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Simply connected 4-manifolds of second betti number 1 bounded by homology lens spaces

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§1. Introduction.

An oriented closed 3-manifold M is called a *homology lens space* if its first integral homology group $H_1(M; \boldsymbol{Z})$ is isomorphic to a finite cyclic group $\boldsymbol{Z}/p\boldsymbol{Z}$ for some $p \geqq 2$. By Boyer [2], for any such M, simply connected *topological* 4-manifolds of second betti number 1 bounded by M are classified by certain equivalence classes of elements of $H_1(M; \boldsymbol{Z})$ together with the Kirby-Siebenmann obstructions to smoothing (see §2). Then it arises the question which of these topological 4-manifolds cannot be given any *smooth structures*. In this paper we consider this question and give a partial answer. We mainly concern ourselves with such 4-manifolds bounded by what is called Dehn surgered 3-manifold.

Let K be a smooth knot in S^3. Then we denote by $M(K; p/q)$ the oriented closed 3-manifold obtained by p/q-Dehn surgery on K, where p and q are integers with $\gcd(p, q)=1$ and $q>0$ (see [18]). It is easy to see that the Dehn surgered 3-manifold $M(K; p/q)$ is a homology lens space with $H_1(M(K; p/q); \boldsymbol{Z})$ isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z}$. If $q=1$, we can attach a 2-handle to the 4-ball D^4 along K with p-framing and we denote the resulting handlebody by $V(K; p)$. Note that $V(K; p)$ is a simply connected smooth 4-manifold of second betti number 1 whose boundary $\partial V(K; p)$ is diffeomorphic to $M(K; p/1)$. Then one of our main results of this paper is the following.

COROLLARY 3.4. *Let K be a slice knot and $p>2$ an integer, where p is even or p has some prime factor p' with $p' \equiv 3 \pmod 4$. Suppose V is a simply connected topological 4-manifold of second betti number 1 bounded by the Dehn surgered 3-manifold $M(K; p/1)$. Then V admits a smooth structure if and only if it is homeomorphic to the handlebody $V(K; p)$.*

We also show, using the topological classification due to Boyer, that there do exist many simply connected topological 4-manifolds of second betti number

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1 bounded by $M(K; p/1)$ (see Corollary 3.11). Corollary 3.4 shows, however, that among these there is only one 4-manifold that can admit a smooth structure.

For the proof of Corollary 3.4 we construct a smooth closed 4-manifold whose certain 2-dimensional homology class is represented by a smoothly embedded 2-sphere and then use results of Kuga [**11**] and Lawson [**12**]. Using our techniques we also show in § 5 that there exist infinitely many rational numbers p/q such that for any knot K in S^3 the Dehn surgered 3-manifold $M(K; p/q)$ bounds a simply connected *topological* 4-manifold of second betti number 1 but never bounds a *smooth* one. Using this example we exhibit infinitely many homology lens spaces which, though their invariants defined by Fukuhara [**7**] vanish, cannot be obtained from S^3 by integral Dehn surgery on knots.

Throughout the paper all homology groups are with integral coefficients.

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§ 2. Topological classification.

DEFINITION 2.1. A compact orientable (topological) 4-manifold is said to be a *homotopy* S^2 (or *homology* S^2) if it has the same homotopy type (resp. homology groups) as S^2.

Note that a simply connected 4-manifold with a connected boundary is a homotopy S^2 if and only if its second betti number is 1. Furthermore, if V is a homology S^2, $H_1(\partial V) \cong \boldsymbol{Z}/p\boldsymbol{Z}$ for some $p \geqq 0$.

We have the following characterization of homology lens spaces which bound topological homotopy S^2.

PROPOSITION 2.2. *Let M be a homology lens space with $H_1(M)$ isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z}$ ($p \geqq 2$). Then there exists a topological homotopy S^2 bounded by M if and only if there exists a homology class $\alpha \in H_1(M)$ such that* $\mathrm{lk}(\alpha, \alpha) = \pm 1/p$, *where* $\mathrm{lk}: H_1(M) \times H_1(M) \to \boldsymbol{Q}/\boldsymbol{Z}$ *denotes the linking pairing of M.*

PROOF. Let V be a homotopy S^2 bounded by M and let $\gamma \in H_2(V, \partial V) \cong \boldsymbol{Z}$ be a generator. Then $\partial\gamma \in H_1(M)$ satisfies $\mathrm{lk}(\partial\gamma, \partial\gamma) = \pm 1/p$, where $\partial: H_2(V, \partial V) \to H_1(\partial V)$ ($= H_1(M)$) is the boundary homomorphism (see, for example, [8]).

Conversely, suppose the existence of $\alpha \in H_1(M)$ with $\mathrm{lk}(\alpha, \alpha) = \pm 1/p$. Then by Proposition 2.5 below, M bounds a homotopy S^2. Alternatively, Fukuhara [**7**] shows that M is obtained by an integral surgery on a knot in some homology 3-sphere. By Freedman [**6**], every homology 3-sphere bounds a contractible 4-manifold; hence, M bounds a 4-manifold V obtained by attaching a 2-handle to a contractible 4-manifold. Obviously, V is a homotopy S^2. □

EXAMPLE 2.3. Let K be a knot in S^3 and consider the Dehn surgered 3-manifold $M=M(K;p/q)$. Let $\alpha\in H_1(M)$ be the homology class represented by a meridian of K (Figure 1). Then we have $\mathrm{lk}(\alpha, \alpha)=-q/p$. (Here we adopt the same sign convention as in [8].) Thus by Proposition 2.2, $M(K;p/q)$ bounds a topological homotopy S^2 if and only if $\pm q$ is a quadratic residue mod p.

