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Realizations of factor representations of finite type with emphasis on their characters for wreath products of compact groups with the infinite symmetric group

By

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Abstract

Characters of factor representations of finite type of the wreath products $G = \mathfrak{S}_\infty(T)$ of any compact groups T with the infinite symmetric group $\mathfrak{S}_\infty$ were explicitly given in [HH4]–[HH6], as the extremal continuous positive definite class functions f_A on G determined by a parameter A. In this paper, we give a special kind of realization of a factor representation π^A associated to f_A. This realization is better than the Gelfand-Raikov realization $\pi_f, f = f_A$, in [GR] at least at the point where a matrix element $\langle \pi^A(g)v_0, v_0\rangle$ of π^A for a cyclic vector v_0 can be calculated explicitly, which is exactly equal to the character f_A (and so π^A has a trace-element v_0). So the positive-definiteness of class functions f_A given in [HH4]–[HH6] is automatically guaranteed, a proof of which occupies the first half of [HH6] in the case of T infinite. The case where T is abelian contains the cases of infinite Weyl groups and the limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r, 1, n)$ of complex reflexion groups.

Introduction

Let A be a datum which determines a character f_A of the wreath product group $\mathfrak{S}_\infty(T)$ of compact group T with the infinite symmetric group $\mathfrak{S}_\infty$. We mean by a character an extremal continuous positive definite class function on the group. The precise parametrization through A is recalled in Section 2. The aim of this paper is to construct a nice realization of a factor representation of finite type of $\mathfrak{S}_\infty(T)$ for any A which yields f_A as its matrix element.

The character formula for $\mathfrak{S}_\infty$ was established by Thoma in [Tho2]. Later in [VK1], Vershik-Kerov characterized the Thoma parameters as asymptotic frequencies of growing Young diagrams and showed that the characters of $\mathfrak{S}_\infty$

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are expressed as pointwise limits of the normalized irreducible characters of $\mathfrak{S}_n$, the symmetric group of degree n. Hirai captured the Thoma characters in [Hir] by using a different kind of approximation procedure. This method has an advantage that it is applicable to general wreath product groups including the infinite Weyl groups of other types. In a series of works [HH1]–[HH6], Hirai-Hirai obtained a complete character formula for the wreath product $\mathfrak{S}_\infty(T)$ of any compact group T with the infinite symmetric group $\mathfrak{S}_\infty$.

On the other hand, Vershik-Kerov constructed in [VK2] a factor representation of finite type of $\mathfrak{S}_\infty$ which realizes the Thoma character as its matrix element. It is useful to give such a nice realization of the factor representation. Among its applications, let us mention here two cases. In [BG], Bożejko-Guţă obtained a class of generalized Brownian motions associated with the Thoma characters. A positive definite function on $\mathcal{P}_2(\infty)$, the set of the pair partitions, is needed to introduce a Gaussian state of the algebra generated by the field operators on a certain Fock space. They used the realization due to Vershik-Kerov to extend the Thoma character on $\mathfrak{S}_\infty$ to $\mathcal{P}_2(\infty)$. Another example is due to Biane in [Bia] concerning asymptotic concentration which is observed in irreducible decomposition of some representations of $\mathfrak{S}_n$ as $n \to \infty$. For example, in irreducible decomposition of the regular representation of $\mathfrak{S}_n$, we see that a typical irreducible component occupies a dominant size (the so-called limit shape of Young diagrams) under appropriate scaling limit. Biane showed in [Bia] that such a concentration phenomenon is observed in a sequence of the Vershik-Kerov factor representations and that the typical irreducible component is characterized by using free probability theory. See also [Hor] for a survey on this concentration phenomenon and free probability.

Motivated by these facts in the above paragraphs, we are led to construct those realizations analogous to Vershik-Kerov's for the explicitly given characters of $\mathfrak{S}_\infty(T)$. Apart from expected similar applications to the case of $\mathfrak{S}_\infty$, we note that our realization gives an alternative simpler proof of the positive-definiteness for f_A in [HH4]–[HH6], which is given at first as a class function on the group by a formula (cf. the right hand side of (2.6) below), and which should be proved to be positive definite and extremal, and then to cover all characters of factor representations of finite type of the group.

The paper is organized as the table of contents. After reviewing the character formula for $\mathfrak{S}_\infty(T)$, we construct our realization of the factor representation step by step.

1. Wreath product $\mathfrak{S}_\infty(T)$ of a compact group T with the infinite symmetric group $\mathfrak{S}_\infty$

1.1. Wreath product $\mathfrak{S}_\infty(T)$ of a compact group T with $\mathfrak{S}_\infty$

A permutation σ on a set J is called *finite* if its support $\mathrm{supp}(\sigma) := \{\, j \in J \,;\, \sigma(j) \neq j \,\}$ is finite, and we denote by $\mathfrak{S}_J$ the group of all finite permutations on J. The *infinite symmetric group* $\mathfrak{S}_\infty$ is the permutation group $\mathfrak{S}_{\boldsymbol{N}}$ on the set of natural numbers $\boldsymbol{N}$.

Let T be a compact group. We consider a wreath product group $\mathfrak{S}_J(T)$ of T with a permutation group $\mathfrak{S}_J$ as follows:

$$(1.1)\qquad \mathfrak{S}_J(T) = D_J(T) \rtimes \mathfrak{S}_J\,,\ D_J(T) = \textstyle\prod'_{j\in J} T_j\,,\ \ T_j = T\ (j \in J),$$

where the symbol $\prod'$ means the restricted direct product or, for $d = (t_j)_{j\in J} \in D_J(T)$, $t_i = e_T$ the identity element of T, except a finite number of $i \in J$. An element $\sigma \in \mathfrak{S}_J$ acts on $D_J(T)$ as

$$(1.2)\qquad D_J(T) \ni d = (t_j)_{j\in J} \overset{\sigma}{\longmapsto} \sigma(d) = (t'_j)_{j\in J} \in D_J(T),$$

where $t'_j = t_{\sigma^{-1}(j)}$ $(j \in J)$. Identifying groups $D_J(T)$ and $\mathfrak{S}_J$ with their images in the semidirect product $\mathfrak{S}_J(T)$, we have $\sigma\, d\, \sigma^{-1} = \sigma(d)$. The groups $\mathfrak{S}_{I_n}, D_{I_n}(T)$ and $\mathfrak{S}_{I_n}(T)$ for $I_n = \{1, 2, \ldots, n\} \subset \boldsymbol{N}$ are denoted by $\mathfrak{S}_n, D_n(T)$ and $\mathfrak{S}_n(T)$ respectively, then $G := \mathfrak{S}_\infty(T)$ is an inductive limit of compact groups $G_n := \mathfrak{S}_n(T) = D_n(T) \rtimes \mathfrak{S}_n$. An inductive system $(H_n)_{n\geq 1}$ is called in [TSH] a *countable LCG inductive system* if each H_n is locally compact and each homomorphism $H_n \to H_{n+1}$ is homeomorphic. Introducing in the inductive limit $H := \lim_{n\to\infty} H_n$ the inductive limit topology τ_{ind}, we get a topological group [TSH, Theorem 5.7], and it has sufficiently many continuous positive definite functions and so continuous unitary representations [TSH, Section 5]. The present system $(G_n)_{n\geq 1}$ is an example of a countable LCG inductive system.

When T is a non-trivial finite group, the topology τ_{ind} on $G = \mathfrak{S}_\infty(T)$ is discrete, and all the characters of factor representations of finite type were given in [HH1]–[HH2].

When T is infinite, τ_{ind} is neither discrete nor locally compact, and all such characters for G were given in [HH4] and [HH6].

A natural subgroup of $G = \mathfrak{S}_\infty(T)$ is given as a wreath product of T with the alternating group $\mathfrak{A}_\infty$ as $G' := \mathfrak{A}_\infty(T) = D_\infty(T) \rtimes \mathfrak{A}_\infty$.

In the case where T is abelian, let $S \subset T$ be a subgroup, and assume that S is open in T or equivalently the index $[T : S]$ is finite. We define a subgroup $G^S := \mathfrak{S}^S_\infty(T) = D^S_\infty(T) \rtimes \mathfrak{S}_\infty$ as follows: put $P(d) = \prod_{j\in\boldsymbol{N}} t_j$ for $d = (t_j)_{j\in\boldsymbol{N}} \in D_\infty(T)$, and

$$(1.3)\qquad \mathfrak{S}^S_\infty(T) := D^S_\infty(T) \rtimes \mathfrak{S}_\infty$$

with $D^S_\infty(T) := \{\, d = (t_j)_{j\in\boldsymbol{N}}\,;\, P(d) \in S\,\}$. Then G^S is a normal subgroup with a finite index $[G : G^S] = [T : S]$.

For the groups G', G^S and $G'^S := \mathfrak{A}^S_\infty(T) := D^S_\infty(T) \rtimes \mathfrak{A}_\infty$, there hold also the similar character formulas for factor representations of finite type.

This kind of groups $\mathfrak{S}_\infty(T)$ and $\mathfrak{S}^{\{e_T\}}_\infty(T)$ with T abelian, contain the infinite Weyl groups of classical types, $W_{\mathbf{A}_\infty} = \mathfrak{S}_\infty$ of type $\mathbf{A}_\infty$, $W_{\mathbf{B}_\infty} = \mathfrak{S}_\infty(\boldsymbol{Z}_2)$ of type $\mathbf{B}_\infty/\mathbf{C}_\infty$, and $W_{\mathbf{D}_\infty} = \mathfrak{S}^e_\infty(\boldsymbol{Z}_2)$ of type $\mathbf{D}_\infty$, and moreover the inductive limits $\mathfrak{S}_\infty(\boldsymbol{Z}_r) = \lim_{n\to\infty} G(r, 1, n)$ of complex reflexion groups $G(r, 1, n) = \mathfrak{S}_n(\boldsymbol{Z}_r)$ (cf. [AK], [Kaw], [Sho]).

1.2. Standard decomposition of elements and conjugacy classes

An element $g = (d, \sigma) \in G = \mathfrak{S}_\infty(T)$ is called *basic* in the following two cases:

CASE 1: σ is cyclic and $\mathrm{supp}(d) := \{j \in \boldsymbol{N}; t_j \neq e_T\} \subset \mathrm{supp}(\sigma)$;
CASE 2: $\sigma = \mathbf{1}$ and for $d = (t_i)_{i \in \boldsymbol{N}}$, $t_q \neq e_T$ only for one $q \in \boldsymbol{N}$.

The element $(d, \mathbf{1})$ in Case 2 is denoted by ξ_q, and put $\mathrm{supp}(\xi_q) := \mathrm{supp}(d) = \{q\}$.

For a cyclic permutation σ of ℓ integers, we define its *length* as $\ell(\sigma) = \ell$, and for the identity permutation $\mathbf{1}$, put $\ell(\mathbf{1}) = 1$ for convenience. In this connection, ξ_q is also denoted by $(t_q, (q))$ with a trivial cyclic permutation (q) of length 1. In Cases 1 and 2, put $\ell(g) = \ell(\sigma)$ for $g = (d, \sigma)$, and $\ell(\xi_q) = 1$.

An arbitrary element $g = (d, \sigma) \in G$, is expressed as a product of basic elements as

$$g = \xi_{q_1} \xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m \tag{1.4}$$

with $g_j = (d_j, \sigma_j)$ in Case 1, in such a way that the supports of these components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) = \mathrm{supp}(\sigma_j)$ $(1 \leq j \leq m)$, are mutually disjoint. This expression of g is unique up to the orders of ξ_{q_k}'s and g_j's, and is called *standard decomposition* of g. Note that $\ell(\xi_{q_k}) = 1$ for $1 \leq k \leq r$ and $\ell(g_j) = \ell(\sigma_j) \geq 2$ for $1 \leq j \leq m$, and that, for $\mathfrak{S}_\infty$-components, $\sigma = \sigma_1 \sigma_2 \cdots \sigma_m$ gives the cycle decomposition of σ.

To write down conjugacy class of g, we introduce some notations. Denote by $[t]$ the conjugacy class of $t \in T$, and by $T/\!\sim$ the set of all conjugacy classes of T, and $t \sim t'$ denotes that $t, t' \in T$ are mutually conjugate in T. For a basic component $g_j = (d_j, \sigma_j)$ of g, let $\sigma_j = (i_{j,1} \;\; i_{j,2} \;\; \ldots \;\; i_{j,\ell_j})$ and put $K_j := \mathrm{supp}(\sigma_j) = \{ i_{j,1}, i_{j,2}, \ldots, i_{j,\ell_j} \}$ with $\ell_j = \ell(\sigma_j)$. For $d_j = (t_i)_{i \in K_j}$, we put

$$P_{\sigma_j}(d_j) := \big[t'_{\ell_j} t'_{\ell_j - 1} \cdots t'_2 t'_1\big] \in T/\!\sim \quad \text{with} \quad t'_k = t_{i_{j,k}} \; (1 \leq k \leq \ell_j). \tag{1.5}$$

Lemma 1.1. *Let $\sigma \in \mathfrak{S}_\infty$ be a cycle, and put $K = \mathrm{supp}(\sigma)$ and $G_K = \mathfrak{S}_K(T)$.*

(i) *An element $g = (d, \sigma) \in G_K$ is conjugate in it to $g' = (d', \sigma) \in G_K$ with $d' = (t'_i)_{i \in K}, t'_i = e_T \; (i \neq i_0), [t'_{i_0}] = P_\sigma(d)$ for any $i_0 \in K$ arbitrarily fixed.*

(ii) *Identify $\tau \in \mathfrak{S}_\infty$ with its image in $G = \mathfrak{S}_\infty(T)$. Then we have, for $g = (d, \sigma)$,*

$$\tau g \tau^{-1} = (\tau(d), \tau \sigma \tau^{-1}) \;\; (=: (d', \sigma') \; (put) \,),$$

and $P_{\sigma'}(d') = P_\sigma(d)$.

Theorem 1.2. *For an element $g \in G = \mathfrak{S}_\infty(T)$, let its standard decomposition into basic elements be $g = \xi_{q_1} \xi_{q_2} \cdots \xi_{q_r} g_1 g_2 \cdots g_m$ in* (1.4)*, with $\xi_{q_k} = (t_{q_k}, (q_k))$ and $g_j = (d_j, \sigma_j)$, σ_j cyclic,* $\mathrm{supp}(d_j) \subset \mathrm{supp}(\sigma_j)$. *Then the conjugacy class of g is determined by*

$$[t_{q_k}] \in T/\!\sim \quad (1 \leq k \leq r) \quad \textit{and} \quad (P_{\sigma_j}(d_j), \ell(\sigma_j)) \quad (1 \leq j \leq m), \tag{1.6}$$

where $P_{\sigma_j}(d_j) \in T/\!\sim$ and $\ell(\sigma_j) \geq 2$.

1.3. The case where T is abelian

Assume T be abelian. Then the set $T/\!\sim$ of conjugacy classes is equal to T itself. Take a $g \in G$ and take its standard decompositon (1.4). For $g_j = (d_j, \sigma_j)$, put $g'_j := (d'_j, \sigma_j)$, where $d'_j = (t'_i)_{i\in\boldsymbol{N}}$ with $t'_{i_0} = P(d_j) = \prod_{i\in K_j} t_i$ for some $i_0 \in K_j := \mathrm{supp}(\sigma_j)$, and $t'_i = e_T$ elsewhere.

Lemma 1.3. *Let T be abelian. For a $g \in G = \mathfrak{S}_\infty(T)$, let its standard decomposition be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$ in (1.4). Define g'_j $(1 \le j \le m)$ as above and put $g' = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g'_1 g'_2 \cdots g'_m$. Then, g and g' are mutually conjugate in G.*

Corollary 1.4. *A complete set of parameters of the conjugacy classes of non-trivial elements $g \in G = \mathfrak{S}_\infty(T)$ is given by*

$$\{\, t'_1, t'_2, \ldots, t'_r \,\} \quad \text{and} \quad \{\, (u_j, \ell_j)\ ;\ 1 \le j \le m \,\}, \tag{1.7}$$

where $t'_k = t_{q_k} \in T^* := T \setminus \{\, e_T \,\}$, $u_j = P(d_j) \in T$, $\ell_j \ge 2$, and $r + m > 0$.

2. Characters of $\mathfrak{S}_\infty(T)$ with T compact and of $\mathfrak{S}^S_\infty(T)$ with $S \subset T$ abelian compact

2.1. Character formula for factor representations of finite type of $\mathfrak{S}_\infty(T)$

Let $\widehat{T}$ be the dual of T consisting of all equivalence classes of continuous irreducible unitary representations (= IURs). We identify every equivalence class with one of its representative. Thus $\zeta \in \widehat{T}$ is an IUR and denote by χ_ζ its character: $\chi_\zeta(t) = \mathrm{tr}(\zeta(t))$ $(t \in T)$, then $\dim\zeta = \chi_\zeta(e_T)$.

