of simple graded Lie algebras of the ν-th kind endowed with grade-reversing Cartan involutions (Theorem 3.14). In the course of this reduction, we make use of a result on the equivalence of pairs of an infinite-dimensional simple graded Lie algebra of type α_0 and a grade-reversing involution (Theorem 3.12). In § 5 we will give a method of constructing noncompact generalized Jordan triple systems, starting from compact ones.

Throughout this paper, all objects are assumed to be defined over the real number field $\boldsymbol{R}$, unless otherwise stated. The complexification of a vector space V (resp. a Lie algebra $\mathfrak{g}$) is denoted by V^C (resp. $\mathfrak{g}^C$). $\boldsymbol{H}$ denotes the field of quaternions. $\boldsymbol{Z}$ denotes the ring of integers.

§ 1. Gradations of real semisimple Lie algebras

1.1. Let $\mathfrak{g} = \sum_{k \in \boldsymbol{Z}} \mathfrak{g}_k$ be a graded Lie algebra (or shortly GLA) over $\boldsymbol{R}$ with finite or infinite dimension. We always assume that $\dim \mathfrak{g}_k < \infty$ for all k and that $\mathfrak{g}_{-1} \neq (0)$. The family of the subspaces $(\mathfrak{g}_k)$ is called a *gradation* in the Lie algebra $\mathfrak{g}$. We say that an element $E \in \mathfrak{g}$ is a *characteristic element* of the GLA $\mathfrak{g}$, if each subspace $\mathfrak{g}_k$ is the eigenspace of the operator $\operatorname{ad} E$ for the eigenvalue k. Note that $E \in \mathfrak{g}_0$. Let $(\mathfrak{g}_k)$ and $(\mathfrak{g}'_k)$ be two gradations in $\mathfrak{g}$. We say that $(\mathfrak{g}_k)$ and $(\mathfrak{g}'_k)$ are *isomorphic*, if the GLA's $\mathfrak{g} = \sum \mathfrak{g}_k$ and $\mathfrak{g} = \sum \mathfrak{g}'_k$ are isomorphic, that is, there exists an element $a \in \operatorname{Aut} \mathfrak{g}$ such that $a(\mathfrak{g}_k) = \mathfrak{g}'_k$ for all k. A GLA $\mathfrak{g} = \sum_{k \in Z} \mathfrak{g}_k$ is said to be *of the ν-th kind* ($\nu > 0$), if $\mathfrak{g}_{\pm\nu} \neq (0)$ and $\mathfrak{g}_k = (0)$ for $|k| > \nu$. A finite-dimensional real semisimple GLA $\mathfrak{g}$ has a grade-reversing Cartan involution [17], and consequently $\mathfrak{g}$ is of the ν-th kind for some ν. Note that the GLA $\mathfrak{g}$ has a unique characteristic element E.

1.2. Let $\mathfrak{g} = \sum \mathfrak{g}_k$ be a real semisimple GLA of the ν-th kind with characteristic element E, and let τ be a grade-reversing Cartan involution of $\mathfrak{g}$. Then we have $\tau(E) = -E$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition by τ, where $\tau|_{\mathfrak{k}} = 1$ and $\tau|_{\mathfrak{p}} = -1$. Let us choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing E. We then have

$$\mathfrak{a} \subset \mathfrak{g}_0 \cap \mathfrak{p} . \tag{1.1}$$

Let Δ be the (restricted) root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We identify Δ with a subset of $\mathfrak{a}$ with respect to the inner product $\langle\ ,\ \rangle$ on $\mathfrak{a}$ induced by the Killing form of $\mathfrak{g}$, and we denote by $\mathfrak{g}^\alpha$ the root space for $\alpha \in \Delta$ in $\mathfrak{g}$. As a direct consequence of the decomposition of $\mathfrak{g}$ into root spaces, we have the following

LEMMA 1.1. *Each graded subspace $\mathfrak{g}_k$ of $\mathfrak{g}$ is expressed as*

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{c}(\mathfrak{a}) + \sum_{\langle\alpha, E\rangle = 0} \mathfrak{g}^\alpha , \\ \mathfrak{g}_k &= \sum_{\langle\alpha, E\rangle = k} \mathfrak{g}^\alpha \qquad (k \neq 0,\ |k| \leqslant \nu) , \end{aligned} \tag{1.2}$$

where $\mathfrak{c}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.

The above lemma shows that the gradation $(\mathfrak{g}_k)$ of $\mathfrak{g}$ gives rise to a partition of the root system Δ:

$$\begin{aligned} \Delta &= \bigcup_{k=-\nu}^{\nu} \Delta_k , \\ \Delta_k &= \{\alpha \in \Delta \colon \langle\alpha, E\rangle = k\} . \end{aligned} \tag{1.3}$$

Also one has

$$(\Delta_k + \Delta_l) \cap \Delta \subset \Delta_{k+l} ; \tag{1.4}$$

in particular Δ_0 is a closed subsystem of Δ. Let us choose a linear order in Δ in such a way that if $\alpha \in \Delta$ is positive then $\langle\alpha, E\rangle$ is non-negative. Let Π be a simple root system of Δ relative to this linear order. Then we have

$$\Pi \subset \bigcup_{k \geqslant 0} \Delta_k . \tag{1.5}$$

Putting $\Pi_k = \Pi \cap \Delta_k$, we get a partition of Π:

$$\Pi = \bigcup_{k=0}^{n} \Pi_k, \tag{1.6}$$

where n is the number such that $\Pi_n \neq \emptyset$, $\Pi_k = \emptyset$ for every $k > n$. Note that Π_1 is not empty, since $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_{-1} \neq 0$. It is easy to see that Π_0 is a fundamental system for the root system Δ_0.

DEFINITION 1.2. Let Π be a fundamental system of the root system Δ. If Π satisfies (1.5), then Π is said to be *compatible* with the gradation $(\mathfrak{g}_k)$.

Let W be the Weyl group of the root system Δ and W_0 be the Weyl group of the subsystem Δ_0. Since the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ is stable under the Cartan involution τ, it is reductive; τ induces a Cartan involution of the derived (semisimple) subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. We have the decomposition

$$\mathfrak{g}_0 = \mathfrak{k} \cap \mathfrak{g}_0 + \mathfrak{p} \cap \mathfrak{g}_0 . \tag{1.7}$$

Since $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{g}_0$, Δ_0 is viewed as the (restricted) root system of $\mathfrak{g}_0$ with respect to $\mathfrak{a}$. Let $G = \mathrm{Ad}\,\mathfrak{g}$. Let G_0 and K^* be the analytic subgroups of G generated by $\mathfrak{g}_0$ and $\mathfrak{k} \cap \mathfrak{g}_0$, respectively. Then we have

$$W_0 = N_{K^*}(\mathfrak{a})/C_{K^*}(\mathfrak{a}), \tag{1.8}$$

where $N_{K^*}(\mathfrak{a})$ (resp. $C_{K^*}(\mathfrak{a})$) is the normalizer (resp. centralizer) of $\mathfrak{a}$ in K^*.

LEMMA 1.3. *Let Π and Π' be two fundamental systems of Δ which are compatible with the gradation $(\mathfrak{g}_k)$ of $\mathfrak{g}$. Let $\Pi = \bigcup_{k=0}^{n} \Pi_k$ and $\Pi' = \bigcup_{j=0}^{m} \Pi'_j$ be the partitions of Π and Π' given in* (1.6). *Then there exists $s \in W_0$ such that $s(\Pi) = \Pi'$; in this case we have $n = m$ and $s(\Pi_i) = \Pi'_i$ $(1 \leqslant i \leqslant n)$.*

Proof. For a fundamental system Ω of Δ, we denote by $\Delta^+(\Omega)$ (resp. $\Delta^-(\Omega)$) the set of positive (resp. negative) roots in Δ with respect to the linear order determined by Ω. We claim first that $\bigcup_{k\geqslant 1} \Delta_k \subset \Delta^+(\Pi)$. Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$. Choose a root $\alpha = \sum_i m_i\alpha_i \in \Delta_k$ $(k \geqslant 1)$, where each m_i is assumed to be non-zero. Since $\alpha \notin \Delta_0$, at least one α_i in the above expression are in $\Pi - \Pi_0$. Suppose $\alpha \in \Delta^-(\Pi)$. Then each m_i is negative, and hence we have $k = \langle \alpha, E\rangle = \sum_j m_j \langle \alpha_j, E\rangle < 0$. But this contradicts the assumption $k \geqslant 1$. Therefore we get $\alpha \in \Delta^+(\Pi)$, which proves our claim. Similarly $\bigcup_{k\geqslant 1} \Delta_k \subset \Delta^+(\Pi')$ holds. Let us choose an element $s \in W_0$ which sends Π_0 to Π_0'. By (1.8), s has a representative in G_0 and so it comes from a grade-preserving inner automorphism of $\mathfrak{g}$. Consequently $s(E) = E$. Therefore, for $\alpha_k \in \Pi_i$ $(0 \leqslant i \leqslant n)$

$$\langle s(\alpha_k), E\rangle = \langle s(\alpha_k),\ s(E)\rangle = \langle \alpha_k, E\rangle = i\,. \tag{1.9}$$

Thus we have seen $s(\Pi_i) \subset \Delta_i$ $(0 \leqslant i \leqslant n)$, which implies that the fundamental system $s(\Pi)$ of Δ is compatible with the gradation $(\mathfrak{g}_k)$. The same argument as for Π shows that $\bigcup_{k\geqslant 1}\Delta_k \subset \Delta^+(s(\Pi))$. Furthermore we see $s(\Pi) \cap \Delta_0 = \Pi' \cap \Delta_0$. Therefore it follows that $\Delta^+(s(\Pi)) = \Delta^+(\Pi')$, which implies $s(\Pi) = \Pi'$. Since s is induced by a grade-preserving automorphism of $\mathfrak{g}$, we get $s(\Delta_i) = \Delta_i$ $(1 \leqslant i \leqslant n)$, and consequently $s(\Pi_i) = \Pi_i'$ $(1 \leqslant i \leqslant n)$.

LEMMA 1.4. *Let τ_1 and τ_2 be two grade-reversing Cartan involutions of the GLA $\mathfrak{g} = \sum_k \mathfrak{g}_k$. Let $\mathfrak{g} = \mathfrak{k}_i + \mathfrak{p}_i$ be the Cartan decomposition by τ_i $(i = 1, 2)$, where $\tau_i|_{\mathfrak{k}_i} = 1$ and $\tau_i|_{\mathfrak{p}_i} = -1$. Then there exists an element $X_0 \in \mathfrak{g}_0 \cap \mathfrak{p}_1 \cap \mathfrak{p}_2$ such that*

$$(\exp X_0)\tau_2(\exp(-X_0)) = \tau_1\,, \tag{1.10}$$

where the exponentials are taken in G.

Proof. Let B be the Killing form of $\mathfrak{g}$. Put $B_i(X, Y) = -B(X, \tau_i(Y))$, $i = 1, 2$. Then B_1 and B_2 are $\operatorname{Aut}\mathfrak{g}$-invariant inner products on $\mathfrak{g}$. Let $\operatorname{Pos}(\mathfrak{g}, B_2)$ (resp. $\operatorname{Sym}(\mathfrak{g}, B_2)$) be the totality of positive definite symmetric (resp. symmetric) operators on $\mathfrak{g}$ relative to B_2. Since τ_1 and τ_2 are involutive, it follows that $\tau_1\tau_2 \in \operatorname{Pos}(\mathfrak{g}, B_2)$. Let $C(E)$ be the algebraic subgroup of $\operatorname{Aut}\mathfrak{g}$ consisting of all elements $g \in \operatorname{Aut}\mathfrak{g}$ which commute with $\operatorname{ad} E$. Since τ_i is grade-reversing, we have $\tau_i(\operatorname{ad} E) = -(\operatorname{ad} E)\tau_i$. Therefore we have $\tau_1\tau_2 \in C(E) \cap \operatorname{Pos}(\mathfrak{g}, B_2)$. By a result of Neher [11], there exists $X_0 \in \operatorname{Lie} C(E) \cap \operatorname{Sym}(\mathfrak{g}, B_2) = \mathfrak{g}_0 \cap \operatorname{Sym}(\mathfrak{g}, B_2)$ such that

$$\tau_i(\operatorname{ad} X_0)\tau_i = -\operatorname{ad} X_0\,, \qquad i = 1, 2, \tag{1.11}$$

$$(\exp X_0)\tau_2(\exp(-X_0)) = \tau_1\,. \tag{1.12}$$

In view of (1.11), we get $X_0 \in \mathfrak{g}_0 \cap \mathfrak{p}_1 \cap \mathfrak{p}_2$.

1.3. Let $\mathfrak{g}$ be a real semisimple Lie algebra and Π be a fundamental system of a restricted root system of $\mathfrak{g}$. By a *partition* of Π we mean a disjoint union $\Pi = \bigcup_{k=0}^n \Pi_k$ such that Π_1 and Π_n are not empty. Some of the subsets Π_i may be empty. The partition is sometimes denoted by $(\Pi_0, \Pi_1, \cdots, \Pi_n)$.

DEFINITION 1.5. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be real semisimple Lie algebras, and let Π and Π' be fundamental systems of restricted root systems of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Partitions $(\Pi_0, \cdots, \Pi_n)$ of Π and $(\Pi'_0, \cdots, \Pi'_m)$ of Π' are said to be *equivalent*, if $n = m$ and if there exists an isomorphism φ of the Dynkin diagram of Π to that of Π' sending Π_i to Π'_i $(0 \leqslant i \leqslant n)$.

The following theorem is a criterion as to whether two real semisimple GLA's are isomorphic.

