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ESTIMATE FOR INDEX OF CLOSED MINIMAL HYPERSURFACES IN SPHERES

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Abstract

The aim of this work is to deal with index of closed orientable non-totally geodesic minimal hypersurface Σ^n of the Euclidean unit sphere $\mathbf{S}^{n+1}$ whose second fundamental form has squared norm bounded from below by n. In this case we shall show that the index of stability, denoted by Ind_Σ, is greater than or equal to $n+3$, with equality occurring at only Clifford tori $\mathbf{S}^k\left(\frac{k}{n}\right) \times \mathbf{S}^{n-k}\left(\sqrt{\frac{(n-k)}{n}}\right)$. Moreover, we shall prove also that, besides Clifford tori, we have the following gap: $Ind_\Sigma \geq 2n+5$.

1. Introduction

One fundamental paper on the theory of minimal hypersurfaces of the Euclidean sphere $\mathbf{S}^{n+1}$ is the seminal work of Simons [11]. Among many interesting results, it was proved that, besides totally geodesic spheres, closed minimal hypersurfaces $\Sigma^n \subset \mathbf{S}^{n+1}$ whose squared norm of the second fundamental form satisfies $|A|^2 \leq n$, must have $|A|^2 = n$. Moreover, he also proved that, the index of such hypersurfaces are greater than or equal to one and equality occurs at only totally geodesic spheres, which yields instability of such class of hypersurfaces. After that, one celebrated result concerning the equality on the above pinching was obtained, independently, by Chern-do Carmo-Kobayashi [6] and Lawson [8], where they proved that Clifford tori are the unique minimal hypersurfaces where $|A|^2 = n$. Besides totally geodesic spheres, Clifford tori are the most simple known examples of compact minimal hypersurfaces in $\mathbf{S}^{n+1}$. Then a question raises from these results: What is the index of a closed oriented minimal hypersurface $\Sigma^n \subset \mathbf{S}^{n+1}$? Moreover, among such hypersurfaces, do Clifford tori have the lower index? By using the *Gauß* map as test function we may show that, besides totally geodesic spheres, $Ind_\Sigma \geq n+3$, see e.g. [11], [7] and [10]. Moreover, it is easy to check that minimal Clifford tori have index $n+3$. Hence, it

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was conjectured that in fact, $Ind_\Sigma \geq n+3$, and it is reached only for Clifford tori. For dimension two, Urbano [12] settled this conjecture by showing that for non-totally geodesic sphere, a closed orientable minimal surface $\Sigma^2 \subset \mathbf{S}^3$, has index greater than or equal to 5 occurring equality only for the Clifford torus $\mathbf{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbf{S}^1\left(\frac{1}{\sqrt{2}}\right)$. For high dimension, some few progress has been made in the matter of this problem. One of the first result is due to Brasil-Delgado-Guadalupe [5], where they proved the conjecture, under the additional hypothesis that the scalar curvature of Σ^n is constant, but, essentially this is a direct consequence of a result contained in the work of Nomizu and Smyth [9]. Recently, Perdomo [10] also proved such result for hypersurfaces with some type of symmetries. Since, closed minimal hypersurfaces of the Euclidean sphere with $|A| \leq \sqrt{n}$ are well known, here we shall prove this conjecture provided its second fundamental form is bounded from below by $\sqrt{n}$. More exactly, we shall prove.

Theorem 1. *Let $x : \Sigma^n \looparrowright \mathbf{S}^{n+1}$ be a non-totally geodesic minimal isometric immersion of a closed oriented Riemannian manifold Σ^n with norm of the second fundamental form bounded from below by $\sqrt{n}$. Then $Ind_\Sigma \geq n+3$ with equality occurring at only Clifford tori $\mathbf{S}^k\left(\frac{k}{n}\right) \times \mathbf{S}^{n-k}\left(\sqrt{\frac{(n-k)}{n}}\right)$.*

We point out that for $|A|$ constant the result is also a combination of our one jointly with the theorem of Chern-do Carmo- Kobayashi [6] and Lawson [8]. On the other hand Perdomo [10] conjectured that, besides totally geodesic spheres and Clifford tori, closed minimal hypersurfaces of $\mathbf{S}^{n+1}$ have $Ind_\Sigma \geq 2n+5$. Under the same hypothesis of Theorem 1 we shall prove this conjecture.

