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ON THE MEAN CURVATURE EVOLUTION OF TWO-CONVEX HYPERSURFACES

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Abstract

We study the mean curvature evolution of smooth, closed, two-convex hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 3$. Within this framework we effect a reconciliation between the flow with surgeries—recently constructed by Huisken and Sinestrari in [**HS3**]—and the well-known weak solution of the level-set flow: we prove that the two solutions agree in an appropriate limit of the surgery parameters and in a precise quantitative sense. Our proof relies on geometric estimates for certain L^p-norms of the mean curvature which are of independent interest even in the setting of classical mean curvature flow. We additionally show how our construction can be used to pass these estimates to limits and produce regularity results for the weak solution.

1. Introduction

Consider a smooth, closed hypersurface immersion $F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$. The solution of mean curvature flow starting from $\mathcal{M}_0 \equiv F_0(\mathcal{M}^n)$ is the one-parameter family $F : \mathcal{M}^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\text{(MCF)} \qquad \begin{cases} \dfrac{\partial F}{\partial t}(p,t) = -H(p,t)\nu(p,t), \quad p \in \mathcal{M}^n,\ t \geq 0, \\ F(\cdot, 0) = F_0, \end{cases}$$

where H and ν denote the mean curvature and outward-pointing unit normal, respectively. According to our choice of signs, the right-hand side is the mean curvature vector $\vec{H}$ and the mean curvature of the round sphere is positive. We write $M_t \equiv F(\cdot, t)(\mathcal{M}^n)$.

The non-linear evolution (MCF) generates curvature blow-up in finite time, motivating a detailed analysis of the geometric structure of the surface in high-curvature regions. We discuss the case $n \geq 2$.

Huisken [**H1**] showed that any convex hypersurface of dimension at least two must contract smoothly to a point in finite time and in an asymptotically round fashion. It is well known, however, that surfaces which are not uniformly positively curved can develop local singularities

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before they vanish. See [**H2**, **H3**, **HS1**, **HS2**] for a classification of all such singularities in the mean-convex setting.

Of course, the onset of local singularities precludes even a formal definition of the subsequent evolution in the language of differential geometry. For topological applications, this is fatal. However, for the class of *two-convex* surfaces with $n \geq 3$, Huisken and Sinestrari [**HS3**] have recently succeeded in extending the classical flow in a topologically controlled way using a surgery-based approach inspired by [**Ha**]. We recall that a surface is by definition two-convex if, at each point, the sum of any two of the principal curvatures is non-negative.

According to the algorithm in [**HS3**], the smooth evolution is interrupted shortly before the singular time by a surgery procedure which replaces a *neck* with two regions diffeomorphic to disks. A neck can be thought of as a piece of the surface which can be represented (up to a homothety) as a graph over a cylinder with small C^k-norm for a suitable integer k. Any connected components of known topology are also discarded at the surgery time. The smooth flow is then restarted and the whole process is repeated. Huisken and Sinestrari established that this algorithm terminates after a finite number of surgery times.

Mean curvature flow with surgeries does not constitute a weak solution of (MCF) in the traditional sense since it relies on a non-canonical modification of the surface at each chosen surgery time. It is controlled by a set of parameters $1 << H_0 < H_1 < H_2 < H_3 < \infty$ which determine when and where surgery is performed. H_0 represents a control scale beyond which the second fundamental form and its derivatives match one of several possible canonical profiles. The smooth evolution is stopped when the curvature reaches $H_3 >> H_0$, and surgery is performed away from the point of maximum curvature at a smaller scale $H_1 = \xi_1 H_3$ ($\xi_1 < 1$) such that the maximum of the curvature drops by a fixed amount to $H_2 = \xi_2 H_3$ ($\xi_1 < \xi_2 < 1$). The constants H_0, ξ_1, ξ_2 are determined by $\mathcal{M}_0$, and H_3 provides a uniform upper bound on the curvature for all times and across all surgeries.

The starting point for the work in this paper is the observation from [**HS3**] that the parameters H_1, H_2, H_3 are not unique; they can in fact be made arbitrarily large. We are therefore prompted to consider an increasing sequence of parameters $\{H_1^i, H_2^i, H_3^i\}_{i \geq 1}$, corresponding to a whole sequence of mean curvature flows with surgeries, along which the surgery times grow and the necks cut out during surgery become increasingly thin. As the parameters increase, more surgeries may be required. However, for each set of finite parameters H_1^i, H_2^i, H_3^i, only finitely many surgeries are required.

This motivates the following question: how does the object produced by the limit $H_1^i, H_2^i, H_3^i \to \infty$ relate to the unique *weak solution* of mean curvature flow? In this paper we combine a geometric barrier construction with new estimates on the required number of surgeries and Brakke's *clearing out lemma* to show that the two concepts agree in a precise quantitative sense; see Theorem 4.3. We refer the reader to Section 4 for the relevant definitions and a full description of the argument.

In addition to further endorsing the geometric relevance of the weak solution, this result provides a new framework within which to investigate regularity properties of the level-set flow. We also provide the first example of an application of this kind (see Section 5).

We point out that a different version of Theorem 4.3 was independently obtained by Lauer in [**L**].

The results in this paper are laid out as follows. Our first object of study in Section 2 is classical mean curvature flow. In the two-convex setting, we find an upper bound—which behaves like $t^{-1/2}$ for small t—on the L^p-norms of the mean curvature for all $p < n-1$; see Theorem 2.4. This result should be compared with the estimates obtained by Ecker and Huisken in [**EH**]. The critical exponent $p = n-1$ arises naturally from the two-convex geometry via an application of the *roundness estimate* from [**HS3**]. These estimates play a crucial role in the proof of our main theorem. In addition, they have an interesting application to the regularity theory developed in [**E2**]; see Remark 2.6.

In Section 3 we begin our analysis of mean curvature flow with surgeries. We briefly review the construction in [**HS3**] and use it to show that every L^p-norm of the mean curvature is non-increasing at the surgery times. We establish as a corollary that the smooth estimates on the classical evolution are preserved—in a weak sense—by the flow with surgeries. This leads us to a new bound on the required number of surgeries which strengthens the corresponding result in [**HS3**] and which is essential for our primary application.

Our main theorem appears in Section 4. We suitably interpret the flow with surgeries in the language of weak solutions and provide a formal statement of the convergence result in Theorem 4.3. Our proof relies on a combination of global barrier arguments and quantitative local techniques which are well-suited to the study of necks. The tools employed here are the familiar avoidance principle, Brakke's clearing out lemma, and the bound from Section 3 on the required number of surgeries. We obtain quantitative estimates on the rate of convergence in terms of the surgery parameters.

Finally, in Section 5, we refine our convergence theorem using techniques developed in [**S**, **MS**]. This allows us to pass the integral estimates from Sections 2 and 3 to limits.

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2. Integral estimates for classical mean curvature flow

In this section we bound the L^p-norms of the mean curvature for $p < n-1$ under smooth mean curvature flow in the two-convex setting. In the next section we show how these ideas can be applied to obtain a new bound on the number of surgeries required for the flow constructed by Huisken and Sinestrari in [**HS3**].

Notation. We denote by $g = \{g_{ij}\}$ the induced metric on the hypersurface $\mathcal{M}_t = F(\cdot, t)(\mathcal{M}^n)$. We then denote by $d\mu$ the surface measure, by $A = \{h_{ij}\}$ the second fundamental form, and by $\lambda_1 \leq \cdots \leq \lambda_n$ the ordered principal curvatures at the space-time point $(p,t) \in \mathcal{M}^n \times [0,T]$.

We assume throughout that the dimension n of the hypersurface is at least 3 and that the initial surface is two-convex, i.e. $\lambda_1 + \lambda_2 \geq 0$ everywhere on $\mathcal{M}_0$. Following Definition 2.5 in [**HS3**], we introduce some notation which will clarify our exposition. Recall that if $\lambda_1 + \lambda_2 \geq 0$ on $\mathcal{M}_0$, then by the strong maximum principle $\lambda_1 + \lambda_2 > 0$ on $\mathcal{M}_t$ for all $t > 0$.

Definition 2.1 (Class of Two-Convex Surfaces, [**HS3**])**.** We denote by $\mathcal{C}(R, \alpha)$, $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, the class of smooth, closed hypersurface immersions $F : \mathcal{M}^n \to \mathbb{R}^{n+1}$ satisfying

$$\lambda_1 + \lambda_2 \geq \alpha_0 H, \quad H \geq \alpha_1 R^{-1}, \quad \mathcal{H}^n(\mathcal{M}^n) \leq \alpha_2 R^n, \tag{2.1}$$

for some constants $R, \alpha_0, \alpha_1, \alpha_2 > 0$.

The parameter R is chosen such that $\sup_{\mathcal{M}_0} |A|^2 = R^{-2}$. Proposition 2.6 in [**HS3**] establishes that any smooth, closed, strictly two-convex surface belongs to $\mathcal{C}(R, \alpha)$ and satisfies $|A|^2 \leq R^{-2}$ for some R, α. Furthermore, $\mathcal{C}(R, \alpha)$ is invariant under both smooth mean curvature flow and *standard surgery* as defined in Section 3.

Note that R represents the scale of the surface and will therefore feature explicitly in the estimates below. For later reference we point out that the surgery parameters H_k $(k = 1, 2, 3)$ can be written in the form $\tilde{H}_k R^{-1}$ where each $\tilde{H}_k$ depends only on scale-free properties of $\mathcal{M}_0$; see Section 3. We stress that the inequality $|A|^2 \leq R^{-2}$ is not invariant under mean curvature flow and pertains only to the initial data. This assumption is made throughout but will not be repeated.