THEOREM 1.6. *Let* $\mathfrak{g} = \sum_k \mathfrak{g}_k$ *and* $\mathfrak{g}' = \sum_k \mathfrak{g}'_k$ *be real semisimple GLA's of the* ν*-th kind, and let* Π *and* Π' *be, respectively, fundamental systems of restricted root systems of* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *compatible with the gradations. Let* $(\Pi_0, \cdots, \Pi_n)$ *and* $(\Pi'_0, \cdots, \Pi'_m)$ *be the partitions of* Π *and* Π' *given in* (1.6). *If the two GLA's are isomorphic, then the above two partitions are equivalent. The converse is true, if the Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are isomorphic.*

Proof. Let E (resp. E') be the characteristic element of the GLA $\mathfrak{g}$ (resp. $\mathfrak{g}'$). Let Δ (resp. Δ') be the restricted root system of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) with Π (resp. Π') as a fundamental system. Since Π and Π' are compatible with the gradations, we can suppose that Δ (resp. Δ') is the root system with respect to a maximal abelian subspace $\mathfrak{a}$ (resp. $\mathfrak{a}'$) of the (-1)-eigenspace $\mathfrak{p}$ (resp. $\mathfrak{p}'$) in $\mathfrak{g}$ (resp. $\mathfrak{g}'$) under a grade-reversing Cartan involution τ (resp. τ') of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) satisfying $E \in \mathfrak{a}$ (resp. $E' \in \mathfrak{a}'$). Let $\mathfrak{k}$ and $\mathfrak{k}'$ be the $(+1)$-eigenspaces in $\mathfrak{g}$ and $\mathfrak{g}'$ under τ and τ', respectively. Now let φ be a grade-preserving isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$. $\varphi\tau\varphi^{-1}$ is a grade-reversing Cartan involution of $\mathfrak{g}'$. By Lemma 1.4, one can find an element $X_0 \in \mathfrak{g}'_0$ such that $\varphi_1 := (\exp X_0)\varphi$ satisfies

$$\varphi_1\tau = \tau'\varphi_1, \tag{1.13}$$

where the exponential is taken in the adjoint group of $\mathfrak{g}'$. φ_1 is a grade-preserving isomorphism, since $X_0 \in \mathfrak{g}'_0$. By (1.13) we have

$$\varphi_1(\mathfrak{k}) = \mathfrak{k}', \qquad \varphi_1(\mathfrak{p}) = \mathfrak{p}'. \tag{1.14}$$

Next we claim that there exists an element $X_1 \in \mathfrak{k}' \cap \mathfrak{g}'_0$ such that $\varphi_2 := (\exp X_1)\varphi_1$ satisfies

$$\varphi_2(\mathfrak{a}) = \mathfrak{a}'. \tag{1.15}$$

The reductive subalgebra $\mathfrak{g}'_0$ of $\mathfrak{g}'$ can be decomposed by τ':

$$\mathfrak{g}'_0 = \mathfrak{k}' \cap \mathfrak{g}'_0 + \mathfrak{p}' \cap \mathfrak{g}'_0. \tag{1.16}$$

Since φ_1 is grade-preserving, both $\varphi_1(\mathfrak{a})$ and $\mathfrak{a}'$ are maximal abelian subspaces of $\mathfrak{p}' \cap \mathfrak{g}_0'$ (cf. (1.1)). Therefore, for the decomposition (1.16), one can find $X_1 \in \mathfrak{k}' \cap \mathfrak{g}_0'$ such that $\varphi_2(\mathfrak{a}) = (\exp X_1)\varphi_1(\mathfrak{a}) = \mathfrak{a}'$. Note that φ_2 is still grade-preserving. Let $\Delta = \cup_{k=-\nu}^{\nu} \Delta_k$ and $\Delta' = \cup_{k=-\nu}^{\nu} \Delta_k'$ be the partitions given by (1.3). By (1.15) we have $\varphi_2(\Delta) = \Delta'$ and moreover

$$\varphi_2(\Delta_k) = \Delta_k', \qquad |k| \leqslant \nu. \tag{1.17}$$

From this it follows that the fundamental system $\varphi_2(\Pi)$ of Δ' is compatible with the gradation $(\mathfrak{g}_k')$. By (1.8) and Lemma 1.3, there exists an element $X_2 \in \mathfrak{k}' \cap \mathfrak{g}_0'$ such that $\varphi_3 := (\exp X_2)\varphi_2$ sends Π to Π' and Π_i to Π_i' $(0 \leqslant i \leqslant n = m)$.

To prove the converse, let ψ be an isomorphism of $(\Pi_0, \cdots, \Pi_n)$ to $(\Pi_0', \cdots, \Pi_n')$. Under the assumption, ψ extends to an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$, denoted again by ψ. Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ and $\Pi' = \{\beta_1, \cdots, \beta_l\}$. The characteristic elements E and E' are uniquely determined by the equations

$$\langle E, \alpha_i \rangle = k, \qquad \langle E', \beta_j \rangle = k \tag{1.18}$$

for $\alpha_i \in \Pi_k$, $0 \leqslant k \leqslant n$, $1 \leqslant i \leqslant l$, and for $\beta_j \in \Pi_k'$, $0 \leqslant k \leqslant n$, $1 \leqslant j \leqslant l$, where $\langle\ ,\ \rangle$ denotes the inner products defined by the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}'$. We may assume $\psi(\alpha_i) = \beta_i$ $(1 \leqslant i \leqslant l)$ by renumbering roots in Π'. Then we have

$$\langle \psi(E), \beta_i \rangle = \langle \psi(E), \psi(\alpha_i) \rangle = \langle E, \alpha_i \rangle = k \tag{1.19}$$

for $\alpha_i \in \Pi_k$, or equivalently $\beta_i \in \Pi_k'$. Comparing (1.19) with (1.18), we conclude $\psi(E) = E'$.

The following is a classification theorem for gradations in a semisimple Lie algebra.

THEOREM 1.7. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and Π be a fixed fundamental system of a fixed restricted root system Δ of $\mathfrak{g}$. Let $\mathscr{G}$ be the set of isomorphism classes of gradations in $\mathfrak{g}$ and let $\mathscr{P}$ be the set of equivalence classes of partitions of Π under the automorphism group of the Dynkin diagram of Π. Then there exists a bijection Φ of $\mathscr{G}$ to $\mathscr{P}$.*

Proof. Choose a gradation $(\mathfrak{g}_k)$ of $\mathfrak{g}$. To this gradation one can associate a compatible fundamental system $\Pi^{(1)}$ (cf. Definition 1.2) and the compatible partition $(\Pi_0^{(1)}, \cdots, \Pi_n^{(1)})$ of $\Pi^{(1)}$ (cf. (1.6)). According to Satake [13], there exists $a \in \mathrm{Ad}\,\mathfrak{g}$ such that $a(\Pi^{(1)}) = \Pi$. Put $\Pi_k = a(\Pi_k^{(1)})$. Then

a gives an equivalence between the two partitions $(\Pi_0^{(1)}, \cdots, \Pi_n^{(1)})$ and $(\Pi_0, \cdots, \Pi_n)$. Choose another gradation $(\mathfrak{g}'_k)$ of $\mathfrak{g}$ which is isomorphic to $(\mathfrak{g}_k)$ under $\varphi \in \mathrm{Aut}\,\mathfrak{g}$. To $(\mathfrak{g}'_k)$ there correspond a compatible fundamental system $\Pi^{(2)}$ and the compatible partition $(\Pi_0^{(2)}, \cdots, \Pi_m^{(2)})$. There exists $b \in \mathrm{Ad}\,\mathfrak{g}$ such that $b(\Pi^{(2)}) = \Pi$. Let $\Pi'_k = b(\Pi_k^{(2)})$. Then b gives an equivalence of $(\Pi_0^{(2)}, \cdots, \Pi_m^{(2)})$ to $(\Pi'_0, \cdots, \Pi'_m)$. From Theorem 1.6 and its proof it follows that $m = n$ and that φ can be modified to give an isomorphism $\tilde{\varphi}$ (still contained in $\mathrm{Aut}\,\mathfrak{g}$) of $\Pi^{(2)}$ to $\Pi^{(1)}$ which sends $(\Pi_0^{(2)}, \cdots, \Pi_n^{(2)})$ to $(\Pi_0^{(1)}, \cdots, \Pi_n^{(1)})$. Therefore $b\tilde{\varphi}^{-1}a^{-1}$ induces an automorphism of the Dynkin diagram of Π which sends $(\Pi_0, \cdots, \Pi_n)$ to $(\Pi'_0, \cdots, \Pi'_n)$. Thus we can define the mapping Φ by putting

$$\Phi([(\mathfrak{g}_k)]) = [(\Pi_0, \cdots, \Pi_n)], \tag{1.20}$$

where [ ] denotes the isomorphism (or equivalence) class. That Φ is injective follows from Theorem 1.6. We want to prove the surjectivity of Φ. Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ and let $(\Pi_0, \cdots, \Pi_n)$ be a partition of Π. We write $\alpha \in \Delta$ in the form $\alpha = \sum_{i=1}^{l} m_i(\alpha)\alpha_i$. For the partition $(\Pi_0, \cdots, \Pi_n)$, let us define an integer-valued function h_Π on Δ in the following way:

$$h_\Pi(\alpha) = \sum_{\alpha_i \in \Pi_1} m_i(\alpha) + 2 \sum_{\alpha_j \in \Pi_2} m_j(\alpha) + \cdots + n \sum_{\alpha_k \in \Pi_n} m_k(\alpha). \tag{1.21}$$

Let $\alpha, \beta \in \Delta$. If $\alpha + \beta \in \Delta$, then

$$h_\Pi(\alpha + \beta) = h_\Pi(\alpha) + h_\Pi(\beta). \tag{1.22}$$

Let us put

$$\mathfrak{g}_p = \sum_{h_\Pi(\alpha) = p} \mathfrak{g}^\alpha, \qquad p \neq 0,\ p \in \boldsymbol{Z}, \tag{1.23}$$

$$\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{a}) + \sum_{h_\Pi(\alpha) = 0} \mathfrak{g}^\alpha,$$

where $\mathfrak{a}$ is the abelian subspace of $\mathfrak{g}$ on which Δ is defined. Then we have $\mathfrak{g} = \sum \mathfrak{g}_p$. By (1.22), $(\mathfrak{g}_p)$ is a gradation of $\mathfrak{g}$. Put

$$\Delta_p = \{\alpha \in \Delta\colon h_\Pi(\alpha) = p\}, \qquad p \in \boldsymbol{Z}. \tag{1.24}$$

Then we get $\Pi_p = \Pi \cap \Delta_p$. This implies that $\Phi([(\mathfrak{g}_p)]) = [(\Pi_0, \cdots, \Pi_n)]$.

Remark 1.8. The partition (1.6) has been considered by Kac [5] for the complex semisimple case and is called the *characteristic* of the gradation $(\mathfrak{g}_k)$. We will use this terminology for the real semisimple case.

DEFINITION 1.9. The gradation given in (1.23) is called the gradation defined by the partition $(\Pi_0, \cdots, \Pi_n)$.

Remark 1.10. Let $\mathfrak{g} = \sum \mathfrak{g}_k$ be a real semisimple GLA of the ν-th kind with characteristic element E, and let Π be a fundamental system compatible with the gradation $(\mathfrak{g}_k)$. Let $\Pi = \cup_{k=0}^{n} \Pi_k$ be the partition given in (1.6). Then it follows easily that $h_\Pi(\alpha) = \langle \alpha, E \rangle$ holds, and hence the gradation $(\mathfrak{g}_k)$ coincides with the gradation defined by the partition $(\Pi_0, \cdots, \Pi_n)$.

§ 2. Gradations of type α_0

2.1. Definition 2.1. Let $\mathfrak{g} = \sum_{k \in \mathbf{Z}} \mathfrak{g}_k$ be a real GLA with $\dim \mathfrak{g} \leqslant \infty$. We say that $\mathfrak{g}$ is *of type* α_0 if the following conditions are satisfied:

$$\mathfrak{g}_{-k-1} = [\mathfrak{g}_{-k}, \mathfrak{g}_{-1}], \quad \mathfrak{g}_{k+1} = [\mathfrak{g}_k, \mathfrak{g}_1] \quad (k \geqslant 1). \tag{2.1}$$

Lemma 2.2. *Let* $\mathfrak{g} = \sum \mathfrak{g}_k$ *be a real semisimple GLA of the* ν*-th kind, and let* $\Pi = \cup_{k=0}^{n} \Pi_k$ *be the characteristic of the gradation* $(\mathfrak{g}_k)$. *If* $(\mathfrak{g}_k)$ *is of type* α_0, *then* $\Pi_k = \varnothing$ *for* $k \geqslant 2$.

Proof. Let E be the characteristic element of the gradation $(\mathfrak{g}_k)$. Note that $\Pi_1 \neq \varnothing$ (cf. 1.2). Choose a root $\alpha_i \in \Pi - \Pi_0$. Suppose that $\langle \alpha_i, E \rangle = k > 1$. Since $(\mathfrak{g}_k)$ is of type α_0, we have

$$\mathfrak{g}^{\alpha_i} \subset \mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}_1] = \textstyle\sum' [\mathfrak{g}^\beta, \mathfrak{g}^\gamma], \tag{2.2}$$

where the sum $\sum'$ is taken over the roots β and γ such that $\langle \beta, E \rangle = k - 1$, $\langle \gamma, E \rangle = 1$. If $[\mathfrak{g}^\beta, \mathfrak{g}^\gamma] \neq (0)$, then $\beta + \gamma$ is a root. Taking account of (1.2), we conclude that there exist two positive roots β, γ such that $\alpha_i = \beta + \gamma$, $\langle \beta, E \rangle = k - 1$ and $\langle \gamma, E \rangle = 1$. This contradicts the fact that α_i is simple. Therefore we have $k = 1$, or equivalently $\alpha_i \in \Pi_1$.

Lemma 2.3. *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra, and* Π *be a fundamental system of a root system* Δ *of* $\mathfrak{g}$. *Let* (Π_0, Π_1) *be a partition of* Π. *Then the gradation* $(\mathfrak{g}_k)$ *defined by* (Π_0, Π_1) *(cf. Definition* 1.9) *is of type* α_0.

Proof. Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ and $\Pi_1 = \{\alpha_{i_1}, \cdots, \alpha_{i_s}\}$. We write $\alpha \in \Delta$ in the form $\sum_{i=1}^{l} m_i(\alpha)\alpha_i$. Then the function h_Π in (1.21) for the partition (Π_0, Π_1) is given by

$$h_\Pi(\alpha) = \sum_{k=1}^{s} m_{i_k}(\alpha). \tag{2.3}$$

We define the function h on $\varDelta$ by

$$h(\alpha) = \sum_{i=1}^{l} m_i(\alpha) . \tag{2.4}$$

Let $\varDelta = \cup_k \varDelta_k$ be the partition (1.3) induced by the gradation $(\mathfrak{g}_k)$. If we put $\varDelta_p^{(k)} = \{\alpha \in \varDelta_p \colon h(\alpha) = k\}$, then we have a partition of $\varDelta_p$:

$$\varDelta_p = \bigcup_{k \geqslant p} \varDelta_p^{(k)} \qquad (p > 0) . \tag{2.5}$$

We claim

$$\mathfrak{g}^\beta \subset [\mathfrak{g}_1, \mathfrak{g}_{p-1}] , \quad \beta \in \varDelta_p \quad (p \geqslant 2) . \tag{2.6}$$

We want to prove this by induction on k in (2.5). Let us take $\beta \in \varDelta_p^{(p)}$ first. We have then $h(\beta) = h_\Pi(\beta) = p$, which implies that $m_i(\beta) = 0$ for $i \neq i_1, \cdots , i_s$. Hence one can write β as

$$\beta = \sum_{k=1}^{s} m_{i_k}(\beta) \alpha_{i_k} . \tag{2.7}$$

Since β is in $\varDelta_p$ $(p \geqslant 2)$, β is positive but not simple. Consequently, there exists a root $\alpha_i \in \Pi$ such that $\beta - \alpha_i$ is a root. Therefore, in view of the expression (2.7), we have $\alpha_i \notin \Pi_0$, that is, α_i coincides with one of α_{i_t} $(1 \leqslant t \leqslant s)$, say α_{i_k}. Set $\gamma = \beta - \alpha_{i_k} \in \varDelta$. We have then $h_\Pi(\gamma) = h_\Pi(\beta) - 1 = p - 1$. This implies $\gamma \in \varDelta_{p-1}$. Therefore we obtain $\mathfrak{g}^\beta = [\mathfrak{g}^{\alpha_{i_k}}, \mathfrak{g}^\gamma] \subset [\mathfrak{g}_1, \mathfrak{g}_{p-1}]$, which proves (2.6) for $\beta \in \varDelta_p^{(p)}$.