For a $g \in G = \mathfrak{S}_\infty(T)$, let its standard decomposition into basic components be

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m, \tag{2.1}$$

where the supports of components, $q_1, q_2, \ldots, q_r$, and $\mathrm{supp}(g_j) := \mathrm{supp}(\sigma_j)$ $(1 \le j \le m)$, are mutually disjoint. Furthermore, $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \ne e_T$, with $\ell(\xi_{q_k}) = 1$ for $1 \le k \le r$, and σ_j is a cycle of length $\ell(\sigma_j) \ge 2$ and $\mathrm{supp}(d_j) \subset K_j = \mathrm{supp}(\sigma_j)$. For $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put $P_{\sigma_j}(d_j)$ as in (1.5).

For one-dimensional charcters of $\mathfrak{S}_\infty$, we introduce simple notation as

$$\chi_\varepsilon(\sigma) := \mathrm{sgn}_{\mathfrak{S}}(\sigma)^\varepsilon \quad (\sigma \in \mathfrak{S}_\infty\ ;\ \varepsilon = 0, 1). \tag{2.2}$$

As a parameter for characters of G, we prepare a set

$$\alpha_{\zeta,\varepsilon}\ (\zeta \in \widehat{T}, \varepsilon \in \{0,1\}) \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta\in\widehat{T}}\,, \tag{2.3}$$

of decreasing sequences of non-negative real numbers $\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,p})_{p\in\boldsymbol{N}}$,

$$\alpha_{\zeta,\varepsilon,1} \ge \alpha_{\zeta,\varepsilon,2} \ge \alpha_{\zeta,\varepsilon,3} \ge \cdots \ge 0\,,$$

and a set of non-negative $\mu_\zeta \geq 0$ $(\zeta \in \widehat{T})$, which altogether satisfy the condition

$$\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1, \tag{2.4}$$
$$\text{with } \|\alpha_{\zeta,\varepsilon}\| = \sum_{p\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,p}, \quad \|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta.$$

Theorem 2.1 ([HH4]–[HH6]). *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a compact group T with $\mathfrak{S}_\infty$. Then, for a parameter*

$$A := \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \ ; \ \mu \Big), \tag{2.5}$$

in (2.3)–(2.4)*, the following formula determines a character f_A of G: for an element $g \in G$, let* (2.1) *be its standard decomposition, then*

(2.6)
$$f_A(g) = \prod_{1\leq k\leq r} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{p\in\boldsymbol{N}} \frac{\alpha_{\zeta,\varepsilon,p}}{\dim\zeta} + \frac{\mu_\zeta}{\dim\zeta} \right) \chi_\zeta(t_{q_k}) \right\}$$
$$\times \prod_{1\leq j\leq m} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{p\in\boldsymbol{N}} \left(\frac{\alpha_{\zeta,\varepsilon,p}}{\dim\zeta} \right)^{\ell(\sigma_j)} \chi_\varepsilon(\sigma_j) \right) \chi_\zeta\big(P_{\sigma_j}(d_j)\big) \right\},$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$.

Conversely any character of G is given in the form of f_A.

Remark 2.1. The case of $\mathfrak{S}_\infty$ itself can be considered as a special case of $\mathfrak{S}_\infty(T)$ with the trivial $T = \{ e_T \}$. In this case, we have in [Tho2] a parameter (α, β) with decreasing sequences of non-negative real numbers $\alpha = (\alpha_p)_{p\in\boldsymbol{N}}, \beta = (\beta_p)_{p\in\boldsymbol{N}}$ satisfying $\|\alpha\| + \|\beta\| \leq 1$. Take the trivial representation $\mathbf{1}_T$ of $T = \{e_T\}$ superfluously and put $\mu = (\mu_{\mathbf{1}_T})$ with $\mu_{\mathbf{1}_T} = 1 - \|\alpha\| - \|\beta\|$. Then we have the corresponding parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \ ; \ \mu \Big)$, satisfying the equality condition (2.4).

Remark 2.2. Assume T be finite. Put $\widehat{T}^* := \widehat{T} \setminus \{ \mathbf{1}_T \}$ with the trivial representation $\mathbf{1}_T$ of T and $T^* = T \setminus \{e_T\}$. Then, $\sum_{\zeta\in\widehat{T}} (\dim\zeta)\, \chi_\zeta = 0$ and $1 = \chi_{\mathbf{1}_T} = -\sum_{\zeta\in\widehat{T}^*} (\dim\zeta)\, \chi_\zeta$ on T^*. By this linear dependence between characters χ_ζ, we may accept the parameter A for f_A not necessarily under the equality condition (2.4) but under the weaker inequality condition $\sum_{\zeta\in\widehat{T}} \sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| \leq 1$, loosing the validity of the formula of f_A for $t_{q_k} = e_T$ and accordingly for $g = e$ (cf. [HH2]). However we insist here to keep the condition (2.4), called (MAX) condition, and keep the uniqueness of the parameter A and the validity of the character formula even for $t_{q_k} = e_T$ and $g = e$.

Remark 2.3. For a $g = (d, \sigma) \in G = D_\infty(T) \rtimes \mathfrak{S}_\infty$, put $\mathrm{supp}(g) := \mathrm{supp}(\sigma) \subset \boldsymbol{N}$. Let $K(G)$ denote the set of continuous positive definite class functions on G and $K_1(G)$ the normalized ones in $K(G)$. An $f \in K(G)$ is called *factorizable* if $f(g_1 g_2) = f(g_1)f(g_2)$ for any g_1, g_2 such that $\mathrm{supp}(g_1) \cap \mathrm{supp}(g_2) = \emptyset$. The set of all factorizable $f \in K_1(G)$ is denoted by $F(G)$, and that of all extremal $f \in K_1(G)$ or characters of G is denoted by $E(G)$. It is proved in [HH6, Section 4] that $f \in K_1(G)$ is extremal if and only if it is factorizable, that is, $E(G) = F(G)$. This important fact helps us to analyse situations and to calculate matrix elements in the succeeding sections.

Note that, in the first half of [HH6], it is proved that the class function f_A given by the formula (2.6) is positive definite if the parameter A in (2.5) is given by (2.3)–(2.4), and that, in the second half of [HH6], it is proved that the set $E'(G)$ of such functions f_A is exactly equal to the set $F(G)$ of normalized *factorizable* positive definite class functions: $E'(G) = F(G)$. Since $E(G) = F(G)$, we have $E'(G) = F(G) = E(G)$.

2.2. Characters of $\mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$, T abelian

When T is abelian, the general character formula (2.6) for $G = \mathfrak{S}_\infty(T) = D_\infty(T) \rtimes \mathfrak{S}_\infty$ with a compact group T has a simplified form. In this abelian case, $\widehat{T}$ is nothing but the dual group consisting of all one-dimensional characters of T, and for each $\zeta \in \widehat{T}$, its character χ_ζ is identified with ζ itself.

For a $g \in G$, let its standard decomposition be as in (2.1),

$$g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m,$$

with $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \neq e_T$, for $1 \le k \le r$, and $g_j = (d_j, \sigma_j)$ for $1 \le j \le m$. Put $K_j = \mathrm{supp}(\sigma_j)$, and for $d_j = (t_i)_{i \in K_j} \in D_{K_j}(T) \hookrightarrow D_\infty(T)$, put

$$P_{K_j}(d_j) = \prod_{i \in K_j} t_i, \quad \zeta(d_j) := \zeta(P_{K_j}(d_j)) = \prod_{i \in K_j} \zeta(t_i). \tag{2.7}$$

As a parameter for characters of G, we prepare a set

$$\alpha_{\zeta,\varepsilon} \ (\zeta \in \widehat{T}, \varepsilon \in \{0,1\}) \quad \text{and} \quad \mu = (\mu_\zeta)_{\zeta \in \widehat{T}}, \tag{2.8}$$

of decreasing sequences of non-negative real numbers $\alpha_{\zeta,\varepsilon} = (\alpha_{\zeta,\varepsilon,p})_{p \in \boldsymbol{N}}$, and a set of non-negative $\mu_\zeta \ge 0$ ($\zeta \in \widehat{T}$), which satisfy the condition

$$\sum_{\zeta \in \widehat{T}} \sum_{\varepsilon \in \{0,1\}} \|\alpha_{\zeta,\varepsilon}\| + \|\mu\| = 1. \tag{2.9}$$

Theorem 2.2 ([HH1], [HH4]–[HH6]). *Let $G = \mathfrak{S}_\infty(T)$ be a wreath product group of a compact abelian group T with $\mathfrak{S}_\infty$. Then, for a parameter $A := \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \ ; \ \mu\Big)$ in (2.8)–(2.9), the following formula determines a character f_A of G: for an element $g \in G$, let its standard decomposition be* *as above, then*

$$f_A(g) = \prod_{1\le k\le r} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{p\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,p} \;+\; \mu_\zeta \right) \zeta(t_{q_k}) \right\}$$
$$\times \prod_{1\le j\le m} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{p\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,p})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \right) \zeta(d_j) \right\}, \tag{2.10}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, *and* $\zeta(d_j)$ *as in* (2.7).

Conversely any character of G *is given in the form of* f_A.

2.3. Characters of the subgroup $\mathfrak{S}_\infty^S(T) \subset \mathfrak{S}_\infty(T)$ with $S \subset T$ abelian

Let T be abelian and $S \subset T$ an open subgroup. Let $G^S = \mathfrak{S}_\infty^S(T) = D_\infty^S(T) \rtimes \mathfrak{S}_\infty$ be the natural subgroup defined in (1.3). Then it has a general character formula similar to that for $G = \mathfrak{S}_\infty(T)$.

Take an element $g \in G^S = \mathfrak{S}_\infty^S(T)$ and let its standard decomposition as an element of $G \supset G^S$ be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r} g_1 g_2 \cdots g_m$ with $\xi_{q_k} = (t_{q_k}, (q_k))$ and $g_j = (d_j, \sigma_j), d_j = (t_i)_{i\in K_j}, K_j = \mathrm{supp}(\sigma_j)$. Note that each component ξ_{q_k} does not necessarily belong to G^S, and that the component $g_j = (d_j, \sigma_j)$ belongs to G^S if and only if $P(d_j) = \prod_{i\in K_j} t_i \in S$. Even so, we have the following character formula for the subgroup G^S, deduced from Theorem 2.2.

Theorem 2.3 ([HH5], [HH6]). *Let* $G^S = \mathfrak{S}_\infty^S(T)$ *be the subgroup of* $G = \mathfrak{S}_\infty(T)$ *given by* (1.3) *with* T *abelian and compact and* $S \subset T$ *an open subgroup. Then, for a parameter* $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$ *in* (2.8)–(2.9), *the following formula determines a character* f_A^S *of* G^S*: for an element* $g \in G^S$, *let its standard decomposition be as above, then*

$$f_A^S(g) = \prod_{1\le k\le r} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{p\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,p} \;+\; \mu_\zeta \right) \zeta(t_{q_k}) \right\}$$
$$\times \prod_{1\le j\le m} \left\{ \sum_{\zeta\in\widehat{T}} \left(\sum_{\varepsilon\in\{0,1\}} \sum_{p\in\boldsymbol{N}} (\alpha_{\zeta,\varepsilon,p})^{\ell(\sigma_j)} \cdot \chi_\varepsilon(\sigma_j) \right) \zeta(d_j) \right\}, \tag{2.11}$$

where $\chi_\varepsilon(\sigma_j) = \mathrm{sgn}_{\mathfrak{S}}(\sigma_j)^\varepsilon = (-1)^{\varepsilon(\ell(\sigma_j)-1)}$, *and* $\zeta(d_j)$ *as in* (2.7).

Conversely any character of G^S *is given in the form of* f_A^S.

For the proofs, see [HH5, Section 17] and [HH6, Section 14].

The parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$ for f_A^S is not unique even under the normalization condition (2.9). Define a translation $R(\zeta_0)$ on A by an element $\zeta_0 \in \widehat{T}$ as

$$R(\zeta_0)A := \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,R(\zeta_0)\mu\Big) \tag{2.12}$$

with $\alpha'_{\zeta,\varepsilon} = \alpha_{\zeta\zeta_0^{-1},\varepsilon}\ \big((\zeta,\varepsilon) \in \widehat{T}\times\{0,1\}\big)$; $R(\zeta_0)\mu = (\mu'_\zeta)_{\zeta\in\widehat{T}}$, $\mu'_\zeta = \mu_{\zeta\zeta_0^{-1}}$.

Proposition 2.4. *Assume that two parameters for characters*

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big) \quad and \quad A' = \Big((\alpha'_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu'\Big)$$

both satisfy the normalization condition (2.9)*. Then, they determine the same character on* G^S*, or* $f^S_A = f^S_{A'}$*, if and only if* $A' = R(\zeta_0)A$ *for some* $\zeta_0 \in \widehat{T}$ *which is trivial on* S*.*

In this case, as characters on the bigger group $G \supset G^S$*, we have* $f_{A'}(g) = \pi_{\zeta_0}(g)\cdot f_A(g)$ $(g \in G)$*, where* π_{ζ_0} *is a one-dimensional character of* G *defined as* $\pi_{\zeta_0}(g) := \zeta_0(P(d))$ *for* $g = (d,\sigma) \in G$*. Thus each character of finite type on* G^S *has at most and in general* $|T/S|$ *number of different extensions as characters on* G*.*

3. Special realization of factor representations of $\mathfrak{S}_\infty(T)$, T abelian compact

Let T be abelian compact. Take a parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$ in (2.8)–(2.9), and consider the character f_A in (2.10) of $G = \mathfrak{S}_\infty(T)$ corresponding to it. In this section, we construct a factor representation of finite type π^A of $G = \mathfrak{S}_\infty(T)$ such that in its representation space $\mathcal{H}^A$ there exists a cyclic unit vector w_0 such that $\langle \pi^A(g)w_0, w_0\rangle = f_A(g)$ $(g \in G)$. So w_0 is a trace-element of π^A.

First put

$$\mathcal{X} = \Bigg(\bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \mathcal{N}_{\zeta,\varepsilon}\Bigg) \bigsqcup \Bigg(\bigsqcup_{\zeta\in\widehat{T}} \Xi_\zeta\Bigg), \tag{3.1}$$

where $\mathcal{N}_{\zeta,\varepsilon} = \{(\zeta,\varepsilon,p); p \in \boldsymbol{N}\} \cong \boldsymbol{N}$, $\Xi_\zeta = \{(\zeta,\xi); \xi \in [0,\mu_\zeta]\} \cong [0,\mu_\zeta]$ $(\zeta \in \widehat{T})$, and put $\mathcal{X}_{disc} = \bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \mathcal{N}_{\zeta,\varepsilon}$ and $\mathcal{X}_{cont} = \bigsqcup_{\zeta\in\widehat{T}} \Xi_\zeta$, then $\mathcal{X} = \mathcal{X}_{disc} \bigsqcup \mathcal{X}_{cont}$.

Let ν be a probability measure on $\mathcal{X}$ given by A through

$$\nu(\{(\zeta,\varepsilon,p)\}) = \alpha_{\zeta,\varepsilon,p} \ \ (p \in \boldsymbol{N}), \qquad d\nu((\zeta,\xi)) = d\xi \ \ (\xi \in [0,\mu_\zeta]), \tag{3.2}$$

where $d\xi$ denotes the Lebesgues measure on the interval $[0,\mu_\zeta]$.

Put $I = I_N$ for $N = 1,2,\ldots,\infty$, where $I_N = \{1,2,\ldots,N\}$ for $N < \infty$ and $I_\infty = \boldsymbol{N}$, and further put $G_I = \mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I$ as in Section 1. Then $G_I = \mathfrak{S}_N(T)$ for $I = I_N$. We put $\mathcal{X}^I = \{\boldsymbol{x} = (x_i)_{i\in I}, x_i \in \mathcal{X}_i = \mathcal{X}\ (i \in I)\}$ and take the product measure $\nu^I = \prod_{i\in I}\nu_i$ with $\nu_i = \nu$ on $\mathcal{X}_i = \mathcal{X}$. Then the permutation group $\mathfrak{S}_I$ acts on $\mathcal{X}^I = \prod_{i\in I}\mathcal{X}_i$ as $\sigma(\boldsymbol{x}) = (x_{\sigma^{-1}(i)})$ for $\sigma \in \mathfrak{S}_I$, and leaves invariant ν^I.