Our starting point is the following well-known result from [**H1**].

Lemma 2.2 (Evolution Equations, [**H1**])**.** *Let $\mathcal{M}_t$ be a smooth solution of mean curvature flow. The surface measure and mean curvature* *satisfy the evolution equations*

$$\frac{\partial}{\partial t} d\mu = -H^2 d\mu, \quad \frac{\partial}{\partial t} H = \Delta H + |A|^2 H$$

as long as $\mathcal{M}_t$ remains smooth.

Let $\mathcal{M}_t$ be the smooth solution of mean curvature flow starting from an initial surface $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ for some R, α. Using Lemma 2.2 we compute

$$\frac{d}{dt} \int_{\mathcal{M}_t} H^p \, d\mu = -p(p-1) \int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} \, d\mu + \int_{\mathcal{M}_t} H^p \left(p|A|^2 - H^2 \right) d\mu$$

for any $p \in \mathbb{R}$. We consider $p > 0$ and use the two-convex geometry to deal with the mixed term on the right-hand side. We appeal to the following *roundness estimate* from [**HS3**] (for later reference we state the full result for the flow with surgeries, which is defined in the next section).

Theorem 2.3 (Roundness Estimate, [**HS3**]). *i) Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ for some R, α and consider the smooth solution $\mathcal{M}_t$ of mean curvature flow starting from $\mathcal{M}_0$. For any $\eta > 0$ there exists a constant $C_\eta = C_\eta(n, \alpha) > 0$ such that*

$$|A|^2 - \frac{H^2}{n-1} \leq \eta H^2 + C_\eta R^{-2} \tag{2.2}$$

on $\mathcal{M}_t$ as long as the solution remains smooth.

ii) There exists $\tilde{\eta} = \tilde{\eta}(n) > 0$, where $0 < \tilde{\eta} < 1$, such that the following holds. The parameters controlling the surgery procedure can be chosen such that for all $0 < \eta < \tilde{\eta}$ the estimate (2.2) holds on the solution of mean curvature flow with surgeries starting from $\mathcal{M}_0$.

We therefore restrict our attention to $p < n-1$. To make this concrete, let $0 < \varepsilon \leq 1$, fix $p = n - 1 - \varepsilon$, and choose $\eta = \eta(n, \varepsilon) > 0$ sufficiently small with respect to ε, for example

$$\eta_\varepsilon \equiv \frac{\varepsilon}{2(n-1)(n-1-\varepsilon)}.$$

We henceforth suppress the subscript ε on η and write C_ε in place of C_{η_ε} for ease of notation. Now applying Theorem 2.3 with this choice of η, we obtain

$$\begin{aligned}\frac{d}{dt} \int_{\mathcal{M}_t} H^p \, d\mu \leq & - p(p-1) \int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} \, d\mu \\ & - \frac{\varepsilon}{2(n-1)} \int_{\mathcal{M}_t} H^{p+2} \, d\mu + pC_\varepsilon R^{-2} \int_{\mathcal{M}_t} H^p \, d\mu.\end{aligned}$$

Using Hölder's inequality, Lemma 2.2, and Definition 2.1, we conclude

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}_t} H^p\,d\mu \le -p(p-1)&\int_{\mathcal{M}_t}|\nabla H|^2 H^{p-2}\,d\mu\\ &-\frac{\varepsilon}{2(n-1)}\alpha_2^{-\frac{2}{p}}R^{\frac{-2n}{p}}\left(\int_{\mathcal{M}_t}H^p\,d\mu\right)^{\frac{p+2}{p}}\\ &+pC_\varepsilon R^{-2}\int_{\mathcal{M}_t}H^p\,d\mu.\end{aligned}$$

Define

$$\varphi = \exp\left(-\frac{pC_\varepsilon}{R^2}t\right)\int_{\mathcal{M}_t}H^p\,d\mu.$$

We have proved that φ is non-increasing under the smooth evolution. In particular, the L^p-norm of the mean curvature is bounded for all $p < n-1$ under smooth mean curvature flow on any finite time interval. In fact, by solving the ODE

$$\frac{d}{dt}\varphi \le -\frac{\varepsilon}{2(n-1)}\alpha_2^{-\frac{2}{p}}R^{\frac{-2n}{p}}\exp\left(\frac{2C_\varepsilon}{R^2}t\right)\varphi^{\frac{p+2}{p}},$$

we obtain an L^p-estimate for the mean curvature which behaves like $t^{-1/2}$ for small t (this agrees with the scaling of the interior estimates obtained by Ecker and Huisken in [**EH**]).

Proposition 2.4 (Smooth L^p Estimate)**.** *Let $\mathcal{M}_0 \in \mathcal{C}(R,\alpha)$ for some R,α and consider the smooth solution $\mathcal{M}_t$ of mean curvature flow starting from $\mathcal{M}_0$. For any $0 < \varepsilon \le 1$ there exists a constant $C_\varepsilon = C_\varepsilon(n,\alpha) > 0$ such that, setting $p = n-1-\varepsilon$, we have*

$$\|H\|_{L^p(\mathcal{M}_t)} \le C(n)\,\alpha_2^{1/p}R^{(n-p)/p}\left(\frac{pC_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}}\exp\left(\frac{C_\varepsilon}{R^2}t\right)\left(\exp\left(\frac{2C_\varepsilon}{R^2}t\right)-1\right)^{-\frac{1}{2}}$$

for all $t > 0$ as long as the solution remains smooth. Here $C(n) > 0$ is a constant depending only on n.

Note that the additional estimate in the second part of Theorem 2.3 has not yet been used. In the next section, we will apply this calculation to a solution of mean curvature flow with surgeries, and in this setting it will be necessary to call upon the roundness estimate (with the same constants) before and after surgeries.

We now make an informal remark on the critical exponent $p = n-1$.

Remark 2.5 (Two-Convexity)**.** The coefficient of H on the left-hand side of the roundness estimate (2.2) determines the values of p which are susceptible to our approach. The factor $(n-1)^{-1}$, and therefore the critical exponent $p = n-1$ (with $\varepsilon = 0$), arise from the property that on the round cylinder $|A|^2 - (n-1)^{-1}H^2 \equiv 0$. It is in this way that the two-convex geometry declares itself.

In the setting of two-convex surfaces, the mean curvature controls the full norm of the second fundamental form: $(1/n)H^2 \le |A|^2 \le nH^2$. Proposition 2.4 is therefore equivalent to a bound on the L^p-norms of A for $p < n-1$.

In particular, if we denote by $T < \infty$ the extinction time of the smooth solution $\mathcal{M}_t$, we obtain

$$\int_0^T \int_{\mathcal{M}_t} |A|^{n+1-\varepsilon}\, d\mu\, dt \le C_1\varepsilon^{-1} \exp\Big(C_2 R^{-2} T\Big) \int_{\mathcal{M}_0} H^{n-1-\varepsilon}\, d\mu \tag{2.3}$$
$$\le C_3 R^{1+\varepsilon}, \tag{2.4}$$

where the constants $C_1, C_2, C_3 > 0$ depend only on n, ε, α. Here we have used Definition 2.1 and the well-known bound $T \le (n/2)\alpha_1^{-2}R^2$ from Lemma 3.8 below.

Remark 2.6 (Size of the Singular Set)**.** The estimate (2.4) is relevant to the regularity theory developed by Ecker in [**E2**]. As above we use T to denote the extinction time of the smooth evolution. Let $\operatorname{sing}(\mathcal{M}_T) \subset \mathbb{R}^{n+1}$ be the set of points $x \in \mathbb{R}^{n+1}$ such that the space-time point (x, T) is singular. Theorem 1.1 in [**E2**] asserts that the integrability condition

$$\int_0^T \int_{\mathcal{M}_t} |A|^q\, d\mu\, dt < \infty, \quad q \ge 2,$$

implies $\mathcal{H}^{n+2-q}(\operatorname{sing}(\mathcal{M}_T)) = 0$ for all $2 \le q \le n+2$ and $\operatorname{sing}(\mathcal{M}_T) = \emptyset$ for $q \ge n+2$. Hence, if $\mathcal{M}_0 \in \mathcal{C}(R,\alpha)$ for some R, α, and $\mathcal{M}_t$, $0 \le t < T$, is the maximal smooth mean curvature evolution of $\mathcal{M}_0$, we can apply (2.4) and conclude that $\mathcal{H}^{1+\varepsilon}\big(\operatorname{sing}(\mathcal{M}_T)\big) = 0$ for all $\varepsilon > 0$. That is to say, the Hausdorff dimension $\dim_{\mathcal{H}}(\operatorname{sing}(\mathcal{M}_T))$ of the singular set satisfies $\dim_{\mathcal{H}}(\operatorname{sing}(\mathcal{M}_T)) \le 1$; compare Corollary 1.7 in [**E2**].

In the next section we begin our analysis of the surgery construction developed by Huisken and Sinestrari.

3. MCF with surgeries and the number of surgeries

This section is devoted to the task of finding a quantitative bound on the number of surgeries required by the algorithm constructed in [**HS3**]. The key step is to adapt the smooth calculation from the previous section to this discontinuous setting. The resultant estimate, Corollary 3.9, is the essential ingredient in the proof of the convergence result in Section 4.

We begin with the necessary definitions from the theory developed by Huisken and Sinestrari in [**HS3**].