Suppose next that (2.6) is valid for all $\alpha \in \varDelta_p^{(m)}$, and choose $\beta \in \varDelta_p^{(m+1)}$. By the same reason as above, β is positive but not simple. We have one of the following two situations: a) There exists $\alpha_{i_k} \in \Pi_1$ such that $\beta - \alpha_{i_k} \in \varDelta$, b) there exists $\alpha_j \in \Pi_0$ such that $\beta - \alpha_j \in \varDelta$. In the case a) we proceed as above to get the assertion (2.6). Suppose that b) occurs. Let $\gamma = \beta - \alpha_j \in \varDelta$. Then we have $h_\Pi(\gamma) = h_\Pi(\beta) = p$ and $h(\gamma) = h(\beta) - 1 = m$. Hence $\gamma \in \varDelta_p^{(m)}$. By the assumption of the induction, we get $\mathfrak{g}^\gamma \subset [\mathfrak{g}_1, \mathfrak{g}_{p-1}]$. Therefore we obtain

$$\mathfrak{g}^\beta = [\mathfrak{g}^{\alpha_j}, \mathfrak{g}^\gamma] \subset [\mathfrak{g}^{\alpha_j}, [\mathfrak{g}_1, \mathfrak{g}_{p-1}]] \subset [\mathfrak{g}_0, [\mathfrak{g}_1, \mathfrak{g}_{p-1}]] \subset [\mathfrak{g}_1, \mathfrak{g}_{p-1}] . \tag{2.8}$$

Thus we have proved (2.6). The second equality of (2.1) is a direct consequence of (2.6). The first one in (2.1) is immediately obtained from the second by applying a grade-reversing Cartan involution.

2.2. Let $\mathfrak{g} = \sum \mathfrak{g}_k$ be a real semisimple GLA of the ν-th kind. We come back to the situation in 1.2 and preserve notations there. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then one can write $\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{a}$, where $\mathfrak{h}^+ = \mathfrak{h} \cap \mathfrak{k}$. Let $\tilde{\Delta}$ be the root system of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}^C$ (= the complexification of $\mathfrak{h}$). $\mathfrak{h}_0 := i\mathfrak{h}^+ + \mathfrak{a}$ is the real part of $\mathfrak{h}^C$. We identify $\tilde{\Delta}$ with a subset of $\mathfrak{h}_0$ via the inner product $\langle\ ,\ \rangle$ defined by the Killing form of $\mathfrak{g}^C$. Put $\tilde{\Delta}_\bullet = \tilde{\Delta} \cap i\mathfrak{h}^+$. Let σ be the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$, and let $\tilde{\Pi}$ be the σ-fundamental system of $\tilde{\Delta}$ [13]. Let ϖ be the orthogonal projection of $\mathfrak{h}_0$ onto $\mathfrak{a}$ with respect to $\langle\ ,\ \rangle$. $\Delta := \varpi(\tilde{\Delta} - \tilde{\Delta}_\bullet)$ is a restricted root system (or the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$). Let us consider the complexified GLA of $\mathfrak{g}$:

$$\mathfrak{g}^C = \sum_k \mathfrak{g}_k^C\,, \tag{2.9}$$

where $\mathfrak{g}_k^C$ is the complexification of $\mathfrak{g}_k$.

LEMMA 2.4. (i) *The characteristic element E of $\mathfrak{g}$ is also that of $\mathfrak{g}^C$.* (ii) *The following partition is valid:*

$$\tilde{\Delta} = \bigcup_k \tilde{\Delta}_k\,, \tag{2.10}$$

where $\tilde{\Delta}_k = \{\alpha \in \tilde{\Delta} : \langle \alpha, E\rangle = k\}$. (iii) *We have*

$$\tilde{\Delta}_0 = (\varpi^{-1}(\Delta_0) \cap \tilde{\Delta}) \cup \tilde{\Delta}_\bullet\,, \tag{2.11}$$

$$\tilde{\Delta}_k = \varpi^{-1}(\Delta_k) \cap \tilde{\Delta} \qquad (k \neq 0)\,. \tag{2.12}$$

(iv) *The subspace $\mathfrak{g}_k^C$ are expressed as*

$$\mathfrak{g}_0^C = \mathfrak{h}^C + \sum_{\alpha \in \tilde{\Delta}_0} \tilde{\mathfrak{g}}^\alpha\,, \tag{2.13}$$

$$\mathfrak{g}_k^C = \sum_{\alpha \in \tilde{\Delta}_k} \tilde{\mathfrak{g}}^\alpha \qquad (k \neq 0)\,, \tag{2.14}$$

where $\tilde{\mathfrak{g}}^\alpha$ is the root space in $\mathfrak{g}^C$ corresponding to $\alpha \in \tilde{\Delta}$.

Proof. The assertions (i), (ii) are immediate. Let $\alpha \in \tilde{\Delta}_0$. $\varpi(\alpha) = 0$ if and only if $\alpha \in \tilde{\Delta}_\bullet$. If $\varpi(\alpha) \neq 0$, then $\varpi(\alpha) \in \Delta$. Hence $\langle \varpi(\alpha), E\rangle = \langle \alpha, E\rangle = 0$, which implies $\varpi(\alpha) \in \Delta_0$. Thus the inclusion $\subset$ in (2.11) was obtained. Similarly we have the converse inclusion. (2.12) can be proved analogously. By (1.2) we have

$$\mathfrak{g}_0^C = \mathfrak{c}(\mathfrak{a})^C + \sum_{\gamma \in \Delta_0} (\mathfrak{g}^\gamma)^C\,. \tag{2.15}$$

From the expression of $\mathfrak{c}(\mathfrak{a})$ in terms of the roots in $\tilde{\Delta}_\bullet$ (cf. [15]), we see that $\mathfrak{c}(\mathfrak{a})^C = \mathfrak{h}^C + \sum_{\alpha \in \tilde{\Delta}_\bullet} \tilde{\mathfrak{g}}^\alpha$. On the other hand we have $(\mathfrak{g}^\gamma)^C = \sum' \tilde{\mathfrak{g}}^\alpha$, where the sum $\sum'$ is taken over the roots $\alpha \in \tilde{\Delta}$ satisfying $\varpi(\alpha) = \gamma$. Therefore, from (2.11) and (2.15) we obtain (2.13). Similarly we have (2.14).

We can choose a σ-fundamental system $\tilde{\Pi} = \{\alpha_1, \cdots, \alpha_s\}$ of $\tilde{\Delta}$ in such a way that the relation

$$\langle \alpha_i, E \rangle \geqslant 0 \qquad (1 \leqslant i \leqslant s) \tag{2.16}$$

is satisfied. $\tilde{\Pi}$ is compatible with the gradation $(\mathfrak{g}_k^C)$ of $\mathfrak{g}^C$. Let $\tilde{\Pi}_\bullet = \tilde{\Pi} \cap \tilde{\Delta}_\bullet$. Then $\Pi = \varpi(\tilde{\Pi} - \tilde{\Pi}_\bullet)$ is a fundamental system of Δ. By (2.16) we have $\langle \gamma_i, E \rangle \geqslant 0$ for each $\gamma_i \in \Pi$, and so Π is compatible with the gradation $(\mathfrak{g}_k)$ of $\mathfrak{g}$. Let $\tilde{\Pi}_i = \tilde{\Pi} \cap \tilde{\Delta}_i$ and $\Pi_i = \Pi \cap \Delta_i$. Then the following lemma is easily seen.

LEMMA 2.5.
$$\tilde{\Pi}_0 = (\varpi^{-1}(\Pi_0) \cap \tilde{\Pi}) \cup \tilde{\Pi}_\bullet,$$
$$\tilde{\Pi}_k = \varpi^{-1}(\Pi_k) \cap \tilde{\Pi} \qquad (k \neq 0).$$

In particular, the number of $\tilde{\Pi}_i$'s in the partition $\tilde{\Pi} = \cup_i \tilde{\Pi}_i$ is equal to the number of Π_i's in the partition $\Pi = \cup_i \Pi_i$.

The next theorem gives a characterization of the gradations of type α_0 in terms of their characteristics.

THEOREM 2.6. *Let $\mathfrak{g} = \sum \mathfrak{g}_k$ be a real semisimple GLA of the ν-th kind, and $\Pi = \cup_{k=0}^n \Pi_k$ be the characteristic of the gradation $(\mathfrak{g}_k)$. Then $(\mathfrak{g}_k)$ is of type α_0 if and only if $\Pi_k = \varnothing$ for every $k \geqslant 2$.*

Proof. Suppose that $\Pi_k = \varnothing$ $(k \geqslant 2)$. Consider the GLA $\mathfrak{g}^C = \sum \mathfrak{g}_k^C$, and let $\tilde{\Pi} = \cup_{k=0}^n \tilde{\Pi}_k$ be the characteristic of the gradation $(\mathfrak{g}_k^C)$. Then, by Lemma 2.5 and Theorem 1.6, we get $\tilde{\Pi}_k = \varnothing$ $(k \geqslant 2)$. Lemma 2.3 and Remark 1.10 now imply that $(\mathfrak{g}_k^C)$ is of type α_0. Hence, for $k \geqslant 1$, we have $\mathfrak{g}_{k+1}^C = [\mathfrak{g}_1^C, \mathfrak{g}_k^C] = [\mathfrak{g}_1, \mathfrak{g}_k]^C$, from which it follows that $\mathfrak{g}_{k+1} = [\mathfrak{g}_1, \mathfrak{g}_k]$. The converse assertion has been proved in Lemma 2.2.

Let Π be a fundamental system of a restricted root system of a real semisimple Lie algebra $\mathfrak{g}$. Two subsets $\Omega_1, \Omega_2 \subset \Pi$ are said to be *equivalent*, if there exists an automorphism a of the Dynkin diagram of Π sending Ω_1 to Ω_2. The equivalence class of Ω_1 is denoted by $[\Omega_1]$. Combining Theorem 2.6 with Theorem 1.7 we have the following

THEOREM 2.7. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, and let Π be a fundamental system of a restricted root system of $\mathfrak{g}$. Let $\mathscr{G}_{\alpha_0}$ be the set* *of isomorphism classes of gradations of type* α_0 *in* $\mathfrak{g}$, *and let* $\mathscr{P}_{\alpha_0}$ *be the set of equivalence classes of all non-empty subsets of* Π. *Then there exists a bijection* Φ_{α_0} *of* $\mathscr{G}_{\alpha_0}$ *to* $\mathscr{P}_{\alpha_0}$.

Proof. Let $[(\mathfrak{g}_k)]$ denote the isomorphism class of the gradation $(\mathfrak{g}_k)$ of $\mathfrak{g}$. By Theorem 2.6, the characteristic of $(\mathfrak{g}_k)$ is of the form $\Pi = \Pi_0 \cup \Pi_1$ (See also Theorem 1.6). By Theorem 1.6 we may define Φ_{α_0} to be

$$\Phi_{\alpha_0}([(\mathfrak{g}_k)]) = [\Pi_1]\,. \tag{2.17}$$

Then Theorem 1.7 shows that Φ_{α_0} is bijective.

Now we will assume $\mathfrak{g}$ to be real simple. Let $\mathscr{G}$ be the same as in Theorem 1.7, and let $\mathscr{G}^{(\nu)}$ be the subset of $\mathscr{G}$ consisting of isomorphism classes of all gradations of the ν-th kind in $\mathfrak{g}$. Let $\mathscr{G}^{(\nu)}_{\alpha_0} = \mathscr{G}^{(\nu)} \cap \mathscr{G}_{\alpha_0}$.

LEMMA 2.8. $\mathscr{G}^{(\nu)} = \mathscr{G}^{(\nu)}_{\alpha_0}$ *holds for* $\nu = 1, 2$.

Proof. The case $\nu = 1$ is trivial. For $\nu = 2$, see Tanaka [16].

Let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be a fundamental system of a restricted root system Δ of $\mathfrak{g}$, and ϑ be the dominant root. We write $\vartheta = \sum_{i=1}^{l} m_i(\vartheta)\alpha_i$.

Let

$$\begin{aligned} \mathscr{P}^{(1)} &= \{[\{\alpha_i\}]\colon m_i(\vartheta) = 1 \quad (1 \leqslant i \leqslant l)\}\,, \\ \mathscr{G}^{(2)} &= \{[\{\alpha_i\}]\colon m_i(\vartheta) = 2 \quad (1 \leqslant i \leqslant l)\} \\ &\qquad \cup \{[\{\alpha_i, \alpha_j\}]\colon m_i(\vartheta) = m_j(\vartheta) = 1 \quad (1 \leqslant i \neq j \leqslant l)\}\,. \end{aligned} \tag{2.18}$$

The next theorem gives the classification of real simple GLA's of the ν-th kind ($\nu = 1, 2$).

THEOREM 2.9. *For* $\nu = 1, 2$, *there exists a bijection* $\Phi^{(\nu)}$ *of* $\mathscr{G}^{(\nu)}$ *to* $\mathscr{P}^{(\nu)}$.

Proof. We define the map $\Phi^{(\nu)}$ to be the map Φ_{α_0} in (2.17). Choose an element $[(\mathfrak{g}_k)] \in \mathscr{G}^{(\nu)}$, and let $\Pi = \Pi_0 \cup \Pi_1$ be the characteristic of $(\mathfrak{g}_k)$ (cf. Lemma 2.8). By a property of the dominant root, we see $\vartheta \in \Delta_\nu$. Therefore $\langle \vartheta, E \rangle = \sum_{i=1}^{l} m_i(\vartheta) = \nu$, which implies that the cardinality of Π_1 is less than or equal to ν. So the theorem is a direct consequence of Theorem 2.7.

2.3. Let $[\Pi_1] \in \mathscr{P}^{(\nu)}$, $\nu = 1, 2$ and put $\Pi_0 = \Pi - \Pi_1$. For a partition (Π_0, Π_1) of Π, the function h_Π in (1.21) is given by $h_\Pi(\alpha) = \sum_{\alpha_i \in \Pi_1} m_i(\alpha)$ for $\alpha \in \Delta$. Let us put $\Delta_k = \{\alpha \in \Delta\colon h_\Pi(\alpha) = k\}$, $k \in \boldsymbol{Z}$. Then, by Theorem

2.9, we have the partition $\Delta = \bigcup_{k=-\nu}^{\nu} \Delta_k$. In this paragraph we will enumerate this kind of partitions for each irreducible classical root system Δ which give rise to all gradations of the ν-th kind ($\nu = 1, 2$) of classical real simple Lie algebras. We only give Δ_{-k} $(0 \leqslant k \leqslant \nu)$, since $\Delta_k = -\Delta_{-k}$ holds.