Theorem 2. *Let $x : \Sigma^n \looparrowright \mathbf{S}^{n+1}$ be a non-totally geodesic minimal isometric immersion of a closed oriented Riemannian manifold Σ^n with norm of the second fundamental form bounded from below by $\sqrt{n}$. Then, besides Clifford tori $\mathbf{S}^k\left(\frac{k}{n}\right) \times \mathbf{S}^{n-k}\left(\sqrt{\frac{(n-k)}{n}}\right)$, $Ind_\Sigma \geq 2n+5$.*

2. The index of minimal hypersurface

Unless stated otherwise, all manifold considered on this work will be connected, while closed means compact without boundary. During this section we shall present a briefly introduction of some well known facts concerning to the stability of orientable hypersurfaces in the Euclidean unit sphere that will be used on our work. They may be found on the literature, essentially in [1], [3], [4] and [11]. Given an isometric immersion $x : \Sigma^n \looparrowright \mathbf{S}^{n+1}$ of a compact oriented Riemannian manifold Σ^n into the Euclidean unit sphere $\mathbf{S}^{n+1}$ we shall denote its second fundamental form by A while its mean curvature H will be given by $nH = tr\,A$. In particular, if $k_1, \ldots, k_n$ are the principal curvatures of A we have

(2.1)
$$|A|^2 + 2S_2 = S_1^2,$$

where S_1 and S_2 are respectively the first and the second symmetric functions of the principal curvatures $k_1, \dots, k_n$.

On the other hand, given a differentiable function $f \in C^\infty(\Sigma)$ there exists a normal variation x_t of the x, with variational normal field fN, such that

$$\frac{d}{dt}A(t)_{|t=0} = -n\int_\Sigma fH\, d\Sigma,$$

where $A(t)$ is the area of each immersion x_t and $d\Sigma$ stands for the volume element of Σ. Therefore, minimal hypersurfaces of unit sphere are critical points for the area functional. In order to understand the behavior of such critical points it is fundamental to compute the second derivative of $A(t)$ that is given by:

$$\frac{d^2}{dt^2}A(t)_{|t=0} = -\int_\Sigma fJf\, d\Sigma,$$

where $J = \Delta + |A|^2 + n$ is the stability operator, which is also called Jacobi operator. Hence, we may associate to J a quadratic form $Q(f) = -\int_\Sigma fJf\, d\Sigma$, for $f \in C^\infty(\Sigma)$. The index of stability of a minimal hypersurface as above, denoted by Ind_Σ, is defined as the maximum dimension of a subspace $V \subset C^\infty(\Sigma)$ for which this quadratic form is negative definite. For instance, the constant functions always have such propriety. Indeed,

$$Q(1) = -\int_\Sigma (|A|^2 + n)\, d\Sigma \le -n\, Area(\Sigma) < 0.$$

Then any compact oriented minimal hypersurface $\Sigma^n \subset \mathbf{S}^{n+1}$ has $Ind_\Sigma \ge 1$, which means that it is unstable. Moreover, as stated in the introduction Simons [11] proved that $Ind_\Sigma = 1$ only for totally geodesic spheres.

3. Analysis of support functions

Given a hypersurface $x : \Sigma^n \hookrightarrow \mathbf{S}^{n+1}$ the support functions $l_v = \langle x, v\rangle$ and $f_v = \langle N, v\rangle$, where $v \in \mathbf{R}^{n+2}$ and N stands for the *Gau*ß map, play an important role on the theory of immersions. Since $V = \{l_v : v \in \mathbf{R}^{n+2}\}$ and $W = \{f_v : v \in \mathbf{R}^{n+2}\}$ are linear subspaces of the vector space $C^\infty(\Sigma)$, by using *Gau*ß and Codazzi equations we may obtain many interesting properties for this couple of functions, see e.g. Alias [1].

(3.1)
$$\nabla l_v = v^\top,$$

(3.2)
$$Hess\, l_v(X, Y) = -l_v\langle X, Y\rangle + f_v\langle AX, Y\rangle$$

(3.3)
$$\nabla f_v = -A(v^\top),$$

(3.4)
$$Hess\, f_v(X, Y) = -\langle \nabla A(v^\top, X), Y\rangle + l_v\langle AX, Y\rangle - f_v\langle AX, AY\rangle,$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here $v^{\top}$ stands for the projection of v on the tangent bundle $T\Sigma$.

In particular, on the minimal case we have a well known lemma.