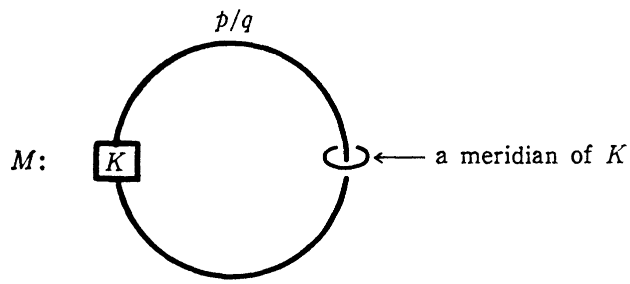


Figure 1.

Now we recall the classification of topological homotopy S^2 bounded by a homology lens space due to Boyer [2].

DEFINITION 2.4. Let V be a homology S^2 bounded by a homology lens space M and let $\gamma\in H_2(V, \partial V)\cong \boldsymbol{Z}$ be a generator. Then we define the homology class $\beta(V)\in H_1(M)$ by $\beta(V)=\partial\gamma$, where $\partial: H_2(V, \partial V)\to H_1(\partial V)$ $(=H_1(M))$ is the boundary homomorphism. We call $\beta(V)$ the *boundary class* of V, which is well-defined up to a multiple of ± 1.

PROPOSITION 2.5 (Boyer [2]). *Let M be a homology lens space with $H_1(M)$ isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z}$ $(p>2)$. Then the following holds.*

(1) *For every $\beta\in H_1(M)$ with $\mathrm{lk}(\beta, \beta)=\pm 1/p$, M bounds a topological homotopy S^2 whose boundary class is equal to $\pm\beta$. If p is even, such a 4-manifold is unique. If p is odd, there are exactly two such 4-manifolds, which are distinguished by their Kirby-Siebenmann obstructions.*

(2) *Let V and V' be topological homotopy S^2 bounded by M. Then V and V' are homeomorphic if and only if there exists a homeomorphism $h: M\to M$ such that $h_*(\beta(V))=\pm\beta(V')$ (and the Kirby-Siebenmann obstructions of V and V' agree if p is odd).*

REMARK 2.6. For $p=2$, we have a slightly different classification.

REMARK 2.7. In Proposition 2.5 (2), we do not consider the orientations of V and V'. If they are given orientations compatible with that of M and we consider their oriented homeomorphism types, then the homeomorphism h above should be orientation preserving.

EXAMPLE 2.8. Consider the lens space $L(p, q)$ with $p\geqq 2$ even. Suppose q is a quadratic residue mod p. Then homeomorphism classes of topological homotopy S^2 bounded by $L(p, q)$ are in one-to-one correspondence with elements of $\{n\in(\boldsymbol{Z}/p\boldsymbol{Z})^{\times}; n^2q\equiv\pm 1\}/\{\pm 1\}$ (if $q^2\not\equiv\pm 1 \pmod p$) or $\{n\in(\boldsymbol{Z}/p\boldsymbol{Z})^{\times}; n^2q\equiv\pm 1\}/\{\pm 1, \pm q\}$ (if $q^2\equiv\pm 1 \pmod p$). Here, for a commutative ring R, $R^{\times}$ denotes the group consisting of all units in R. For details see [**2**]. Note that we can compute the number of elements in these sets as in Corollary 3.11 below. See also Remark 3.12.

§ 3. Main results.

In this section we state our main theorem (Theorem 3.3) and show some of its consequences. The proof of Theorem 3.3 will be given in § 4.

DEFINITION 3.1. Let M_0 and M_1 be oriented closed 3-manifolds. Then M_0 and M_1 are *homology cobordant* if there exists a smooth oriented 4-manifold W such that $\partial W=M_0\cup(-M_1)$ and the inclusions induce isomorphisms $H_*(M_i)\to H_*(W)$ $(i=0, 1)$.

DEFINITION 3.2. Let M be a homology lens space homology cobordant to $L(p, 1)$. Note that $L(p, 1)$ is homeomorphic to the Dehn surgered 3-manifold $M(K; p/1)$, where K is the trivial knot. Let $\alpha\in H_1(L(p, 1))$ be the homology class represented by a meridian of K. Then we define $\alpha(M)\in H_1(M)$ to be the homology class corresponding to $\alpha\in H_1(L(p, 1))$ through a homology cobordism. We call $\alpha(M)$ a *canonical generator* of $H_1(M)$. Note that $\mathrm{lk}(\alpha(M), \alpha(M))=-1/p$.

Note that $\alpha(M)$ possibly depends on the choice of a homology cobordism. In the following, $\alpha(M)$ will be a canonical generator with respect to a fixed homology cobordism. (In fact, $\alpha(M)$ is unique in our situation. See Remark 3.9.)

THEOREM 3.3. *Let M be a homology lens space homology cobordant to $L(p, 1)$. Suppose V is a topological homology S^2 bounded by M such that $\beta(V)\neq\pm\alpha(M)$ (and $\mathrm{lk}(\beta(V), \beta(V))=\mathrm{lk}(\alpha(M), \alpha(M))$ if p is odd), where $\beta(V)$ is the boundary class of V. Then V cannot admit any smooth structures.*

A smooth knot K in S^3 $(=\partial D^4)$ is called *slice* if it bounds a smoothly embedded 2-disk in D^4. As a corollary to Theorem 3.3, we have the following.