Mean curvature flow with surgeries. Section 2 in [**HS3**] defines the solution of mean curvature flow with surgeries starting from a smooth hypersurface immersion $F_0 : \mathcal{M}_1 \to \mathbb{R}^{n+1}$ in $\mathcal{C}(R,\alpha)$ for some R, α. It consists of a sequence of smooth mean curvature flows $F_t^i : \mathcal{M}_i \to \mathbb{R}^{n+1}$, $t \in [T_{i-1}, T_i]$, such that the initial hypersurface for the family F^1 is given by $F_0 : \mathcal{M}_1 \to \mathbb{R}^{n+1}$, and such that the initial hypersurface for the flow $F_t^i : \mathcal{M}_i \to \mathbb{R}^{n+1}$ on $[T_{i-1}, T_i]$, $2 \leq i \leq N$, is obtained from $F_{T_{i-1}}^{i-1}$ by:

i) performing *standard surgery* on finitely many disjoint *necks*, replacing each of them with two spherical caps (see below for definitions); and
ii) removing finitely many disconnected components.

We again write $\mathcal{M}_t$ for the solution of mean curvature flow with surgeries. The surgery time T_N is the extinction time of $\mathcal{M}_t$ if all connected components of $\mathcal{M}_{T_N}$ can be identified as copies of $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$, or alternatively if this can be achieved after performing finitely many surgeries on $\mathcal{M}_{T_N}$.

Necks. In order to control the L^p-norms of the mean curvature under mean curvature flow with surgeries, we therefore require estimates on the curvature of necks before and after surgery. The following definition is independent of mean curvature flow.

Definition 3.1 (Hypersurface Neck, [**HS3**])**.** Let $F : \mathcal{M}^n \to \mathbb{R}^{n+1}$ be a smooth hypersurface with induced metric g and Weingarten map W, and let $\mathcal{N} : \mathbb{S}^{n-1} \times [a, b] \to (\mathcal{M}^n, g) \subset \mathbb{R}^{n+1}$ be a local diffeomorphism. Then $\mathcal{N}$ is an (ϵ, k)-hypersurface neck if

$$|r^{-2}(z)g - \bar{g}|_{\bar{g}} \leq \epsilon, \quad |\bar{D}^j(r^{-2}(z)g)|_{\bar{g}} \leq \epsilon, \quad \text{and} \quad |(d/dz)^j \log r(z)| \leq \epsilon$$

uniformly for $1 \leq j \leq k$, and if in addition

$$|W(q) - r^{-1}(z)\bar{W}| \leq \epsilon r^{-1}(z) \quad \text{and} \quad |\nabla^l W(q)| \leq \epsilon r^{-l-1}(z)$$

for $1 \leq l \leq k$ and for all $q \in \mathbb{S}^{n-1} \times \{z\}$ and all $z \in [a, b]$. Here $\bar{g}$ is the standard metric on the cylinder and $r : [a, b] \to \mathbb{R}$ is the average radius of the cross-section $\mathcal{N}(\mathbb{S}^{n-1} \times \{z\})$ with respect to the pullback of g on $\mathcal{M}^n$.

Remark 3.2 (Maximal Normal Neck)**.** In order to deal with overlapping necks, it is necessary to introduce the concept of a *maximal normal* hypersurface neck. Definitions 3.8 and 3.11 in [**HS3**]—see also [**Ha**]—use constant mean curvature slices and harmonic mappings to guarantee uniqueness of the neck up to isometries.

Definition 3.3 (Length Parameter, [**HS3**])**.** We define the length of a hypersurface neck $\mathcal{N} : \mathbb{S}^{n-1} \times [a, b] \to \mathbb{R}^{n+1}$ to be $b - a$.

According to this definition, length is a scale-free quantity. The length of the neck plays a crucial role in our analysis and will depend only on the dimension n. Recall that the distance (with respect to the metric) between the two ends of the standard embedded cylinder of length

L and radius r_0 is $r_0 L$. The control scale H_0 depends on our choice of L.

Surgery. Let $\mathcal{N} : \mathbb{S}^{n-1} \times [a,b] \to \mathcal{M}$ be a normal neck of length $L \geq 8\Lambda$ (Λ will be chosen later) and let $z_0 \in [a,b]$ such that $[z_0 - 4\Lambda, z_0 + 4\Lambda] \subset [a,b]$. Section 3 of [**HS3**] defines the following *standard surgery* with parameters $0 < \tau < 1$ and $B > 10\Lambda$ at the cross-section $\mathcal{N}_{z_0} = \mathcal{N}(\mathbb{S}^{n-1} \times \{z_0\})$. To the left of z_0, it leaves the collar $\mathbb{S}^{n-1} \times [a, z_0 - 3\Lambda]$ untouched, and replaces $\mathcal{N}(\mathbb{S}^{n-1} \times [z_0 - 3\Lambda, z_0])$ with a ball attached smoothly to $\mathcal{N}_{z_0 - 3\Lambda}$. The other direction is similarly modified.

For convenience we set $z_0 - 4\Lambda = 0$ and consider $\mathcal{N} : \mathbb{S}^{n-1} \times [0, 4\Lambda] \to \mathcal{M}$. We denote by $\bar{C}_{z_0} : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ the straight cylinder best approximating the surface at the cross-section $\mathcal{N}_{z_0}$.

i) (*Pinching*) Let $u(z) \equiv r_0 \exp(-B/(z - \Lambda))$ on $[\Lambda, 3\Lambda]$ for $B > 10\Lambda$. Given $0 < \tau < 1$, define $\tilde{\mathcal{N}}(\omega, z) \equiv \mathcal{N}(\omega, z) - \tau u(z)\nu(\omega, z)$.
ii) (*Symmetrising*) Denote by $\varphi : [0, 4\Lambda] \to \mathbb{R}^+$ a fixed smooth transition function with $\varphi = 1$ on $[0, 2\Lambda]$, $\varphi = 0$ on $[3\Lambda, 4\Lambda]$, and $\varphi' \leq 0$. In addition, let $\tilde{C}_{z_0} : \mathbb{S}^{n-1} \times [0, 4\Lambda] \to \mathbb{R}^{n+1}$ denote the bent cylinder defined by $\tilde{C}_{z_0}(\omega, z) \equiv \bar{C}_{z_0}(\omega, z) - \tau u(z)\nu_{\bar{C}}(\omega, z)$. Now interpolate to obtain an axially symmetric surface
$$\hat{\mathcal{N}}(\omega, z) \equiv \varphi(z)\tilde{\mathcal{N}}(\omega, z) + (1 - \varphi(z))\tilde{C}_{z_0}(\omega, z).$$
iii) (*Capping Off*) Finally, on $[3\Lambda, 4\Lambda]$ change u to a function $\hat{u}$ such that $\tau\hat{u}(z) \to r(z_0) = r_0$ as $z \to z_1$ for some $z_1 \in (3\Lambda, 4\Lambda]$. We can assume that $\tilde{C}_{z_0}([3\Lambda, 4\Lambda])$ is a smoothly attached, axially symmetric and uniformly convex cap. There is a fixed upper bound on the curvature and each of its derivatives, independent of Λ, τ, B.

Mean curvature flow with surgeries is governed by the choice of surgery procedure, the neck parameters ϵ, k, L, and the algorithm parameters H_0, H_1, H_2, H_3.

Integral estimates for necks. We are now ready to control the L^p-norms of the mean curvature across surgery. The next lemma is independent of mean curvature flow and will be combined with the smooth calculation from the previous section in Theorem 3.6.

In Lemma 3.4 we consider an (ϵ, k)-hypersurface neck $\mathcal{N} : \mathbb{S}^{n-1} \times [0, L] \to \mathcal{M}^-$ contained in $\mathcal{M}^- \in \mathcal{C}(R, \alpha)$ for some R, α, and a single standard surgery at $z_0 = L/2$. As before, r_0 will be the mean radius of the cross-section $\mathcal{N}_{z_0}$. We refer to r_0 as the *scale* of the neck and we write $\mathcal{M}^+$ ($\mathcal{M}^-$) for the surface before (after) surgery. For ease of notation we write $\mathcal{U}^- \subset \mathcal{M}^-$ for the subset of $\mathcal{M}^-$ altered by the surgery and $\mathcal{U}^+$ for the subset of $\mathcal{M}^+$ replacing $\mathcal{U}^-$.