Lemma 1. *Let $x : \Sigma^n \looparrowright \mathbf{S}^{n+1}$ be a minimal isometric immersion of an oriented Riemannian manifold Σ^n. Then we have*

(1) $\Delta l_v = -n l_v$,

(2) $\Delta f_v = 2S_2 f_v$,

(3) *In addition, if Σ^n is closed, then $\int_\Sigma l_v \, d\Sigma = 0$ and $\int_\Sigma S_2 f_v \, d\Sigma = 0$.*

Proceeding on the analysis of such functions we shall present a series of lemmas. Some of them already appear on literature, but for readers convenience we shall give their proofs. Henceforth we shall choose a suitable basis $\{e_1, \dots, e_{n+2}\}$ on the Euclidean space $\mathbf{R}^{n+2}$.

Lemma 2. *Let $x : \Sigma^n \looparrowright \mathbf{S}^{n+1}$ be an isometric immersion of a closed oriented Riemannian manifold Σ^n. Then we have*

(1) *If $x(\Sigma^n)$ is not a totally geodesic sphere, $\dim V = n+2$,*

(2) *Given a non null vector $u \in \mathbf{R}^{n+2}$. Then, either $\{l_u, 1\}$ is an independent set or $x(\Sigma^n)$ is a geodesic sphere,*

(3) *If $x(\Sigma^n)$ is not a geodesic sphere, $\dim \overline{V} = n+3$, where $\overline{V} = \{a + l_v : a \in \mathbf{R}, v \in \mathbf{R}^{n+2}\}$.*

Proof. The proof of (1) is direct. In fact, let us suppose that $\{l_{e_1}, \dots, l_{e_{n+2}}\}$ is a dependent set. Hence there exist non null real constants $a_1, \dots, a_{n+2}$ satisfying

$$\sum_{i=1}^{n+2} a_i l_{e_i} = 0.$$

Thus considering $u = \sum_{i=1}^{n+2} a_i e_i$ we conclude that $l_u = \langle x, u \rangle = 0$. Then $x(\Sigma^n) = \mathbf{S}^n$ is a totally geodesic sphere that finishes (1).

Now let us suppose that $\{l_u, 1\}$ is a dependent set. Then there exists a real constant a such that $l_u = \langle x, u \rangle = a$. Thus, if $\Gamma_u = \{p \in \mathbf{R}^{n+2} : \langle p, u \rangle = a\}$ is the hyperplane then $x(\Sigma^n) \subset \mathbf{S}^{n+1} \cap \Gamma_u$, i.e. $x(\Sigma^n) = \mathbf{S}^n(r)$ is a geodesic sphere with center $c = a\dfrac{u}{|u|^2}$ and radius $r = \dfrac{\sqrt{|u|^2 - a^2}}{|u|}$, which finishes item (2).

The assertion of (3) follows from (2). $\square$

On the other hand Nomizu and Smyth [9] proved the following theorem.

Theorem [Nomizu-Smyth]. *Let M be a complete orientable Riemannian manifold of dimension $n \geq 2$ isometrically immersed in $\mathbf{S}^{n+1}$ and let ϕ be the associated Gauß mapping.*

i) *If* $\phi(M)$ *is contained in a great hypersphere of* $\mathbf{S}^{n+1}$ *then* M *is imbedded as a great hypersphere and so* $\phi(M)$ *is a single point.*
ii) *If* $\phi(M)$ *is contained in a small hypersphere of* $\mathbf{S}^{n+1}$ *but is not a single point, then* M *is imbedded as a small hypersphere and* $\phi(M)$ *is a full small hypersphere.*

Now we notice that this theorem yields a direct proof of the next lemma.

LEMMA 3. *Let* $x : \Sigma^n \looparrowright \mathbf{S}^{n+1}$ *be an isometric immersion of a closed oriented Riemannian manifold* Σ^n. *Then we have*
(1) *If* $x(\Sigma^n)$ *is not a totally geodesic sphere, dim* $W = n+2$,
(2) *If* $x(\Sigma^n)$ *is not a geodesic sphere, dim* $\overline{W} = n+3$, *where* $\overline{W} = \{a + f_v : a \in \mathbf{R}, v \in \mathbf{R}^{n+2}\}$.

Proof. For item (1) we address to item i) of the above theorem. While for item (2) we use item ii) of the cited theorem. Indeed, let us suppose that $\{f_{e_1}, \ldots, f_{e_{n+2}}, 1\}$ is a dependent set. Hence there exist non null real constants $a_1, \ldots, a_{n+2}$ satisfying

$$\sum_{i=1}^{n+2} a_i f_{e_i} = 1.$$

Thus considering the hyperplane $\Gamma_u = \{p \in \mathbf{R}^{n+2} : \langle p, u\rangle = 1\}$, $u = \sum_{i=1}^{n+2} a_i e_i$ we conclude that $N(\Sigma^n) \subset \mathbf{S}^{n+1} \cap \Gamma_u$, i.e. $N(\Sigma^n) = \mathbf{S}^n(r)$ is a geodesic sphere with center $c = \dfrac{u}{|u|^2}$ and radius $r = \dfrac{\sqrt{|u|^2 - 1}}{|u|}$. Therefore, we may apply item ii) of the above theorem to conclude that $x(\Sigma^n) = \mathbf{S}^n$ is a geodesic sphere. From here we have that $\{f_{e_1}, \ldots, f_{e_{n+2}}, 1\}$ is a basis for $\overline{W}$ up to geodesic spheres, which concludes the proof of lemma. □