COROLLARY 3.4. *Let K be a slice knot and $p>2$ an integer, where p is even or p has some prime factor p' with $p'\equiv 3 \pmod 4$. Suppose V is a topological homotopy S^2 bounded by the Dehn surgered 3-manifold $M(K; p/1)$. Then V admits a smooth structure if and only if it is homeomorphic to the handlebody $V(K; p)$.*

REMARK 3.5. Corollary 3.4 and Corollary 3.11 below hold in general for pseudo-p-shake-slice knots defined by Boyer [**1**].

PROOF OF COROLLARY 3.4. Since K is slice, a generator of $H_2(V(K;p))\cong \boldsymbol{Z}$ is represented by a smoothly embedded 2-sphere S. Set $W=V(K;p)-\operatorname{Int}N(S)$, where $N(S)$ denotes a tubular neighborhood of S. Then it is easy to see that W is a homology cobordism between $M=M(K;p/1)$ and $L(p, 1)$. Furthermore the boundary class $\beta(V(K;p))$ coincides with the canonical generator $\alpha(M)$ of M.

Next we show that there is no $\beta\in H_1(M)$ with $\operatorname{lk}(\beta, \beta)=-\operatorname{lk}(\alpha(M), \alpha(M))=1/p$ if p is odd. Suppose β is such a homology class. Since $\alpha(M)$ is a generator of $H_1(M)$, $\beta=r\alpha(M)$ for some $r\in\boldsymbol{Z}$. Then $1/p\equiv\operatorname{lk}(\beta, \beta)\equiv r^2\operatorname{lk}(\alpha(M), \alpha(M))\equiv -r^2/p \pmod{\boldsymbol{Z}}$; hence, we have $r^2\equiv -1 \pmod{p}$. However, there is no such r if p has some prime factor p' with $p'\equiv 3 \pmod 4$ (for details see the proof of Corollary 3.11 below).

Now suppose V is a smooth homotopy S^2 bounded by $M(K;p/1)$. Then by Theorem 3.3, we have $\beta(V)=\pm\alpha(M)$. Hence V is homeomorphic to the handlebody $V(K;p)$ by Proposition 2.5. □

REMARK 3.6. In Theorem 3.3 (or Corollary 3.4), we cannot in general omit the assumption that $\operatorname{lk}(\beta(V), \beta(V))=\operatorname{lk}(\alpha(M), \alpha(M))$ (resp. p has some prime factor p' with $p'\equiv 3 \pmod 4$) if p is odd. For example, let K and K' be the trivial knot and the (left-hand) trefoil knot respectively. Then $M(K';-5/1)$ is homeomorphic to the lens space $L(5, 1)\cong M(K;5/1)$ ([**16**]). Set $V=V(K';-5)$. Then V is a *smooth* homotopy S^2 bounded by $L(5, 1)$. It is easy to see that V and $V(K;5)$ are (orientation reversingly) homotopy equivalent relative to boundaries (cf. [**7**, §7]). However, they are not homeomorphic to each other, since $\beta(V)\neq\pm\alpha(L(5, 1))$. Note that $\operatorname{lk}(\beta(V), \beta(V))=-\operatorname{lk}(\alpha(L(5, 1)), \alpha(L(5, 1)))$ and that 5 has no prime factor p' with $p'\equiv 3 \pmod 4$.

REMARK 3.7. Corollary 3.4 does not hold for non-slice knots in general. For example, let K be the $(3, -11)$-torus knot. Then $M(K;34/1)$ is homeomorphic to the lens space $L(34, 9)$ ([**16**]), which bounds exactly two topological homotopy S^2 (see Example 2.8). Both of them admit smooth structures, since they are homeomorphic to the handlebodies $V(K;34)$ and $V(K';-34)$ respectively, where K' is the $(5, 7)$-torus knot (see [**16**]). Thus K is not pseudo-34-shake-slice (see Remark 3.5), though K has zero Arf invariant.

Using Theorem 3.3, we obtain the following result, which is already known for the lens spaces $L(p, 1)$ $(p\geqq 2)$.

COROLLARY 3.8. *Let M be a homology lens space homology cobordant to $L(p, 1)$, where $p\geqq 2$ is even or p has some prime factor p' with $p'\equiv 3 \pmod 4$. Then for any homeomorphism $h: M\to M$, h_* acts on $H_1(M)$ by the multiplication of ± 1.*

PROOF. Suppose there exists a homeomorphism $h: M \to M$ with $h_* \neq \pm 1$ on $H_1(M)$. By [**15**] we may assume h is a diffeomorphism. Then using a homology cobordism, we can construct a smooth homology S^2 bounded by M whose boundary class is different from $\pm\alpha(M)$. This contradicts Theorem 3.3. $\square$

REMARK 3.9. By a similar argument we can show that a canonical generator is uniquely determined up to a multiple of ± 1, provided p satisfies the condition in Corollary 3.8.

EXAMPLE 3.10. Let K be the (3, 5)-torus knot and set $M = M(K; -16/1)$. Then by [**16**] M is homeomorphic to the lens space $L(16, 7)$, which admits a self-homeomorphism h with $h_* \neq \pm 1$ on $H_1(L(16, 7))$. Thus M is not homology cobordant to $L(16, 1)$ by Corollary 3.8. In particular, the generator of $H_2(V(K; -16)) \cong \boldsymbol{Z}$ cannot be represented by a smoothly embedded 2-sphere; i.e., K is not pseudo-(-16)-shake-slice.