Lemma 3.4 (L^p Estimate across Surgery)**.** *Let $p \geq 0$. We can choose $L = L(n, p)$ sufficiently large such that the following property holds. Let*

$\mathcal{M}^-$, $\mathcal{M}^+$, $\mathcal{U}^-$, $\mathcal{U}^+$, and r_0 be as described immediately above. There exists a constant $C(n,p) > 0$ depending only on n, p such that

$$\int_{\mathcal{U}^-} H^p\, d\mu - \int_{\mathcal{U}^+} H^p\, d\mu \geq C(n,p) L (r_0)^{n-p}.$$

Proof. Fix $p \geq 0$ and consider the neck $\mathcal{N} : \mathbb{S}^{n-1} \times [0, L] \to \mathcal{M}^-$ with scale r_0. From Definition 3.1 we have $|\lambda_1(q)| \leq \epsilon r(z)^{-1}$ and $|\lambda_j(q) - r(z)^{-1}| \leq \epsilon r(z)^{-1}$, $j = 2, \ldots, n$, for all $q \in \mathbb{S}^{n-1} \times \{z\}$ and for each $z \in [0, L]$. In addition, $|(d/dz)^l \log r(z)| \leq \epsilon$, $1 \leq l \leq k$, on $[0, L]$. Hence there exists $\epsilon_0 = \epsilon_0(n, L) > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ we have

$$\int_{\mathcal{U}^-} H^p\, d\mu \geq \left(\frac{9}{10}\frac{n-1}{r_0}\right)^p \left(\frac{9}{10}\omega_{n-1}\tilde{L}(r_0)^n\right), \tag{3.1}$$

where $\tilde{L} = L - 2\Lambda$. The replacement region $\mathcal{U}^+$ consists of two copies of $\mathcal{U}_1^+ \cup \mathcal{U}_2^+$ where $\mathcal{U}_2^+$ denotes the convex cap attached to $\mathcal{N}_{3\Lambda}$ as described in step iii) of the surgery procedure and $\mathcal{U}_1^+$ corresponds to the modified cylinder $\hat{\mathcal{N}}$ in between. Using Remark 3.20 in [**HS3**] and estimating as we did in (3.1), we can choose $\tau = \tau(n) > 0$ sufficiently small such that

$$\int_{\mathcal{U}_1^+} H^p\, d\mu \leq \omega_{n-1}(n-1)^p (2\Lambda)(r_0)^{n-p}\left(\frac{11}{10}\right)^{p+1}. \tag{3.2}$$

It is clear that the final step of the surgery construction can also be adapted such that

$$\int_{\mathcal{U}_2^+} H^p\, d\mu \leq \omega_n n^p (r_0)^{n-p}\left(\frac{11}{10}\right)^{p+1}. \tag{3.3}$$

We can therefore choose $L = C(n,p) + 8\Lambda$ sufficiently large in terms of n and p such that the result follows from (3.1), (3.2), and (3.3). q.e.d.

We henceforth assume that a fixed choice has been made for the parameters τ, B as well as the transition function and convex cap in steps ii) and iii) of the surgery procedure such that Lemma 3.4 holds.

L^p estimate for mean curvature flow with surgeries. We now combine the smooth calculation from the previous section with Lemma 3.4 to prove that Proposition 2.4 applies—in a weak sense—to the flow with surgeries. In what follows: $\mathcal{M}_t$ denotes the solution of mean curvature flow with surgeries; T_j, $j = 1, 2, \ldots, N$, are the surgery times; and $\mathcal{M}_{T_j^-}$ ($\mathcal{M}_{T_j^+}$) denotes the surface at time T_j before (after) surgery. The solution $\mathcal{M}_t$ is determined by a set of parameters H_1, H_2, H_3 which control the choice of surgery times and locations. We recall the main result from [**HS3**].

Theorem 3.5 (Existence & Finite Extinction, [**HS3**])**.** *Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ for some R, α. There exist constants $\omega_1, \omega_2, \omega_3 > 1$ depending* *only on α such that the following holds. If we set $H_2 = \omega_2 H_1$ and $H_3 = \omega_3 H_2$, then for any $H_1 \geq \omega_1 R^{-1}$ there exists an associated mean curvature flow with surgeries $\mathcal{M}_t$, $0 \leq t \leq T_N < \infty$, starting from $\mathcal{M}_0$ and such that:*

- i) *each surgery is performed at the earliest time T_j such that the curvature reaches $\max H(\cdot, T_j^-) = H_3$;*
- ii) *after the two-step surgery procedure, $\max H(\cdot, T_j^+) \leq H_2$;*
- iii) *all surgeries start from a cross-section of a normal hypersurface neck with mean radius $r_0 = (n-1)/H_1$;*
- iv) *$N < \infty$.*

We emphasize that Theorem 3.5 holds for any choice of length parameter $L \geq 20 + 8\Lambda$. We refine the choice of L such that Lemma 3.4 holds for all $p < n-1$. This leads to the following estimate for the flow with surgeries.

Theorem 3.6 (L^p Estimate for Flow with Surgeries)**.** *Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ for some R, α. We can choose $L = L(n)$ sufficiently large such that the following property holds. Consider the solution $\mathcal{M}_t$, $0 \leq t \leq T_N < \infty$, of mean curvature flow with surgeries starting from $\mathcal{M}_0$. For any $0 < \varepsilon \leq 1$ there exists a constant $C_\varepsilon = C_\varepsilon(n, \alpha) > 0$ such that*

$$\begin{aligned}\exp\left(pC_\varepsilon R^{-2} t_0\right) \int_{\mathcal{M}_0} H^p \, d\mu \geq p(p-1) \int_0^{t_0} \int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} \, d\mu \, dt \\ + \int_{\mathcal{M}_{t_0}} H^p \, d\mu + \frac{\varepsilon}{2(n-1)} \int_0^{t_0} \int_{\mathcal{M}_t} H^{p+2} \, d\mu \, dt,\end{aligned}$$

for any $0 < t_0 \leq T_N$ and where $p = n - 1 - \varepsilon$.

Proof. First recall that Proposition 2.4 relies only on Theorem 2.3. Since the estimate (2.2) survives surgery without any modifications to the constants, we conclude that the smooth calculation applies to each smooth time interval $[0, T_1], [T_1, T_2], \ldots, [T_m, t_0]$. Note that the one-sided time derivatives exist at each surgery time T_j. We can therefore integrate on each time interval $[T_j, T_{j+1}]$ and sum the $m+1$ contributions. Furthermore $\exp\left(-pC_\varepsilon R^{-2} t\right)$ is continuous in t, and from Lemma 3.4 we have

$$\int_{\mathcal{M}_{T_{j+1}^-}} H^p \, d\mu > \int_{\mathcal{M}_{T_{j+1}^+}} H^p \, d\mu$$

for each $p \geq 0$ and for all $j \geq 0$. Here we simply disregard any contribution made by the components discarded at the surgery time. This completes the proof. q.e.d.

Remark 3.7 (Brakke Flow)**.** Recall that, in the language of smooth surfaces, Brakke's definition of a weak mean curvature evolution [**B**] requires

$$\int_{\mathcal{M}_0} \phi \, d\mu \geq \int_{\mathcal{M}_{t_0}} \phi \, d\mu + \int_0^{t_0} \int_{\mathcal{M}_t} \phi H^2 + H \langle \nabla \phi, \nu \rangle \, d\mu \, dt$$

for all non-negative $\phi = \phi(x) \in C_c^2(\mathbb{R}^{n+1})$. Lemma 3.4 reveals that the flow with surgeries does not satisfy Brakke's definition, but the error term introduced by each standard surgery satisfies the estimate

$$\int_{\mathcal{U}^+} \phi \, d\mu - \int_{\mathcal{U}^-} \phi \, d\mu \leq C(n) L(r_0)^n \sup_{\mathcal{U}^+} \phi, \tag{3.4}$$

where $\mathcal{U}^-$ and $\mathcal{U}^+$ are defined immediately before Lemma 3.4 above. The constant $C(n) > 0$ depends only on n.

Number of surgeries. Theorem 3.5 above asserts that the flow with surgeries terminates after finitely many surgery times. In fact, the proof of this result in [**HS3**] produces quantitative estimates on N in terms of the parameters H_1, H_2, H_3 (note that this does appear explicitly in [**HS3**] as it is not required). Our goal is to strengthen this result using the higher order L^p-estimates in Theorem 3.6.

The following lemma is well known in the context of smooth mean curvature flow; see for example Proposition 2.7 in [**HS3**].

Lemma 3.8 (Finite Extinction Time). *Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ for some R, α and consider the solution $\mathcal{M}_t$, $0 \leq t \leq T_N < \infty$, of mean curvature flow with surgeries starting from $\mathcal{M}_0$. We have*

$$T_N \leq \min \left\{ \frac{n}{2} \alpha_1^{-2} R^2, \frac{1}{2n} \big(\operatorname{diam}(\mathcal{M}_0) \big)^2 \right\},$$

where T_N denotes the extinction time of $\mathcal{M}_t$ and $\operatorname{diam}(\mathcal{M}_0)$ *denotes the diameter of $\mathcal{M}_0$.*

Proof. The first estimate can be found in the proof of Corollary 4.7 in [**HS3**]. In the smooth setting, it follows from the trivial inequality $\partial_t H \geq \Delta H + (1/n) H^3$, an ODE comparison argument, and Definition 2.1. Corollary 3.21 in [**HS3**] ensures that surgery does not reduce the mean curvature, so the same estimate applies to the flow with surgeries.

The second bound follows from Jung's theorem and the avoidance principle. This argument applies to the flow with surgeries since the surgery procedure does not interfere with the spherical barrier and the avoidance principle can be applied on each smooth time interval. q.e.d.

Combining Lemma 2.2 this time with the two-convex inequality $|A|^2 \leq nH^2$ gives $\partial_t H \leq \Delta H + nH^3$. Comparison with the associated ODE in this case yields a uniform lower bound $\delta T \geq C(n, \alpha)(H_2)^{-2}$ on the time interval δT separating any two consecutive surgery times; see Remark 7.17 in [**HS3**]. Recall that the mean curvature has to increase from H_2

to $H_3 = \omega_3 H_2 > H_2$ during this time. Hence the number of surgery times satisfies the bound

$$N \leq C(n, \alpha) R^2 (H_2)^2. \tag{3.5}$$

This guarantees that the flow with surgeries must terminate after a finite number of surgery times for any finite H_1, H_2, H_3. However, the estimate (3.5) is not strong enough for our application in the next section. We therefore use an alternative argument—inspired by [**HS3**, **Ha**]—in combination with Theorem 3.6 to establish a new bound on N.