1. Type A_{n-1} $(n \geqslant 2)$
$\Delta = \{\pm(x_i - x_j)\colon 1 \leqslant i < j \leqslant n\}$,
$\Pi = \{\alpha_1, \cdots, \alpha_{n-1}\}$, $\alpha_i = x_{i+1} - x_i$ $(1 \leqslant i \leqslant n-1)$,
$\vartheta = \alpha_1 + \cdots + \alpha_{n-1}$.

a) The sets $\{\alpha_p\}$ $(1 \leqslant p \leqslant [n/2])$ are complete representatives of $\mathscr{P}^{(1)}$. The partition of Δ for $\Pi_1 = \{\alpha_p\}$ is given by

$$(2.19) \quad \begin{aligned} \Delta_0 &= \{\pm(x_i - x_j)\colon 1 \leqslant i < j \leqslant p \text{ or } p+1 \leqslant i < j \leqslant n\}, \\ \Delta_{-1} &= \{x_i - x_j\colon 1 \leqslant i \leqslant p,\ p+1 \leqslant j \leqslant n\}. \end{aligned}$$

b) The sets $\{\alpha_p, \alpha_{p+q}\}$ $(1 \leqslant p \leqslant [n/2],\ 1 \leqslant q \leqslant n-2p)$ are complete repesentatives of $\mathscr{P}^{(2)}$. The partition of Δ for $\Pi_1 := \{\alpha_p, \alpha_{p+q}\}$ is given by

$$(2.20) \quad \begin{aligned} \Delta_0 &= \{\pm(x_i - x_j)\colon 1 \leqslant i < j \leqslant p \text{ or } p+1 \leqslant i < j \leqslant p+q \\ &\qquad \text{or } p+q+1 \leqslant i < j \leqslant n\}, \\ \Delta_{-1} &= \{x_i - x_j\colon 1 \leqslant i \leqslant p,\ p+1 \leqslant j \leqslant p+q \\ &\qquad \text{or } p+1 \leqslant i \leqslant p+q,\ p+q+1 \leqslant j \leqslant n\}, \\ \Delta_{-2} &= \{x_i - x_j\colon 1 \leqslant i \leqslant p,\ p+q+1 \leqslant j \leqslant n\}. \end{aligned}$$

2. Type B_n $(n \geqslant 2)$
$\Delta = \{\pm(x_i \pm x_j)\ (1 \leqslant i < j \leqslant n),\ \pm x_i\ (1 \leqslant i \leqslant n)\}$,
$\Pi = \{\alpha_1, \cdots, \alpha_n\}$, $\alpha_i = x_{i+1} - x_i$ $(1 \leqslant i \leqslant n-1)$, $\alpha_n = -x_n$,
$\vartheta = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$.
The automorphism group of Π is trivial in this case.

c) $\mathscr{P}^{(1)}$ consists of a single set $\{\alpha_1\}$. The corresponding partition of Δ is given by

$$(2.21) \quad \begin{aligned} \Delta_0 &= \{\pm(x_i \pm x_j)\ (2 \leqslant i < j \leqslant n),\ \pm x_i\ (2 \leqslant i \leqslant n)\}, \\ \Delta_{-1} &= \{x_1 \pm x_j\ (2 \leqslant j \leqslant n),\ x_1\}. \end{aligned}$$

d) $\mathscr{P}^{(2)}$ consists of the sets $\{\alpha_2\}, \cdots, \{\alpha_n\}$. The partition of Δ for $\Pi_1 = \{\alpha_k\}$ $(2 \leqslant k \leqslant n)$ is given by

$$(2.22) \quad \begin{aligned} \Delta_0 = \{&\pm(x_i - x_j)\ (1 \leqslant i < j \leqslant k),\ \pm(x_i \pm x_j)\ (k+1 \leqslant i < j \leqslant n), \\ &\pm x_i\ (k+1 \leqslant i \leqslant n)\}, \end{aligned}$$

$$\Delta_{-1} = \{x_i \pm x_j \ (1 \leqslant i \leqslant k,\ k+1 \leqslant j \leqslant n),\ x_i \ (1 \leqslant i \leqslant k)\},$$
$$\Delta_{-2} = \{x_i + x_j \ (1 \leqslant i < j \leqslant k)\}.$$

3. Type C_n $(n \geqslant 3)$
$\Delta = \{\pm (x_i \pm x_j) \ (1 \leqslant i < j \leqslant n),\ \pm 2x_i \ (1 \leqslant i \leqslant n)\}$,
$\Pi = \{\alpha_1, \cdots, \alpha_n\}$, $\alpha_i = x_{i+1} - x_i$ $(1 \leqslant i \leqslant n-1)$, $\alpha_n = -2x_n$,
$\vartheta = 2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n$.
The automorphism group of Π is trivial in this case.

e) $\mathscr{P}^{(1)}$ consists of a single set $\{\alpha_n\}$. The corresponding partition of Δ is given by

$$\text{(2.23)} \quad \Delta_0 = \{\pm (x_i - x_j) : 1 \leqslant i < j \leqslant n\},$$
$$\Delta_{-1} = \{x_i + x_j \ (1 \leqslant i < j \leqslant n),\ 2x_i \ (1 \leqslant i \leqslant n)\}.$$

f) $\mathscr{P}^{(2)}$ consists of the sets $\{\alpha_1\}, \cdots, \{\alpha_{n-1}\}$. The partition of Δ for $\Pi_1 = \{\alpha_k\}$ $(1 \leqslant k \leqslant n-1)$ is given by

$$\text{(2.24)} \quad \Delta_0 = \{\pm (x_i - x_j) \ (1 \leqslant i < j \leqslant k),\ \pm (x_i \pm x_j) \ (k+1 \leqslant i < j \leqslant n),$$
$$\pm 2x_i \ (k+1 \leqslant i \leqslant n)\},$$
$$\Delta_{-1} = \{x_i \pm x_j \ (1 \leqslant i \leqslant k,\ k+1 \leqslant j \leqslant n)\},$$
$$\Delta_{-2} = \{x_i + x_j \ (1 \leqslant i < j \leqslant k),\ 2x_i \ (1 \leqslant i \leqslant k)\}.$$

4. Type BC_n $(n \geqslant 1)$
$\Delta = \{\pm (x_i \pm x_j) \ (1 \leqslant i < j \leqslant n),\ \pm x_i,\ \pm 2x_i \ (1 \leqslant i \leqslant n)\}$,
$\Pi = \{\alpha_1, \cdots, \alpha_n\}$, $\alpha_i = x_{i+1} - x_i$ $(1 \leqslant i \leqslant n-1)$, $\alpha_n = -x_n$,
$\vartheta = 2\alpha_1 + \cdots + 2\alpha_n$.
The automorphism group of Π is trivial in this case. $\mathscr{P}^{(1)}$ is empty.

g) $\mathscr{P}^{(2)}$ consists of the sets $\{\alpha_1\}, \cdots, \{\alpha_n\}$. The partition of Δ for $\Pi_1 = \{\alpha_k\}$ $(1 \leqslant k \leqslant n)$ is given by

$$\text{(2.25)} \quad \Delta_0 = \{\pm (x_i - x_j) \ (1 \leqslant i < j \leqslant k),\ \pm (x_i \pm x_j) \ (k+1 \leqslant i < j \leqslant n),$$
$$\pm x_i,\ \pm 2x_i \ (k+1 \leqslant i \leqslant n),$$
$$\Delta_{-1} = \{x_i \pm x_j \ (1 \leqslant i \leqslant k,\ k+1 \leqslant j \leqslant n),\ x_i \ (1 \leqslant i \leqslant k)\},$$
$$\Delta_{-2} = \{x_i + x_j \ (1 \leqslant i < j \leqslant k),\ 2x_i \ (1 \leqslant i \leqslant k)\}.$$

5. Type D_n $(n \geqslant 4)$
$\Delta = \{\pm (x_i \pm x_j) : 1 \leqslant i < j \leqslant n\}$,
$\Pi = \{\alpha_1, \cdots, \alpha_n\}$, $\alpha_i = x_{i+1} - x_i$ $(1 \leqslant i \leqslant n-1)$, $\alpha_n = -x_{n-1} - x_n$,
$\vartheta = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$.
The sets $\{\alpha_1\}$, $\{\alpha_n\}$ are complete representatives of $\mathscr{P}^{(1)}$ for $n \neq 4$, while

$\{\alpha_1\}$ is for $n=4$. The sets $\{\alpha_2\}, \cdots, \{\alpha_{n-2}\}, \{\alpha_1, \alpha_n\}, \{\alpha_{n-1}, \alpha_n\}$ are complete representatives of $\mathscr{P}^{(2)}$ for $n \neq 4$, while $\{\alpha_2\}, \{\alpha_3, \alpha_4\}$ are for $n=4$.

h) The partition of Δ for $\Pi_1 = \{\alpha_1\}$ is given by

$$(2.26)\quad \Delta_0 = \{\pm(x_i \pm x_j)\colon 2 \leqslant i < j \leqslant n\}, \qquad \Delta_{-1} = \{x_1 \pm x_j\colon 2 \leqslant j \leqslant n\}.$$

i) The partition of Δ for $\Pi_1 = \{\alpha_n\}$ is given by

$$(2.27)\quad \Delta_0 = \{\pm(x_i - x_j)\colon 1 \leqslant i < j \leqslant n\}, \qquad \Delta_{-1} = \{x_i + x_j\colon 1 \leqslant i < j \leqslant n\}.$$

j) The partition of Δ for $\Pi_1 = \{\alpha_k\}$ $(2 \leqslant k \leqslant n-2)$ is given by

$$(2.28)\quad \begin{aligned} \Delta_0 &= \{\pm(x_i - x_j)\ (1 \leqslant i < j \leqslant k),\ \pm(x_i \pm x_j)\ (k+1 \leqslant i < j \leqslant n)\}, \\ \Delta_{-1} &= \{x_i \pm x_j\colon 1 \leqslant i \leqslant k,\ k+1 \leqslant j \leqslant n\}, \\ \Delta_{-2} &= \{x_i + x_j\colon 1 \leqslant i < j \leqslant k\}. \end{aligned}$$

k) The partition of Δ for $\Pi_1 = \{\alpha_{n-1}, \alpha_n\}$ is given by

$$(2.29)\quad \begin{aligned} \Delta_0 &= \{\pm(x_i - x_j)\colon 1 \leqslant i < j \leqslant n-1\}, \\ \Delta_{-1} &= \{x_i \pm x_n\colon 1 \leqslant i \leqslant n-1\}, \\ \Delta_{-2} &= \{x_i + x_j\colon 1 \leqslant i < j \leqslant n-1\}. \end{aligned}$$

l) The partition of Δ for $\Pi_1 = \{\alpha_1, \alpha_n\}$ is given by

$$(2.30)\quad \begin{aligned} \Delta_0 &= \{\pm(x_i - x_j)\colon 2 \leqslant i < j \leqslant n\}, \\ \Delta_{-1} &= \{x_1 - x_j\ (2 \leqslant j \leqslant n),\ x_i + x_j\ (2 \leqslant i < j \leqslant n)\}, \\ \Delta_{-2} &= \{x_1 + x_j\ (2 \leqslant j \leqslant n)\}. \end{aligned}$$

Remark 2.10. Let $\mathfrak{g}$ be a classical real simple Lie algebra, and let $(\mathfrak{g}, \Pi_1)$ denote the gradation of the ν-th kind in $\mathfrak{g}$ corresponding to Π_1. By using a)–l) above, we see that $(\mathfrak{g}, \Pi_1)$ satisfies $\nu = 2$ and $\dim \mathfrak{g}_{-2} = 1$, if and only if $(\mathfrak{g}, \Pi_1)$ is one of the followings: $(\mathfrak{sl}(n, \boldsymbol{R}), \{\alpha_1, \alpha_{n-1}\})$ $(n \geqslant 3)$, $(\mathfrak{su}(p, q), \{\alpha_1\})$ $(1 \leqslant p \leqslant q)$, $(\mathfrak{so}(p, q), \{\alpha_2\})$ $(2 \leqslant p \leqslant q)$, $(\mathfrak{sp}(n, \boldsymbol{R}), \{\alpha_1\})$ $(n \geqslant 3)$, $(\mathfrak{so}^*(2n), \{\alpha_1\})$ $(n \geqslant 4)$. This reproduces the Cheng's result stated in Introduction for $\mathfrak{g}$ classical.

§ 3. The Lie algebra $\mathscr{L}(B)$ and GJTS's

3.1. For the later use we will mention some properties of a universal graded Lie algebra and the GLA $\mathscr{L}(B)$ both due to Kantor [6], [7]. For convenience, we denote by $a_1 \circ a_2$ the commutator product $[a_1, a_2]$ in a Lie algebra, and define $a_1 \circ \cdots \circ a_m$ inductively by $(a_1 \circ \cdots \circ a_{m-1}) \circ a_m$. Let $\mathscr{U}_{-1}$ be a finite-dimensional vector space, and $\mathscr{U}_-$ be the free Lie algebra generated by $\mathscr{U}_{-1}$ [2]. Let $\mathscr{U}_{-m}$ be the subspace of $\mathscr{U}_{-}$ which is spanned by elements of the form $a_1 \circ \cdots \circ a_m$, where $a_1, \cdots, a_m \in \mathscr{U}_{-1}$. Then one can write $\mathscr{U}_{-}$ in the form of a GLA:

$$\mathscr{U}_{-} = \sum_{i=1}^{\infty} \mathscr{U}_{-i}. \tag{3.1}$$

Let $\mathscr{U}_{-1}^{*}$ be the dual space of $\mathscr{U}_{-1}$ and let

$$\mathscr{U}_n = (\otimes^{n+1} \mathscr{U}_{-1}^{*}) \otimes \mathscr{U}_{-1}, \tag{3.2}$$

whose elements are viewed as $\mathscr{U}_{-1}$-valued $(n + 1)$-linear operators on $\mathscr{U}_{-1}$. Put

$$\mathscr{U}_{+} = \sum_{i=0}^{\infty} \mathscr{U}_i, \tag{3.3}$$

$$\mathscr{U} = \mathscr{U}_{-} + \mathscr{U}_{+} = \sum_{i=-\infty}^{\infty} \mathscr{U}_i.$$

Then it is known [6] that, with respect to suitably defined bracket relations, $\mathscr{U}$ becomes a GLA in which $\mathscr{U}_{-}$ is a graded subalgebra. The GLA is called the *universal graded Lie algebra* (or simply UGLA) generated by $\mathscr{U}_{-1}$. The assignment $\mathscr{U}_{-1} \mapsto \mathscr{U}$ has a functorial property in the following sense: Let φ be a linear isomorphism of $\mathscr{U}_{-1}$ onto another vector space $\mathscr{V}_{-1}$. Then it is easy to see that φ naturally extends to a grade-preserving Lie isomorphism $\hat{\varphi}$ of the UGLA $\mathscr{U}$ generated by $\mathscr{U}_{-1}$ onto the UGLA $\mathscr{V}$ generated by $\mathscr{V}_{-1}$. Here $\hat{\varphi}|_{\mathscr{U}_{-}}$ is the mapping, induced by φ, between the free Lie algebras $\mathscr{U}_{-}$ and $\mathscr{V}_{-}$; $\hat{\varphi}|_{\mathscr{U}_{+}}$ comes from the mapping, induced by φ, between the tensor algebras over $\mathscr{U}_{-1}$ and over $\mathscr{V}_{-1}$.

DEFINITION 3.1. A GLA $U = \sum_{i=-\infty}^{\infty} U_i$ is said to be *of type* α if it is of type α_0 and if $U_{+} = \sum_{i\geqslant 0} U_i$ contains no ideal of U expect (0) and $\sum_{i\geqslant 2} U_{-i}$ contains no graded ideal of U other than (0).

DEFINITION 3.2. Let $V = \sum_{i=-\infty}^{\infty} V_i$ be a GLA. The subspace $V_{-1} + V_0 + V_1$ is called the *local part* of V and is denoted by $\operatorname{loc}(V)$. Let $U = \sum_{i=-\infty}^{\infty} U_i$ be another GLA. A linear map φ of $\operatorname{loc}(V)$ into $\operatorname{loc}(U)$ is called a *homomorphism* between the local parts, if it satisfies

$$\varphi([x_i, x_j]) = [\varphi(x_i), \varphi(x_j)] \qquad x_i \in V_i,\ x_j \in V_j, \tag{3.4}$$

where $(i, j) = (0, 0), (-1, 0), (-1, 1)$ and $(0, 1)$. Moreover, if φ is bijective, then it is called an *isomorphism.*

3.2. Let U_{-1} be a (finite-dimensional) vector space and B: $U_{-1} \times U_{-1} \times U_{-1} \to U_{-1}$ be a trilinear mapping. Then the pair (U_{-1}, B) (sometimes denoted by B for brevity) is called a triple system. We shall often write (xyz) instead of $B(x, y, z)$. A triple system (U_{-1}, B) is called a *generalized Jordan triple system* (or shortly GJTS), if the equality

$$(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz)) \tag{3.5}$$

is valid for $u, v, x, y, z \in U_{-1}$. Furthermore, if the additional condition

$$(xyz) = (zyx) \qquad x, y, z \in U_{-1} \tag{3.6}$$

is satisfied, then B is called a *Jordan triple system* (or simply JTS).