Using Corollary 3.8, we can completely determine how many topological homotopy S^2 a homology lens space as in Corollary 3.4 bounds.

COROLLARY 3.11. *Let K and p be as in Corollary* 3.4. *Furthermore let* $p = 2^e p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $(e \geqq 0,\ e_i > 0)$ *be the prime decomposition of p. Then the number of homeomorphism classes of topological homotopy S^2 bounded by $M(K; p/1)$ is equal to*

$$\begin{cases} 2^{r-1} & (\textit{if } e=1 \textit{ and } p_i \equiv 3 \pmod 4 \textit{ for some } p_i) \textit{ or} \\ 2^r & (\textit{if } e=0, 2 \textit{ or } e=1 \textit{ and } p_i \not\equiv 3 \pmod 4 \textit{ for every } p_i) \textit{ or} \\ 2^{r+1} & (\textit{if } e \geqq 3). \end{cases}$$

PROOF. By Proposition 2.5 and Corollary 3.8, we see that such homeomorphism classes are in one-to-one correspondence with elements of $\{n \in (\boldsymbol{Z}/p\boldsymbol{Z})^\times;\ n^2 \equiv \pm 1\}/\{\pm 1\}$ (if p is even) or $(\{n \in (\boldsymbol{Z}/p\boldsymbol{Z})^\times;\ n^2 \equiv \pm 1\}/\{\pm 1\}) \times \boldsymbol{Z}/2\boldsymbol{Z}$ (if p is odd). It is well-known that $(\boldsymbol{Z}/p\boldsymbol{Z})^\times \cong (\boldsymbol{Z}/2^e\boldsymbol{Z})^\times \times (\boldsymbol{Z}/p_1^{e_1}\boldsymbol{Z})^\times \times \cdots \times (\boldsymbol{Z}/p_r^{e_r}\boldsymbol{Z})^\times$. Furthermore one has

$$(3.1) \qquad (\boldsymbol{Z}/2^e\boldsymbol{Z})^\times \cong \begin{cases} 0 & (e=1) \\ \boldsymbol{Z}/2\boldsymbol{Z} & (e=2) \\ \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2^{e-2}\boldsymbol{Z} & (e \geqq 3) \end{cases}$$

and

$$(3.2) \qquad (\boldsymbol{Z}/p_i^{e_i}\boldsymbol{Z})^\times \cong \boldsymbol{Z}/p_i^{e_i-1}(p_i-1)\boldsymbol{Z},$$

where $-1 \bmod 2^e$ corresponds to $(1, 0) \in \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2^{e-2}\boldsymbol{Z}$ in the third isomorphism of (3.1) and $-1 \bmod p_i^{e_i}$ corresponds to the unique element of order 2 in $\boldsymbol{Z}/p_i^{e_i-1}(p_i-1)\boldsymbol{Z}$ in (3.2). (See, for example, Chapter 3 of [**17**].) Using these isomorphisms, we easily obtain

$$\#\{n\in(\boldsymbol{Z}/p\boldsymbol{Z})^{\times}\,;\,n^2\equiv 1\}=\begin{cases}2^r & (e\leqq 1)\\ 2^{r+1} & (e=2)\\ 2^{r+2} & (e\geqq 3)\end{cases}$$

and

$$\#\{n\in(\boldsymbol{Z}/p\boldsymbol{Z})^{\times}\,;\,n^2\equiv -1\}=\begin{cases}0 & (e\geqq 2 \text{ or } p_i\equiv 3 \pmod 4 \text{ for some } p_i)\\ 2^r & (\text{otherwise}).\end{cases}$$

Here the symbol # denotes the number of elements in the set. Now the result follows immediately. □

REMARK 3.12. If K is the unknot, then $M(K;p/1)$ is the lens space $L(p, 1)$ and the above result holds for every $p\geqq 2$. See Example 2.8.

Finally we note that there exists a *smooth* homotopy S^2 not homeomorphic to a handlebody $V(K;p)$. For example, let K be a smooth knot in S^3 and p an integer such that the Dehn surgered 3-manifold $M(K;p/1)$ has no lens spaces as its connected summands. Furthermore, let Σ ($\neq S^3$) be a homology 3-sphere which bounds a smooth contractible 4-manifold Δ. Then define V to be the 4-manifold obtained by the boundary connected sum of $V(K;p)$ and Δ. Then V is a smooth homotopy S^2 with $\partial V\cong M(K;p/1)\#\Sigma$. However V is not homeomorphic to a handlebody, since the boundary 3-manifold of V cannot be obtained by Dehn surgery on knots by a recently announced result of Gordon-Luecke [**9**].

§4. Proof of Theorem 3.3.

Let M be a homology lens space homology cobordant to $L(p, 1)$ and suppose V is a *smooth* homology S^2 bounded by M such that $\beta(V)\neq\pm\alpha(M)$ (and $\mathrm{lk}(\beta(V), \beta(V))=\mathrm{lk}(\alpha(M), \alpha(M))$ if p is odd). Let W be a homology cobordism between M and $L(p, 1)$. Set $X=V\cup_M W\cup_{L(p,1)} D(p)$, where $D(p)$ is the 2-disk bundle over S^2 with euler number p. (In other words, $D(p)$ is diffeomorphic to the handlebody $V(K;p)$, where K is the trivial knot.) Note that $\partial D(p)\cong L(p, 1)$. Then X is an orientable smooth closed 4-manifold with $H_1(X)=0$ and $H_2(X)\cong \boldsymbol{Z}\oplus\boldsymbol{Z}$.