Corollary 3.9 (Number of Surgery Times)**.** *We can choose $L = L(n)$ sufficiently large such that the following property holds. Let $\mathcal{M}_0 \in \mathcal{C}(R, \alpha)$ for some R, α and consider the solution $\mathcal{M}_t$, $0 \leq t \leq T_N < \infty$, of mean curvature flow with surgeries starting from $\mathcal{M}_0$ and with parameters H_1, H_2, H_3. For any $0 < \varepsilon \leq 1$ there exists a constant $C_\varepsilon = C_\varepsilon(n, \alpha) > 0$ depending only on n, ε, α such that*

$$N \leq C_\varepsilon L^{-1} R^{1+\varepsilon} (H_1)^{1+\varepsilon},$$

where N denotes the number of surgery times associated with $\mathcal{M}_t$.

Proof. Let $0 < \varepsilon \leq 1$. Theorem 3.6, Lemma 3.8, and Definition 2.1 combine to produce a uniform constant $C(n, \varepsilon, \alpha) > 0$ depending only on n, ε, α such that

$$\int_{\mathcal{M}_t} H^{n-1-\varepsilon} \, d\mu \leq C(n, \varepsilon, \alpha) R^{1+\varepsilon} \tag{3.6}$$

on $[0, T_N]$. In addition, Lemma 3.4 guarantees that each surgery consumes

$$\int_{\mathcal{U}^-} H^{n-1-\varepsilon} \, d\mu - \int_{\mathcal{U}^+} H^{n-1-\varepsilon} \, d\mu \geq C(n, \varepsilon) L (H_1)^{-1-\varepsilon}.$$

Note that we can again ignore the contribution made by the components discarded at the surgery time. Hence there exists a constant $C > 0$ depending only on n, ε, α such that the total number of surgeries $S(T_N)$ performed on the time interval $[0, T_N]$ cannot exceed $C L^{-1} R^{1+\varepsilon} (H_1)^{1+\varepsilon}$. In addition, the number of connected components at any given time $t_0 \in [0, T_N]$ is bounded above by $C_0 + S(t_0)$, where C_0 is the number of components at time $t = 0$ and $S(t_0)$ represents the total number of surgeries performed on the time interval $[0, t_0)$. Recall that at each surgery time the curvature has to drop from H_3 to H_2 via the two-step surgery procedure described in Section 2.2, and C_0 is bounded courtesy of Definition 2.1. The number of surgery times N therefore satisfies the desired estimate. q.e.d.

We next use Corollary 3.9 to reconcile the solution produced by the non-canonical surgery construction with the unique weak solution of mean curvature flow introduced in [**B**, **CGG**, **ES1**].

4. Approximating weak solutions using MCF with surgeries

The flow with surgeries and the so-called weak solution are heretofore independent interpretations of a global mean curvature evolution of $\mathcal{M}_0$. In this section we investigate the relationship between the two: Theorem 4.3 below asserts that the solution of the flow with surgeries converges to the weak solution in an appropriate limit of the surgery parameters. We obtain quantitative estimates on the rate of convergence using Corollary 3.9 and Brakke's clearing out lemma.

In order to set up a precise statement of the result, we rapidly recall some basic definitions and results from the theory of weak solutions. There are several comprehensive treatments of weak solutions available in the literature—for further details we refer to [**B**, **CGG**, **ES1**, **I**, **W**]. Here we adopt the viewpoint taken in [**CGG**, **ES1**, **MS**]; see also [**HI**, **S**].

Weak solutions. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set with mean-convex boundary $\partial\Omega$ and consider the classical solution of (MCF) starting from $\partial\Omega$. The smoothly evolving surfaces $\mathcal{M}_t$ can therefore be represented as the level-sets of a continuous scalar "time" function $u : \bar{\Omega} \to \mathbb{R}$ satisfying the degenerate elliptic boundary value problem

$$(\star) \qquad \begin{cases} \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = -\dfrac{1}{|Du|}, \\ \qquad u\Big|_{\partial\Omega} \;=\; 0. \end{cases}$$

A weak concept of solutions to $(\star)$ can be defined using a variational approach; compare [**HI**, **MS**, **S**]. For more general formulations we refer to [**CGG**, **ES1**] and to [**B**, **I**].

Definition 4.1 (Weak Solution)**.** Given $u \in C^{0,1}(\bar{\Omega})$ such that $|Du|^{-1} \in L^1(\Omega)$, $u > 0$ on Ω, and $\{u = 0\} = \partial\Omega$, we say that u is a weak solution of $(\star)$ on Ω if

$$\int_\Omega \left(|Du| - \frac{u}{|Du|}\right) dx \leq \int_\Omega \left(|Dv| - \frac{v}{|Du|}\right) dx$$

for any Lipschitz continuous function v on Ω such that $\{u \neq v\} \subset\subset \Omega$.

There exists a unique weak solution of $(\star)$ on Ω; see [**CGG**, **ES1**]. We hereafter write $\bar{u}$ for the weak solution and we define

$$\Gamma_t \equiv \begin{cases} \partial\left\{x \in \Omega \;\middle|\; \bar{u}(x) > t\right\} & \text{for all } t \leq \bar{T} \\ \emptyset & \text{for all } t > \bar{T} \end{cases}$$

where $\bar{T} \equiv \sup_\Omega(\bar{u})$. We have $\bar{T} \leq \min\{C(n,\alpha)R^2, C(n)(\operatorname{diam}(\mathcal{M}_0))^2\}$; see Theorem 4.1 in [**ES2**]. We also write $\Omega_t \equiv \{\bar{u} > t\}$ for the regions enclosed by the level-sets Γ_t.

With the preceding definitions in hand, we now turn to the geometric properties of the weak solution. Recall from [**ES1**] that Γ_t agrees with the smooth evolution $\mathcal{M}_t$ starting from $\partial\Omega$ if and so long as the latter exists. In addition, it satisfies the *avoidance principle*: if $\mathcal{M}_t$, $t_0 \le t \le t_1$, is any smooth, compact mean curvature flow with positive mean curvature, and if $\mathcal{M}_{t_0} \cap \Gamma_{t_0} = \emptyset$, then $\mathcal{M}_t \cap \Gamma_t = \emptyset$ for all $t_0 < t \le t_1$. Equivalently, $\frac{d}{dt}\mathrm{dist}(\mathcal{M}_t, \Gamma_t) \ge 0$.

We adopt the notation $\partial^* A$ for the reduced boundary of a set $A \subset \mathbb{R}^{n+1}$.

Definition 4.2 (Minimising Hull)**.** Let $U \subset \mathbb{R}^{n+1}$ be an open set. We say that the set $E \subset \mathbb{R}^{n+1}$ is a minimising hull in U if $\mathcal{H}^n(\partial^* E \cap K) \le \mathcal{H}^n(\partial^* F \cap K)$ for any $F \supset E$ such that $F \setminus E \subset\subset U$ and any compact set $K \supset (F \setminus E)$.

It is well known that the sets Ω_t are minimising hulls in Ω; see [**W**] and [**HI**, **MS**]. These fundamental geometric features of the weak solution play an important role in our construction.

It was established in [**ES1**, **CGG**] that $\bar{u}$ can be approximated uniformly in C^0 by smooth, non-compact solutions of mean curvature flow satisfying an appropriate "regularised" boundary value problem; see also [**HI**, **MS**]. (Note that in the next section we will in fact make use of the theory developed in [**MS**].) We now put forth a new approximation scheme in the two-convex setting using solutions of mean curvature flow with surgeries.

Mean curvature flow with surgeries. We restrict our attention to domains Ω in $\mathbb{R}^{n+1}$, $n \ge 3$, such that $\partial\Omega \in \mathcal{C}(R, \alpha)$ for some R, α. Let $\mathcal{M}_t$, $t \in [0, T_N]$, be the corresponding flow with surgeries. We now construct the level-set function u associated with the evolution $\mathcal{M}_t$. If T_j is a surgery time, we write $E_{T_j^-}$ for the closed domain in $\mathbb{R}^{n+1}$ bounded by $\mathcal{M}_{T_j^-}$ and $F_{T_j^+}$ for the open set in $\mathbb{R}^{n+1}$ enclosed by $\mathcal{M}_{T_j^+}$. Note that the surgery procedure gives rise to points $x \in \Omega$ such that $x \notin \mathcal{M}_t$ for any t.

Define
$$u(x) \equiv \begin{cases} t & \text{for all } x \in \mathcal{M}_t \\ T_j & \text{for all } x \in E_{T_j^-} \setminus F_{T_j^+},\ j = 1, \dots, N, \end{cases}$$
where $E_{T_j^-} \setminus F_{T_j^+}$ is the region overlooked as a result of the alterations made at the surgery time T_j. These regions are by definition plateaus in $\mathrm{graph}(u) \subset \mathbb{R}^{n+1} \times \mathbb{R}$ and the corresponding level-sets $\{u = T_j\}$ may not be smooth hypersurfaces. Observe that $\mathcal{M}_{T_j^-} = \partial\left(\mathrm{int}\{u \ge T_j\}\right)$ and $\mathcal{M}_{T_j^+} = \partial\{u > T_j\}$. For convenience we define the sets $\Sigma_t \equiv \{u > t\}$ and $\tilde{\Sigma}_t \equiv \mathrm{int}\{u \ge t\}$ so that $\mathcal{M}_{T_j^+} = \partial\Sigma_{T_j}$ and $\mathcal{M}_{T_j^-} = \partial\tilde{\Sigma}_{T_j}$.

For all $t \notin \{T_1, \dots, T_N\}$ we have $\mathcal{M}_t = \{u = t\}$. The function u is smooth in these regions and in addition $u \in C^{0,1}(\bar{\Omega})$.