DEFINITION 3.3. Let (U_{-1}, B) and (V_{-1}, B') be two GJTS's. We say that a linear map φ of U_{-1} into V_{-1} is a *homomorphism* if φ satisfies

$$\varphi(B(x, y, z)) = B'(\varphi(x), \varphi(y), \varphi(z)) \qquad x, y, z \in U_{-1}\,. \tag{3.7}$$

Moreover, if φ is bijective, then φ is called an *isomorphism*. In this case, (U_{-1}, B) and (V_{-1}, B') are said to be *isomorphic*.

DEFINITION 3.4. Let (U_{-1}, B) be a GJTS. A subspace V of U_{-1} is called an *ideal* (resp. *K-ideal*) if

$$\begin{gathered} B(V, U_{-1}, U_{-1}) + B(U_{-1}, V, U_{-1}) + B(U_{-1}, U_{-1}, V) \subset V \\ (\text{resp. } B(V, U_{-1}, U_{-1}) + B(U_{-1}, U_{-1}, V) \subset V) \end{gathered} \tag{3.8}$$

is valid. (U_{-1}, B) is called *simple* (resp. *K-simple*), if B is not a zero map and if (U_{-1}, B) has no non-trivial ideal (resp. K-ideal). We say that (U_{-1}, B) satisfies the *condition* (A) if $B(U_{-1}, a, U_{-1}) = 0$ implies $a = 0$.

Obviously K-simplicity implies simplicity, but the converse is not always true (cf. [8], [1]). It is known [1] that simplicity implies the condition (A).

Now let (U_{-1}, B) be a GJTS and let $\mathscr{U} = \sum_{i=-\infty}^{\infty} \mathscr{U}_i$ be the UGLA generated by $\mathscr{U}_{-1} := U_{-1}$. We put

$$\begin{aligned} L_{ab}(x) &= B(a, b, x) = (abx)\,, \\ R_{ab}(x) &= B(x, a, b) = (xab)\,, \\ B_a(x, y) &= B(x, a, y) = (xay)\,. \end{aligned} \tag{3.9}$$

Note that $L_{ab}, R_{ab} \in \mathscr{U}_0$ and $B_a \in \mathscr{U}_1$. Let U_1 be the subspace of $\mathscr{U}_1$ consisting of all operators B_a, $a \in U_{-1}$, and let U_0 be the subspace of $\mathscr{U}_0$

spanned by operators L_{ab}, where $a, b \in U_{-1}$. In the UGLA $\mathscr{U}$ we have

$$[B_a, b] = L_{ba} \qquad a, b \in U_{-1}, \tag{3.10}$$

$$[L_{ab}, B_c] = -B_{(bac)}, \tag{3.11}$$

$$[L_{ab}, L_{cd}] = L_{(abc)d} - L_{c(bad)}. \tag{3.12}$$

Hence we get

$$[U_{-1}, U_1] = U_0, \quad [U_0, U_0] \subset U_0, \quad [U_0, U_1] \subset U_1. \tag{3.13}$$

Let $\mathscr{L}_0(B)$ be the (graded) subalgebra of $\mathscr{U}$ generated by the subspaces U_{-1} and U_1. $\mathscr{L}_0(B)$ can be written as

$$\mathscr{L}_0(B) = \mathscr{U}_- + \sum_{i \geqslant 0} U_i. \tag{3.14}$$

It is of type α_0. Furthermore it can be seen (Lemma 5 [6]) that $\sum_{i \geqslant 0} U_i$ contains no ideal of $\mathscr{L}_0(B)$ other than zero. Let D be a maximal graded ideal of $\mathscr{L}_0(B)$ contained in $\sum_{i \geqslant 2} \mathscr{U}_{-i}$. Note that such an ideal D is unique. We define the GLA $\mathscr{L}(B) = \sum_{i=-\infty}^{\infty} V_i$ to be

$$\mathscr{L}(B) = \mathscr{L}_0(B)/D, \tag{3.15}$$

which is of type α (not necessarily of finite dimension) [7]. $\mathscr{L}(B)$ is uniquely determined by the given GJTS (U_{-1}, B). We call $\mathscr{L}(B)$ the *Kantor algebra* for B. By the definition, the subalgebra $\mathscr{L}(B)_+ = \sum_{i \geqslant 0} V_i$ is canonically isomorphic to $\mathscr{L}_0(B)_+ = \sum_{i \geqslant 0} U_i$. So, in the sequel, we will regard $\mathscr{L}(B)_+$ as a subalgebra of $\mathscr{U}$ via the above isomorphism. We need the following

THEOREM 3.5 ([7], [6]). (i) *Let (U_{-1}, B) be a GJTS. If (U_{-1}, B) is K-simple and if* $\dim \mathscr{L}(B) < \infty$, *then the Lie algebra $\mathscr{L}(B)$ is simple.* (ii) *Conversely, let $V = \sum_{i=-\infty}^{\infty} V_i$ be a simple GLA of type α_0 and let τ be a grade-reversing involutive automorphism of V. If we define a trilinear map B_τ by*

$$B_\tau(y, x, z) = [[\tau(x), y], z], \qquad x, y, z \in V_{-1}, \tag{3.16}$$

then (V_{-1}, B_τ) is a K-simple GJTS.

3.3. In this paragraph, we construct a grade-reversing involutive automorphism τ_B of $\mathscr{L}(B)$. This was done originally by Kantor [7], but the proof given there is rather sketchy; so we give it rigorously by relaxing the condition "center-free" to the condition (A). Let (U_{-1}, B) be a GJTS satisfying the condition (A), and let $\mathscr{L}(B) = \sum_{i=-\infty}^{\infty} U_i$. Consider the mapping $\tau\colon U_{-1} \to U_1$:

$$\tau(a) = B_a . \tag{3.17}$$

The condition (A) implies that τ is a linear isomorphism of U_{-1} onto U_1. Let $\mathscr{L}'(B) = \sum_{i=-\infty}^{\infty} V_i$ be the GLA which is obtained from $\mathscr{L}(B)$ by reversing the gradation, that is, by putting $V_i = U_{-i}$ for each i. Let $\mathscr{U} = \sum_{i=-\infty}^{\infty} \mathscr{U}_i$ and $\mathscr{V} = \sum_{i=-\infty}^{\infty} \mathscr{V}_i$ be the UGLA's generated by $\mathscr{U}_{-1} = U_{-1}$ and $\mathscr{V}_{-1} = V_{-1} = U_1$, respectively. Let us consider the two subalgebras $\mathscr{L}(B)^{(1)} = \sum_{i\geqslant 1} U_i \subset \mathscr{L}(B)$ and $\mathscr{L}'(B)^{(1)} = \sum_{i\geqslant 1} V_i$. $\mathscr{L}(B)^{(1)}$ is viewed as a graded subalgebra of $\mathscr{U}$ (cf. the statement just before Theorem 3.5). Let $a \in V_i$ $(i \geqslant 1)$ and let

$$F(a)(x_1, \cdots, x_{i+1}) = a \circ x_1 \circ \cdots \circ x_{i+1} , \tag{3.18}$$

where $x_1, \cdots, x_{i+1} \in V_{-1} = \mathscr{V}_{-1}$. Then $F(a)$ is a V_{-1}-valued $(i+1)$-linear form on V_{-1}, and hence $F(a) \in \mathscr{V}_i$. We extend F linearly to the whole $\mathscr{L}'(B)^{(1)}$.

LEMMA 3.6. *The mapping F is an injective grade-preserving homomorphism of $\mathscr{L}'(B)^{(1)}$ into $\mathscr{V}_+ = \sum_{i\geqslant 0} \mathscr{V}_i$.*

Proof. It is known [6] that F is a grade-preserving homomorphism. We claim first that $F|_{V_1}$ is injective. Let $a \in V_1$ and suppose that $F(a) = 0$. Then, from (3.10), (3.11), and (3.18), we have

$$0 = F(a)(\tau(u), \tau(v)) = [[a, B_u], B_v] = B_{(uav)} = \tau((uav)) \tag{3.19}$$

for $\tau(u), \tau(v) \in \mathscr{V}_{-1} = V_{-1} = U_1$. Since $\tau\colon U_{-1} \to U_1$ is a linear isomorphism, we have that $(uav) = 0$ for all $u, v \in U_{-1}$, and hence, by the condition (A), $a = 0$. Therefore Ker F is contained in $\sum_{i\geqslant 2} V_i$. It can be seen (cf. the proof of Lemma 4 [6]) that Ker F is an ideal of the whole $\mathscr{L}'(B)$, that is, Ker F is an ideal of $\mathscr{L}(B)$ contained in $\sum_{i\leqslant -2} U_i$. Since $\mathscr{L}(B)$ is of type α, we have Ker $F = (0)$.

By Lemma 3.6, we may identify $\mathscr{L}'(B)^{(1)}$ with its F-image in $\mathscr{V}_+$. The linear isomorphism τ of $\mathscr{U}_{-1}$ onto $\mathscr{V}_{-1}$ naturally extends to a grade-preserving isomorphism $\hat{\tau}$ of $\mathscr{U}$ onto $\mathscr{V}$ (cf. 3.1).

LEMMA 3.7. *$\hat{\tau}|_{U_1} = \tau^{-1}$ is valid. Furthermore $\hat{\tau}$ sends $\mathscr{L}(B)^{(1)}$ to $\mathscr{L}'(B)^{(1)}$.*

Proof. Choose $B_a \in U_1 \subset \mathscr{U}_1$ $(a \in U_{-1})$ and let $\hat{\tau}(B_a) = B' \in \mathscr{V}_1$. Then, by the naturality of $\hat{\tau}$ we have

$$B'(\tau(u), \tau(v)) = \tau(B_a(u, v)) \qquad u, v \in U_{-1}, \tag{3.20}$$

from which it follows that

$$B'(B_u, B_v) = \tau((uav)) = B_{(uav)} = F(a)(B_u, B_v) = a(B_u, B_v). \tag{3.21}$$

B_u and B_v being arbitrary, we get $\hat{\tau}(B_a) = B' = a$, which implies that $\hat{\tau}|_{U_1} = \tau^{-1}$. Hence $\hat{\tau}(U_1) = \tau^{-1}(U_1) = U_{-1} = V_1$. Since $\mathscr{L}(B)$ is of type α, the subalgebra $\mathscr{L}(B)^{(1)}$ (resp. $\mathscr{L}'(B)^{(1)}$) is generated by U_1 (resp. V_1) in $\mathscr{U}$ (resp. $\mathscr{V}$). Therefore we conclude that $\hat{\tau}$ sends $\mathscr{L}(B)^{(1)}$ to $\mathscr{L}'(B)^{(1)}$.

Now we define τ_B as

$$\tau_B = \begin{cases} \hat{\tau} & \text{on } \mathscr{L}(B)^{(1)}, \\ \hat{\tau}^{-1} & \text{on } \mathscr{L}(B)_- := \sum_{i \leqslant -1} U_i = \mathscr{L}'(B)^{(1)}. \end{cases} \tag{3.22}$$

From Lemma 3.7, we have that τ_B is an involutive linear endomorphism of $\mathscr{L}(B)_- + \mathscr{L}(B)^{(1)}$ and that τ_B is a Lie homomorphism both on $\mathscr{L}(B)^{(1)}$ and on $\mathscr{L}(B)_-$. We extend τ_B to an involutive linear endomorphism (denoted again by τ_B) of the whole $\mathscr{L}(B)$ by putting

$$\tau_B(L_{ab}) = -L_{ba} \qquad L_{ab} \in U_0. \tag{3.23}$$

The following proposition is a variation of Proposition 6′ in Kantor [7], in which we relax an assumption in the original one.

PROPOSITION 3.8. *Let* (U_{-1}, B) *be a GJTS satisfying the condition* (A). *Then the linear endomorphism* τ_B *defined by* (3.22) *and* (3.23) *is a grade-reversing involutive automorphism of* $\mathscr{L}(B)$.

Proof. By using (3.22) and (3.23), and by following Kantor [7] (p. 428), we can show that τ_B is a homomorphism.

τ_B is called the *grade-reversing canonical involution* of $\mathscr{L}(B)$.

3.4. Let (U_{-1}, B) be a GJTS satisfying the condition (A). If the GLA $\mathscr{L}(B)$ is of the ν-th kind, then we say that (U_{-1}, B) is *of the ν-th kind.* Note that B is a JTS if and only if $\nu = 1$ [7]. The symmetric bilinear form γ_B [19] on U_{-1} defined by

$$\gamma_B(x, y) = \tfrac{1}{2}\operatorname{Tr}(2R_{xy} + 2R_{yx} - L_{xy} - L_{yx}) \tag{3.24}$$

is called the *trace form* of the GJTS (U_{-1}, B). If γ_B is positive definite, then (U_{-1}, B) is said to be *compact*. When $\nu = 1$, our definition of "compactness" is the same as that for JTS's. Suppose that (U_{-1}, B) is a GJTS of the first or the second kind satisfying the condition (A). Then it is known [1] that (U_{-1}, B) is compact if and only if $\mathscr{L}(B)$ is semisimple and τ_B is a Cartan involution. Suppose that (U_{-1}, B) is compact of the first or the second kind. Then (U_{-1}, B) is simple if and only if it is K-simple ([1]).

3.5. In this paragraph we treat infinite-dimensional simple GLA's. Let $\mathfrak{g} = \sum_{i=-\infty}^{\infty} \mathfrak{g}_i$ be a simple GLA of type α_0 and τ be a grade-reversing involutive automorphism of $\mathfrak{g}$. Then $(\mathfrak{g}_{-1}, B_\tau)$ is a K-simple GJTS (cf. Theorem 3.5), and hence it satisfies the condition (A) (cf. 3.2). Therefore the Kantor algebra $\mathscr{L}(B_\tau)$ admits the grade-reversing canonical involution τ_{B_τ}. Consider the UGLA $\mathscr{U} = \sum_{i=-\infty}^{\infty} \mathscr{U}_i$ generated by $\mathscr{U}_{-1} = \mathfrak{g}_{-1}$ and consider the subalgebra $\mathscr{L}_0(B_\tau)$ of $\mathscr{U}$ given in (3.14). We then **have**

LEMMA 3.9. $\operatorname{loc}(\mathfrak{g}) \cong \operatorname{loc}(\mathscr{L}_0(B_\tau))$.