Next we define homology classes θ and τ in $H_2(X)$ as follows. Let S be the zero section of $D(p)$, which is a smoothly embedded 2-sphere. We consider S to be embedded in X and define $\theta=[S]\in H_2(X)$. Let f be a 2-chain in $D(p)$ represented by a fiber of $D(p)$, then by the definition of $\alpha(M)$, $[\partial f]\in H_1(L(p, 1))$ and $\alpha(M)\in H_1(M)$ are homologous in W. Let c be a 2-chain in W with $\partial c= a\cup(-\partial f)$, where a is a 1-cycle in M with $[a]=\alpha(M)$. Let γ be a generator of $H_2(V, \partial V)\cong \boldsymbol{Z}$ ($\partial\gamma=\beta(V)$). Since $\alpha(M)$ is a generator of $H_1(M)$, we have $\beta(V)=r\alpha(M)$ for some $r\in\boldsymbol{Z}$. (Note that $r\not\equiv\pm 1 \pmod p$ by the assumption.) Then there exists a 2-chain c' in V such that $[c']=-\gamma$ and $\partial c'=-ra$. Then define $\tau=[c'\cup r(c\cup f)]\in H_2(X)$. (See Figure 2.) Note that $\theta\cdot\theta=p$ and $\theta\cdot\tau=r$.

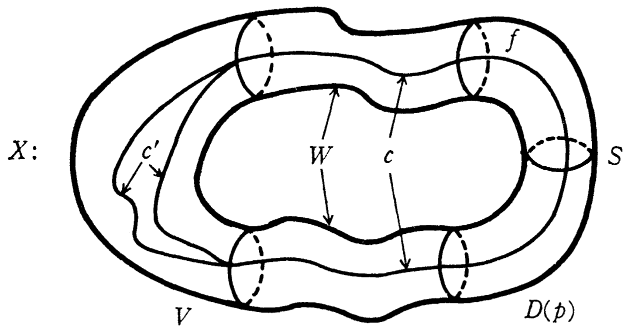


Figure 2.

Lemma 4.1. *$H_2(X)$ is generated by θ and τ.*

Proof. There is a generator δ of $H_2(V)$ such that $i_*\delta=-p\gamma$, where $i_*: H_2(V)\to H_2(V, \partial V)$ is the homomorphism induced by the inclusion. Then we have

$$p\tau = [pc'\cup rp(c\cup f)] = j_*\delta+r\theta \qquad \text{in } H_2(X), \tag{4.1}$$

where $j: V\to X$ is the inclusion map. Let G_1 be the subgroup of $H_2(X)$ generated by $j_*\delta$ and θ, and let G_2 be the one generated by $j_*\delta$, θ and τ. Then by (4.1) $[G_2: G_1]=p$, since $j_*\delta$ and θ are linearly independent in $H_2(X)$. Furthermore by the Mayer-Vietoris exact sequence

$$0 \longrightarrow H_2(V)\oplus H_2(W\cup D(p)) \longrightarrow H_2(X) \longrightarrow H_1(M) \longrightarrow 0,$$

we have $[H_2(X): G_1]=p$. Hence we obtain $G_2=H_2(X)$. Since $j_*\delta=p\tau-r\theta$ by (4.1), $H_2(X)$ is generated by θ and τ. $\square$

Since $H_2(X)$ is of rank 2, there exist generators ξ and η of $H_2(X)$ with respect to which the intersection matrix of X is one of the following forms:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

(See [**14**, p. 19].) Set $\theta=a\xi+b\eta$ and $\tau=c\xi+d\eta$ ($a, b, c, d\in \boldsymbol{Z}$).

Case (1). We have

$$2ab = p\,, \tag{4.2}$$

$$ad+bc = r\,, \tag{4.3}$$

$$ad-bc = \pm 1\,, \tag{4.4}$$

where (4.2) and (4.3) are equivalent to $\theta\cdot\theta=p$ and $\theta\cdot\tau=r$ respectively, and (4.4) is equivalent to that θ and τ generate $H_2(X)$.

Since θ is represented by a smoothly embedded 2-sphere S, we may assume $|a|\leqq 1$ by [**11**] or [**5**]. (Kuga [**11**] proves the result assuming X is simply connected. However, this assumption can be eliminated by a recent result of Donaldson [**3**].) By (4.2) $a\neq 0$; hence, $a=\pm 1$ and $2b=\pm p$. By (4.3) and (4.4) we have $2bc=r\pm 1$; hence, $r\equiv\pm 1 \pmod p$. Since $\beta(V)=r\alpha(M)$, this is a contradiction.

Case (2). We have

$$a^2-b^2 = p\,, \tag{4.5}$$

$$ac-bd = r\,, \tag{4.6}$$

$$ad-bc = \pm 1 \tag{4.7}$$

as in Case (1). Changing orientations of ξ and/or η if necessary, we may assume $a\geqq 0$ and $b\geqq 0$. Using (4.5), (4.6) and (4.7), we have $c-d=(1/p)(a-b)(r\pm 1)$. Thus, if $a-b=1$, we have $r\equiv\pm 1 \pmod p$, which contradicts the assumption. Hence by (4.5) $a-b\geqq 2$. Furthermore a, $b>0$ by (4.7). Hence, $\theta=a\xi+b\eta$ cannot be represented by a smoothly embedded 2-sphere by [**12**], which is a contradiction.