Convergence. Theorem 3.5 provides three constants $\omega_1, \omega_2, \omega_3$ which depend only on α and which produce a flow with surgeries starting from $\partial\Omega$ for any choice $H_1 \geq \omega_1 R^{-1}$ with $H_2 = \omega_2 H_1$ and $H_3 = \omega_3 H_2$. Now consider an *increasing sequence of parameters* $\{H_1^i, H_2^i, H_3^i\}_{i\geq 1}$ corresponding to a sequence $\{\mathcal{M}_t^i\}_{i\geq 1}$ of mean curvature flows with surgeries. This generates an associated sequence of level-set functions $\{u_i\}_{i\geq 1}$. The ratios ω_2, ω_3 are fixed along the sequence; that is, $H_2^i = \omega_2 H_1^i$ and $H_3^i = \omega_3 H_2^i$ for each i. We are now able to formulate a precise statement of our convergence result.

Theorem 4.3 (Convergence to Weak Solution)**.** *Suppose $n \geq 3$ and let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in \mathcal{C}(R,\alpha)$ for some R, α. We can choose $L = L(n)$ sufficiently large such that the following holds. Let $\bar{u} \in C^{0,1}(\bar{\Omega})$ be the weak solution generated by Ω, and consider the solution $u_i \in C^{0,1}(\bar{\Omega})$ of the flow with surgeries starting from $\partial\Omega$ and with parameters H_1^i, H_2^i, H_3^i. For any $0 < \varepsilon \leq 1$ there exists a constant $C_\varepsilon = C_\varepsilon(n,\alpha) > 0$ depending only on n, ε, α such that*

$$\max_{x\in\bar{\Omega}} |u_i(x) - \bar{u}(x)| \leq C_\varepsilon L R^{1+\varepsilon} (H_1^i)^{-1+\varepsilon}.$$

Theorem 4.3 follows from Corollary 3.9, Lemma 4.4, and Proposition 4.6. As a first step we prove that u_i sits below $\bar{u}$ for each i. We then show that $\bar{u}$ can also be used as a lower barrier to u_i after an appropriate "time"-translation which depends explicitly on H_1^i; see step 2 below.

Step 1. We record a global barrier result controlling the height of thelevel-set functions u_i relative to $\bar{u}$.

Lemma 4.4 (Upper Barrier)**.** *Suppose $n \geq 3$ and let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in \mathcal{C}(R,\alpha)$ for some R, α. Let $\bar{u}$ be the weak solution generated by Ω, and denote by u_i the solution of the flow with surgeries starting from $\partial\Omega$ and with parameters H_1^i, H_2^i, H_3^i. We have $u_i(x) \leq \bar{u}(x)$ for all $x \in \bar{\Omega}$.*

Before proceeding with the proof, we recall the concept of the *solid tube* enclosed by a hypersurface neck; see Proposition 3.25 in [**HS3**].

Proposition 4.5 (Solid Tube, [**HS3**])**.** *There exists $\epsilon_0 = \epsilon_0(n) > 0$ depending only on n such that the following holds. Given a normal (ϵ, k)-hypersurface neck $\mathcal{N} : \mathbb{S}^{n-1} \times [0, L] \to \mathcal{M}^n \subset \mathbb{R}^{n+1}$ with parameters $L \geq 20 + 8\Lambda \geq 100$, $0 < \epsilon \leq \epsilon_0$, and $k \geq 2$, there exists a unique local diffeomorphism $G : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$ such that: i) G (when restricted to the cylinder) agrees with $\mathcal{N}$; ii) each cross-section $G(\bar{B}_1^n \times \{z_0\}) \subset \mathbb{R}^{n+1}$ is an embedded area minimising hypersurface; iii) G is a harmonic* *diffeomorphism when restricted to each slice $\bar{B}_1^n \times \{z_0\}$; and iv) G is ϵ-close in C^{k+1}-norm to the standard isometric embedding of a solid cylinder in $\mathbb{R}^{n+1}$.*

Proof of Lemma 4.4. Since $\mathcal{M}_0^i = \Gamma_0 = \partial\Omega$, we have $\mathcal{M}_\delta^i \subset\subset \Omega$ for all small $\delta > 0$. Hence $\frac{d}{dt}\mathrm{dist}(\mathcal{M}_{t+\delta}^i, \Gamma_t) \geq 0$ as long as $\mathcal{M}_{t+\delta}^i$ remains smooth. However, it is straightforward to see that this property is preserved by the surgery construction. Each standard surgery is performed on a neck $\mathcal{N}_0$ of length L which encloses a solid tube $G_0 : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$. As in Lemma 3.4, we denote by $\mathcal{U}^+$ the two caps introduced by each standard surgery. By construction, $\mathcal{U}^+ \subset G_0(\bar{B}_1^n \times [0, L])$ and therefore $\mathcal{M}_{T_j^+}^i \subset E_{T_j^-}$. This property is clearly respected by step two of the surgery procedure in which finitely many components are discarded. Therefore $\mathrm{dist}(\mathcal{M}_{t+\delta}^i, \Gamma_t)$ is non-decreasing across each surgery time. Since there are only finitely many surgery times, we therefore have $\mathcal{M}_{t+\delta}^i \subset\subset \Omega_t$ for all $t \geq 0$. The desired result then follows from the fact that the level-set functions $u_i, \bar{u}$ are continuous. q.e.d.

Step 2: "Time-Shifting" the Weak Solution. Our goal is now to translate $\bar{u}$ vertically (in "time") until it sits below u_i; this will complete the proof of Theorem 4.3. We use Brakke's clearing out lemma and Corollary 3.9 to control the process in a quantitative way. The heuristic idea can be described as follows.

The solutions $\mathcal{M}_t^i$ and Γ_t agree—and coincide with the classical evolution—up to and including the first surgery time T_1 (more precisely T_1^-). We now freeze $\mathcal{M}_{T_1^-}^i$ and run the weak solution a short time longer until $\Gamma_{T_1+t_w} \subset \Sigma_{T_1}^i$. That is, we give the weak solution enough time to vacate the regions modified by surgery. This must happen for some finite $t_w < \infty$ in light of the curvature assumption on $\partial\Omega$; we claim that t_w can be controlled explicitly in terms of the surgery parameters with the expected parabolic scaling.

We then perform surgery on $\mathcal{M}_{T_1^-}^i$, after which $\mathcal{M}_{T_1^+}^i \cap \Gamma_{T_1+t_w} = \emptyset$. This allows us to restart the two evolutions and apply the avoidance principle until the next surgery time. We iterate the argument at each subsequent surgery time.

The fundamental quantity to control is therefore the combined scaling of the estimates on t_w and N. The length parameter L plays an important role.

Proposition 4.6 (Lower Barrier). *Suppose $n \geq 3$ and let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in \mathcal{C}(R, \alpha)$ for some R, α. We can choose $L = L(n)$ sufficiently large such that the following holds. Let $\bar{u}$ be the weak solution generated by Ω, and denote by u_i the solution of the flow with surgeries starting from $\partial\Omega$ and with parameters H_1^i, H_2^i, H_3^i. There exists a constant $C(n, \alpha) > 0$ depending only on n, α such that*

for all $x \in \bar{\Omega}$ *we have*
$$\bar{u}(x) \le u_i(x) + N t_w,$$
where $t_w \le C(n,\alpha) L^2 (H_1^i)^{-2}$ *and* N *is the number of surgery times associated with* u_i.

We adapt the statement of the famous *clearing out lemma* from [**B**] to our setting; see Theorem 7.3 in [**ES2**].

Theorem 4.7 (Clearing Out Lemma, [**B**])**.** *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be an open, bounded set with mean-convex boundary* $\partial\Omega$*, and consider the weak solution* $\bar{u}$ *generated by* Ω*. There exist constants* $C(n), \theta(n) > 0$ *depending only on* n *such that the following holds. For any* $x_0 \in \mathbb{R}^{n+1}$ *and* $\rho > 0$ *the estimate*
$$\mathcal{H}^n\big(\Gamma_{t_0} \cap B_\rho(x_0)\big) \le \theta(n)\rho^n$$
implies $\Gamma_t \cap B_{\rho/2}(x_0) = \emptyset$*, where* $t - t_0 \le C(n)\rho^2$.

The parabolic scaling of the estimate on the waiting time $t - t_0$ will lead to a refined upper bound on the time t_w required by the weak solution to vacate the regions modified by surgery. Proposition 4.6 will follow from the next two results, which we state and prove separately, in combination with the avoidance principle.

Remark 4.8 (Proof of Prop. 4.6)**.** Points modified at a surgery time belong either to hypersurface necks or to components of known topology which are subsequently discarded. Discarded components may contain regions which are not cylindrical; these will be discussed in Lemma 4.10.

We first deal with regions directly affected by the surgery procedure itself. In what follows we say that the point $x \in \mathbb{R}^{n+1}$ is *modified by the surgery procedure* if x belongs to the part of a solid tube G which is changed by surgery. In the language of Proposition 4.5, this implies that $x \in G(\bar{B}_1^n \times [\Lambda, L - \Lambda])$.