Proof. Let $\mathscr{L}_0(B_\tau) = \mathscr{U}_- + \sum_{i\geqslant 0} U_i$. Let φ_{-1} be the identity map of $\mathfrak{g}_{-1}$ onto $\mathscr{U}_{-1}$. We define φ_1: $\mathfrak{g}_1 \to U_1$ be $\varphi_1(\tau(u)) = (B_\tau)_u$ for $u \in \mathfrak{g}_{-1}$. Then, since $\tau_{B_\tau}(u) = (B_\tau)_u$ (cf. (3.22)), we have $\varphi_1 = \tau_{B_\tau}\tau^{-1}$. τ_{B_τ} is a bijection of $\mathfrak{g}_{-1} = \mathscr{U}_{-1}$ to U_1, and so φ_1 is a linear isomorphism of $\mathfrak{g}_1$ onto U_1. Since $\mathfrak{g}$ is simple, we easily see $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$. We define φ_0: $\mathfrak{g}_0 \to U_0$ by putting $\varphi_0([\tau(u), v]) = L_{vu}$ $(u, v \in \mathfrak{g}_{-1})$. We claim that φ_0 is a bijection. First note that the representation ρ: $\mathfrak{g}_0 \to \mathrm{ad}_{\mathfrak{g}_{-1}}\mathfrak{g}_0$ is faithful. Indeed, suppose the contrary; choose a non-zero element $x \in \mathfrak{g}_0$ such that $[x, \mathfrak{g}_{-1}] = 0$. Then $\sum_{k,l\geqslant 0} (\mathrm{ad}\, \mathfrak{g}_1)^k (\mathrm{ad}\, \mathfrak{g}_0)^l x$ is a non-zero ideal of $\mathfrak{g}$. But this is impossible, since $\mathfrak{g}$ is simple. That φ_0 is surjective is trivial. Choose an element $a = \sum_{i=1}^{s} \lambda_i[\tau(u_i), v_i] \in \mathfrak{g}_0$ $(\lambda_i \in \boldsymbol{R})$, and suppose $\varphi_0(a) = 0$. Then, for every $x \in \mathfrak{g}_{-1} = \mathscr{U}_{-1}$, we have $[a, x] = \sum_i \lambda_i [[\tau(u_i), v_i], x] = \sum_i \lambda_i B_\tau(v_i, u_i, x) = \sum_i \lambda_i L_{v_i u_i}(x) = \varphi_0(a)x = 0$. Since ρ is faithful, we get $a = 0$. Thus φ_0 is a bijection. We shall prove that the linear bijection $\varphi = \varphi_{-1} \times \varphi_0 \times \varphi_1$ is an isomorphism of $\operatorname{loc}(\mathfrak{g})$ onto $\operatorname{loc}(\mathscr{L}_0(B_\tau))$. Let $x \in \mathfrak{g}_{-1}$ and $y = [\tau(u), v] \in \mathfrak{g}_0$, where $u, v \in \mathfrak{g}_{-1}$. Then we have

$$\begin{aligned}\varphi_{-1}([y, x]) &= [y, x] = [[\tau(u), v], x] = B_\tau(v, u, x) = L_{vu}(x)\\ &= [L_{vu}, x] = [\varphi_0(y), \varphi_{-1}(x)]\,.\end{aligned}$$

Let $u, v \in \mathfrak{g}_{-1}$. Then, for $x \in \mathfrak{g}_{-1} = \mathscr{U}_{-1}$ we get

$$[\varphi_1(\tau(u)), \varphi_{-1}(v)](x) = [(B_\tau)_u, v](x) = B_\tau(v, u, x) = L_{vu}(x) = \varphi_0([\tau(u), v])(x) .$$

Let $w \in \mathfrak{g}_{-1}$. Then we have

$$\varphi_1([y, \tau(w)]) = \varphi_1([[\tau(u), v], \tau(w)]) = \varphi_1(\tau([[u, \tau(v)], w])) = (B_\tau)_{[[u,\tau(v)],w]} .$$

On the other hand, by using (3.11), we get

$$[\varphi_0(y), \varphi_1(\tau(w))] = [\varphi_0([\tau(u), v]), \varphi_1(\tau(w))] = [L_{vu}, (B_\tau)_w] = -(B_\tau)_{(uvw)} .$$

Furthermore $-(uvw) = -B_\tau(u, v, w) = [[u, \tau(v)], w]$. Thus we have proved $\varphi_1([y, \tau(w)]) = [\varphi_0(y), \varphi_1(\tau(w))]$. Using the above equalities, one can show that $\varphi_0([x, y]) = [\varphi_0(x), \varphi_0(y)]$ for $x, y \in \mathfrak{g}_0$.

LEMMA 3.10. *Let $\mathfrak{g} = \sum_{i=-\infty}^{\infty} \mathfrak{g}_i$ be a real simple GLA of type α_0. Let τ be a grade-reversing involutive automorphism of $\mathfrak{g}$, and let $\mathscr{L}(B_\tau)$ be the Kantor algebra for the GJTS $(\mathfrak{g}_{-1}, B_\tau)$. Then there exists a grade-preserving isomorphism φ of $\mathfrak{g}$ onto $\mathscr{L}(B_\tau)$ such that*

$$\varphi\tau = \tau_{B_\tau}\varphi . \tag{3.25}$$

Proof. Let $\mathscr{U} = \sum_{i=-\infty}^{\infty} \mathscr{U}_i$ be the UGLA such that $\mathscr{U}_{-1} = \mathfrak{g}_{-1}$. For a GLA $\mathfrak{h} = \sum_{i=-\infty}^{\infty} \mathfrak{h}_i$, we put $\mathfrak{h}_+ = \sum_{i\geqslant 0} \mathfrak{h}_i$ and $\mathfrak{h}_- = \sum_{i\leqslant -1} \mathfrak{h}_i$. We define a map F of $\mathfrak{g}_+$ to $\mathscr{U}_+$ quite analogously as in (3.18). Note that (3.18) is meaningful for $i = 0$. Since $\mathfrak{g}$ is simple of type α_0, the map F is an injective grade-preserving homomorphism [6]. An easy computation shows that F coincides with $\varphi_0 \times \varphi_1$ on $\mathfrak{g}_0 + \mathfrak{g}_1$. Let $\mathscr{L}(B_\tau) = \sum_{i=-\infty}^{\infty} U_i$. We identify $\mathscr{L}(B_\tau)_+$ with $\mathscr{L}_0(B_\tau)_+ \subset \mathscr{U}_+$. Then, as is seen in the proof of Lemma 3.9, we have $F(\mathfrak{g}_0 + \mathfrak{g}_1) = U_0 + U_1$. Considering that $\mathfrak{g}$ and $\mathscr{L}(B_\tau)$ are both of type α_0, we conclude that the GLA's $\mathfrak{g}_+$ and $\mathscr{L}(B_\tau)_+$ are isomorphic under F. The map $F' := \tau_{B_\tau}F\tau$ is a grade-preserving isomorphism of $\mathfrak{g}_-$ onto $\mathscr{L}(B_\tau)_-$. We define a map φ to be F on $\mathfrak{g}_+$ and to be F' on $\mathfrak{g}_-$. Then φ is a grade-preserving linear isomorphism of $\mathfrak{g}$ onto $\mathscr{L}(B_\tau)$. φ is an isomorphism between $\operatorname{loc}(\mathfrak{g})$ and $\operatorname{loc}(\mathscr{L}(B_\tau))$, and it is also bracket-preserving on $\mathfrak{g}_+$ and on $\mathfrak{g}_-$. By using these properties we can conclude inductively that φ is bracket-preserving on the whole $\mathfrak{g}$. In order to show (3.25), it is enough to verify it for an element of $\operatorname{loc}(\mathfrak{g})$. Let $a = [\tau(u), v] \in \mathfrak{g}_0$, where $u, v \in \mathfrak{g}_{-1}$. In view of (3.23) and the definition of φ, we have

$\varphi\tau([\tau(u), v]) = -\varphi([\tau(v), u]) = -L_{uv} = \tau_{B_\tau}(L_{vu}) = \tau_{B_\tau}\varphi([\tau(u), v])$. We can easily see (3.25) for the case $a \in \mathfrak{g}_1$ or $a \in \mathfrak{g}_{-1}$.

LEMMA 3.11. *Let (U_{-1}, B) and (U'_{-1}, B') be two GJTS's satisfying the condition* (A). *Then an isomorphism ψ of (U_{-1}, B) onto (U'_{-1}, B') induces a grade-preserving isomorphism $\mathscr{L}(\psi)$ of $\mathscr{L}(B)$ onto $\mathscr{L}(B')$. Furthermore*

$$\mathscr{L}(\psi)\tau_B = \tau_{B'}\mathscr{L}(\psi)\,, \tag{3.26}$$

where τ_B and $\tau_{B'}$ are the grade-reversing canonical involutions of $\mathscr{L}(B)$ and $\mathscr{L}(B')$ respectively.

Proof. Let $\mathscr{U} = \sum_{i=-\infty}^{\infty}\mathscr{U}_i$ and $\mathscr{U}' = \sum_{i=-\infty}^{\infty}\mathscr{U}'_i$ be the UGLA's such that $\mathscr{U}_{-1} = U_{-1}$ and $\mathscr{U}'_{-1} = U'_{-1}$. ψ extends to an isomorphism $\hat{\psi}$ of $\mathscr{U}$ onto $\mathscr{U}'$ (cf. 3.1). Let $\mathscr{L}_0(B) = \mathscr{U}_- + \sum_{i\geqslant 0} U_i$ and $\mathscr{L}_0(B') = \mathscr{U}'_- + \sum_{i\geqslant 0} U'_i$ be the subalgebras in (3.14). All objects in $\operatorname{loc}(\mathscr{L}_0(B'))$ are denoted by the same notations as the corresponding ones in $\operatorname{loc}(\mathscr{L}_0(B))$ but with primes. Since ψ is an isomorphism between the two GJTS's, we have $(\hat{\psi}(B_a))(\psi(u), \psi(v)) = \psi(B_a(u, v)) = B'_{\psi(a)}(\psi(u), \psi(v))$, $u, v \in U_{-1}$. This implies $\hat{\psi}(B_a) = B'_{\psi(a)}$. Also we get $\hat{\psi}(L_{ab}) = L'_{\psi(a)\psi(b)}$, $a, b \in U_{-1}$. These arguments show that $\hat{\psi}$ sends $\operatorname{loc}(\mathscr{L}_0(B))$ to $\operatorname{loc}(\mathscr{L}_0(B'))$. Noting that $\mathscr{L}_0(B)$ and $\mathscr{L}_0(B')$ are of type α_0, we have that $\hat{\psi}$ sends $\mathscr{L}_0(B)$ to $\mathscr{L}_0(B')$. Therefore $\hat{\psi}$ induces a grade-preserving isomorphism $\mathscr{L}(\psi)$ of $\mathscr{L}(B)$ onto $\mathscr{L}(B')$ (cf. (3.15)). To see (3.26) it suffices to check it on $\operatorname{loc}(\mathscr{L}(B))$. Let us identify $\operatorname{loc}(\mathscr{L}(B))$ with $\operatorname{loc}(\mathscr{L}_0(B))$ etc. For $L_{ab} \in U_0$, we have $\mathscr{L}(\psi)\tau_B(L_{ab}) = \mathscr{L}(\psi)(-L_{ba}) = -L'_{\psi(b)\psi(a)} = \tau_{B'}(L'_{\psi(a)\psi(b)}) = \tau_{B'}\mathscr{L}(\psi)(L_{ab})$. The remaining cases are also easily derived.

THEOREM 3.12. *Let $\mathfrak{g} = \sum_{i=-\infty}^{\infty}\mathfrak{g}_i$ and $\mathfrak{g}' = \sum_{i=-\infty}^{\infty}\mathfrak{g}'_i$ be two real simple GLA's of type α_0. Let τ and τ' be grade-reversing involutive automorphisms of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. Then the GJTS's $(\mathfrak{g}_{-1}, B_\tau)$ and $(\mathfrak{g}'_{-1}, B_{\tau'})$ are isomorphic if and only if there exists a grade-preserving isomorphism θ of $\mathfrak{g}$ onto $\mathfrak{g}'$ such that*

$$\theta\tau = \tau'\theta\,. \tag{3.27}$$

Proof. As is seen from what was pointed out at the beginning of this paragraph, $\mathscr{L}(B_\tau)$ and $\mathscr{L}(B_{\tau'})$ admit the grade-reversing canonical involutions τ_{B_τ} and $\tau_{B_{\tau'}}$, respectively. Suppose that there exists an isomorphism ψ of $(\mathfrak{g}_{-1}, B_\tau)$ onto $(\mathfrak{g}'_{-1}, B_{\tau'})$. By Lemma 3.11, we obtain the grade-preserving isomorphism $\mathscr{L}(\psi)$ of $\mathscr{L}(B_\tau)$ onto $\mathscr{L}(B_{\tau'})$ satisfying

$$\mathscr{L}(\psi)\tau_{B_\tau} = \tau_{B_{\tau'}}\mathscr{L}(\psi)\,. \tag{3.28}$$

On the other hand, by Lemma 3.10, one can find grade-preserving isomorphisms $\varphi\colon \mathfrak{g} \to \mathscr{L}(B_\tau)$ and $\varphi'\colon \mathfrak{g}' \to \mathscr{L}(B_{\tau'})$ which satisfy the conditions $\varphi\tau = \tau_{B_\tau}\varphi$ and $\varphi'\tau' = \tau_{B_{\tau'}}\varphi'$. Consequently the composite map $\theta = \varphi'^{-1}\mathscr{L}(\psi)\varphi$ is seen to be the desired one. The converse assertion is easily seen.

3.6. We apply the results in 3.5 to the finite-dimensional case.

DEFINITION 3.13. Let $\mathfrak{g} = \sum_k \mathfrak{g}_k$ be a real simple GLA of the first or the second kind, and τ be a grade-reversing Cartan involution of $\mathfrak{g}$. The pair $(\mathfrak{g}, \tau)$ is called an *admissible pair.* We say that two admissible pairs $(\mathfrak{g}, \tau)$ and $(\mathfrak{g}', \tau')$ are *isomorphic,* if there exists a grade-preserving isomorphism φ of $\mathfrak{g}$ onto $\mathfrak{g}'$ such that $\varphi\tau = \tau'\varphi$.

We have the following classification theorem for compact GJTS's of the first or the second kind.

THEOREM 3.14. *Let $\mathscr{B}$ be the set of isomorphism classes of compact real simple GJTS's of the first or the second kind, and let $\mathscr{A}$ be the set of isomorphism classes of admissible pairs. Then there exists a bijection Ψ of $\mathscr{A}$ onto $\mathscr{B}$.*

Proof. As was mentioned in 3.4, for a compact real GJTS of the first or the second kind, simplicity and K-simplicity are identical. Let $(\mathfrak{g}, \tau)$ be an admissible pair, and let $\mathfrak{g} = \sum_{i=-\nu}^{\nu} \mathfrak{g}_i$ $(\nu = 1, 2)$. The condition $\nu = 1, 2$ implies that $\mathfrak{g}$ is of type α_0 (cf. Lemma 2.8). Hence, by Theorem 3.5 and Lemma 3.10, the pair $(\mathfrak{g}_{-1}, B_\tau)$ is a simple GJTS of the ν-th kind $(\nu = 1, 2)$. Furthermore, since τ is a Cartan involution, $(\mathfrak{g}_{-1}, B_\tau)$ is compact (cf. 3.4). We put

$$\Psi([(\mathfrak{g}, \tau)]) = [(\mathfrak{g}_{-1}, B_\tau)]\,, \tag{3.29}$$

where [ ] denotes an isomorphism class. From Theorem 3.12 it follows that $\Psi\colon \mathscr{A} \to \mathscr{B}$ is well-defined and injective. Now choose an element $[(U_{-1}, B)] \in \mathscr{B}$. Since (U_{-1}, B) is simple of the ν-th kind, $\mathscr{L}(B)$ is simple of the ν-th kind (cf. 3.4 and Theorem 3.5). $\mathscr{L}(B)$ admits the grade-reversing canonical involution τ_B, which is a Cartan involution by the assumption for (U_{-1}, B). Consequently the pair $(\mathscr{L}(B), \tau_B)$ is admissible. Furthermore, for $x, y, z \in U_{-1}$ we have

$$B(y, x, z) = B_x(y, z) = [[B_x, y], z] = [[\tau_B(x), y], z]\,, \tag{3.30}$$

which implies $B = B_{\tau_B}$. Therefore we get $\Psi([(\mathscr{L}(B), \tau_B)]) = [(U_{-1}, B)]$.