Case (3). We have

$$a^2+b^2 = p\,, \tag{4.8}$$

$$ac+bd = r\,, \tag{4.9}$$

$$ad-bc = \pm 1\,. \tag{4.10}$$

Set $\tau\cdot\tau=t$, then the intersection matrix of X with respect to the generators θ and τ is $\begin{pmatrix} p & r \\ r & t\end{pmatrix}$. Since the intersection form of X is definite and unimodular, $\det\begin{pmatrix} p & r \\ r & t\end{pmatrix}=pt-r^2=1$; hence, $r^2\equiv -1 \pmod p$. If p is odd, we have $-r^2/p\equiv \mathrm{lk}(\beta(V), \beta(V))\equiv \mathrm{lk}(\alpha(M), \alpha(M))\equiv -1/p \pmod{\boldsymbol{Z}}$ by the assumption; hence, $r^2\equiv 1 \pmod p$. Thus p must be even. Then by (4.8) and (4.10), both a and b are odd and $\theta=a\xi+b\eta$ is a characteristic homology class. Furthermore $|ab|\neq 1$, since otherwise $p=2$ and $r\equiv\pm 1 \pmod p$. Thus θ cannot be represented by a smoothly embedded 2-sphere by [**12**, Theorem 3 (ii)] (see also [**3**]), which is a contradiction. Hence the 4-manifold V cannot be smooth. This completes the proof of Theorem 3.3. □

REMARK 4.2. We do not know whether the assumption that $\mathrm{lk}(\beta(V), \beta(V)) = \mathrm{lk}(\alpha(M), \alpha(M))$ is necessary when $p \neq 5$. (See Remark 3.6 in § 3.) For example, if a homology lens space M homology cobordant to $L(13, 1)$ bounds a smooth homology S^2 V with $\mathrm{lk}(\beta(V), \beta(V)) \neq \mathrm{lk}(\alpha(M), \alpha(M))$, then for some smooth closed 4-manifold X with the homology of $\boldsymbol{CP}^2 \# \boldsymbol{CP}^2$ the homology class corresponding to $(2, 3) \in \boldsymbol{Z} \oplus \boldsymbol{Z} \cong H_2(X)$ can be represented by a smoothly embedded 2-sphere.

§ 5. Homology lens spaces which cannot bound smooth homology S^2.

In [7] Fukuhara has shown that there exist infinitely many lens spaces $L(p, q)$ which, though each q is a quadratic residue mod p, cannot bound a smooth homology S^2. He showed this using a Rohlin-type invariant for certain homology lens spaces. In this section we show the existence of infinitely many homology lens spaces with the same property which cannot be detected by Fukuhara's invariant. First we prove the following.

THEOREM 5.1. *Let m and n be odd integers with $n \geqq 3$ and K be a knot in S^3. Set $p = mn^2 + 1$ and $q = n^2$. Let $\alpha \in H_1(M(K; p/q))$ be the homology class represented by a meridian of K. Suppose V is a homology S^2 bounded by the Dehn surgered 3-manifold $M(K; p/q)$ with $\beta(V) = \pm mn\alpha$. Then V cannot admit any smooth structures.*

REMARK 5.2. There does exist such a *topological* 4-manifold as above by Proposition 2.5.

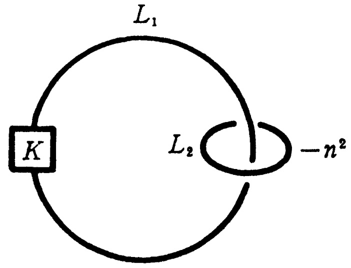


Figure 3.

PROOF OF THEOREM 5.1. Let W be the handlebody obtained by attaching two 2-handles to the 4-ball D^4 along the link $L = L_1 \cup L_2$ as in Figure 3 using the indicated framings. Then ∂W is homeomorphic to $M(K; p/q)$ (see, for example, Lemma 2.1 of [13]). Furthermore $\alpha \in H_1(M(K; p/q))$ corresponds to the homology class of $H_1(\partial W)$ represented by a meridian of L_1.

Suppose V is a *smooth* homology S^2 bounded by $M=M(K;p/q)$ with $\beta(V)=\pm mn\alpha$. Set $X=W\cup_M(-V)$, which is a smooth closed 4-manifold with $H_1(X)=0$ and $H_2(X)\cong \boldsymbol{Z}\oplus\boldsymbol{Z}\oplus\boldsymbol{Z}$. Next we define the homology classes θ, τ and ω in $H_2(X)$ as follows. Let S_i be the PL embedded 2-sphere in W which is the union of the core of the 2-handle attached along L_i and the cone over L_i embedded in D^4 $(i=1, 2)$. Note that S_2 is smoothly embedded, while S_1 is not necessarily locally flat. We consider S_i to be embedded in X and define $\theta=[S_1]$ and $\tau=[S_2]$. Let c be a 2-chain in W represented by a cocore of the 2-handle attached along L_1. We may asume $[\partial c]=\alpha$ in $H_1(\partial W)$. Let $\gamma\in H_2(V,\partial V)\cong\boldsymbol{Z}$ be a generator, then $\partial\gamma=\beta(V)=r\alpha$, where $r=\pm mn$. Hence there is a 2-chain c' in V such that $[c']=-\gamma$ and $\partial c'=-r(\partial c)$. Then we define $\omega=[c'\cup rc]$.

Using the same argument as in Lemma 4.1, we see that θ, τ and ω generate $H_2(X)$. Note that τ is represented by a smoothly embedded 2-sphere S_2.