Lemma 4.9 (Regions Modified by Surgery Procedure)**.** *Suppose* $n \ge 3$ *and let* $\Omega \subset \mathbb{R}^{n+1}$ *be an open, bounded set such that* $\partial\Omega \in \mathcal{C}(R,\alpha)$ *for some* R, α*. We can choose* $L = L(n)$ *sufficiently large such that the following holds. Let* $\bar{u}$ *be the weak solution generated by* Ω*, and denote by* u_i *the solution of the flow with surgeries starting from* $\partial\Omega$ *and with parameters* H_1^i, H_2^i, H_3^i*. Let* T_j*,* $j \in \{1, \ldots, N\}$*, be a surgery time for* u_i*, and assume that* $t_0 > T_j \ge 0$ *is such that* $\mathcal{H}^n(\Gamma_{t_0}) = \mathcal{H}^n(\partial^*\Omega_{t_0})$ *and* $\Gamma_{t_0} \subset \tilde{\Sigma}^i_{T_j}$*. There exist constants* $C_1, C_2 > 0$ *depending only on* n *such that* $\Gamma_{t_0+\bar{t}} \cap B_{\rho_0/2}(x) = \emptyset$ *for all* $x \in \tilde{\Sigma}^i_{T_j}$ *modified by the surgery procedure, where* $\rho_0 = C_1 L (H_1^i)^{-1}$ *and* $\bar{t} \le C_2 L^2 (H_1^i)^{-2}$.

Proof. Consider any $x \in G_0(\bar{B}_1^n \times [\Lambda, L - \Lambda])$, where $G_0 : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$ is the solid tube enclosed by a neck $\mathcal{N}_0$ with scale r_0^i and center

p_0. By Definition 3.1, there exists $\epsilon_1 = \epsilon_1(n, L) > 0$ such that for all $0 < \epsilon \leq \epsilon_1$ and each such x we have $\mathcal{H}^n\big(\mathcal{N}_0 \cap B_{(\Lambda r_0^i)}(x)\big) \leq 4\Lambda\omega_{n-1}(r_0^i)^n$. We refine our choice of $\Lambda(n) > 0$ (if necessary) so that

$$\mathcal{H}^n\big(\mathcal{N}_0 \cap B_{(\Lambda r_0^i)}(x)\big) \leq \theta(n)\Lambda^n(r_0^i)^n, \tag{4.1}$$

where $\theta(n) > 0$ is the constant from Theorem 4.7. We therefore define $\rho_0 \equiv (n-1)\Lambda(H_1^i)^{-1}$, and we claim that (4.1) implies $\mathcal{H}^n\big(\Gamma_{t_0} \cap B_{\rho_0}(x)\big) \leq \theta(n)(\rho_0)^n$. The result then follows from the clearing out lemma.

We first use the minimising hull property of the weak solution to show that $\mathcal{H}^n\big((\Gamma_{t_0} \cap G_0) \cap B_{\rho_0}(x)\big) \leq \theta(\rho_0)^n$. In fact, direct comparison of the set Ω_{t_0} with the perturbation $\Omega_{t_0} \cup G_0$ yields the estimate $\mathcal{H}^n\big((\Gamma_{t_0} \cap G_0) \cap B_{\rho_0}(x)\big) \leq \mathcal{H}^n\big(\mathcal{N}_0 \cap B_{\rho_0}(x)\big)$. This is the step which requires the assumption $\mathcal{H}^n(\Gamma_{t_0}) = \mathcal{H}^n(\partial^*\Omega_{t_0})$.

It is then necessary to confirm that no other part of the surface can intersect $B_{\rho_0}(x)$—that is, $B_{\rho_0}(x) \cap (\tilde{\Sigma}^i_{T_j} \setminus G_0) = \emptyset$. To this end we exploit the fact that the neck is close to a shrinking cylinder on an earlier time interval. We require additional machinery from [**HS3**].

Let $\mathcal{B}_{g(t)}(p, l) \equiv \{q \in \mathcal{M}^n \mid d_{g(t)}(p, q) \leq l\}$. Lemmas 7.4 and 7.9 in [**HS3**] assert that the backward parabolic neighborhood

$$\begin{aligned}\mathcal{P}(p_0, T_j, &r_0^i L, (r_0^i)^2\omega) \\ &\equiv \Big\{(q, s) \;\Big|\; q \in \mathcal{B}_{g(T_j)}(p_0, r_0^i L), s \in [T_j - (r_0^i)^2\omega, T_j]\Big\}\end{aligned}$$

can be made arbitrarily close to a portion of an exact cylinder evolving by mean curvature flow over the same time interval. It is clear that we can therefore choose Λ, $L = C(n) + 8\Lambda$ and $\omega = C(n)L^2$ large enough such that Lemma 3.4 and (4.1) hold, and in addition such that $B_{\rho_0}(x)$ is completely contained within the solid tube enclosed by $\mathcal{N}_0$ at the earlier time $T_j - \omega(r_0^i)^2$. Surgery cannot interfere with the neck before time T_j since the curvature is below the surgery scale H_1^i.

Since each point $x \in \mathbb{R}^{n+1}$ satisfies $x \in \mathcal{M}_t^i$ for at most one t, this ensures that the ball does not touch any part of the weak solution which lies outside the neck $\mathcal{N}_0$. Hence $\mathcal{H}^n\big(\Gamma_{t_0} \cap B_{\rho_0}(x)\big) \leq \theta(\rho_0)^n$ as claimed, and the assertion then follows directly from Theorem 4.7. q.e.d.

We point out that the barriers constructed by Ecker in [**E1**] can also be used to obtain the estimate in Lemma 4.9. Next we discuss the components discarded at the surgery times.

Canonical neighborhoods. The proof of Theorem 8.1 in [**HS3**] establishes that, at each surgery time, all points with mean curvature exceeding H_2^i are contained in one of finitely many disjoint regions $\mathcal{A}_l$. Let $r_\partial^i \equiv 2(n-1)/H_1^i$. Each $\mathcal{A}_l$ must assume one of five possible structures (only the final two have non-empty boundary):

i) $\mathcal{A}_l$ is uniformly convex and diffeomorphic to $\mathbb{S}^n$;

ii) $\mathcal{A}_l$ is the union of a neck $\mathcal{N}_0$ with two disks and forms a connected component diffeomorphic to $\mathbb{S}^n$;
iii) $\mathcal{A}_l$ is a maximal hypersurface neck $\mathcal{N}_0$ which covers an entire connected component of $\mathcal{M}^i_{T_j^-}$ and is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$;
iv) $\mathcal{A}_l$ is the union of a neck $\mathcal{N}_0$ with a region diffeomorphic to a disk, and has one boundary component with mean radius r^i_∂;
v) $\mathcal{A}_l$ is a neck $\mathcal{N}_0$ with two boundary components (each of which has mean radius r^i_∂) and is therefore diffeomorphic to $\mathbb{S}^{n-1} \times [0,1]$.

Components of known topology are discarded at the surgery time. In addition, one standard surgery is performed at the cross-section nearest to each boundary component with mean radius $r^i_0 \equiv (n-1)/H^i_1$, forming a component diffeomorphic to $\mathbb{S}^n$ which is also discarded.

It is therefore necessary to deal with the points affected by step two of the surgery procedure. Let T_j be a surgery time. Consider any $\mathcal{A}_l \subset \mathcal{M}^i_{T_j}$ and the corresponding domain $\mathcal{G}_l \subset \mathbb{R}^{n+1}$ enclosed by $\mathcal{A}_l$. Let $\mathcal{S} \subset \mathcal{G}_l$ be the open set in $\mathbb{R}^{n+1}$ enveloped by a component removed at the surgery time T_j. We state and prove a result which gives an upper bound on the extinction time $T_\mathcal{S} \equiv \sup\{\, t \geq 0 \mid \Gamma_t \neq \emptyset \,\}$ of the weak solution generated by $\mathcal{S}$.

Lemma 4.10 (Discarded Components)**.** *Suppose $n \geq 3$ and let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in \mathcal{C}(R,\alpha)$ for some R, α. Let $\mathcal{M}^i_t$ be the solution of the flow with surgeries starting from $\partial\Omega$ and with parameters H^i_1, H^i_2, H^i_3. In addition, let T_j be any surgery time for $\mathcal{M}^i_t$, and consider any discarded component $\partial\mathcal{S}$ produced by the solution $\mathcal{M}^i_t$ at time T_j. Denote by $\mathcal{S} \subset \mathbb{R}^{n+1}$ the open set enveloped by $\partial\mathcal{S}$ and let $u_\mathcal{S} : \bar{\mathcal{S}} \to \mathbb{R}$ be the weak solution generated by the domain $\mathcal{S}$. There exists a constant $C(n,\alpha) > 0$ depending only on n, α such that $T_\mathcal{S} \leq C(n,\alpha)L^2(H^i_1)^{-2}$, where $T_\mathcal{S}$ denotes the extinction time of $u_\mathcal{S}$.*

Proof. If $\partial\mathcal{S}$ is uniformly convex, we can use the curvature bound from Theorem 7.14 in [**HS3**] in combination with Myers' theorem and a standard spherical barrier argument to obtain the desired estimate. In the remaining cases the discarded component contains a neck.

The neck either covers $\partial\mathcal{S}$—in which case $\partial\mathcal{S}$ is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$—or separates two regions diffeomorphic to disks (this implies that $\partial\mathcal{S}$ is again diffeomorphic to $\mathbb{S}^n$). In addition, each disk must either be uniformly convex or correspond to the region introduced by a previous surgery; see Lemma 7.12 in [**HS3**]. In these cases, the estimate follows from an argument similar to the proof of Lemma 4.9. q.e.d.