3.1. Fundamental representations of $G_I = \mathfrak{S}_I(T)$

3.1.1. Representation $\Pi'_{\mathcal{X}}$

For each $x \in \mathcal{X}$, we prepare a T-module as follows. For $x = (\zeta,\varepsilon,p) \in \mathcal{X}_{disc}$ with $(\zeta,\varepsilon) \in \widehat{T}\times\{0,1\}, p \in \boldsymbol{N}$, put $V(x) = V(\zeta,\varepsilon,p) = V(\zeta,\varepsilon)$ and $Z_x = Z_{\zeta,\varepsilon}$,

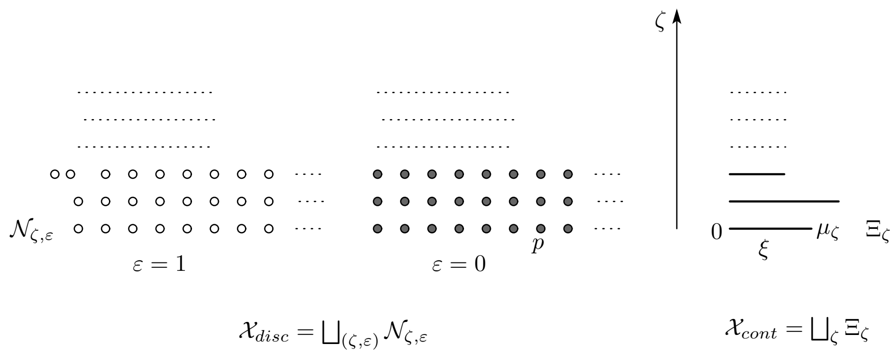

Figure 1. Image of $\mathcal{X} = \mathcal{X}_{disc} \sqcup \mathcal{X}_{cont}$

where $V(\zeta, \varepsilon) \cong \boldsymbol{C}$ is a one-dimensional T-module indexed by $\varepsilon = 0, 1$ such that

$$Z_{\zeta,\varepsilon}(t)v = \zeta(t)v \qquad (v \in V(\zeta, \varepsilon),\ t \in T) \tag{3.3}$$

For $x = (\zeta, \xi) \in \mathcal{X}_{cont}$ with $\zeta \in \widehat{T}$, put $V(x) = V(\zeta, \xi) = V(\zeta)$ and $Z_x = \zeta$, where $V(\zeta) \cong \boldsymbol{C}$ is a one-dimensional T-module such that

$$Z_\zeta(t)v = \zeta(t)v \quad (v \in V(\zeta),\ t \in T). \tag{3.4}$$

Denote by $\boldsymbol{V}(\mathcal{X})$ the sum of a direct sum and a direct integral of $V(x)$'s as

$$\boldsymbol{V}(\mathcal{X}) = \sum_{x \in \mathcal{X}_{disc}}^{\oplus} V(x) \bigoplus \int_{\mathcal{X}_{cont}}^{\oplus} V(x)\, d\nu(x) = \int_{\mathcal{X}}^{\oplus} V(x)\, d\nu(x).$$

For a measurable vector field $\boldsymbol{v} = (\boldsymbol{v}(x))_{x \in \mathcal{X}}, \boldsymbol{v}(x) \in V(x)$, on $\mathcal{X}$, define its norm as

$$\|\boldsymbol{v}\|^2 = \int_{\mathcal{X}} \|\boldsymbol{v}(x)\|^2\, d\nu(x).$$

Then the vector field $\mathbf{1}_{\mathcal{X}} = (1_x)$ with $1_x = 1 \in V(x) \cong \boldsymbol{C}$ is a unit vector since $\|\mathbf{1}_{\mathcal{X}}\|^2 = \int_{\mathcal{X}} d\nu(x) = 1$. The Hilbert space $\boldsymbol{V}(\mathcal{X})$ consists of measurable vector fields $\boldsymbol{v}$ with $\|\boldsymbol{v}\| < \infty$, and on it we have a T-module structure as

$$\big(Z_{\mathcal{X}}(t)\boldsymbol{v}\big)(x) := Z_x(t)\boldsymbol{v}(x) \quad (x \in \mathcal{X}). \tag{3.5}$$

Take copies $\boldsymbol{V}(\mathcal{X})_i = \boldsymbol{V}(\mathcal{X}_i) = \boldsymbol{V}(\mathcal{X})$ for $i \in I$ and make tensor product

$$\boldsymbol{W}(\mathcal{X}) = \otimes_{i \in I} \boldsymbol{V}(\mathcal{X})_i = \otimes_{i \in I} \boldsymbol{V}(\mathcal{X}_i)$$

with respect to a reference vector $(\mathbf{1}_{\mathcal{X}_i})_{i \in I}$. This means that the space is spanned by the set of decomposable elements $\otimes_{i \in I} \boldsymbol{v}_i$ such that $\boldsymbol{v}_i = \mathbf{1}_{\mathcal{X}_i}$ for sufficiently large $i \in I$.

We give a unitary representation $\Pi'_{\mathcal{X}}$ of $G_I = D_I(T) \rtimes \mathfrak{S}_I$ on $\boldsymbol{W}(\mathcal{X})$ as follows. First we put for $d = (t_i)_{i\in I} \in D_I(T)$ as

$$\Pi'_{\mathcal{X}}(d)\big(\otimes_{i\in I}\, \boldsymbol{v}_i\big) := \otimes_{i\in I}\big(Z_{\mathcal{X}_i}(t_i)\boldsymbol{v}_i\big)$$

for $\boldsymbol{w} = \otimes_{i\in I}\boldsymbol{v}_i \in \boldsymbol{W}(\mathcal{X}) = \otimes_{i\in I}\,\boldsymbol{V}(\mathcal{X}_i)$ with $\boldsymbol{v}_i \in \boldsymbol{V}(\mathcal{X}_i)$, and put for $\sigma \in \mathfrak{S}_I$, $\kappa(\sigma)\boldsymbol{w} := \otimes_{i\in I}\boldsymbol{v}_{\sigma^{-1}(i)}$. Then we have $\kappa(\sigma)\cdot\Pi'(d)\cdot\kappa(\sigma^{-1}) = \Pi'(\sigma(d))$ with $\sigma(d) = (t_{\sigma^{-1}(i)})_{i\in I}$. From this we get the following result.

Lemma 3.1. *For $g = (d,\sigma) \in G_I = D_I(T) \rtimes \mathfrak{S}_I$, put*

$$\Pi'_{\mathcal{X}}(g)\boldsymbol{w} := \Pi'_{\mathcal{X}}(d)\kappa(\sigma)\boldsymbol{w}$$

for $\boldsymbol{w} \in \boldsymbol{W}(\mathcal{X}) = \otimes_{i\in I}\,\boldsymbol{V}(\mathcal{X}_i)$. Then $\Pi'_{\mathcal{X}}$ is a unitary representation of G_I.

3.1.2. Representation $\Pi_{\mathcal{X}}$

Let us rewrite the above representation using vector fields on $\mathcal{X}^I = \prod_{i\in I}\mathcal{X}_i$ and intoduce a multiplier coming from 1-cocycle for $(\mathfrak{S}_I, \mathcal{X}^I)$. A decomposable element $\boldsymbol{w} = \otimes_{i\in I}\boldsymbol{v}_i \in \boldsymbol{W}(\mathcal{X}) = \otimes_{i\in I}\,\boldsymbol{V}(\mathcal{X}_i)$ can be considered as a measurable vector field on $\mathcal{X}^I$ with values $\boldsymbol{w}(\boldsymbol{x}) = \otimes_{i\in I}\boldsymbol{v}_i(x_i) \in \otimes_{i\in I}V(x_i)$ at $\boldsymbol{x} = (x_i)_{i\in I} \in \mathcal{X}^I$, where the last tensor product is taken with respcet to the reference vector $(1_{x_i})_{i\in I}$, $1_{x_i} = 1 \in V(x_i) \cong \boldsymbol{C}$ when $I = \boldsymbol{N}$.

For the action of $\sigma \in \mathfrak{S}_I$, the value of $\kappa(\sigma)\boldsymbol{w}$ at $\boldsymbol{x} = (x_i)_{i\in I} \in \mathcal{X}^I$ is

$$\begin{aligned}(3.6)\qquad \big(\kappa(\sigma)\boldsymbol{w}\big)(\boldsymbol{x}) = \otimes_{i\in I}\boldsymbol{v}_{\sigma^{-1}(i)}(x_i) &= \kappa'(\sigma)\big(\otimes_{i\in I}\,\boldsymbol{v}_i(x_{\sigma(i)})\big) = \\ &= \kappa'(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}))\big),\end{aligned}$$

where $\kappa'(\sigma)$ at the right hand sides denotes an action similar to $\kappa(\sigma)$ on the spaces of values given as

$$\kappa'(\sigma)\,:\,\otimes_{i\in I}V(x_i) \ni \otimes_{i\in I}v_i \longmapsto \otimes_{i\in I}v_{\sigma^{-1}(i)} \in \otimes_{i\in I}V(x_{\sigma^{-1}(i)}).$$

We remark that in the present case $\dim V(x_i) = 1$ for all $i \in I$, and so we can omit $\kappa'(\sigma)$ if we identify canonically each $\otimes_{i\in I}V(x_i)$ with $\boldsymbol{C}$. However we treat in Section 4 the case where $\dim V(x_i) > 1$, and for $I = I_N, N < \infty$, $\kappa'(\sigma)$ is a linear map from $V_1 \otimes V_2 \otimes \cdots \otimes V_N$ to $V_{\sigma^{-1}(1)} \otimes V_{\sigma^{-1}(2)} \otimes \cdots \otimes V_{\sigma^{-1}(N)}$ with $V_i = V(x_i)$.

Now we can define an operator $\Pi_{\mathcal{X}}(g)$ for $g = (d,\sigma) \in G_I$. Denote by $\mathcal{H}(\mathcal{X})$ the Hilbert space of measurable vector fields $\boldsymbol{w} = (\boldsymbol{w}(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{X}^I}, \boldsymbol{x} = (x_i)_{i\in I} \in \mathcal{X}^I$ with norm $\|\boldsymbol{w}\|^2 := \int_{\mathcal{X}^I}\|\boldsymbol{w}(\boldsymbol{x})\|^2\,d\nu^I(\boldsymbol{x})$, then we put

$$(3.7)\qquad \big(\Pi_{\mathcal{X}}(g)\boldsymbol{w}\big)(\boldsymbol{x}) = (-1)^{j(\sigma,\boldsymbol{x})}\,Z_{\boldsymbol{x}}(d)\kappa'(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}))\big),$$

where $Z_{\boldsymbol{x}}(d) = \prod_{i\in I}Z_{x_i}(t_i)$ for $d = (t_i)_{i\in I}$, and $j(\sigma,\boldsymbol{x})$ is the number of inversions in $(\sigma^{-1}(i))_{i\in J_1(\boldsymbol{x})}$ with $J_1(\boldsymbol{x}) = \{i \in I; x_i \in \bigsqcup_{\zeta\in\widehat{T}}\mathcal{N}_{\zeta,1}\}$.

Proposition 3.2. *The formula* (3.7) *gives a unitary representation of G_I on $\mathcal{H}(\mathcal{X})$.*

Proof. For the factor $(-1)^{j(\sigma,\boldsymbol{x})}$, the 1-cocycle condition

$$(-1)^{j(\tau\sigma,\boldsymbol{x})} = (-1)^{j(\tau,\boldsymbol{x})}(-1)^{j(\sigma,\tau^{-1}(\boldsymbol{x}))} \tag{3.8}$$

should be guranteed. Put $J = J_1(\boldsymbol{x})$ for simplicity, then $J_1(\tau^{-1}(\boldsymbol{x})) = \tau^{-1}J$, since $\tau^{-1}(\boldsymbol{x}) = (x_{\tau(i)})_{i\in I}$. Let z_i $(i \in I)$ be independent variables and put $z = (z_i)_{i\in I}$, $\sigma(z) = (z_{\sigma^{-1}(i)})_{i\in I}$, and for a subset $K \subset I$, put $\nabla_K(z) = \prod_{j<k;j,k\in K}(z_j - z_k)$. Then,

$$\nabla_K(\sigma(z)) = \prod_{j<k;j,k\in K}(z_{\sigma^{-1}(j)} - z_{\sigma^{-1}(k)}) = (-1)^m \nabla_{\sigma^{-1}K}(z),$$

where m is the number of inversions in $\{\sigma^{-1}(k); k \in K\}$. Hence,

$$\nabla_J(\tau(z)) = (-1)^{j(\tau,\boldsymbol{x})}\nabla_{\tau^{-1}J}(z), \qquad \nabla_J((\tau\sigma)(z)) = (-1)^{j(\tau\sigma,\boldsymbol{x})}\nabla_{(\tau\sigma)^{-1}J}(z).$$

On the ther hand, the latter can be calculated in another way as

$$\begin{aligned}\nabla_J((\tau\sigma)(z)) &= \prod_{j<k;j,k\in J}(z_{\sigma^{-1}(\tau^{-1}(j))} - z_{\sigma^{-1}(\tau^{-1}(k))}) \\ &= (-1)^{j(\tau,\boldsymbol{x})} \prod_{j'<k';j',k'\in\tau^{-1}J}(z_{\sigma^{-1}(j')} - z_{\sigma^{-1}(k')}) \\ &= (-1)^{j(\tau,\boldsymbol{x})}(-1)^{j(\sigma,\tau^{-1}(\boldsymbol{x}))}\nabla_{(\tau\sigma)^{-1}J}(z),\end{aligned}$$

where we put $j' = \tau^{-1}(j)$ etc. and take into account $\tau^{-1}J = J_1(\tau^{-1}(\boldsymbol{x}))$. Therefore we get the 1-cocycle condition (3.8).

Since $d \to \Pi_{\mathcal{X}}(d)$ and $\sigma \to \Pi_{\mathcal{X}}(\sigma)$ are unitary representations of $D_I(T)$ and $\mathfrak{S}_I$ respectively, it is enough for us to verify the relation $\Pi_{\mathcal{X}}(\sigma^{-1})\Pi_{\mathcal{X}}(d)\Pi_{\mathcal{X}}(\sigma) = \Pi_{\mathcal{X}}(\sigma^{-1}(d))$. From the formula (3.7) we have

$$\begin{aligned}\big(\Pi_{\mathcal{X}}(\sigma^{-1})\Pi_{\mathcal{X}}(d)\Pi_{\mathcal{X}}(\sigma)\boldsymbol{w}\big)(\boldsymbol{x}) &= (-1)^{j(\sigma,\boldsymbol{x})}(-1)^{j(\sigma^{-1},\sigma(\boldsymbol{x}))}\cdot Z_{\sigma(\boldsymbol{x})}(d)\boldsymbol{w}(\boldsymbol{x}) \\ &= \big(\Pi_{\mathcal{X}}(\sigma^{-1}(d))\boldsymbol{w}\big)(\boldsymbol{x}).\end{aligned}$$

In fact, we have $(-1)^{j(\sigma,\boldsymbol{x})}(-1)^{j(\sigma^{-1},\sigma(\boldsymbol{x}))} = (-1)^{j(\sigma^{-1}\sigma,\boldsymbol{x})} = 1$ from (3.8) and $Z_{\sigma(\boldsymbol{x})}(d) = \prod_{i\in I} Z_{x_{\sigma^{-1}(i)}}(t_i) = \prod_{i\in I} Z_{x_i}(t_{\sigma(i)}) = Z_{\boldsymbol{x}}(\sigma^{-1}(d))$. □

Remark 3.1. For the symmetric group $\mathfrak{S}_I$, a unitary representation of it is defined in [VK2] by $h(x,y) \to \mathrm{sign}(\sigma, x)\, h(\sigma x, y)$, where $x = (x_i)_{i\in I}$ and $\mathrm{sign}(\sigma, x) := (-1)^r$ with $r = i(\sigma, x)$ the number of inversions in $\{\sigma(j); j \in J := J_1(x)\}$. Here we have $i(\sigma, x) = j(\sigma^{-1}, \boldsymbol{x})$ with $\boldsymbol{x} := x$, and the action of $\sigma \in \mathfrak{S}_I$ on $x = (x_i)_{i\in I}$ in [VK2] should be understood as $\sigma x := (x_{\sigma(i)})_{i\in I}$ and so $(\tau\sigma)x = \sigma(\tau x)$. On the other hand, in the formula (3.13) in [BG], the unitary representation of $\mathfrak{S}_I$ is translated from [VK2] as $h(x,y) \to (-1)^{i(\sigma,x)}h(\sigma^{-1}x, y)$. However, the multiplier $(-1)^{i(\sigma,x)}$ should be $(-1)^{i(\sigma^{-1},x)} = (-1)^{j(\sigma,\boldsymbol{x})}$ with $\boldsymbol{x} := x$, because the action $\sigma x := (x_{\sigma^{-1}(i)})_{i\in I}$ here is different from that in [VK2].