§ 4. Classification of compact classical simple GJTS's

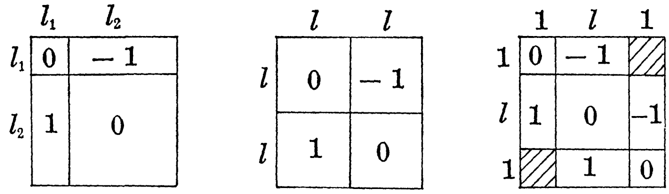
4.1. A real GJTS (U_{-1}, B) is called *classical simple*, if $\mathscr{L}(B)$ is classical simple. In order to classify all compact classical simple GJTS's of the first or the second kind, we have to determine the set $\mathscr{A}$ (cf. Theorem 3.14). This will be carried out by

(4.1) finding all gradations of the ν-th kind ($\nu = 1, 2$) in each real simple Lie algebra $\mathfrak{g}$ up to isomorphisms and by

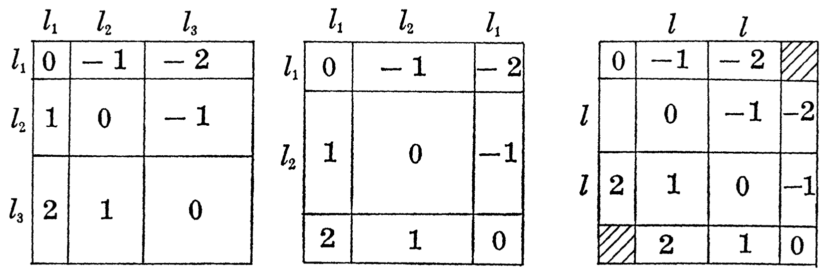
(4.2) classifying all grade-reversing Cartan involutions for each gradation in $\mathfrak{g}$, up to conjugacy under automorphisms of the gradation.

(4.2) has been settled in Lemma 1.4; there exists a single conjugacy class of grade-reversing Cartan involutions for each gradation. (4.1) will be settled in 4.2 by using the results in 2.3. The next task is to determine $\mathfrak{g}_{-1}$ explicitly for each gradation $(\mathfrak{g}_k)$ and to find explicitly a grade-reversing Cartan involution. Thus we will be able to compute the GJTS's B_τ we are seeking (cf. (3.29)), by means of (3.16).

4.2. We refer to Takeuchi [15] for the realizations of classical real simple Lie algebras $\mathfrak{g}$ and the choices of the maximal abelian subspaces $\mathfrak{a}$, on which the root systems Δ in 2.3 are defined. We can then compute a root vector corresponding to each $\alpha \in \Delta$; by virtue of (1.23), (1.24) and



$(\mathrm{I}1)_{l_1, l_2}$ $(\mathrm{I}2)_l$ $(\mathrm{I}3)_l$



$(\mathrm{II}1)_{l_1, l_2, l_3}$ $(\mathrm{II}2)_{l_1, l_2}$ $(\mathrm{II}3)_l$

2.3 one can find all the graded subspaces of $\mathfrak{g}$. It turns out by case-by-case checking that, for classical real simple Lie algebras $\mathfrak{g}$, every gradation of the first or the second kind falls into one of the six types given in the page 106. For each diagram above the big square indicates an element of $\mathfrak{g}$, a matrix. The integer k put in each divided portion indicates that that portion lies in the graded subspace $\mathfrak{g}_k$. The number beside each edge denotes the size of the portion. A shaded portion indicates that that portion does not belong to $\mathfrak{g}$.

We use the following notations for the matrices: $X^* = {}^t\overline{X}$, where X is a real or complex matrix. E_l denotes the unit matrix of degree l. $J_l = (a_{ij})$ is the $l \times l$ matrix with $a_{ij} = \delta_{i,l+1-j}$.

$$\tilde{J}_l = \begin{pmatrix} 0 & J_l \\ -J_l & 0 \end{pmatrix}, \quad \hat{J}_l = E_l \otimes \tilde{J}_1, \quad J'_l = J_l \otimes E_2,$$

$$K_1 = \operatorname{diag}(\sqrt{-1}, -\sqrt{-1}), \ K_l = J_l \otimes K_1,$$

$$A_{p,q} = \begin{pmatrix} 0 & 0 & J_p \\ 0 & E_{q-p} & 0 \\ J_p & 0 & 0 \end{pmatrix} (p \leqslant q), \qquad A'_{p,q} = A_{p,q} \otimes E_2.$$

We also use the following notations for the vector spaces: $M_{p,q}(\boldsymbol{K})$ denotes the vector space of $p \times q$ matrices with entries in $\boldsymbol{K}$, where $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$.

$M_{p,q}(\boldsymbol{H}) = \{X \in M_{2p,2q}(\boldsymbol{C})\colon \overline{X}\hat{J}_q = \hat{J}_p X\}$,

$SH'_n(\boldsymbol{C}) = \{X \in \mathfrak{gl}(n, \boldsymbol{C})\colon J_n X^* J_n = -X\}$,

$SH'_n(\boldsymbol{H}) = \{X \in M_{n,n}(\boldsymbol{H})\colon J'_n X^* J'_n = -X\}$,

$Alt'_n(\boldsymbol{K}) = \{X \in \mathfrak{gl}(n, \boldsymbol{K})\colon J_n {}^tX J_n = -X\}$, $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$,

$H'_n(\boldsymbol{H}) = \{X \in \mathfrak{gl}(2n, \boldsymbol{C})\colon \overline{X}\hat{J}_n = \hat{J}_n X,\ X^*K_n + K_n X = 0\}$,

$Sym'_n(\boldsymbol{K}) = \{X \in \mathfrak{gl}(n, \boldsymbol{K})\colon J_n {}^tX J_n = X\}$, $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$.

Table I (resp. Table II) is the list of all possible gradations of the first (resp. second) kind (up to isomorphisms) in classical real simple Lie algebras $\mathfrak{g}$ and the corresponding graded subspaces $\mathfrak{g}_{-1}$. Note that every gradation of the first or the second kind is determined by Π_1 (cf. 2.3).

Let $(\mathfrak{g}_k)$ be an arbitrary gradation in a real simple Lie algebra $\mathfrak{g}$ listed in Table I or II. It is easy to see that if we put $\tau(X) = -X^*$ $(X \in \mathfrak{g})$, then τ is a grade-reversing Cartan involution of the GLA $\mathfrak{g} = \sum_k \mathfrak{g}_k$. Table III (resp. Table IV) is the list of JTS's (resp. GJTS's of the second

Table I

	$\mathfrak{g}$	Π_1	gradation	$\mathfrak{g}_{-1}$
1	$\mathfrak{sl}(n, \boldsymbol{C}),\ n \geqslant 3$	$\{\alpha_p\},\ 1 \leqslant p \leqslant [\frac{n}{2}]$	$(\mathrm{I1})_{p,n-p}$	$M_{p,n-p}(\boldsymbol{C})$
2	$\mathfrak{sl}(n, \boldsymbol{R}),\ n \geqslant 3$	$\{\alpha_p\},\ 1 \leqslant p \leqslant [\frac{n}{2}]$	$(\mathrm{I1})_{p,n-p}$	$M_{p,n-p}(\boldsymbol{R})$
3	$\mathfrak{sl}(n, \boldsymbol{H}),\ n \geqslant 3$	$\{\alpha_p\},\ 1 \leqslant p \leqslant [\frac{n}{2}]$	$(\mathrm{I1})_{2p,2n-2p}$	$M_{p,n-p}(\boldsymbol{H})$
4	$\mathfrak{so}(2n+1, \boldsymbol{C}),\ n \geqslant 2$	$\{\alpha_1\}$	$(\mathrm{I3})_{2n-1}$	$M_{1,2n-1}(\boldsymbol{C})$
5	$\mathfrak{so}(p, q),\ 1 \leqslant p < q$	$\{\alpha_1\}$	$(\mathrm{I3})_{p+q-2}$	$M_{1,p+q-2}(\boldsymbol{R})$
6	$\mathfrak{sp}(n, \boldsymbol{C}),\ n \geqslant 3$	$\{\alpha_n\}$	$(\mathrm{I2})_n$	$Sym'_n(\boldsymbol{C})$
7	$\mathfrak{sp}(n, \boldsymbol{R}),\ n \geqslant 3$	$\{\alpha_n\}$	$(\mathrm{I2})_n$	$Sym'_n(\boldsymbol{R})$
8	$\mathfrak{su}(n, n),\ n \geqslant 3$	$\{\alpha_n\}$	$(\mathrm{I2})_n$	$SH'_n(\boldsymbol{C})$
9	$\mathfrak{sp}(n, n),\ n \geqslant 2$	$\{\alpha_n\}$	$(\mathrm{I2})_{2n}$	$SH'_n(\boldsymbol{H})$
10	$\mathfrak{so}^*(4n),\ n \geqslant 3$	$\{\alpha_n\}$	$(\mathrm{I2})_{2n}$	$H'_n(\boldsymbol{H})$
11	$\mathfrak{so}(2n, \boldsymbol{C}),\ n \geqslant 4$	$\{\alpha_1\}$	$(\mathrm{I3})_{2n-2}$	$M_{1,2n-2}(\boldsymbol{C})$
12	$\mathfrak{so}(2n, \boldsymbol{C}),\ n \geqslant 5$	$\{\alpha_n\}$	$(\mathrm{I2})_n$	$Alt'_n(\boldsymbol{C})$
13	$\mathfrak{so}(n, n).\ n \geqslant 4$	$\{\alpha_1\}$	$(\mathrm{I3})_{2n-2}$	$M_{1,2n-2}(\boldsymbol{R})$
14	$\mathfrak{so}(n, n),\ n \geqslant 5$	$\{\alpha_n\}$	$(\mathrm{I2})_n$	$Alt'_n(\boldsymbol{R})$

Table II

	$\mathfrak{g}$	Π_1	gradation	$\mathfrak{g}_{-1}$
1	$\mathfrak{sl}(n, C)$, $n \geqslant 3$	$\{\alpha_p, \alpha_{p+q}\}$, $1 \leqslant p \leqslant [\frac{n}{2}]$, $1 \leqslant q \leqslant n-2p$	$(\text{II}1)_{p,q,n-p-q}$	$M_{p,q}(C) \times M_{q,n-p-q}(C)$
2	$\mathfrak{sl}(n, R)$, $n \geqslant 3$	$\{\alpha_p, \alpha_{p+q}\}$, $1 \leqslant p \leqslant [\frac{n}{2}]$, $1 \leqslant q \leqslant n-2p$	$(\text{II}1)_{p,q,n-p-q}$	$M_{p,q}(R) \times M_{q,n-p-q}(R)$
3	$\mathfrak{sl}(n, H)$, $n \geqslant 3$	$\{\alpha_p, \alpha_{p+q}\}$, $1 \leqslant p \leqslant [\frac{n}{2}]$, $1 \leqslant q \leqslant n-2p$	$(\text{II}1)_{2p,2q,2n-2p-2q}$	$M_{p,q}(H) \times M_{q,n-p-q}(H)$
4	$\mathfrak{so}(2n+1, C)$, $n \geqslant 2$	$\{\alpha_k\}$, $2 \leqslant k \leqslant n$	$(\text{II}2)_{k,2n-2k+1}$	$M_{k,2n-2k+1}(C)$
5	$\mathfrak{so}(p, q)$, $2 \leqslant p < q$	$\{\alpha_k\}$, $2 \leqslant k \leqslant p$	$(\text{II}2)_{k,p+q-2k}$	$M_{k,p+q-2k}(R)$
6	$\mathfrak{sp}(n, C)$, $n \geqslant 3$	$\{\alpha_k\}$, $1 \leqslant k \leqslant n-1$	$(\text{II}2)_{k,2n-2k}$	$M_{k,2n-2k}(C)$
7	$\mathfrak{sp}(n, R)$, $n \geqslant 3$	$\{\alpha_k\}$, $1 \leqslant k \leqslant n-1$	$(\text{II}2)_{k,2n-2k}$	$M_{k,2n-2k}(R)$
8	$\mathfrak{su}(n, n)$, $n \geqslant 3$	$\{\alpha_k\}$, $1 \leqslant k \leqslant n-1$	$(\text{II}2)_{k,2n-2k}$	$M_{k,2n-2k}(C)$
9	$\mathfrak{sp}(n, n)$, $n \geqslant 2$	$\{\alpha_k\}$, $1 \leqslant k \leqslant n-1$	$(\text{II}2)_{2k,4n-4k}$	$M_{k,2n-2k}(H)$
10	$\mathfrak{so}^*(4n)$, $n \geqslant 3$	$\{\alpha_k\}$, $1 \leqslant k \leqslant n-1$	$(\text{II}2)_{2k,4n-4k}$	$M_{k,2n-2k}(H)$
11	$\mathfrak{su}(p, q)$, $1 \leqslant p < q$	$\{\alpha_k\}$, $1 \leqslant k \leqslant p$	$(\text{II}2)_{k,p+q-2k}$	$M_{k,p+q-2k}(C)$
12	$\mathfrak{sp}(p, q)$, $1 \leqslant p < q$	$\{\alpha_k\}$, $1 \leqslant k \leqslant p$	$(\text{II}2)_{2k,2p+2q-4k}$	$M_{k,p+q-2k}(H)$
13	$\mathfrak{so}^*(4n+2)$, $n \geqslant 2$	$\{\alpha_k\}$, $1 \leqslant k \leqslant n$	$(\text{II}2)_{2k,4n+2-4k}$	$M_{k,2n-2k+1}(H)$
14	$\mathfrak{so}(2n, C)$, $n \geqslant 4$	$\{\alpha_k\}$, $2 \leqslant k \leqslant n-2$	$(\text{II}2)_{k,2n-2k}$	$M_{k,2n-2k}(C)$
14′	$\mathfrak{so}(2n, C)$, $n \geqslant 4$	$\{\alpha_{n-1}, \alpha_n\}$	$(\text{II}2)_{n-1,2}$	$M_{n-1,2}(C)$
15	$\mathfrak{so}(2n, C)$, $n \geqslant 5$	$\{\alpha_1, \alpha_n\}$	$(\text{II}3)_{n-1}$	$M_{1,n-1}(C) \times Alt'_{n-1}(C)$
16	$\mathfrak{so}(n, n)$, $n \geqslant 4$	$\{\alpha_k\}$, $2 \leqslant k \leqslant n-2$	$(\text{II}2)_{k,2n-2k}$	$M_{k,2n-2k}(R)$
16′	$\mathfrak{so}(n, n)$, $n \geqslant 4$	$\{\alpha_{n-1}, \alpha_n\}$	$(\text{II}2)_{n-1,2}$	$M_{n-1,2}(R)$
17	$\mathfrak{so}(n, n)$, $n \geqslant 5$	$\{\alpha_1, \alpha_n\}$	$(\text{II}3)_{n-1}$	$M_{1,n-1}(R) \times Alt'_{n-1}(R)$