Set $s=\omega\cdot\omega$. Then the intersection matrix of X with respect to generators θ, τ and ω is

$$Q=\begin{pmatrix} m & 1 & r \\ 1 & -n^2 & 0 \\ r & 0 & s \end{pmatrix}.$$

Set $\varepsilon=\det Q$ $(=\pm 1)$, then we easily obtain $s(mn^2+1)=(mn^2+1)(mn^2-1)+1-\varepsilon$. If $\varepsilon=-1$, mn^2+1 must divide 2, which is a contradiction. Hence $\varepsilon=1$, $s=mn^2-1$, and $\operatorname{sign} Q=-1$. Thus the intersection form of X is isomorphic to $(1)\oplus(-1)\oplus(-1)$. Furthermore since $\tau\cdot\theta\equiv\theta\cdot\theta$ (mod 2) and $\tau\cdot\omega\equiv\omega\cdot\omega$ (mod 2), τ is characteristic. Then by the same argument as in [**12**], we see that τ cannot be represented by a smoothly embedded 2-sphere, since $\tau^2=-n^2\leqq -9$. This is a contradiction. This completes the proof. □

As a corollary to Theorem 5.1, we have the following.

COROLLARY 5.3. *For every odd square $q\geqq 9$, there exist infinitely many positive integers p with the following properties.*

(1) *p is prime to q.*

(2) *For every knot K in S^3, the Dehn surgered 3-manifold $M(K;p/q)$ bounds a topological homotopy S^2 but never bounds a smooth homology S^2. In particular, $M(K;p/q)$ cannot be obtained from S^3 by integral Dehn surgery on knots.*

To prove Corollary 5.3, we need the following.

LEMMA 5.4. *Let $p=2t$, where t is a prime number with $t\equiv 3$ (mod 4). Suppose $r^2q\equiv\pm 1$ (mod p) and $r'^2q\equiv\pm 1$ (mod p), where r, r' and q are integers. Then $r\equiv\pm r'$ (mod p).*

PROOF. Since r is prime to p, there is an integer s with $r'\equiv sr \pmod{p}$. Then $s^2\equiv\pm s^2r^2q\equiv\pm r'^2q\equiv\pm 1 \pmod{p}$. Then by the assumption on p, we have $s\equiv\pm 1 \pmod{p}$. (See [7] or the proof of Corollary 3.11.) $\square$

PROOF OF COROLLARY 5.3. We have $q=n^2$ for some odd integer $n\geqq 3$. Set $n=2n'+1$ and $p'_k=q(8k+5)+1$ $(k=1, 2, 3, \cdots)$. Then we have $p'_k/2=4qk+10n'(n'+1)+3\equiv 3 \pmod{4}$. Since $4q=4(4n'^2+4n'+1)$ and $10n'(n'+1)+3$ are relatively prime, there are infinitely many prime numbers in the sequence $\{p'_k/2\}_{k=1}^{\infty}$ by Dirichlet's theorem. Let $\{p_k\}$ be an infinite subsequence of $\{p'_k\}$ such that each $p_k/2$ is a prime number. We show that each p_k satisfies the required condition.

First, note that $M(K; p/q)$ $(p=p_k)$ bounds a topological homotopy S^2, since $q=n^2$ is a quadratic residue mod p (see Example 2.3). Suppose $M(K; p/q)$ bounds a homology S^2 V with $\beta(V)=r\alpha$, where $\alpha\in H_1(M(K; p/q))$ is the homology class represented by a meridian of K. By the above construction, $p=mn^2+1$ for some odd integer m. We have $\mathrm{lk}(\beta(V), \beta(V))\equiv -r^2q/p\equiv\pm 1/p \pmod{\boldsymbol{Z}}$; hence, $r^2q\equiv\pm 1 \pmod{p}$. By Lemma 5.4 such an integer r modulo p is determined up to a multiple of ± 1; hence, $r\equiv\pm mn \pmod{p}$. Thus, by Theorem 5.1, V cannot admit any smooth structures. $\square$

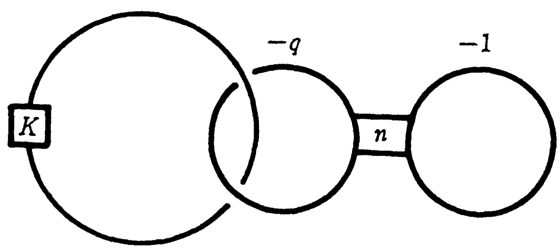
We now recall the definition of Fukuhara's invariant $\tilde{\mu}$ for certain homology lens spaces ([7]). Let M be a homology lens space with $H_1(M)$ isomorphic to $\boldsymbol{Z}/p\boldsymbol{Z}$, where $p=2t^e$ $(e\geqq 1)$ with t an odd prime congruent to 3 modulo 4. Furthermore we assume that there is a homology class $\beta\in H_1(M)$ with $\mathrm{lk}(\beta, \beta)=\pm 1/p$. Then Fukuhara shows that M can be obtained by an integral surgery on a knot in some homology 3-sphere Σ. Fukuhara's invariant $\tilde{\mu}(M)$ is defined to be the Rohlin invariant of Σ. If p is of the form $2t^e$, this is well-defined. Furthermore, if M bounds a smooth homology S^2, $\tilde{\mu}(M)$ must vanish.

REMARK 5.5. Such a homology lens space as above bounds a topological homotopy S^2 and, by the assumption that p is of the form $2t^e$, such a 4-manifold is unique. Then $\tilde{\mu}(M)$ coincides with the Kirby-Siebenmann obstruction of the homotopy S^2.