3.2. Calculation of matrix elements of $\Pi_{\mathcal{X}}$

Let us compute a matrix element of the representation $\Pi_{\mathcal{X}}$ with the parameter A. Take a $g = (d,\sigma) \in G_I = \mathfrak{S}_I(T)$ and let its standard decomposition into basic components be $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}g_1g_2\cdots g_m$ as in (2.1) with $\xi_{q_k} = (t_{q_k},(q_k)), t_{q_k} \neq e_T$, $g_j = (d_j,\sigma_j)$, σ_j a cycle. Put $Q = \{q_1,q_2,\ldots,q_r\}$, $K_j = \mathrm{supp}(\sigma_j)$ and $K = \mathrm{supp}(\sigma) = \bigsqcup_{1\le j\le m} K_j$. For the parameter A, we put

$$p_\zeta = \|\alpha_{\zeta,0}\| + \mu_\zeta, \quad q_\zeta = \|\alpha_{\zeta,1}\| \quad (\zeta \in \widehat{T}) \quad \text{with} \quad \|\alpha_{\zeta,\varepsilon}\| = \sum_{p\in \boldsymbol{N}} \alpha_{\zeta,\varepsilon,p}.$$

Let $\boldsymbol{w}^0 = \otimes_{i\in I}\mathbf{1}_{\mathcal{X}_i} \in \mathcal{H}(\mathcal{X})$ and let us compute the matrix element $\phi(g) = \langle \Pi_{\mathcal{X}}(g)\boldsymbol{w}^0, \boldsymbol{w}^0\rangle$. This is not so simple but we get a summation formula for $\phi(g)$ in general. The calculations here give us helpful indications for our later task to obtain the character f_A as a matrix element of a cyclic representation π^A corresponding to a unit cyclic vector $\mathbf{1}_\Delta$. From the formula (3.7) we get

$$\phi(g) = \int_{\mathcal{X}^I} (-1)^{j(\sigma,\boldsymbol{x})}\left(\prod_{i\in I} Z_{x_i}(t_i)\right) d\nu^I(\boldsymbol{x}). \tag{3.9}$$

Let $[K_j]$ be the smallest interval in $\boldsymbol{N}$ containing K_j, and put $\overline{K} = \bigcup_{1\le j\le m}[K_j]$, $\boldsymbol{x}_{\overline{K}} = (x_i)_{i\in\overline{K}}$, and $\boldsymbol{x}_K = (x_i)_{i\in K}$. Let $j(\sigma,\boldsymbol{x}_{\overline{K}})$ be the number of inversions in $(\sigma^{-1}(i))$ for $i \in J_1(\boldsymbol{x}_{\overline{K}}) := \{i \in \overline{K}; x_i \in \bigsqcup_{\zeta\in\widehat{T}}\mathcal{N}_{\zeta,1}\}$, and similarly for $j(\sigma,\boldsymbol{x}_K)$. Then we have $(-1)^{j(\sigma,\boldsymbol{x})} = (-1)^{j(\sigma,\boldsymbol{x}_{\overline{K}})}$, but not necessarily $(-1)^{j(\sigma,\boldsymbol{x})} = (-1)^{j(\sigma,\boldsymbol{x}_K)}$. This complicates our calculations.

Lemma 3.3. *Let $g = (d,\sigma) \in G_I = \mathfrak{S}_I(T)$ be with standard decomposition as above. Put for $d = (t_i)_{i\in I}$, $d_{\overline{K}} = (t_i)_{i\in\overline{K}}$ and $Z_{\boldsymbol{x}_{\overline{K}}}(d_{\overline{K}}) := \prod_{i\in\overline{K}} Z_{x_i}(d_i)$ and $\nu^{\overline{K}}$ the product of $\nu_i = \nu$ on $\mathcal{X}_i = \mathcal{X}, i \in \overline{K}$. Assume $Q\cap\overline{K} = \emptyset$ for g, then,*

$$\phi(g) = \prod_{q\in Q}\Big(\sum_{\zeta\in\widehat{T}}(p_\zeta+q_\zeta)\,\zeta(t_q)\Big)\int_{\mathcal{X}^{\overline{K}}}(-1)^{j(\sigma,\boldsymbol{x}_{\overline{K}})}\,Z_{\boldsymbol{x}_{\overline{K}}}(d_{\overline{K}})\,d\nu^{\overline{K}}(\boldsymbol{x}_{\overline{K}}).$$

Further assume for σ that the multiplicative factor $(-1)^{j(\sigma,\boldsymbol{x})}$ has the property

$$(-1)^{j(\sigma,\boldsymbol{x})} = \prod_{1\le j\le m}(-1)^{j(\sigma_j,\boldsymbol{x}_{K_j})} \quad (\boldsymbol{x}\in\mathcal{X}^I). \tag{3.10}$$

Then, with $F(d_j,\sigma_j) = \displaystyle\int_{\mathcal{X}^{K_j}}(-1)^{j(\sigma_j,\boldsymbol{x}_{K_j})}\,Z_{\boldsymbol{x}_{K_j}}(d_j)\,d\nu^{K_j}(\boldsymbol{x}_{K_j})$,

$$\phi(g) = \prod_{q\in Q}\Big(\sum_{\zeta\in\widehat{T}}(p_\zeta+q_\zeta)\,\zeta(t_q)\Big)\times\prod_{1\le j\le m}F(d_j,\sigma_j),$$

where $Z_{\boldsymbol{x}_{K_j}}(d_{K_j}) = \prod_{i\in K_j} Z_{x_i}(t_i)$, and ν^{K_j} is the product of $\nu_i = \nu$ on $\mathcal{X}_i = \mathcal{X}, i \in K_j$.

Proof. If $Q \cap \overline{K} = \emptyset$, then for any $q \in Q$ the variable x_q in $\boldsymbol{x}$ does not change the value $j(\sigma, \boldsymbol{x})$, and so we can perform independently the integration with respect to $d\nu(x_q)$ in the integral expression (3.9) of $\phi(g)$. Then we get

$$\int_{\mathcal{X}_q} Z_{x_q}(t_q)\, d\nu(x_q) = \sum_{\zeta \in \widehat{T}} (\|\alpha_{\zeta,0}\| + \mu_\zeta + \|\alpha_{\zeta,1}\|)\, \zeta(t_q) = \sum_{\zeta \in \widehat{T}} (p_\zeta + q_\zeta)\zeta(t_q).$$

The second assertion is straightforward from the assumption. □

Note that the assumptions in Lemma 3.3 are satisfied if all $K_j = \mathrm{supp}(\sigma_j)$ are intervals in $\boldsymbol{N}$. For the integration $F(d_j, \sigma_j)$ on $\mathcal{X}^{K_j}$, we examine two cases here.

Case 1: $\sigma_j = (1\ 2\ 3\ \ldots\ \ell)^{-1} = (\ell\ \ell{-}1\ \ldots\ 2\ 1)$ with $\ell = \ell_j$ the length of σ_j.

In this case, $K_j = \{1, 2, \ldots, \ell\}$. If $J_1(\boldsymbol{x}_{K_j})$ does not contain ℓ, then $j(\sigma_j, \boldsymbol{x}_{K_j}) = 0$, and the partial integration corresponding to this case gives us

$$\prod_{1 \le i < \ell} \Big(\sum_{\zeta_i \in \widehat{T}} (p_{\zeta_i} + q_{\zeta_i})\, \zeta_i(t_i) \Big) \times \Big(\sum_{\zeta_\ell \in \widehat{T}} p_{\zeta_\ell}\, \zeta_\ell(t_\ell) \Big).$$

If $J_1(\boldsymbol{x}_{K_j})$ contains ℓ, then $j(\sigma_j, \boldsymbol{x}_{K_j}) = |J_1(\boldsymbol{x}_{K_j})| - 1$. Therefore the partial integration corresponding to this case is

$$\prod_{1 \le i < \ell} \Big(\sum_{\zeta_i \in \widehat{T}} (p_{\zeta_i} - q_{\zeta_i})\, \zeta_i(t_i) \Big) \times \Big(\sum_{\zeta_\ell \in \widehat{T}} q_{\zeta_\ell}\, \zeta_\ell(t_\ell) \Big).$$

Therefore we get

$$F(d_j, \sigma_j) = \sum_{\zeta_1, \zeta_2, \ldots, \zeta_\ell \in \widehat{T}} \left\{ \prod_{1 \le i < \ell} (p_{\zeta_i} + q_{\zeta_i}) \cdot p_{\zeta_\ell} + \prod_{1 \le i < \ell} (p_{\zeta_i} - q_{\zeta_i}) \cdot q_{\zeta_\ell} \right\} \prod_{1 \le i \le \ell} \zeta_i(t_i).$$

Case 2: $\sigma_j = (1\ 2\ 3\ \ldots\ \ell)$ with $\ell = \ell_j = \ell(\sigma_j)$.

In this case, $K_j = \{1, 2, \ldots, \ell\}$. If $J_1(\boldsymbol{x}_{K_j})$ does not contain 1, then $j(\sigma_j, \boldsymbol{x}_{K_j}) = 0$, and if $J_1(\boldsymbol{x}_{K_j})$ contains 1, then $j(\sigma_j, \boldsymbol{x}_{K_j}) = |J_1(\boldsymbol{x}_{K_j})| - 1$. Therefore similar calculations as above give us

$$F(d_j, \sigma_j) = \sum_{\zeta_1, \zeta_2, \ldots, \zeta_\ell \in \widehat{T}} \left\{ p_{\zeta_1} \cdot \prod_{2 \le i \le \ell} (p_{\zeta_i} + q_{\zeta_i}) + q_{\zeta_1} \cdot \prod_{2 \le i \le \ell} (p_{\zeta_i} - q_{\zeta_i}) \right\} \prod_{1 \le i \le \ell} \zeta_i(t_i).$$

Example 3.1. Let $\sigma_1 = (1\ 3), \sigma_2 = (2\ 4)$. For $\boldsymbol{x} \in \mathcal{X}^I$, assume that $J_1(\boldsymbol{x}) = \{1, 4\}$, $\{2, 3\}$, then $j(\sigma_1, \boldsymbol{x}_{K_1}) = j(\sigma_2, \boldsymbol{x}_{K_2}) = 0,\ j(\sigma, \boldsymbol{x}) = 1$, whence the equality (3.10) does not hold. Moreover if $J_1(\boldsymbol{x}) = \{1,2,4\}, \{3,2,4\}$ (resp. $\{1,3,2\}, \{1,3,4\}$), then $j(\sigma_1, \boldsymbol{x}_{K_1}) = 0$ (resp. 1), $j(\sigma_2, \boldsymbol{x}_{K_2}) = 1$ (resp. 0), $j(\sigma, \boldsymbol{x}) = 2$, and (3.10) does not hold. Otherwise (3.10) holds for $J_1(\boldsymbol{x}) \subset \{1,2,3,4\}$. From this, we can write down the matrix element $\phi(g)$ for $g = (d, \sigma)$ with $d = (t_1, t_2, t_3, t_4, e_T, e_T, \ldots)$ in a certain sum.

Suggested by this example, we can give a general summation formula for $\phi(g)$ as follows. Devide $\mathcal{X}$ as

$$\mathcal{X} = \mathcal{X}^0 \sqcup \mathcal{X}^1 \quad \text{with} \quad \mathcal{X}^0 := \bigsqcup_{\zeta \in \widehat{T}} (\mathcal{N}_{\zeta,0} \sqcup \Xi_\zeta),\ \mathcal{X}^1 := \bigsqcup_{\zeta \in \widehat{T}} \mathcal{N}_{\zeta,1},$$

and accordingly for $\mathcal{X}_i = \mathcal{X}$ $(i \in I)$, $\mathcal{X}_i = \mathcal{X}_i^0 \sqcup \mathcal{X}_i^1$ with $\mathcal{X}_i^0 = \mathcal{X}^0, \mathcal{X}_i^1 = \mathcal{X}^1$. For a subset $\mathcal{I} \subset \overline{K}$, let $j(\sigma, \mathcal{I})$ be the number of inversions in $\sigma^{-1}(i), i \in \mathcal{I}$, and put $\mathcal{X}^{\overline{K},\mathcal{I}} := \big(\prod_{i \in \overline{K} \setminus \mathcal{I}} \mathcal{X}_i^0\big) \times \big(\prod_{i \in \mathcal{I}} \mathcal{X}_i^1\big)$, Then $(-1)^{j(\sigma,\boldsymbol{x})} = (-1)^{j(\sigma,\mathcal{I})}$ $(\boldsymbol{x} \in \mathcal{X}^{\overline{K},\mathcal{I}})$ and $\mathcal{X}^{\overline{K}} = \bigsqcup_{\mathcal{I} \subset \overline{K}} \mathcal{X}^{\overline{K},\mathcal{I}}$, and

$$\int_{\mathcal{X}^{\overline{K},\mathcal{I}}} Z_{\boldsymbol{x}_{\overline{K}}}(d_{\overline{K}})\, d\nu^{\overline{K}}(\boldsymbol{x}_{\overline{K}}) = \sum_{\zeta_i \in \widehat{T}\ (i \in \overline{K})} \Big(\prod_{i \in \overline{K} \setminus \mathcal{I}} p_{\zeta_i}\Big)\Big(\prod_{i \in \mathcal{I}} q_{\zeta_i}\Big) \prod_{i \in (Q \cap \overline{K}) \cup K} \zeta_i(t_i).$$

Proposition 3.4. *For $g \in G = \mathfrak{S}_\infty(T)$, put $\overline{K} = \bigcup_{1 \le j \le m} [K_j]$. Then*

(3.11)

$$\begin{aligned}\phi(g) &= \prod_{q \in Q \setminus \overline{K}} \Big(\sum_{\zeta \in \widehat{T}} (p_\zeta + q_\zeta)\, \zeta(t_q)\Big) \int_{\mathcal{X}^{\overline{K}}} (-1)^{j(\sigma, \boldsymbol{x}_{\overline{K}})}\, Z_{\boldsymbol{x}_{\overline{K}}}(d_{\overline{K}})\, d\nu^{\overline{K}}(\boldsymbol{x}_{\overline{K}}) \\ &= \prod_{q \in Q \setminus \overline{K}} \Big(\sum_{\zeta \in \widehat{T}} (p_\zeta + q_\zeta)\, \zeta(t_q)\Big) \times \\ &\quad \times \sum_{\mathcal{I} \subset \overline{K}} (-1)^{j(\sigma,\mathcal{I})} \sum_{\zeta_i \in \widehat{T}\ (i \in \overline{K})} \Big(\prod_{i \in \overline{K} \setminus \mathcal{I}} p_{\zeta_i}\Big)\Big(\prod_{i \in \mathcal{I}} q_{\zeta_i}\Big) \prod_{i \in (Q \cap \overline{K}) \cup K} \zeta_i(t_i).\end{aligned}$$

3.3. Construction of factor representations of finite type π^A of G_I

Starting formally from the fundamental representation $\Pi_{\mathcal{X}}$ given above, we construct bigger representation of G_I. For that, we introduce a new variable $\boldsymbol{y} \in \mathcal{X}^I$ controling multiplicities of representations and construct a unitary representation Π whose certain subrepresentation π^A gives a factor representation corresponding to f_A.

For $\boldsymbol{x} = (x_i)_{i \in I} \in \mathcal{X}^I$, take a tensor product $\boldsymbol{W}(\boldsymbol{x}) = \otimes_{i \in I} V(x_i)$ with respect to a reference vector $(1_{x_i})_{i \in I}$ with $1_{x_i} = 1 \in V(x_i) \cong \boldsymbol{C}$, and take a measurable vector field $\boldsymbol{w}$ on $\mathcal{X}^I \times \mathcal{X}^I$ such that $\boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{W}(\boldsymbol{x})$ for $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X}^I \times \mathcal{X}^I$. We define $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x} = \tau(\boldsymbol{y})$ for some $\tau \in \mathfrak{S}_I$. The norm of $\boldsymbol{w}$ is defined by

$$\|\boldsymbol{w}\|^2 = \int_{\mathcal{X}^I} \sum_{\boldsymbol{y} \sim \boldsymbol{x}} \|\boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y})\|^2\, d\nu^I(\boldsymbol{x}), \tag{3.12}$$

and this gives us a Hilbert space $\mathcal{H}$. The action of $g = (d, \sigma) \in G_I$ is defined through $\Pi_{\mathcal{X}}(g)$ acting on $\boldsymbol{x}$-side as

$$(3.13)\qquad \big(\Pi(g)\boldsymbol{w}\big)(\boldsymbol{x}, \boldsymbol{y}) = (-1)^{j(\sigma, \boldsymbol{x})} Z_{\boldsymbol{x}}(d)\, \kappa'(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}), \boldsymbol{y})\big).$$

Similarly as Proposition 3.2, we can prove that $(\Pi, \mathcal{H})$ is a unitary representation of G_I.