Table III

	$\mathfrak{g}_{-1}$	$B_\tau(Y, X, Z)$
1	$M_{p,n-p}(\boldsymbol{C})$	$YX^*Z + ZX^*Y$
2	$M_{p,n-p}(\boldsymbol{R})$	$Y^tXZ + Z^tXY$
3	$M_{p,n-p}(\boldsymbol{H})$	$YX^*Z + ZX^*Y$
4	$M_{1,2n-1}(\boldsymbol{C})$	$YX^*Z + ZX^*Y - ZJ_{2n-1}{}^tY\overline{X}J_{2n-1}$
5	$M_{1,p+q-2}(\boldsymbol{R})$	$Y^tXZ + Z^tXY - ZA_{p-1,q-1}{}^tYXA_{p-1,q-1}$
6	$Sym'_n(\boldsymbol{C})$	$YX^*Z + ZX^*Y$
7	$Sym'_n(\boldsymbol{R})$	$Y^tXZ + Z^tXY$
8	$SH'_n(\boldsymbol{C})$	$YX^*Z + ZX^*Y$
9	$SH'_n(\boldsymbol{H})$	$YX^*Z + ZX^*Y$
10	$H'_n(\boldsymbol{H})$	$YX^*Z + ZX^*Y$
11	$M_{1,2n-2}(\boldsymbol{C})$	$YX^*Z + ZX^*Y - ZJ_{2n-2}{}^tY\overline{X}J_{2n-2}$
12	$Alt'_n(\boldsymbol{C})$	$YX^*Z + ZX^*Y$
13	$M_{1,2n-2}(\boldsymbol{R})$	$Y^tXZ + Z^tXY - ZJ_{2n-2}{}^tYXJ_{2n-2}$
14	$Alt'_n(\boldsymbol{R})$	$Y^tXZ + Z^tXY$

Table IV

	$\mathfrak{g}_{-1}$	$B_\tau(Y, X, Z)$
1	$M_{p,q}(\boldsymbol{C}) \times M_{q,n-p-q}(\boldsymbol{C})$	$\begin{cases} Y_1X_1^*Z_1 + Z_1X_1^*Y_1 - Z_1Y_2X_2^* \\ Y_2X_2^*Z_2 + Z_2X_2^*Y_2 - X_1^*Y_1Z_2 \end{cases}$
2	$M_{p,q}(\boldsymbol{R}) \times M_{q,n-p-q}(\boldsymbol{R})$	$\begin{cases} Y_1{}^tX_1Z_1 + Z_1{}^tX_1Y_1 - Z_1Y_2{}^tX_2 \\ Y_2{}^tX_2Z_2 + Z_2{}^tX_2Y_2 - {}^tX_1Y_1Z_2 \end{cases}$
3	$M_{p,q}(\boldsymbol{H}) \times M_{q,n-p-q}(\boldsymbol{H})$	$\begin{cases} Y_1X_1^*Z_1 + Z_1X_1^*Y_1 - Z_1Y_2X_2^* \\ Y_2X_2^*Z_2 + Z_2X_2^*Y_2 - X_1^*Y_1Z_2 \end{cases}$
4	$M_{k,2n-2k+1}(\boldsymbol{C})$	$YX^*Z + ZX^*Y - ZJ_{2n-2k+1}{}^tY\overline{X}J_{2n-2k+1}$
5	$M_{k,p+q-2k}(\boldsymbol{R})$	$Y^tXZ + Z^tXY - ZA_{p-k,q-k}{}^tYXA_{p-k,q-k}$
6	$M_{k,2n-2k}(\boldsymbol{C})$	$YX^*Z + ZX^*Y + Z\tilde{J}_{n-k}{}^tY\overline{X}\tilde{J}_{n-k}$
7	$M_{k,2n-2k}(\boldsymbol{R})$	$Y^tXZ + Z^tXY + Z\tilde{J}_{n-k}{}^tYX\tilde{J}_{n-k}$
8	$M_{k,2n-2k}(\boldsymbol{C})$	$YX^*Z + ZX^*Y - ZJ_{2n-2k}Y^*XJ_{2n-2k}$
9	$M_{k,2n-2k}(\boldsymbol{H})$	$YX^*Z + ZX^*Y - ZJ'_{2n-2k}Y^*XJ'_{2n-2k}$
10	$M_{k,2n-2k}(\boldsymbol{H})$	$YX^*Z + ZX^*Y + ZK_{2n-2k}Y^*XK_{2n-2k}$
11	$M_{k,p+q-2k}(\boldsymbol{C})$	$YX^*Z + ZX^*Y - ZA_{p-k,q-k}Y^*XA_{p-k,q-k}$

12	$M_{k,p+q-2k}(H)$	$YX^*Z + ZX^*Y - ZA'_{p-k,q-k}Y^*XA'_{p-k,q-k}$
13	$M_{k,2n-2k+1}(H)$	$YX^*Z + ZX^*Y + ZK_{2n-2k+1}Y^*XK_{2n-2k+1}$
14	$M_{k,2n-2k}(C)$	$YX^*Z + ZX^*Y - ZJ_{2n-2k}{}^tY\bar{X}J_{2n-2k}$
15	$M_{1,n-1}(C) \times Alt'_{n-1}(C)$	$\begin{cases} Y_1X_1^*Z_1 + Z_1X_1^*Y_1 - Z_1Y_2X_2^* \\ Y_2X_2^*Z_2 + Z_2X_2^*Y_2 - X_1^*Y_1Z_2 - Z_2J_{n-1}{}^tY_1\bar{X}_1J_{n-1} \end{cases}$
16	$M_{k,2n-2k}(R)$	$Y^tXZ + Z^tXY - ZJ_{2n-2k}{}^tYXJ_{2n-2k}$
17	$M_{1,n-1}(R) \times Alt'_{n-1}(R)$	$\begin{cases} Y_1{}^tX_1Z_1 + Z_1{}^tX_1Y_1 - Z_1Y_2{}^tX_2 \\ Y_2{}^tX_2Z_2 + Z_2{}^tX_2Y_2 - {}^tX_1Y_1Z_2 - Z_2J_{n-1}{}^tY_1X_1J_{n-1} \end{cases}$

kind) $(\mathfrak{g}_{-1}, B_\tau)$ which are obtained from the gradations given in Table I (resp. Table II). In Table IV, if $\mathfrak{g}_{-1}$ is a direct product of two vector spaces V_1 and V_2, then an element $X \in \mathfrak{g}_{-1}$ is denoted by (X_1, X_2) or $\begin{cases} X_1, \\ X_2 \end{cases}$ where $X_i \in V_i$.

From Table III we have the following

THEOREM 4.1. *Compact classical real simple JTS's (U_{-1}, B) are classified (up to isomorphisms) as follows*:

(1) $U_{-1} = M_{p,q}(K)$, $K = R, C, H$; $p \leqslant q$,

$$B(Y, X, Z) = YX^*Z + ZX^*Y.$$

(2) $U_{-1} = Sym'_n(K)$, $K = R, C$; $n \geqslant 3$,

$$B(Y, X, Z) = YX^*Z + ZX^*Y.$$

(3) $U_{-1} = Alt'_n(K)$, $K = R, C$; $n \geqslant 5$,

$$B(Y, X, Z) = YX^*Z + ZX^*Y.$$

(4) $U_{-1} = SH'_n(K)$, $K = C, H$; $n \geqslant 3$ *for* $K = C$, $n \geqslant 2$ *for* $K = H$,

$$B(Y, X, Z) = YX^*Z + ZX^*Y.$$

(5) $U_{-1} = H'_n(H)$, $n \geqslant 3$,

$$B(Y, X, Z) = YX^*Z + ZX^*Y.$$

(6) $U_{-1} = M_{1,n}(C)$, $n \geqslant 3$, $n \neq 4$,

$$B(Y, X, Z) = YX^*Z + ZX^*Y - ZJ_n{}^tY\bar{X}J_n.$$

(7) $U_{-1} = M_{1,p+q}(R)$, $0 \leqslant p < q$ *or* $3 \leqslant p = q$,

$$B(Y, X, Z) = Y^tXZ + Z^tXY - ZA_{p,q}{}^tYXA_{p,q}.$$

It is easy to see that the above result is essentially the same as the one in Loos [10]. From Table IV we have

THEOREM 4.2. *Compact classical real simple GJTS's* (U_{-1}, B) *of the second kind are classified* (*up to isomorphisms*) *as follows*:

(1) $U_{-1} = M_{p,q}(\boldsymbol{K})\times M_{q,r}(\boldsymbol{K}),\ \boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H},$

$$1 \leqslant p \leqslant \left[\frac{n}{2}\right],\quad p \leqslant r,\quad p+q+r = n,$$

$$B(Y, X, Z) = (Y_1X_1^*Z_1 + Z_1X_1^*Y_1 - Z_1Y_2X_2^*,\ Y_2X_2^*Z_2 + Z_2X_2^*Y_2 - X_1^*Y_1Z_2),$$

where

$$X = (X_1, X_2),\quad Y = (Y_1, Y_2),\quad Z = (Z_1, Z_2).$$

(2) $U_{-1} = M_{k,p+q-2k}(\boldsymbol{K}),\ \boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C},$

$$2 \leqslant k \leqslant p < q \text{ or } 2 \leqslant k < p = q\ (\geqslant 4) \text{ for } \boldsymbol{K} = \boldsymbol{R},$$
$$1 \leqslant k \leqslant p < q \text{ or } 1 \leqslant k < p = q\ (\geqslant 4) \text{ for } \boldsymbol{K} = \boldsymbol{C},$$

$$B(Y, X, Z) = YX^*Z + ZX^*Y - ZA_{p-k,q-k}Y^*XA_{p-k,q-k}.$$

(3) $U_{-1} = M_{k,p+q-2k}(\boldsymbol{H}),$

$$1 \leqslant k \leqslant p < q \text{ or } 1 \leqslant k < p = q,$$

$$B(Y, X, Z) = YX^*Z + ZX^*Y - ZA'_{p-k,q-k}Y^*XA'_{p-k,q-k}.$$

(4) $U_{-1} = M_{k,m-2k}(\boldsymbol{C}),$

$$2 \leqslant k \leqslant n \quad \text{for } m = 2n+1,$$
$$2 \leqslant k \leqslant n-1 \quad \text{for } m = 2n\ (n \geqslant 4),$$

$$B(Y, X, Z) = YX^*Z + ZX^*Y - ZJ_{m-2k}{}^tY\overline{X}J_{m-2k}.$$

(5) $U_{-1} = M_{k,m-2k}(\boldsymbol{H}),$

$$1 \leqslant k \leqslant n \quad \text{for } m = 2n+1\ (n \geqslant 2),$$
$$1 \leqslant k \leqslant n-1 \quad \text{for } m = 2n\ (n \geqslant 3),$$

$$B(Y, X, Z) = YX^*Z + ZX^*Y + ZK_{m-2k}Y^*XK_{m-2k}.$$

(6) $U_{-1} = M_{k,2n-2k}(\boldsymbol{K}),\ \boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C},$

$$1 \leqslant k \leqslant n-1\ (n \geqslant 3),$$

$$B(Y, X, Z) = YX^*Z + ZX^*Y + Z\tilde{J}_{n-k}{}^tY\overline{X}\tilde{J}_{n-k}.$$

(7) $U_{-1} = M_{1,n-1}(\boldsymbol{K}) \times Alt'_{n-1}(\boldsymbol{K}),\ \boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C};\ n \geqslant 5,$

$$B(Y, X, Z) = (Y_1X_1^*Z_1 + Z_1X_1^*Y_1 - Z_1Y_2X_2^*,\ Y_2X_2^*Z_2 + Z_2X_2^*Y - X_1^*Y_1Z_2 - Z_2J_{n-1}{}^tY_1\bar{X}_1J_{n-1}),$$

where

$$X = (X_1, X_2), \quad Y = (Y_1, Y_2), \quad Z = (Z_1, Z_2).$$

§ 5. ε-modifications of compact simple GJTS's

We will give here a method of constructing *noncompact* simple GJTS's, starting from compact simple GJTS's. Let (U_{-1}, B) be a compact real simple GJTS of the ν-th kind ($\nu = 1, 2$). Then we have the admissible pair $(\mathscr{L}(B), \tau_B)$ (cf. the proof of Theorem 3.14). $\mathscr{L}(B)$ is a simple GLA of type α_0 of the ν-th kind ($\nu = 1,2$). For brevity we put $\mathfrak{g} = \mathscr{L}(B)$ and $\tau = \tau_B$. Let $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ and let E be its characteristic element. Note that $\mathfrak{g}_{-1} = U_{-1}$. τ is a grade-reversing Cartan involution of $\mathfrak{g}$. We choose a maximal abelian subspace $\mathfrak{a}$ containing E satisfying (1.1). Let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Now consider a signature ε of roots in Δ in the sense of Oshima-Sekiguchi [12], and let τ_ε be the ε-modification of τ. τ_ε is also an involutive automorphism of $\mathfrak{g}$.

PROPOSITION 5.1. *Let*

$$B_\varepsilon(Y, X, Z) = [[\tau_\varepsilon(X), Y], Z] \qquad X, Y, Z \in U_{-1}. \tag{5.1}$$

Then (U_{-1}, B_ε) *is a noncompact simple GJTS of the* ν*-th kind* ($\nu = 1, 2$).

Proof. In view of the definition of τ_ε [12], it follows that τ_ε coincides with τ on $\mathfrak{a}$. Hence we have $\tau_\varepsilon(E) = -E$ and consequently τ_ε is a grade-reversing involutive automorphism of $\mathfrak{g}$. By Theorem 3.5, (U_{-1}, B_ε) is a K-simple GJTS and hence it is simple (cf. 3.4). Consequently (U_{-1}, B_ε) satisfies the condition (A). The GLA $\mathscr{L}(B_\varepsilon)$ admits the grade-reversing canonical involution τ_{B_ε} (cf. Proposition 3.8). By Lemma 3.10, there exists a grade-preserving isomorphism φ of $\mathfrak{g}$ onto $\mathscr{L}(B_\varepsilon)$ such that $\varphi\tau_\varepsilon = \tau_{B_\varepsilon}\varphi$. Since τ_ε is not a Cartan involution, τ_{B_ε} is not either. Therefore B_ε is not compact [1].

We say that (U_{-1}, B_ε) is an *ε-modification* of (U_{-1}, B).

Remark 5.2. Let $\mathfrak{g} = \sum_k \mathfrak{g}_k$ be a simple GLA of type α_0 of the ν-th kind ($\nu = 1, 2$). Let τ be a grade-reversing Cartan involution of $\mathfrak{g}$ and τ_ε be an ε-modification of τ. The proof of the above theorem shows that B_ε in (5.1) is a noncompact simple GJTS of the ν-th kind ($\nu = 1, 2$).

EXAMPLE 5.3. Let us consider the gradation $(\mathrm{II}2)_{2,4}$ of $\mathfrak{g}=\mathfrak{su}(4,4)$ (cf. Table II). In this case $\mathfrak{g}_{-1}=M_{2,4}(C)$. A grade-reversing Cartan involution is given by τ such that $\tau(X)=-X^*$, $X\in\mathfrak{su}(4,4)$. The corresponding compact simple GJTS is found in Table IV. Let $\Pi=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ be a fundamental system for Δ compatible with the gradation. We define a signature ε by $\varepsilon(\alpha_1)=-1$ and $\varepsilon(\alpha_i)=1$ for $i=2,3,4$ ([12]). By easy computations we can verify that $\tau_\varepsilon(X)=A\tau(X)A$ for $X\in\mathfrak{su}(4,4)$, where $A=\mathrm{diag}(-1,E_6,-1)$. By direct computations we see for X, Y, $Z\in M_{2,4}(C)$

$$B_\varepsilon(Y,X,Z)=YX^*I_{1,1}Z+ZX^*I_{1,1}Y-ZJ_4Y^*I_{1,1}XJ_4\,, \tag{5.2}$$

where $I_{1,1}=\mathrm{diag}(-1,1)$. The above B_ε provides an example of non-compact simple GJTS's.

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