LEMMA 5.6. *Let p and q be the integers of Corollary* 5.3 *and K be a knot in S^3. Then for the homology lens space $M(K; p/q)$ Fukuhara's invariant is defined and* $\tilde{\mu}(M(K; p/q))\equiv(q-1)/8 \pmod{2}$.

PROOF. Since $p=2t$ for some odd prime t with $t\equiv 3 \pmod{4}$ and q is a quadratic residue mod p, Fukuhara's invariant is defined for $M=M(K; p/q)$. Set $m=(p-1)/q$, which is an odd integer. Let Σ be the homology 3-sphere which has the surgery description as in Figure 4 $(q=n^2)$. Then it is easily seen that the component with framing $-q$ is the characteristic sublink in the sense of

[10]. Then by [10, Theorem 4.2], Σ bounds a spin 4-manifold of signature $q-1$. Note that $M(K; p/q)$ is homeomorphic to the 3-manifold described in Figure 3, which is obtained by integral Dehn surgery on a knot in Σ. Hence, $\bar{\mu}(M)\equiv(q-1)/8 \pmod 2$. □



( stands for n times full-twist.)

Figure 4.

If we consider odd squares $q=n^2$ with $n\equiv\pm1 \pmod 8$ in Corollary 5.3, we obtain the following proposition, using Lemma 5.6.

PROPOSITION 5.7. *There are infinitely many homology lens spaces M with the following properties.*

(1) *M bounds a topological homotopy* S^2.

(2) *Fukuhara's invariant is defined for M and it vanishes.*

(3) *Nevertheless, M cannot bound any smooth homology* S^2. *In particular, M cannot be obtained from* S^3 *by integral Dehn surgery on knots.*

Using our techniques, we can find many lens spaces $L(p, q)$ which, though each q is a quadratic residue mod p, cannot bound any smooth homology S^2. There are at least 41 such lens spaces $L(p, q)$ with $2\leqq p\leqq 100$, which include 12 lens spaces found by Fukuhara [7]. For example, consider the lens space $L(52, 3)$, for which Fukuhara's invariant is not defined. There are exactly two topological homotopy S^2 bounded by $L(52, 3)$ (see Example 2.8), whose Kirby-Siebenmann obstructions vanish. However, we can show that no homology S^2 bounded by $L(52, 3)$ can admit a smooth structure, using the same argument as in the proof of Theorem 5.1.

A computation shows that there are exactly 1401 topological homotopy S^2 bounded by lens spaces $L(p, q)$ with $2\leqq p\leqq 100$. We can show that among these there are at least 701 homotopy S^2 which cannot admit any smooth structures. On the other hand, we can find at least 274 homotopy S^2 which can be given smooth structures, using results on lens spaces obtained by Dehn surgery on knots [16], [4] (see also [7]). For example, there are exactly 4 topological homotopy S^2 bounded by $L(56, 9)$. Two of them admit smooth structures, since they are homeomorphic to the handlebodies $V(K; 56)$ and $V(K'; 56)$ respectively, where K is the $(5, -11)$-torus knot and K' is the $(3, -19)$-torus knot [**16**]. The other two 4-manifolds cannot admit any smooth structures, since their Kirby-Siebenmann obstructions do not vanish.

References

[1] S. Boyer, Shake-slice knots and smooth contractible 4-manifolds, Math. Proc. Cambridge Philos. Soc., **98** (1985), 93-106.
[2] ———, Simply-connected 4-manifolds with a given boundary, Trans. Amer. Math. Soc., **298** (1986), 331-357.
[3] S. Donaldson, The orientation of Yang-Mills moduli spaces and 4-manifold topology, J. Diff. Geom., **26** (1987), 397-428.
[4] R. Fintushel and R. J. Stern, Constructing lens spaces by surgery on knots, Math. Z., **175** (1980), 33-51.
[5] ———, Pseudofree orbifolds, Ann. of Math., **122** (1985), 335-364.
[6] M. Freedman, The topology of 4-manifolds, J. Diff. Geom., **17** (1982), 357-453.
[7] S. Fukuhara, On an invariant of homology lens spaces, J. Math. Soc. Japan, **36** (1984), 259-277.
[8] C. Gordon and R. Litherland, On the signature of a link, Invent. Math., **47** (1978), 53-69.
[9] C. Gordon and J. Luecke, Knots are determined by their complements, preprint.
[10] S. Kaplan, Constructing framed 4-manifolds with given almost framed boundaries, Trans. Amer. Math. Soc., **254** (1979), 237-263.
[11] K. Kuga, Representing homology classes of $S^2 \times S^2$, Topology, **23** (1984), 133-137.
[12] T. Lawson, Representing homology classes of almost definite 4-manifolds, Michigan Math. J., **34** (1987), 85-91.
[13] Y. Matsumoto, On the bounding genus of homology 3-spheres, J. Fac. Sci. Univ. Tokyo, **29** (1982), 287-318.
[14] J. Milnor and D. Husemoller, Symmetric bilinear forms, Springer, New York, 1973.
[15] E. Moise, Affine structures on 3-manifolds, Ann. of Math., **56** (1952), 96-114.
[16] L. Moser, Elementary surgery along a torus knot, Pacific J. Math., **38** (1971), 737-745.
[17] P. Ribenboim, Algebraic numbers, Wiley-Interscience, New York-London-Sydney-Toronto, 1972.
[18] D. Rolfsen, Knots and links, Publish or Perish Inc., Berkeley, 1976.

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