Let Δ be the diagonal subset of $\mathcal{X}^I \times \mathcal{X}^I$ and $\mathbf{1}_\Delta$ its characteristic function, then $\|\mathbf{1}_\Delta\|^2 = \int_{\mathcal{X}^I} d\nu^I(\boldsymbol{x}) = 1$. Let $\mathcal{H}^A$ be the closed linear span of $\Pi(G_I)\mathbf{1}_\Delta$ and π^A be the restriction of Π on the subspace $\mathcal{H}^A$. Summarizing these results, we have the following.

Proposition 3.5. *The set of measurable vector fields $\boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y})$, $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X}^I \times \mathcal{X}^I$, with norm* (3.12) *gives a Hilbert space $\mathcal{H}$, and $\Pi(g)$ $(g \in G_I)$ in* (3.13) *is a unitary representation of G_I on $\mathcal{H}$. Its subrepresentation π^A on $\mathcal{H}^A$ has a cyclic unit vector $\mathbf{1}_\Delta$.*

3.4. Calculation of a matrix element for π^A

We calculate the matrix element for $\mathbf{1}_\Delta$ using integral expression

$$(3.14)\quad \langle \pi^A(g)\mathbf{1}_\Delta, \mathbf{1}_\Delta\rangle = \\ = \int_{\mathcal{X}^I} \sum_{\boldsymbol{y}\sim\boldsymbol{x}} (-1)^{j(\sigma,\boldsymbol{x})} \langle Z_{\boldsymbol{x}}(d)\kappa'(\sigma)\big(\mathbf{1}_\Delta(\sigma^{-1}(\boldsymbol{x}), \boldsymbol{y})\big), \mathbf{1}_\Delta(\boldsymbol{x}, \boldsymbol{y})\rangle\, d\nu^I(\boldsymbol{x}),$$

and get a factorizable positive definite class function on G_I.

Theorem 3.6. (i) *The matrix element $\langle \pi^A(g)\mathbf{1}_\Delta, \mathbf{1}_\Delta\rangle$ is given by the same formula as that for the function f_A in Theorem* 2.2. *In particular, assume $I = \boldsymbol{N}$. Then the matrix element $\langle \pi^A(g)\mathbf{1}_\Delta, \mathbf{1}_\Delta\rangle$ is equal to the extremal positive definite class function f_A on $G_I = \mathfrak{S}_\infty(T)$ in Theorem* 2.2 *corresponding to a parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$.*

(ii) *The cyclic representation π^A generated by $\mathbf{1}_\Delta$ is a factor representation of finite type with normalized character f_A.*

Proof. Denote by $\Phi(g)$ the matrix element in (3.14). For $g = (d, \sigma) \in G_I$, the integrand in (3.14) is not zero only when $\sigma^{-1}(\boldsymbol{x}) = \boldsymbol{y} = \boldsymbol{x}$.

For a general element $g \in G_I$, take its standard decomposition as in (2.1), $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}\, g_1 g_2 \cdots g_m$, with $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \neq e_T$, for $1 \le k \le r$, and $g_j = (d_j, \sigma_j)$ for $1 \le j \le m$. Put $Q = \{q_1, q_2, \ldots, q_r\}$, $K_j = \mathrm{supp}(\sigma_j)$, and for $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T)$, $\zeta(d_j) = \prod_{i\in K_j}\zeta(t_i)$. The condition $\sigma^{-1}(\boldsymbol{x}) = \boldsymbol{x}$ says that, for each j, all x_i $(i \in K_j)$ coincide with each other. So the set of such elements, taking from $\mathcal{X}^{K_j}$, is equal to the set of $x_{K_j} = (x_i)_{i\in K_j}$ given as

$$\mathcal{Z}_{K_j} := \bigsqcup_{(\zeta,\varepsilon)} \bigsqcup_{p\in\boldsymbol{N}} \{x_{K_j}; (\forall i)\ x_i = (\zeta, \varepsilon, p)\} \\ \bigsqcup \bigsqcup_{\zeta\in\widehat{T}} \{x_{K_j}; (\forall i)\ x_i = (\zeta, \xi), \xi \in [0, \mu_\zeta]\},$$

where (ζ, ε) runs over $\widehat{T} \times \{0,1\}$. The point mass of $\{x_{K_j}; (\forall i)\ x_i = (\zeta, \varepsilon, p)\}$ with respect to the product measure ν^{K_j} is equal to $(\alpha_{\zeta,\varepsilon,p})^{|K_j|} = (\alpha_{\zeta,\varepsilon,p})^{\ell(\sigma_j)}$.

The integral in (3.14) can be carried out independently on each component $\mathcal{X}_{q_k}$ and $\mathcal{X}^{K_j}$ because, if $\sigma^{-1}(\boldsymbol{x}) = \boldsymbol{x}$, either $J_1(\boldsymbol{x}) \supset K_j$ or $J_1(\boldsymbol{x}) \cap K_j = \emptyset$ holds and the sign $(-1)^{j(\sigma, \boldsymbol{x})}$ is decomposed into a product as

$$(-1)^{j(\sigma,\boldsymbol{x})} = \prod_{1 \le j \le m} \chi_{\varepsilon_j}(\sigma_j) = \prod_{1 \le j \le m} (-1)^{\varepsilon_j(\ell(\sigma_j)-1)},$$

if, for $1 \le j \le m$, the component of x_{K_j} is given by $(\zeta_j, \varepsilon_j, p_j) \in \mathcal{X}_{disc}$. Thus $\Phi(g)$ is expressed as a product of integrals as

$$\begin{aligned} &\prod_{q \in Q} \int_{\mathcal{X}} Z_x(t_q)\, d\nu(x) \times \\ &\times \prod_{1 \le j \le m} \int_{\mathcal{Z}_{K_j}(\subset \mathcal{X}^{K_j})} (-1)^{j(\sigma_j, x_{K_j})} \prod_{i \in K_j} Z_{x_i}(t_i)\, d\nu^{K_j}(x_{K_j}). \end{aligned} \tag{3.15}$$

For each factor of the first term, we get

$$\int_{\mathcal{X}} Z_x(t_q)\, d\nu(x) = \sum_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \sum_{p \in \boldsymbol{N}} \alpha_{\zeta,\varepsilon,p}\, \zeta(t_q) + \sum_{\zeta \in \widehat{T}} \mu_\zeta\, \zeta(t_q).$$

For each factor of the second term, the integral for the integration subdomain $\bigsqcup_{\zeta \in \widehat{T}} \{x_{K_j};\ (\forall i)\ x_i = (\zeta, \xi), \xi \in [0, \mu_\zeta]\} \subset \mathcal{Z}_{K_j} \subset \mathcal{X}^{K_j}$ is zero because for each $\zeta \in \widehat{T}$ the domain is a one-dimensional subset in $[0, \mu_\zeta]^{|K_j|}$ of dimension $\ell_j = |K_j| \ge 2$. On the other hand, the value of the integrand for the subdomain $\bigsqcup_{p \in \boldsymbol{N}} \{x_{K_j}; (\forall i)\ x_i = (\zeta, \varepsilon, p)\}$ is $\zeta(d_j)\chi_\varepsilon(\sigma_j)$, and so we have

$$\begin{aligned} \int_{\mathcal{Z}_{K_j}} (-1)^{j(\sigma_j, x_{K_j})} \prod_{i \in K_j} Z_{x_i}(t_i)\, d\nu^{K_j}(x_{K_j}) = \\ = \sum_{(\zeta,\varepsilon) \in \widehat{T} \times \{0,1\}} \sum_{p \in \boldsymbol{N}} (\alpha_{\zeta,\varepsilon,p})^{\ell(\sigma_j)} \chi_\varepsilon(\sigma_j)\, \zeta(d_j). \end{aligned}$$

By calculations above, we get the product formula for $\Phi(g)$ as for f_A in (2.10), and in case $I = \boldsymbol{N}$ we get $\Phi = f_A$, and the first assertion of the theorem is proved.

For the second assertion, note that the matrix element corresponding to the cyclic vector $\mathbf{1}_\Delta$ is equal to f_A, whence π^A is equivalent to the Gelfand-Raikov representation π_f associated to $f = f_A$. On the other hand, f_A is known in Theorem 2.2 as a normalized character of a factor representation of finite type. Therefore the Gelfand-Raikov representation π_{f_A} is known to be factorial of finite type and its character is equal to f_A, according to a general theory for the representation of topological groups (Theorem 1.6.2 in [HH3]). □

Remark 3.2. In the paper [HH6], the positive-definiteness of the class functions f_A is proved in the first half. Theorem 3.6 above and Theorem 4.7

below give another proof of the positive-definiteness of the functions f_A. See also Remark 2.3.

Remark 3.3. For $I = \boldsymbol{N}$, the cyclic representation π^A has

$$\langle \pi^A(g)\mathbf{1}_\Delta, \mathbf{1}_\Delta\rangle = f_A(g)$$

as its character, and so the factor $\pi^A(G_I)''$ has $\mathbf{1}_\Delta$ as its *trace-element* in the sense of Definition 3 in [Dix, I.6.3].

3.5. Special cases of Theorem 3.6

Here we expose some details in certain special cases. This is to explain how we could arrive from the study of special cases to Theorem 3.6 in the most general case, and also to become more familiar to the method of constructing factor representations.

3.5.1. Special case: $\sum_{\varepsilon\in\{0,1\}} \|\alpha_{\zeta,\varepsilon}\| = 1$ for a cetain $\zeta \in \widehat{T}$ for A

We fix $\zeta \in \widehat{T}$. Let us examine first the simplest case where $\alpha_{\zeta,\varepsilon} = (1, 0, 0, \ldots)$ for some $\varepsilon \in \{0, 1\}$. For each $i \in I$, let $V(\zeta,\varepsilon)_i$ be a copy of $V(\zeta,\varepsilon)$ in (3.3), and consider a tensor product $\otimes_{i\in I} V(\zeta,\varepsilon)_i$ and make it a G_I-module by an action of $g = (d, \sigma) \in G_I$ with $d = (t_i)_{i\in I}, t_i \in T_i = T$, given as

$$P_{\zeta,\varepsilon}((d,\sigma))(\otimes_{i\in I} v_i) = \chi_\varepsilon(\sigma)\zeta(d)(\otimes_{i\in I} v_{\sigma^{-1}(i)}),$$

where $\zeta(d) = \prod_{i\in I}\zeta(t_i) = \zeta(\prod_{i\in I} t_i)$.

Lemma 3.7. *The above formula gives a one-dimensional unitary representation (unitary character) of G_I. If $I = \boldsymbol{N}$, the parameter*

$$A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$$

corresponding to this factor representation of type II_1 is an extreme case where $\alpha_{\zeta,\varepsilon} = (1, 0, 0, \ldots)$ for a $(\zeta,\varepsilon) \in \widehat{T}\times\{0, 1\}$ and accordingly all other parameters are zero.

Starting from this simplest case, we examine more general case where $\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| = 1$ for a fixed $\zeta \in \widehat{T}$ in $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$, and accordingly all other parameters in A are zero. In this case the measure space $(\mathcal{X}, \nu)$ is given as $\mathcal{X} = \bigsqcup_{\varepsilon\in\{0,1\}} \mathcal{N}_{\zeta,\varepsilon}$ with pointwise measure $\nu(\{(\zeta,\varepsilon,p)\}) = \alpha_{\zeta,\varepsilon,p}$.

(I) Take copies of T-module $V(\zeta,\varepsilon)$ as $V(x) = V(\zeta,\varepsilon)$ for $x = (\zeta,\varepsilon,p), p \in \boldsymbol{N}$, and consider its direct integral or weighted direct sum as follows. For a vector field $\boldsymbol{v} = (v(x))_{x\in\mathcal{X}}, v(x) \in V(x)$, on $\mathcal{X}$, we put $\|\boldsymbol{v}\|^2 = \int_{\mathcal{X}} |v(x)|^2\, d\nu(x) = \sum_{x\in\mathcal{X}} \alpha_x |v(x)|^2$, and denote the space of such vector fields by

$$\boldsymbol{V}(\mathcal{X}) = \int_{\mathcal{X}}^{\oplus} V(x)\, d\nu(x) = \sum_{x\in\mathcal{X}}{}^{\oplus} V(x).$$

Take copies $\boldsymbol{V}(\mathcal{X})_i = \boldsymbol{V}(\mathcal{X}_i), i \in I$, and make tensor product $\boldsymbol{W}(\mathcal{X}) = \otimes_{i\in I} \boldsymbol{V}(\mathcal{X}_i)$ with respect to the reference vector $(\mathbf{1}_{\mathcal{X}_i})_{i\in I}$. On $\boldsymbol{W}(\mathcal{X})$, we have a unitary representation $\Pi'_{\mathcal{X}}$ as in Lemma 3.1.

(II) Consider a Hilbert space $\mathcal{H}(\mathcal{X})$ of vector fields $\boldsymbol{w}(\boldsymbol{x}), \boldsymbol{x} = (x_i) \in \mathcal{X}^I = \prod_{i\in I} \mathcal{X}_i$ with norm $\|\boldsymbol{w}\|^2 := \int_{\mathcal{X}^I} \|\boldsymbol{w}(\boldsymbol{x})\|^2 \, d\nu^I(\boldsymbol{x})$, then we have a unitay representation $\Pi_{\mathcal{X}}$ on it given for $g = (d,\sigma) \in G_I$ as

$$\big(\Pi_{\mathcal{X}}(g)\boldsymbol{w}\big)(\boldsymbol{x}) = (-1)^{j(\sigma,\boldsymbol{x})}\,\zeta(d)\,\kappa'(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x})\big) \quad (\boldsymbol{w} \in \mathcal{H}(\mathcal{X})).$$

(III) Now introduce a new parameter $\boldsymbol{y}$ which controls the multiplicities of representations, and construct a new representation Π. For $\boldsymbol{x} = (x_i)_{i\in I} \in \mathcal{X}^I$, take a tensor product $W(\boldsymbol{x}) = \otimes_{i\in I} V(x_i)$ with respect to a reference vector $(1_{x_i})_{i\in I}$, and take a measurable vector field $\boldsymbol{w}$ on $\mathcal{X}^I \times \mathcal{X}^I$ such that $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y}) \in \boldsymbol{W}(\boldsymbol{x})$ for $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{X}^I \times \mathcal{X}^I$. The norm of $\boldsymbol{w}$ is defined by (3.12), and the Hilbert space consisting of such $\boldsymbol{w}$ that $\|\boldsymbol{w}\| < \infty$ is denoted by $\mathcal{H}$. The action of $g = (d,\sigma) \in G_I$ on $\mathcal{H}$ is defined as

$$\big(\Pi(g)\boldsymbol{w}\big)(\boldsymbol{x},\boldsymbol{y}) = (-1)^{j(\sigma,\boldsymbol{x})}\,\zeta(d)\,\kappa'(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}),\boldsymbol{y})\big).$$

Let Δ be the diagonal subset of $\mathcal{X}^I \times \mathcal{X}^I$ and $\mathbf{1}_\Delta$ its characteristic function, then $\|\mathbf{1}_\Delta\| = 1$. Let $(\pi^A, \mathcal{H}^A)$ be the cyclic representation generated by $\mathbf{1}_\Delta$ under Π. We calculate the matrix element $\Phi(g) = \langle \pi^A(g)\mathbf{1}_\Delta, \mathbf{1}_\Delta\rangle$ for $\mathbf{1}_\Delta$ as

$$\Phi(g) = \\ = \int_{\mathcal{X}^I} \sum_{\boldsymbol{y}\sim\boldsymbol{x}} \langle (-1)^{j(\sigma,\boldsymbol{x})}\,\zeta(d)\,\kappa'(\sigma)\big(\mathbf{1}_\Delta(\sigma^{-1}(\boldsymbol{x}),\boldsymbol{y})\big), \mathbf{1}_\Delta(\boldsymbol{x},\boldsymbol{y})\rangle \, d\nu^I(\boldsymbol{x}), \tag{3.16}$$

and get the following result:

Assume $I = \boldsymbol{N}$. The matrix element $\langle \pi^A(g)\mathbf{1}_\Delta, \mathbf{1}_\Delta\rangle$ is equal to the extremal positive definite class function f_A corresponding to a parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}; \mu\Big)$ with $\|\alpha_{\zeta,0}\| + \|\alpha_{\zeta,1}\| = 1$.