As was pointed out before, besides totally geodesic spheres, Clifford tori are the most simple examples of compact minimal hypersurfaces of the sphere. In particular if $x_1 : \mathbf{S}^k(\rho) \hookrightarrow \mathbf{R}^{k+1}$ and $x_2 : \mathbf{S}^{n-k}(\sqrt{1-\rho^2}) \hookrightarrow \mathbf{R}^{n-k+1}$ are the standard immersions, we may consider $x = (x_1, x_2)$ and $N = \left(-\dfrac{\sqrt{1-\rho^2}}{\rho}x_1, \dfrac{\rho}{\sqrt{1-\rho^2}}x_2\right)$ to describe a Clifford torus and its normal, respectively. Therefore considering $\mathbf{S}^k(\rho) \times \mathbf{S}^{n-k}(\sqrt{1-\rho^2}) \subset \mathbf{R}^{k+1} \oplus \mathbf{R}^{n-k+1}$ it is easy to see that $f_{e_i} = \lambda l_{e_i}$ for $i = 1, \ldots, n+2$, where e_i and λ will be chosen appropriately.

On the other hand, in a recent result due to Alias-Brasil-Perdomo [2] they proved that if a hypersurface $x : \Sigma^n \looparrowright \mathbf{S}^{n+1}$ has constant mean curvature with the support functions satisfying $l_v = \lambda f_v$ for some $v \in \mathbf{R}^{n+2}$ and $\lambda \in \mathbf{R}$ then $x(\Sigma)$ is either a totally umbilical sphere or a Clifford torus. We remark that we may prove a slight modification of this result for minimal hypersurface. More exactly we have the next lemma.

Lemma 4. *Let* $x : \Sigma^n \hookrightarrow \mathbf{S}^{n+1}$ *be a non-totally geodesic minimal isometric immersion of a closed oriented Riemannian manifold* Σ^n. *If* $l_u = \lambda f_v$ *for some non null vectors* $u, v \in \mathbf{R}^{n+2}$ *and a real number* λ, *then* $x(\Sigma^n)$ *is a Clifford torus.*

Proof. Let us suppose that $l_u = \lambda f_v$ for some non null vectors $u, v \in \mathbf{R}^{n+2}$. If $l_u \equiv 0$ then $x(\Sigma^n) = \mathbf{S}^n$ is totally geodesic, which contradicts our assumption. Hence we may assume that $l_u \neq 0$. Taking into account that $l_u = \lambda f_v$ we have $\lambda \neq 0$ and Lemma 1 yields

$$-nl_u = \Delta l_u = \lambda \Delta f_v = 2\lambda S_2 f_v = 2S_2 l_u.$$

From here we conclude that $hl_u = 0$, where $h = 2S_2 + n$. Suppose that there exists $p \in \Sigma^n$ such that $h(p) \neq 0$. By continuity we have a neighborhood $\mathscr{U}$ of p where $h(q) \neq 0$ for all $q \in \mathscr{U}$. Hence $l_u(q) = 0$ on $\mathscr{U}$, i.e. $\langle x, u\rangle(q) = 0$ for all $q \in \mathscr{U}$, this means $x(\Sigma^n \cap \mathscr{U}) \subset \mathbf{S}^n$. Using analyticity of x we deduce $x(\Sigma^n) = \mathbf{S}^n$ is totally geodesic, which contradicts our hypothesis. Thus $h = 2S_2 + n \equiv 0$. By using equation (2.1) we derive that $|A|^2 \equiv n$. Therefore, we are able to use the result due to Chern-doCarmo-Kobayashi [6] or Lawson [8] to conclude that $x(\Sigma^n)$ is a Clifford torus $\mathbf{S}^k\left(\frac{k}{n}\right) \times \mathbf{S}^{n-k}\left(\sqrt{\frac{(n-k)}{n}}\right)$. □