3.5.2. Special case: $\|\mu\| = 1$ for A

Fix a $\zeta \in \widehat{T}$. Define a character of the subgroup $D_I(T) \subset G_I(T)$ by $\zeta_D(d) = \zeta(\prod_{i\in I} t_i)$ for $d = (t_i)_{i\in I}$, and consider the induced representation $\pi_\zeta = \mathrm{Ind}_{D_I(T)}^{G_I(T)} \zeta_D$ realized naturally on the space $\ell^2(\mathfrak{S}_I)$.

Lemma 3.8. *The vector $v^0 = \delta_{\mathbf{1}} \in \ell^2(\mathfrak{S}_I)$ is cyclic, where $\mathbf{1}$ denotes the identity element in $\mathfrak{S}_I$. The matrix element $\langle \pi_\zeta(g)v^0, v^0\rangle$ is equal to $\zeta(d)$ for $g = (d,\mathbf{1})$, and vanishes outside of $D_I(T) \subset G_I(T)$. Assume $I = \boldsymbol{N}$. Then the induced representation π_ζ is factorial of type II_1, and the corresponding parameter $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$ is an extreme case where $\mu_\zeta = 1$ and accordingly all other parameters are zero.*

We give another realization of this factor representation. To be more general, we treat the case where $\|\mu\| = \sum_{\zeta\in\widehat{T}} \mu_\zeta = 1$ in $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;$

$\mu\Big)$. In this case the measure space $(\mathcal{X},\nu)$ is given as $\mathcal{X}=\bigsqcup_{\zeta\in\widehat{T}}\Xi_\zeta$, $\Xi_\zeta=\{(\zeta,\xi_\zeta),\xi_\zeta\in[0,\mu_\zeta]\}$ with the Lebesgue measure $d\xi_\zeta$ on the interval $[0,\mu_\zeta]$. Let $V(\zeta)$ be a one-dimensional T-module given in (3.4), and take copies of it as $V(\zeta,\xi_\zeta)=V(\zeta)$, $\xi_\zeta\in\Xi_\zeta$.

(I) For a measurable vector field $\boldsymbol{v}=(v(x))_{x\in\mathcal{X}}, v(x)\in V(x)$, we put $\|\boldsymbol{v}\|^2=\int_{\mathcal{X}}|v(x)|^2\,d\nu(x)$, and denote by

$$\boldsymbol{V}(\mathcal{X})=\int_{\mathcal{X}}^{\oplus}V(x)\,d\nu(x)=\sum_{\zeta\in\widehat{T}}^{\oplus}\int_{\Xi_\zeta}^{\oplus}V(\zeta,\xi_\zeta)\,d\xi_\zeta$$

the space of such vector fields that $\|\boldsymbol{v}\|<\infty$. Take copies $\boldsymbol{V}(\mathcal{X})_i=\boldsymbol{V}(\mathcal{X}_i)$ and make tensor product $\boldsymbol{W}(\mathcal{X})=\otimes_{i\in I}\boldsymbol{V}(\mathcal{X}_i)$ with respect to the reference vector $(\mathbf{1}_{\mathcal{X}_i})_{i\in I}$. On the space $\boldsymbol{W}(\mathcal{X})$ we have a unitary representation $\Pi'_{\mathcal{X}}$ as in Lemma 3.1.

(II) Going to the form of vector fields, we have another but similar unitary representation $(\Pi_{\mathcal{X}},\mathcal{H}(\mathcal{X}))$ given as follows: for $g=(d,\sigma)\in G_I$

$$\big(\Pi_{\mathcal{X}}(g)\boldsymbol{w}\big)(\boldsymbol{x})=Z_{\boldsymbol{x}}(d)\kappa'(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}))\big)\quad(\boldsymbol{w}\in\mathcal{H}(\mathcal{X})),$$

where $\boldsymbol{x}=(x_i)_{i\in I}\in\mathcal{X}^I$, $Z_{x_i}=\zeta$ for $x_i=(\zeta,\xi_\zeta)\in\mathcal{X}_i$.

(III) Now introduce a new parameter $\boldsymbol{y}$ which controls the multiplicities of representations and construct a new representation. For $\boldsymbol{x}=(x_i)_{i\in I}\in\mathcal{X}^I$, take a tensor product $W(\boldsymbol{x})=\otimes_{i\in I}V(x_i)$ with respect to a reference vector $(1_{x_i})_{i\in I}$, and take a vector field $\boldsymbol{w}$ on $\mathcal{X}^I\times\mathcal{X}^I$ such that $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{y})\in\boldsymbol{W}(\boldsymbol{x})$ for $(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}^I\times\mathcal{X}^I$. The norm of $\boldsymbol{w}$ is given by (3.12). The action of $g=(d,\sigma)\in G_I$ is defined as

$$\big(\Pi(g)\boldsymbol{w}\big)(\boldsymbol{x},\boldsymbol{y})=Z_{\boldsymbol{x}}(d)\kappa'(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}),\boldsymbol{y})\big),\tag{3.17}$$

where $Z_{\boldsymbol{x}}(d)=\prod_{i\in I}Z_{x_i}(t_i)$.

Let Δ be the diagonal subset of $\mathcal{X}^I\times\mathcal{X}^I$ and $\mathbf{1}_\Delta$ its characteristic function, then $\|\mathbf{1}_\Delta\|=1$. For the cyclic representation π^A generated by $\mathbf{1}_\Delta$, we calculate a matrix element and get the following result: *In the case $I=\boldsymbol{N}$, the matrix element $\langle\pi^A(g)\mathbf{1}_\Delta,\mathbf{1}_\Delta\rangle$ is equal to the extremal positive definite class function f_A corresponding to a parameter $A=\Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\,;\,\mu\Big)$ with $\|\mu\|=1$.*

3.6. The case of the subgroup $G_I^S=\mathfrak{S}_I^S(T)$ with $S\subset T$ abelian compact

The natural subgroup $G_I^S=\mathfrak{S}_I^S(T)$ of $G_I=\mathfrak{S}_I(T)$ is defined in (1.3) for T abelian and compact, and it is normal and the index $[G:G^S]$ is finite under the assumption that S is open in T. For $I=\boldsymbol{N}$, characters of $G_I^S=\mathfrak{S}_\infty^S(T)$ are given in Theorem 2.3. Note that, for $S=\{e_T\}\subset T=\boldsymbol{Z}_2$, we have $\mathfrak{S}_\infty^S(\boldsymbol{Z}_2)=W_{\mathbf{D}_\infty}$.

The group G_I is decomposed into G_I^S-cosets as $G_I=\bigsqcup_{t\in T/S}\widetilde{t}\,G_I^S=\bigsqcup_{t\in T/S}G_I^S\,\widetilde{t}$ with $\widetilde{t}=(t,e_T,e_T,\ldots)\in D_I(T)$, where "$t\in T/S$" means that

t runs over a complete system of representatives of T/S. The restriction of inner automorphism $g \mapsto \widetilde{t}\,g\,\widetilde{t}^{-1}$ on G_I^S is denoted by θ_t. Let $\mathcal{H}_0^A$ be the closed span of $\pi^A(G_I^S)\mathbf{1}_\Delta$, then the representation space $\mathcal{H}^A$ of π^A is a sum of $\pi^A(\widetilde{t})\mathcal{H}_0^A$, $t \in T/S$, and is generated under G_I^S by a set of vectors $\{\boldsymbol{w}_t^0 := \pi^A(\widetilde{t})\mathbf{1}_\Delta; t \in T/S\}$.

Denote by π_t^A the representation of G_I^S obtained by restricting π^A onto $\pi^A(\widetilde{t})\mathcal{H}_0^A$. Then it has a cyclic vector $\boldsymbol{w}_t^0$ and the matrix element is given as $\langle \pi_t^A(g)\boldsymbol{w}_t^0, \boldsymbol{w}_t^0\rangle = f_A^S(\theta_{t^{-1}}(g)) = f_A^S(g)$ with $f_A^S = f_A|G_I^S$, because f_A is invariant under inner automorphisms.

In the case $I = \boldsymbol{N}$, we know by Theorem 2.3 that f_A^S is the normalized character of a factor representation of finite type of G_I^S. Therefore we see that each π_t^A is a factor representation of finite type having the same f_A^S as its character, and so they are all quasi-equivalent to each other. Thus we get the following result.

Theorem 3.9. *Assume that T be abelian compact and $S \subset T$ be an open subgroup. Then, for the group $G_I^S = \mathfrak{S}_\infty^S(T)$ with $I = \boldsymbol{N}$, a factor representation of finite type with character f_A^S is realised by π_t^A with $t = e_T \in T$ on the space $\mathcal{H}_0^A$ generated by $\mathbf{1}_\Delta$.*

4. Special realization of factor representations of $\mathfrak{S}_\infty(T)$, T any compact

Let T be a compact group and we study the wreath products $G_I = \mathfrak{S}_I(T) = D_I(T) \rtimes \mathfrak{S}_I$ for $I = I_N$ with $N = 1, 2, \ldots, \infty$ with $I_\infty = \boldsymbol{N}$. Put $G_N = \mathfrak{S}_{I_N}(T)$, $G = \mathfrak{S}_\infty(T)$.

4.1. Factor representations of a compact group T

Denote by $\widehat{T}$ the set of all equivalence classes of irreducible unitary representations of T, and for each $\zeta \in \widehat{T}$ we fix a representative of the class and denote it again by the same symbol ζ. Denote by $V(\zeta)$ the representation space of ζ and by χ_ζ its trace character.

4.1.1. A realization of cyclic factor representations of T

Fix a unit vector $v^0 \in V(\zeta)$. Take a complete orthonormal basis $\{e_j; 1 \le j \le \dim\zeta\}$ in $V(\zeta)$ such that $e_1 = v^0$, and let $\zeta_{jk}(t) = \langle \zeta(t)e_k, e_j\rangle$ be matrix elements with respect to it, then $\zeta_{11}(t) = \langle \zeta(t)v^0, v^0\rangle$. Put

$$v_\zeta^0(s) = \zeta(s)v^0 = \zeta(s)e_1 \quad (s \in T). \tag{4.1}$$

Lemma 4.1. (i) *The elements v^0 and $v_\zeta^0(s)$ are both cyclic in $V(\zeta)$, and $\langle \zeta(t)v_\zeta^0(s_1), v_\zeta^0(s_2)\rangle = \zeta_{11}(s_2^{-1}ts_1)$. Denote by ds ($s \in T$) the normalized Haar measure on T, then*

$$\int_T \zeta_{11}(s^{-1}ts)\,ds = \frac{1}{\dim\zeta}\,\chi_\zeta(t).$$

(ii) *There holds the following integral relation*:

$$
\begin{aligned}
(4.2)\qquad & \int_{T^\ell} \zeta_{11}(s_1^{-1}t_1 s_\ell)\,\zeta_{11}(s_2^{-1}t_2 s_1)\,\cdots\,\zeta_{11}(s_\ell^{-1}t_\ell s_{\ell-1})\,ds_1\,ds_2\,\cdots\,ds_\ell = \\
& = \frac{\chi_\zeta(t_\ell t_{\ell-1}\cdots t_2 t_1)}{(\dim\zeta)^\ell}.
\end{aligned}
$$

Proof. It is sufficient for us to note the following two equalities:

$$
\begin{aligned}
\int_T \zeta_{11}(s^{-1}ts)\,ds &= \sum_{k,\ell=1}^{\dim\zeta} \int_T \zeta_{1k}(s^{-1})\,\zeta_{k\ell}(t)\,\zeta_{\ell 1}(s)\,ds \\
&= \sum_{k,\ell=1}^{\dim\zeta} (\dim\zeta)^{-1}\,\delta_{k\ell}\,\zeta_{k\ell}(t) = (\dim\zeta)^{-1}\,\chi_\zeta(t).
\end{aligned}
$$

$$
\begin{aligned}
&\int_T \zeta_{11}(s_1^{-1}t_1 s_\ell)\,\zeta_{11}(s_2^{-1}t_2 s_1)\,ds_1 \\
&= \sum_{j,k=1}^{\dim\zeta} \int_T \zeta_{1j}(s_1^{-1})\,\zeta_{j1}(t_1 s_\ell)\,\zeta_{1k}(s_2^{-1}t_2)\,\zeta_{k1}(s_1)\;ds_1 \\
&= (\dim\zeta)^{-1} \sum_{k=1}^{\dim\zeta} \zeta_{1k}(s_2^{-1}t_2)\,\zeta_{k1}(t_1 s_\ell) \;=\; (\dim\zeta)^{-1}\,\zeta_{11}(s_2^{-1}t_2 t_1 s_\ell).
\end{aligned}
$$

□

For later use we define a continuous direct integral of the same irreducible representation ζ as follows. For $s \in T$, put $V(\zeta; s) = V(\zeta)$, and the representation space $U(\zeta)$, the operator of representation $\zeta^U(t)$ and a special unit vector u_ζ^0 are defined as follows:

$$
(4.3)\quad U(\zeta) = \int_T^\oplus V(\zeta; s)\,ds, \quad \zeta^U(t) = \int_T^\oplus \zeta(t)\,ds, \quad u_\zeta^0 = \int_T^\oplus v_\zeta^0(s)\,ds.
$$

Note that the above space $U(\zeta)$ is nothing but the $V(\zeta)$-valued L^2-space on (T, ds), denoted by $L^2(T, ds; V(\zeta))$, and the representation ζ^U acts on the space of values $V(\zeta)$. For this representation $(\zeta^U, U(\zeta))$, the unit vector u_ζ^0 is a trace-element for ζ. In fact, we get the normalized character of ζ as a matrix element for u_ζ^0 thanks to Lemma 4.1:

$$
\begin{aligned}
(4.4)\qquad \langle \zeta^U(t) u_\zeta^0, u_\zeta^0 \rangle &= \int_T \langle \zeta(t) v_\zeta^0(s), v_\zeta^0(s) \rangle\,ds = \\
&= \int_T \zeta_{11}(s^{-1}ts)\,ds = \frac{1}{\dim\zeta}\,\chi_\zeta(t).
\end{aligned}
$$

4.1.2. Another realization of cyclic factor representations of T

We also prepare another type of a factor representation ζ^{U_1} for ζ. On the dual space $V(\zeta)'$ of $V(\zeta)$, the adjoint representation ζ' acts as $(\zeta(t)v, v') = (v, \zeta'(t^{-1})v')$ $(t \in T)$ for $v \in V(\zeta), v' \in V(\zeta)'$, where $(v, v') := v'(v)$ denotes the natural pairing between these two spaces. Consider the tensor product $U_1(\zeta) = V(\zeta) \otimes V(\zeta)'$ as a T-module through the action $\zeta^{U_1} = \zeta \otimes \mathbf{1}_T$ with the trivial representation $\mathbf{1}_T$ of T, that is,

$$\zeta^{U_1}(t)(v \otimes v') = (\zeta(t)v) \otimes v' \quad (t \in T, v \in V(\zeta), v' \in V(\zeta)'). \tag{4.5}$$

Note that $(\zeta^{U_1}, U_1(\zeta))$ is canonically equivalent to the part corresponding to $\zeta \in \widehat{T}$ of the right regular representation $(R_T, L^2(T))$, which is spanned by the matrix elements of ζ. Take a complete orhtonormal basis $\{e_j; 1 \le j \le \dim \zeta\}$ in $V(\zeta)$ and its dual basis $\{e'_k; 1 \le k \le \dim \zeta\}$ in $V(\zeta)'$ such as $(e_j, e'_k) = \delta_{jk}$, and define a unit vector in $U_1(\zeta)$ as

$$u^1 = \frac{1}{\sqrt{\dim \zeta}} \sum_{1 \le j \le \dim \zeta} e_j \otimes e'_j. \tag{4.6}$$

Lemma 4.2. (i) *The element u^1 is independent of the orthonormal basis used to define it. In particular, $u^1 = (1/\sqrt{\dim \zeta}) \sum_{1 \le j \le \dim \zeta} (\zeta(s)e_j) \otimes (\zeta'(s)e'_j)$ for any $s \in T$. Any element of $U_1(\zeta) = V(\zeta) \otimes V(\zeta)'$ invariant under $\zeta \otimes \zeta'$ is a scalar multiple of u^1.*

(ii) *The unit vectors u^1 is cyclic under ζ^{U_1}, and*

$$\langle \zeta^{U_1}(t)u^1, u^1 \rangle_{U_1} = \frac{1}{\dim \zeta} \chi_\zeta(t),$$

where $\langle \cdot, \cdot \rangle_{U_1}$ denotes the inner product in $U_1(\zeta)$.

Proof. (i) By calculation, we can prove that u^1 is independent of the choice of $\{e_j\}$. An element $u = \sum_{1 \le j,k \le \dim \zeta} b_{jk}(e_j \otimes e'_k)$ of $U_1(\zeta)$ is invariant if and only if the matrix $B = (b_{jk})$ satisfies $\zeta(s)B\,{}^t(\zeta'(s)) = B$ or $\zeta(s)B = B\zeta(s)$ for any $s \in T$. So the irreducibility of ζ guarantees that B is a scalar matrix, and so u is a scalar multiple of u^1.

We omit the proof of (ii). □

Lemma 4.3.

$$\int_{T^\ell} \chi_\zeta(s_1^{-1}t_1 s_\ell)\, \chi_\zeta(s_2^{-1}t_2 s_1) \cdots \chi_\zeta(s_\ell^{-1}t_\ell s_{\ell-1})\, ds_1\, ds_2 \cdots ds_\ell = \\ = \frac{\chi_\zeta(t_\ell t_{\ell-1} \cdots t_2 t_1)}{(\dim \zeta)^{\ell-1}}. \tag{4.7}$$

Remark 4.1. As a representation of $T \times T$, the representation $T \times T \ni (t, s) \to \zeta(t) \otimes \zeta'(s)$ on $U_1(\zeta) = V(\zeta) \otimes V(\zeta)'$ gives a Hilbert algebra in the sense of [Dix, I.5], and concerning Lemma 4.2, we refer to Proposition 3 in [Dix, I.6.3].

4.2. Fundamental representations of G_I

Let $\mathcal{X} = \mathcal{X}_{disc} \bigsqcup \mathcal{X}_{cont}$, where

$$\mathcal{X}_{disc} = \bigsqcup_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \mathcal{N}_{\zeta,\varepsilon}, \qquad \mathcal{N}_{\zeta,\varepsilon} = \{(\zeta,\varepsilon,p); p \in \boldsymbol{N}\} \cong \boldsymbol{N},$$

$$\mathcal{X}_{cont} = \bigsqcup_{\zeta\in\widehat{T}} \Xi_\zeta, \qquad \Xi_\zeta = \{(\zeta,\xi); \xi \in [0,\mu_\zeta]\} \cong [0,\mu_\zeta],$$

and ν the probability measure on $\mathcal{X}$ given by A as follows:

$$\nu(\{(\zeta,\varepsilon,p)\}) = \alpha_{\zeta,\varepsilon,p} \ \ (p \in \boldsymbol{N}), \ \ d\nu((\zeta,\xi)) = d\xi \ \ (\xi \in [0,\mu_\zeta]). \tag{4.8}$$

We put for $I = I_N$ with $N = 1,2,\ldots,\infty$, $\mathcal{X}^I = \prod_{i\in I}\mathcal{X}_i$ with $\mathcal{X}_i = \mathcal{X}$ $(i \in I)\}$ and $\nu^I = \prod_{i\in I}\nu_i$ with $\nu_i = \nu$. Then, the permutation group $\mathfrak{S}_I$ acts on $\mathcal{X}^I$ as $\sigma(\boldsymbol{x}) = (x_{\sigma^{-1}(i)})_{i\in I}$ for $\sigma \in \mathfrak{S}_I$ and $\boldsymbol{x} = (x_i)_{i\in I}, x_i \in \mathcal{X}_i$.

For each $x \in \mathcal{X}$, we prepare a T-module as follows.

I. First choice: For a discrete parameter $x = (\zeta,\varepsilon,p) \in \mathcal{X}_{disc}$ and also for a continuous parameter $x = (\zeta,\xi) \in \mathcal{X}_{cont}$, we put

$$U(x) = U(\zeta) = \int_T^{\oplus} V(\zeta;s)\,ds$$

with the distinguished unit vector $u_x^0 := u_\zeta^0 = \int_T^{\oplus} v_\zeta^0(s)\,ds$, and the action of $t \in T$ is given as follows. We denote an element $u = \int_T^{\oplus} u(s)\,ds$ with $u(s) \in V(\zeta;s) = V(\zeta)$ simply by $u = (u(s))$ in the form of a vector field, then

$$(Z_x(t)u)(s) := (\zeta^U(t)u)(s) = \zeta(t)\big(u(s)\big), \tag{4.9}$$

II. Second choice: For a discrete parameter $x = (\zeta,\varepsilon,p) \in \mathcal{X}_{disc}$, we put $U(x) = U(\zeta) = \int_T^{\oplus} V(\zeta;s)\,ds$, as above (cf. Remark 4.2). For a continuous parameter $x = (\zeta,\xi) \in \mathcal{X}_{cont}$, put

$$U(x) = U_1(\zeta) = V(\zeta)\otimes V(\zeta)', \qquad Z_x(t) = \zeta^{U_1}(t) \ \ (t \in T), \tag{4.10}$$

and $u_x^0 := u_\zeta^1$ the distinguished cyclic unit vector.

4.2.1. Unitary representation $\Pi'_{\mathcal{X}}$ of G_I

Denote by $\boldsymbol{U}(\mathcal{X})$ the sum of a direct sum and a direct integral of $U(x)$'s as

$$\boldsymbol{U}(\mathcal{X}) = \sum_{x\in\mathcal{X}_{disc}}^{\oplus} U(x) \bigoplus \int_{\mathcal{X}_{cont}}^{\oplus} U(x)\,d\nu(x) = \int_{\mathcal{X}}^{\oplus} U(x)\,d\nu(x).$$

For a measurable vector field $\boldsymbol{u} = (\boldsymbol{u}(x))_{x\in\mathcal{X}}, \boldsymbol{u}(x) \in U(x)$, on $\mathcal{X}$, define its norm as $\|\boldsymbol{u}\|^2 = \int_{\mathcal{X}} \|\boldsymbol{u}(x)\|^2\,d\nu(x)$. Then the vector field $\boldsymbol{u}_{\mathcal{X}}^0 = (u_x^0)_{x\in\mathcal{X}}$ with the distinguished unit vector $u_x^0 \in U(x)$ has norm 1, that is, $\|\boldsymbol{u}_{\mathcal{X}}^0\| = 1$. The Hilbert space $\boldsymbol{U}(\mathcal{X})$ consists of measurable vector fields $\boldsymbol{u}$ with $\|\boldsymbol{u}\| < \infty$, and the T-action on it is given by $\big(Z_{\mathcal{X}}(t)\boldsymbol{u}\big)(x) = Z_x(t)\big(\boldsymbol{u}(x)\big)$ $(x \in \mathcal{X})$

Take copies $\boldsymbol{U}(\mathcal{X})_i = \boldsymbol{U}(\mathcal{X}_i) = \boldsymbol{U}(\mathcal{X})$ for $i \in I$ and make tensor product $\boldsymbol{W}(\mathcal{X}) = \otimes_{i\in I}\,\boldsymbol{U}(\mathcal{X})_i$ with respect to a reference vector $(\boldsymbol{u}^0_{\mathcal{X}_i})_{i\in I}$. Then, on a decomposable elements $\boldsymbol{w} = \otimes_{i\in I}\boldsymbol{u}_i \in \boldsymbol{W}(\mathcal{X})$, elements $d = (t_i)_{i\in I} \in D_I(T)$ with $t_i \in T_i = T$ and $\sigma \in \mathfrak{S}_I$ act respectively as

$$\Pi'_{\mathcal{X}}(d)\boldsymbol{w} := \otimes_{i\in I}\big(Z_{\mathcal{X}_i}(t_i)\boldsymbol{u}_i\big), \qquad \kappa(\sigma)\boldsymbol{w} := \otimes_{i\in I}\boldsymbol{u}_{\sigma^{-1}(i)},$$

and we get a unitary representation $\Pi'_{\mathcal{X}}$ of G_I as follows.

Lemma 4.4. *For $g = (d,\sigma) \in G_I = D_I(T) \rtimes \mathfrak{S}_I$, put*

$$\Pi'_{\mathcal{X}}(g)\boldsymbol{w} := \Pi'_{\mathcal{X}}(d)\kappa(\sigma)\boldsymbol{w} \quad (\boldsymbol{w} \in \boldsymbol{W}(\mathcal{X})).$$

Then this gives a unitary representation of G_I.

4.2.2. Unitary representation $\Pi_{\mathcal{X}}$ of G_I

We rewrite $\Pi'_{\mathcal{X}}$ using vector fields on $\mathcal{X}^I$ with values in $\boldsymbol{W}(\boldsymbol{x}) := \otimes_{i\in I}U(x_i)$ at $\boldsymbol{x} = (x_i)_{i\in I}, x_i \in \mathcal{X}_i$, and introduce a multiplier coming from a 1-cocycle for $(\mathfrak{S}_I, \mathcal{X}^I)$. Let $\mathcal{H}(\mathcal{X})$ be the Hilbert space of measurable vector fields $\boldsymbol{w}(\boldsymbol{x}) \in \boldsymbol{W}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{X}^I$, such that $\|\boldsymbol{w}\| < \infty$, where $\|\boldsymbol{w}\|^2 := \int_{\mathcal{X}^I} \|\boldsymbol{w}(\boldsymbol{x})\|^2\, d\nu^I(\boldsymbol{x})$.

Recall that each space $U(x_i)$ is isomorphic to $L^2(T, ds; V(\zeta_i))$ for $x_i = (\zeta_i, \varepsilon_i, p_i)$ or (ζ_i, ξ_i) in the first choice. Therefore, in turn, the value $\boldsymbol{v} = \boldsymbol{w}(\boldsymbol{x}) \in \otimes_{i\in I}U(x_i)$ can be considered as an L^2-function $\boldsymbol{v}(\boldsymbol{s})$ of $\boldsymbol{s} = (s_i)_{i\in I} \in T^I = \prod_{i\in I} T_i$ with values in $\otimes_{i\in I}V(\zeta_i)$, where the measure $dm(\boldsymbol{s})$ on T^I is the product of normalized Haar measures ds_i on T_i. Here the permutation group $\mathfrak{S}_I$ acts in two ways.

(i) The one is the action on the space of values $\otimes_{i\in I}V(\zeta_i)$ as

$$\kappa'(\sigma) :\ \otimes_{i\in I}V(\zeta_i) \ni \otimes_{i\in I}v_i \ \longmapsto\ \otimes_{i\in I}v_{\sigma^{-1}(i)} \in \otimes_{i\in I}V(\zeta_{\sigma^{-1}(i)}).$$

(ii) The other is the action on the variable $\boldsymbol{s} = (s_i)_{i\in I} \in T^I$ as $\sigma(\boldsymbol{s}) = (s_{\sigma^{-1}(i)})_{i\in I}$.

We denote by $\kappa''(\sigma)$ the simultaneous action of σ of type (i) and (ii).

The action $\kappa''(\sigma)$ is natural in the following point of view. Take a decomposable element $w := \otimes_{i\in I}v_i$ with $v_i \in L^2(T_i, ds_i; V_i)$, where v_i is given as a function in $s_i \in T_i$ with values in V_i. By $\sigma \in \mathfrak{S}_I$, we permute the components of w as $\otimes_{i\in I}v_{\sigma^{-1}(i)}$, then this is a function of $\boldsymbol{s} = (s_i)_{i\in I} \in T^I = \prod_{i\in I} T_i$ given as $\otimes_{i\in I}v_{\sigma^{-1}(i)}(s_{\sigma^{-1}(i)})$, and the last expression gives nothing but $\kappa''(\sigma)(w)$.

For our present case, take a decomposable element $\boldsymbol{w} = \otimes_{i\in I}\boldsymbol{u}_i$ with $\boldsymbol{u}_i \in \boldsymbol{U}(\mathcal{X})_i = \boldsymbol{U}(\mathcal{X}_i)$, and consider $\otimes_{i\in I}\boldsymbol{u}_i$ as a vector field on $\mathcal{X}^I$. Then its value at a point $\boldsymbol{x} = (x_i)_{i\in I}, x_i \in \mathcal{X}_i$, is given by $\otimes_{i\in I}\boldsymbol{u}_i(x_i)$, and then by $\otimes_{i\in I}\boldsymbol{u}_i(x_i)(s_i)$ as a function of $\boldsymbol{s}$. A permutation $\sigma \in \mathfrak{S}_I$ acts as $\boldsymbol{w} = \otimes_{i\in I}\boldsymbol{u}_i \mapsto \otimes_{i\in I}\boldsymbol{u}_{\sigma^{-1}(i)}$. Its value at $\boldsymbol{x}$ is $\otimes_{i\in I}\boldsymbol{u}_{\sigma^{-1}(i)}(x_i)$ and further is $\otimes_{i\in I}\boldsymbol{u}_{\sigma^{-1}(i)}(x_i)(s_{\sigma^{-1}(i)})$ as a function in $\boldsymbol{s}$, which is expressed as $\kappa''(\sigma)\big(\otimes_{i\in I}\boldsymbol{u}_i(x_{\sigma(i)})\big) = \kappa''(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}))\big)$, that is,

$$\begin{aligned}\big(\otimes_{i\in I}\,\boldsymbol{u}_{\sigma^{-1}(i)}\big)(\boldsymbol{x}) = \otimes_{i\in I}\boldsymbol{u}_{\sigma^{-1}(i)}(x_i) = \kappa''(\sigma)\big(\otimes_{i\in I}\,\boldsymbol{u}_i(x_{\sigma(i)})\big) = \\ = \kappa''(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}))\big).\end{aligned}$$

Proposition 4.5. *For $g = (d, \sigma) \in G_I$ and $\boldsymbol{w} \in \mathcal{H}(\mathcal{X})$, put*

$$\big(\Pi_{\mathcal{X}}(g)\boldsymbol{w}\big)(\boldsymbol{x}) = (-1)^{j(\sigma,\boldsymbol{x})}\, Z_{\boldsymbol{x}}(d)\, \kappa''(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}))\big), \tag{4.11}$$

where $\boldsymbol{w} = (\boldsymbol{w}(\boldsymbol{x})), \boldsymbol{x} = (x_i)_{i\in I} \in \mathcal{X}^I$, and $Z_{\boldsymbol{x}}(d) = \prod_{i\in I} Z_{x_i}(t_i)$ with Z_{x_i} in (4.9) *in the case of the first choice, and $j(\sigma, \boldsymbol{x})$ is the number of inversions in $(\sigma^{-1}(i))_{i\in J_1(\boldsymbol{x})}$ with $J_1(\boldsymbol{x}) = \{i \in I; x_i \in \bigsqcup_{\zeta\in\widehat{T}} \mathcal{N}_{\zeta,1}\}$ for $\varepsilon = 1$. Then $\Pi_{\mathcal{X}}$ is a unitary representation of G_I on $\mathcal{H}(\mathcal{X})$.*

The proof is a word for word repetition of that for Proposition 3.2.

Furthermore in the case of the second choice, Z_{x_i} for $x_i = (\zeta_i, \xi_i) \in \mathcal{X}_{cont}$ is chosen as in (4.10), and accordingly the transformation $\kappa''(\sigma) : \boldsymbol{W}(\sigma^{-1}(\boldsymbol{x})) \to \boldsymbol{W}(\boldsymbol{x})$ should be defined to realize the transformation $\boldsymbol{w} = \otimes_{i\in I}\boldsymbol{u}_i \mapsto \otimes_{i\in I}\boldsymbol{u}_{\sigma^{-1}(i)}$ for decomposable elements $\boldsymbol{w} \in \boldsymbol{W}(\mathcal{X}) = \otimes_{i\in I}\boldsymbol{U}(\mathcal{X}_i)$ with $\boldsymbol{u}_i \in \boldsymbol{U}(\mathcal{X}_i)$. Hence $\kappa''(\sigma) : \otimes_{i\in I}\boldsymbol{u}_i(x_{\sigma(i)}) \mapsto \otimes_{i\in I}\boldsymbol{u}_{\sigma^{-1}(i)}(x_i)$. Then the assertion in Proposition 4.5 holds in this case too.

4.3. Construction of factor representations of finite type of G_I

Now we introduce a new variable $\boldsymbol{y} \in \mathcal{X}^I$ controling multiplicities of representations and construct a unitary representation Π whose certain subrepresentation π^A gives a factor representation corresponding to f_A. For $\boldsymbol{x} = (x_i)_{i\in I} \in \mathcal{X}^I$, take a tensor product $\boldsymbol{W}(\boldsymbol{x}) = \otimes_{i\in I}U(x_i)$ with respect to a reference vector $(u^0_{x_i})_{i\in I}$, and take a measurable vector field $\boldsymbol{w}$ on $\mathcal{X}^I \times \mathcal{X}^I$ such that $\boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{W}(\boldsymbol{x})$ for $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{X}^I \times \mathcal{X}^I$. We define $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x} = \tau(\boldsymbol{y})$ for some $\tau \in \mathfrak{S}_I$ as before, and the norm of $\boldsymbol{w}$ is defined by

$$\|\boldsymbol{w}\|^2 = \int_{\mathcal{X}^I} \sum_{\boldsymbol{y}\sim\boldsymbol{x}} \|\boldsymbol{w}(\boldsymbol{x}, \boldsymbol{y})\|^2_{\boldsymbol{W}(\boldsymbol{x})}\, d\nu^I(\boldsymbol{x}). \tag{4.12}$$

Denoted by $\mathcal{H}$ the Hilbert space of measurable vector fields $\boldsymbol{w}$ with finite norms in (4.12).