Lemma 5. *Let* $x : \Sigma^n \hookrightarrow \mathbf{S}^{n+1}$ *be a non-totally geodesic minimal isometric immersion of a closed oriented Riemannian manifold* Σ^n. *Then, for all non null vectors* $u, v \in \mathbf{R}^{n+2}$,

(i) *Either,* $\{l_u, f_v, 1\}$ *is independent,*

(ii) *Or* $x(\Sigma^n)$ *is a Clifford torus.*

Proof. We may suppose that $l_u \neq 0$ and $f_v \neq 0$. As before let us start supposing that $\{l_u, f_v, 1\}$ is a dependent set. Then there exist non null real constants a, b such that

$$l_u = af_v + b. \tag{3.5}$$

If $b = 0$ on equation (3.5) we have $l_u = af_v$ with $a \neq 0$ and Lemma 4 yields that $x(\Sigma^n)$ is a Clifford torus. Otherwise, we suppose $b \neq 0$ and making use of equation (3.5) and Lemma 1 we obtain

$$a(2S_2 + n)f_v = -bn.$$

Since $b \neq 0$, we have that the left hand side of this equation never vanishes. Then, either $f_v > 0$ or $f_v < 0$. But, this tell us that $N(\Sigma^n) \subset (\mathbf{S}^{n+1})^{\circ}$, where $(\mathbf{S}^{n+1})^{\circ}$ denotes an open hemisphere of the Euclidean unit sphere $\mathbf{S}^{n+1}$. Then we may apply Theorem 5.2.1 of Simons [11] to conclude that $x(\Sigma^n) = \mathbf{S}^n$ is totally geodesic, which yields a contradiction. □

4. Proof of Theorems 1 and 2

Proof. In order to prove the theorems we introduce the vector subspace

$$\mathscr{L} = \left\{ f \in C^\infty(\Sigma) : f = a \cdot 1 + \sum_{i=1}^{n+2} b_i l_{e_i} + \sum_{i=1}^{n+2} c_i l_{e_i} = a + l_u + f_v \right\}.$$

Now we shall show that $Q(f) < 0$ for all non zero functions $f \in \mathscr{L}$. First of all we recall that

$$Q(f) = -\int_\Sigma f[\Delta f + |A|^2 f + nf]\, d\Sigma.$$

Given $f = a + l_u + f_v \in \mathscr{L}$ we make use of $\Delta l_u = -nl_u$ and $\Delta f_v = -|A|^2 f_v$ to infer

$$\Delta f + |A|^2 f + nf = a(|A|^2 + n) + |A|^2 l_u + nf_v.$$

Hence a straightforward computation yields

$$\begin{aligned} f[\Delta f + |A|^2 f + nf] = {} & |A|^2(l_u + a)^2 + n(f_v + a)^2 \\ & - a\Delta l_u - a\Delta f_v + (n + |A|^2) l_u f_v. \end{aligned}$$

Taking into account Green's identity we also obtain

$$\int_\Sigma n l_u f_v\, d\Sigma = \int_\Sigma |A|^2 l_u f_v\, d\Sigma.$$

Therefore we arrive at

$$Q(f) = -\int_\Sigma [|A|^2(l_u + a)^2 + n(f_v + a)^2]\, d\Sigma - 2n\int_\Sigma l_u f_v\, d\Sigma.$$

Notice that we may write $l_u f_v = (l_u + a)(f_v + a) - a(l_u + f_v + 2a) + a^2$. Hence, for $|A|^2 \geq n$ we have

$$\begin{aligned} Q(f) &\leq -n\int_\Sigma [(l_u + a)^2 + (f_v + a)^2]\, d\Sigma \\ &\quad - 2n\int_\Sigma [(l_u + a)(f_v + a) - a(l_u + f_v + 2a) + a^2]\, d\Sigma \\ &= -n\int_\Sigma (l_u + f_v + 2a)^2\, d\Sigma - n\int_\Sigma (-2a)(l_u + f_v + 2a)\, d\Sigma - n\int_\Sigma 2a^2\, d\Sigma. \end{aligned}$$

From here we conclude

$$Q(f) \leq -n\int_\Sigma (f^2 + a^2)\, d\Sigma. \tag{4.1}$$

Therefore the quadratic form Q is negative definite on the vector space $\mathscr{L}$. On the other hand, using Lemmas 3 and 5, we obtain

$$dim\ \mathscr{X} \geq n + 3 \tag{4.2}$$

and, if $x(\Sigma)$ is not a Clifford torus

$$dim\ \mathscr{X} \geq 2n + 5. \tag{4.3}$$

Since Q is negative definite on $\mathscr{X}$, using (4.2) and (4.3), we complete the proof of Theorems 1 and 2. □

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