The action on $\mathcal{H}$ of $g = (d, \sigma) \in G_I$ is defined through $\Pi_{\mathcal{X}}(g)$ acting on $\boldsymbol{x}$-side as

$$\big(\Pi(g)\boldsymbol{w}\big)(\boldsymbol{x}, \boldsymbol{y}) = (-1)^{j(\sigma,\boldsymbol{x})} Z_{\boldsymbol{x}}(d)\kappa''(\sigma)\big(\boldsymbol{w}(\sigma^{-1}(\boldsymbol{x}), \boldsymbol{y})\big). \tag{4.13}$$

Proposition 4.6. *The formula* (4.13) *gives a unitary representation of G_I on the space $\mathcal{H}$.*

Let Δ be the diagonal subset of $\mathcal{X}^I \times \mathcal{X}^I$ and $\boldsymbol{u}_\Delta(\boldsymbol{x}, \boldsymbol{y})$ is a vector field supported by Δ such that $\boldsymbol{u}_\Delta(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$ if $\boldsymbol{x} \neq \boldsymbol{y}$, and $\boldsymbol{u}_\Delta(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{u}^0(\boldsymbol{x}) = \otimes_{i\in I}u^0_{x_i} \in \boldsymbol{W}(\boldsymbol{x}) = \otimes_{i\in I}U(x_i)$, then $\|\boldsymbol{u}_\Delta\| = 1$. Denote by $\mathcal{H}^A$ the closed linear span of $\Pi(G_I)\boldsymbol{u}_\Delta$ and by π^A the restriction of Π on the subspace $\mathcal{H}^A$. As will be proved in the following, the cyclic representation $(\pi^A, \mathcal{H}^A)$ is factorial of finite type with normalized character f_A.

4.4. Calculation of a matrix element for π^A

We calculate the matrix element of π^A for the cyclic vector $\boldsymbol{u}_\Delta$ as

(4.14)

$$\langle \pi^A(g)\boldsymbol{u}_\Delta, \boldsymbol{u}_\Delta\rangle = \\ = \int_{\mathcal{X}^I} \sum_{\boldsymbol{y}\sim\boldsymbol{x}} (-1)^{j(\sigma,\boldsymbol{x})} \langle Z_{\boldsymbol{x}}(d)\kappa''(\sigma)\big(\boldsymbol{u}_\Delta(\sigma^{-1}(\boldsymbol{x}),\boldsymbol{y})\big), \boldsymbol{u}_\Delta(\boldsymbol{x},\boldsymbol{y})\rangle_{\boldsymbol{W}(\boldsymbol{x})}\, d\nu^I(\boldsymbol{x}),$$

and get the following result.

Theorem 4.7. (i) *The matrix element $\langle \pi^A(g)\boldsymbol{u}_\Delta, \boldsymbol{u}_\Delta\rangle$ is given by the same formula as for the function f_A in Theorem* 2.1. *In particular, in the case $I = \boldsymbol{N}$, it is equal to the extremal positive definite class function f_A in Theorem* 2.1 *corresponding to a parameter* $A = \Big((\alpha_{\zeta,\varepsilon})_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ ;\ \mu\Big)$.

(ii) *The cyclic representation π^A generated by $\boldsymbol{u}_\Delta$ is a factor representation of finite type with normalized character f_A.*

Proof. Denote by $\Phi(g)$ the matrix element in (4.14). For $g = (d,\sigma) \in G_I$, the integrand in (4.14) is not zero only when $\sigma^{-1}(\boldsymbol{x}) = \boldsymbol{y} = \boldsymbol{x}$.

For a general element $g \in G_I$, take its standard decomposition as in (2.1), $g = \xi_{q_1}\xi_{q_2}\cdots\xi_{q_r}\, g_1g_2\cdots g_m$, with $\xi_{q_k} = (t_{q_k}, (q_k)), t_{q_k} \neq e_T$, for $1 \le k \le r$, and $g_j = (d_j, \sigma_j)$ for $1 \le j \le m$. Put $Q = \{q_1, q_2, \ldots, q_r\}$, $K_j = \mathrm{supp}(\sigma_j)$, and for $d_j = (t_i)_{i\in K_j} \in D_{K_j}(T)$, let $P_{\sigma_j}(d_j)$ be as in (1.5).

The condition $\sigma^{-1}(\boldsymbol{x}) = \boldsymbol{x}$ says that, for each j, all x_i $(i \in K_j)$ coincide with each other. So the set of such elements, cut off in $\mathcal{X}^{K_j}$, is equal to the set of $x_{K_j} = (x_i)_{i\in K_j}$ given as

$$\mathcal{Z}_{K_j} := \bigsqcup_{(\zeta,\varepsilon)} \bigsqcup_{p\in\boldsymbol{N}} \Big\{x_{K_j}; (\forall i)\ x_i = (\zeta,\varepsilon,p)\Big\} \\ \bigsqcup \bigsqcup_{\zeta\in\widehat{T}} \Big\{x_{K_j}; (\forall i)\ x_i = (\zeta,\xi), \xi\in[0,\mu_\zeta]\Big\},$$

where (ζ,ε) runs over $\widehat{T}\times\{0,1\}$. The point mass of $\{x_{K_j};\,(\forall i)\,x_i = (\zeta,\varepsilon,p)\}$ with respect to the product measure ν^{K_j} is equal to $(\alpha_{\zeta,\varepsilon,p})^{|K_j|} = (\alpha_{\zeta,\varepsilon,p})^{\ell(\sigma_j)}$. The mass of the set of continuous parameters

$$\{x_{K_j};\ (\forall i)\, x_i = (\zeta,\xi), \xi\in[0,\mu_\zeta]\} \subset (\Xi_\zeta)^{K_j}$$

with respect to the $|K_j|$-dimensional Lebesgues measure is zero, since it is a one-dimensional subregion of $(\Xi_\zeta)^{K_j} \cong [0,\mu_\zeta]^{|K_j|}$ of dimension $|K_j| = \ell(\sigma_j) \ge 2$, where $\Xi_\zeta = \{(\zeta,\xi); \xi\in[0,\mu_\zeta]\} \subset \mathcal{X}_{cont}$.

The integral in (4.14) can be carried out independently on each component $\mathcal{X}_{q_k}$ and $\mathcal{X}^{K_j}$ because, if $\sigma^{-1}(\boldsymbol{x}) = \boldsymbol{x}$, either $J_1(\boldsymbol{x}) \supset K_j$ or $J_1(\boldsymbol{x}) \cap K_j = \emptyset$ holds and the sign $(-1)^{j(\sigma,\boldsymbol{x})}$ is decomposed into a product as

$$(-1)^{j(\sigma,\boldsymbol{x})} = \prod_{1\le j\le m} \chi_{\varepsilon_j}(\sigma_j) = \prod_{1\le j\le m} (-1)^{\varepsilon_j(\ell(\sigma_j)-1)},$$

if the component of x_{K_j} is given by $(\zeta_j, \varepsilon_j, p_j) \in \mathcal{X}_{disc}$. Thus $\Phi(g)$ is expressed as a product as

$$\Phi(g) = \prod_{q\in Q} f_q(t_q) \times \prod_{1\le j\le m} f_{K_j}((d_j,\sigma_j)), \tag{4.15}$$

$$f_q(t_q) = \int_{\mathcal{X}} \langle Z_x(t_q)u_x^0, u_x^0\rangle_{U(x)}\, d\nu(x), \tag{4.16}$$

(4.17)

$$f_{K_j}((d_j,\sigma_j)) = \\ = \int_{\mathcal{Z}_{K_j}} (-1)^{j(\sigma_j,\, x_{K_j})} \Big\langle \otimes_{i\in K_j} \big(Z_{x_i}(t_i)u^0_{x_{\sigma_j^{-1}(i)}}\big), \otimes_{i\in K_j} u^0_{x_i} \Big\rangle_{\boldsymbol{W}(x_{K_j})} d\nu^{K_j}(x_{K_j}).$$

For the last formula (4.17), we note that for $\sigma_j = (i_1\ i_2\ \cdots\ i_{\ell_j})$, $K_j = \mathrm{supp}(\sigma_j) = \{i_1, i_2, \ldots, i_{\ell_j}\}$. An element in $\boldsymbol{W}(x_{K_j})$ is regarded as a vector field on T^{ℓ_j}, hence a function in $s_1, s_2, \ldots, s_{\ell_j}$, in such a manner as is indicated just before Proposition 4.5 (where ℓ_j denotes $\ell(\sigma_j)$). Namely, on the first argument of the inner product in (4.17), σ_j acted as follows:

$$\Big(\kappa''(\sigma_j)(\otimes_{i\in K_j} u^0_{x_i})\Big)(s_1, s_2, \ldots, s_{\ell_j}) = \kappa'(\sigma_j)\Big((\otimes_{i\in K_j} u^0_{x_i})(s_1, s_2, \ldots, s_{\ell_j})\Big) \\ = \kappa'(\sigma_j)\big(\otimes_{i\in K_j} u^0_{x_i}(s_i)\big) = \otimes_{i\in K_j} u^0_{x_{\sigma_j^{-1}(i)}}(s_{\sigma_j^{-1}(i)}).$$

This remark is essential in the computation of (4.18).

For the factor (4.16), we took $(Z_x(t)u)(s) = (\zeta^U(t)u)(s) = \zeta(t)\big(u(s)\big)$ for a discrete parameter $x = (\zeta, \varepsilon, p)$, and the same for a continuous parameter $x = (\zeta, \xi) \in \mathcal{X}_{cont}$ in the first choice, whereas $Z_x(t) = \zeta^{U_1}(t)$ for $x = (\zeta, \xi) \in \mathcal{X}_{cont}$ in the second choice. Then we get in any choice

$$\int_{\mathcal{X}} \langle Z_x(t_q)u_x^0, u_x^0\rangle\, d\nu(x) = \sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}} \sum_{p\in\boldsymbol{N}} \alpha_{\zeta,\varepsilon,p} \frac{\chi_\zeta(t_q)}{\dim\zeta} + \sum_{\zeta\in\widehat{T}} \mu_\zeta \frac{\chi_\zeta(t_q)}{\dim\zeta}.$$

For the factor (4.17), the integral on the region of continuous parameters $\bigsqcup_{\zeta\in\widehat{T}}\{x_{K_j};\ (\forall i)\, x_i = (\zeta,\xi), \xi \in [0,\mu_\zeta]\}$ is zero, since each of the subregion corresponding to $\Xi_\zeta \subset \mathcal{X}_{cont}$ is a one-dimensional subset in $[0,\mu_\zeta]^{|K_j|}$ and of measure 0.

Put $\ell(\sigma_j) = |K_j| = \ell_j$, and $t'_1 = t_{i_1}, t'_2 = t_{i_2}, \ldots, t'_{\ell_j} = t_{i_{\ell_j}}$ for $\sigma_j = (i_1\ i_2\ \cdots\ i_{\ell_j})$. Then the value of the integrand for the region of discrete parameters $\bigsqcup_{p\in\boldsymbol{N}}\{x_{K_j};\ (\forall i)\, x_i = (\zeta,\varepsilon,p)\}$ is

$$(-1)^{j(\sigma_j,\, x_{K_j})} \left\langle \otimes_{i\in K_j} \left(Z_{x_i}(t_i)u^0_{x_{\sigma_j^{-1}(i)}}\right), \otimes_{i\in K_j} u^0_{x_i} \right\rangle = \\ = (-1)^{\varepsilon(\ell_j-1)} \int_{T^{\ell_j}} \prod_{1\le k\le\ell_j} \langle \zeta(t_{i_k})u^0_\zeta(s_{k-1}), u^0_\zeta(s_k)\rangle\, ds_1\, ds_2 \cdots ds_{\ell_j}, \tag{4.18}$$

where we put $s_{\ell_j+1} = s_1$. By Lemmas 4.1 and 4.2, the above integral is equal to

$$
\begin{aligned}
&(-1)^{\varepsilon(\ell_j-1)} \int_{T^{\ell_j}} \zeta_{11}(s_1^{-1}t'_1 s_{\ell_j})\,\zeta_{11}(s_2^{-1}t'_2 s_1)\,\cdots \\
&\qquad \cdots \zeta_{11}(s_{\ell_j}^{-1}t'_{\ell_j} s_{\ell_j-1})\,ds_1\,ds_2\,\cdots\,ds_{\ell_j} = \qquad (4.19)\\
&\quad = (-1)^{\varepsilon(\ell_j-1)}\,\frac{\chi_\zeta(t'_{\ell_j}t'_{\ell_j-1}\cdots t'_2 t'_1)}{(\dim\zeta)^{\ell_j}} = \chi_\varepsilon(\sigma_j)\,\frac{\chi_\zeta(P_{\sigma_j}(d_j))}{(\dim\zeta)^{\ell_j}},
\end{aligned}
$$

where $P_{\sigma_j}(d_j)$ is defined in (1.5). Hence we have

$$
\begin{aligned}
&\int_{\mathcal{Z}_{K_j}} (-1)^{j(\sigma_j,\,x_{K_j})}\,\Big\langle \otimes_{i\in K_j}\Big(Z_{x_i}(t_i)u^0_{x_{\sigma_j(i)}}\Big),\,\otimes_{i\in K_j}u^0_{x_i}\Big\rangle\,d\nu^{K_j}(x_{K_j})\\
&\quad = \sum_{(\zeta,\varepsilon)\in\widehat{T}\times\{0,1\}}\ \sum_{p\in\boldsymbol{N}} \left(\frac{\alpha_{\zeta,\varepsilon,p}}{\dim\zeta}\right)^{\ell(\sigma_j)} \chi_\varepsilon(\sigma_j)\,\chi_\zeta(P_{\sigma_j}(d_j)).
\end{aligned}
$$

By the calculations above, we get the product formula for $\Phi(g)$ as for f_A in (2.5), and in the case $I = \boldsymbol{N}$, $\Phi = f_A$ as is asserted in the theorem.

For the last assertion in (ii), the proof is similar as in the proof of Theorem 3.6. □

Remark 4.2. From the computation of the integration (4.17) on the region of discrete parameters $\bigsqcup_{(\zeta,\varepsilon)}\bigsqcup_{p\in\boldsymbol{N}}\{x_{K_j};(\forall i)\ x_i = (\zeta,\varepsilon,p)\}$, it can be seen that Lemma 4.3 is an obstacle why we cannot choose $Z_x(t) = \zeta^{U_1}(t)$ in the second choice II in **4.2** for discrete parameters $x = (\zeta,\varepsilon,p) \in \mathcal{X}_{disc}$, whereas we could choose it for continuous parameters $x = (\zeta,\xi)$ which appears actually only in the one-dimensional integration (4.16) on $\mathcal{X}$.

Remark 4.3. Let G be a topological group and K a closed subgroup of G. Then the pair (G,K) is called *spherical* if for any IUR T of G, the subspace $V(T)^K$ of all K-invariant vectors in the space $V(T)$ of T has dimension ≤ 1 [Ols, Definition 23.1]. An IUR T of G is said to be a *spherical* representation of a pair (G,K) if $\dim V(T)^K = 1$.

For a topological group G, put $\mathbb{G} = G\times G$ and let $\Delta(G)$ ($\cong G$) be the subgroup of diagonal elements of G. Then the pair $(\mathbb{G},\Delta(G))$ is always *spherical* [Ols, Corollary 24.4]. For an IUR T of $\mathbb{G}$, put $\pi = T|_{G\times\{e\}}$. Assume T has a unit $\Delta(G)$-invariant vector $v_0 \in V(T)$. Then π is a factor representation of G of finite type, and v_0 is a trace-element of π, or $f(g) = \langle\pi(g)v_0, v_0\rangle$ is the normalized character of π. Moreover, Theorem 24.5 in [Ols] says:

The functor $T \to \pi$ establishes a bijection between the set of equivalence classes of spherical representations T of the pair $(\mathbb{G},\Delta(G))$ and the set of quasi-equivalence classes of factor representations of finite type of the group G.

From this standpoint, we can look back our construction of factor representations of finite type π^A of $G = \mathfrak{S}_\infty(T)$.

(See also [Far], for spherical functions for several spherical pairs (G,K) of infinite classical type, and for characters of factor representations of such G.)

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