Proof of Proposition 4.6. We have $\mathcal{M}^i_0 = \Gamma_0 = \partial\Omega$ and therefore $\Gamma_\delta \subset\subset \Omega$ for all $\delta > 0$. The avoidance principle guarantees that $\operatorname{dist}(\Gamma_{\delta+t}, \mathcal{M}^i_t)$ is non-decreasing in t for all $\delta > 0$ and for all $0 < t \leq T_1^-$ until the first surgery time for $\mathcal{M}^i_t$—that is, as long as $\mathcal{M}^i_t$ remains smooth. Hence we have $\Gamma_{\delta+t} \subset\subset \tilde{\Sigma}_t^i$, $0 \le t \le T_1$, and $\Gamma_{\delta+t} \subset\subset \Sigma_t^i$, $0 \le t < T_1$, for all $\delta > 0$. Let $t_\delta \equiv \delta + T_1$. In addition, using (5.1) below, we have $\mathcal{H}^n(\Gamma_{t_\delta}) = \mathcal{H}^n(\partial^* \Omega_{t_\delta})$ for a.e. $\delta > 0$. We now show that $\Gamma_{t_\delta + t_w} \subset\subset \Sigma_{T_1}^i$ for all $\delta > 0$, where $t_w \le C(n,\alpha)L^2(H_1^i)^{-2}$.

Let $\partial\mathcal{S}_l$ denote the (at most) finitely many components discarded by the solution $\mathcal{M}_t^i$ at time T_1. Applying Lemma 4.9, we obtain $\Gamma_{t_\delta+\bar{t}} \subset\subset \left(\Sigma_{T_1}^i \cup (\cup_l \mathcal{S}_l)\right)$ for all small $\delta > 0$, where $\bar{t} \le C(n)L^2(H_1^i)^{-2}$. The avoidance principle for weak solutions yields $\Gamma_{t_\delta+t_w} \subset\subset \Sigma_{T_1}^i$ for all $\delta > 0$, where $t_w \equiv \bar{t} + \max_l T_{\mathcal{S}_l}$ and $T_{\mathcal{S}_l}$ denotes the extinction time of the weak solution generated by $\mathcal{S}_l$. Using Lemma 4.10, we conclude that $\max_l T_{\mathcal{S}_l} \le C(n,\alpha)L^2(H_1^i)^{-2}$.

We then invoke the avoidance principle on the next smooth time interval and iterate the argument finitely many times. This establishes that $\Gamma_{\tilde{t}_\delta+t} \subset\subset \Sigma_t^i$ for all $t \ge 0$ and for all small $\delta > 0$, where $\tilde{t}_\delta \equiv \delta + N t_w$. The proposition follows from the continuity of the level-set functions u_i, u_L. q.e.d.

Proof of Theorem 4.3. Combine Lemma 4.4, Proposition 4.6, and Corollary 3.9. q.e.d.

5. Regularity estimates for weak solutions

We now assemble some preliminary consequences of our undertaking in the previous sections. Our primary goal is to investigate the finer properties of our convergence statement, Theorem 4.3. In particular, we make precise the sense in which the surfaces $\mathcal{M}_t^i$ approximate the level-sets $\{\bar{u} = t\}$ using methods developed in Sections 5 and 6 of [**S**] and Sections 2 and 3 of [**MS**]. This will allow us to pass the integral estimates from Section 3 to limits; see Lemma 5.4 and Corollary 5.5.

Weak compactness. It is well known that if $\partial\Omega$ is mean-convex, the resultant weak solution $\bar{u}$ generated by Ω cannot *fatten*: $\mathcal{H}^{n+1}(\{D\bar{u} = t\}) = 0$ and in particular $\mathcal{H}^{n+1}(\{\bar{u} = t\}) = 0$ for all $t \ge 0$; see for example Section 2 of [**MS**]. Indeed, since $\bar{u} \in C^{0,1}(\bar{\Omega})$, we have

$$\partial^* \{\bar{u} > t\} = \{\bar{u} = t\} \quad \mathcal{H}^n\text{-a.e.} \tag{5.1}$$

for almost every $0 \le t \le \bar{T}$. We will require the following definitions.

Definition 5.1 (Radon Measures)**.** We define the families of Radon measures $\mu_t^i \equiv \mathcal{H}^n \llcorner \partial\{u_i > t\}$, $\tilde{\mu}_t^i \equiv \mathcal{H}^n \llcorner \partial(\mathrm{int}\{u_i \ge t\})$, and $\bar{\mu}_t \equiv \mathcal{H}^n \llcorner \partial^*\{\bar{u} > t\}$.

The main result in [**MS**] establishes that $\bar{\mu}_t$ is continuous in t. We henceforth write $\bar{\Gamma}_t \equiv \partial^*\{\bar{u} > t\}$ and define $I \subset [0, \bar{T}]$ to be the set of times such that (5.1) holds. In particular, $\bar{\mu}_t = \mathcal{H}^n \llcorner \Gamma_t$ for all $t \in I$. The set I has full $\mathcal{L}^1$ measure, and as a first step we establish convergence for each $t \in I$. It is then straightforward to verify that the result can be extended to all $t \in [0, \bar{T}]$ using Corollary 1.2 in [**MS**].

Recall from Lemma 2.2, Lemma 3.4, and Definition 2.1 that we have the uniform area bound

$$\mathcal{H}^n(\mathcal{M}_t^i) \leq \mathcal{H}^n(\partial\Omega) \leq \alpha_2 R^n \tag{5.2}$$

for all $t \geq 0$. Now fix $t \in [0, \bar{T}]$. By the weak compactness theorem for Radon measures, there exist subsequences $\{\mu_t^{i_j}\}_{j\geq 1}$, $\{\tilde{\mu}_t^{i_j}\}_{j\geq 1}$, and Radon measures μ, $\tilde{\mu}$ on Ω such that $\mu_t^{i_j} \to \mu$ and $\tilde{\mu}_t^{i_j} \to \tilde{\mu}$ as measures. Theorem 4.3 ensures that $\operatorname{spt}(\mu) \subset \{\bar{u} = t\}$ and $\operatorname{spt}(\tilde{\mu}) \subset \{\bar{u} = t\}$.

Assume in addition that $t \in I$. We will use Proposition 5.10 in [**S**] to show that $\mu = \tilde{\mu} = \bar{\mu}_t$ and that the limit is independent of the choice of subsequence. In order to apply the proposition, however, we must first prove that the solution of mean curvature flow with surgeries retains the minimising hull property after surgery. Note that this is not immediate, since the condition is global in nature and surgery acts as the inverse operation of a connected sum.

Lemma 5.2 (Minimising Hull Property)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in \mathcal{C}(R, \alpha)$ for some R, α. Consider the solution $\mathcal{M}_t^i$, $0 \leq t \leq T_N$, of the flow with surgeries starting from $\partial\Omega$ and with parameters H_1^i, H_2^i, H_3^i. For all $0 \leq t \leq T_N$, the sets $\tilde{\Sigma}_t^i \equiv \operatorname{int}\{u_i \geq t\}$ and $\Sigma_t^i \equiv \{u_i > t\}$ are minimising hulls in Ω.*

Proof. We employ a standard finite induction argument. Recall that the classical evolution preserves the minimising hull property. Therefore suppose that at any surgery time T_j the set $\tilde{\Sigma}_{T_j}^i$ is a minimising hull. We want to show that the same is true of $\Sigma_{T_j}^i$.

Since $\tilde{\Sigma}_{T_j}^i$ is a valid comparison set, we first appeal to Lemma 3.4 to obtain

$$\mathcal{H}^n(\mathcal{M}_{T_j^-}^i) \geq C(n)L(H_1^i)^{-n} + \mathcal{H}^n(\mathcal{M}_{T_j^+}^i).$$

The surface $\mathcal{M}_{T_j^+}^i$ is obtained from $\mathcal{M}_{T_j^-}^i$ by performing surgery on finitely many independent necks and by subsequently discarding finitely many components of known topology. Since $\tilde{\Sigma}_{T_j}^i$ is a minimising hull, any perturbation $F \supset \Sigma_{T_j}^i$ with $\mathcal{H}^n(\partial^* F) \leq \mathcal{H}^n(\mathcal{M}_{T_j^+}^i)$ must in turn satisfy $F \subset (\tilde{\Sigma}_{T_j}^i \cup \mathcal{M}_{T_j^-}^i)$ and $\partial^* F$ must agree with $\mathcal{M}_{T_j^-}^i$ outside the regions altered at the surgery time.

Consider any neck $\mathcal{N}_0 \subset \mathcal{M}_{T_j^-}^i$ of length L in normal form enclosing a solid tube G_0, and suppose that $\partial^* F$ minimises area among all surfaces outside $\mathcal{M}_{T_j^+}^i$. Then $\partial^* F \cap \big(G_0 \setminus \bar{\mathcal{U}}^+\big)$ is a properly embedded minimal surface in $G_0 \setminus \bar{\mathcal{U}}^+$, where $\bar{\mathcal{U}}^+$ represents the closed domain bounded by the surgery caps $\mathcal{U}^+$ introduced during the surgery procedure. We therefore choose $L = L(n)$ sufficiently large and apply standard non-existence results for minimal surfaces; see for example [**D**] and [**E1**]. Note that the maximum principle ensures that a minimal surface cannot extend from $\mathcal{U}^+$ into $G_0 \setminus \bar{\mathcal{U}}^+$ without joining the two ends. This completes the proof. q.e.d.

We can therefore follow the proof of Proposition 5.10 in [**S**] to obtain $\mu = \tilde{\mu} = \bar{\mu}_t$, regardless of the choice of subsequence.

The result extends to the remaining times $t \in [0, \bar{T}] \setminus I$ via a standard growth control argument—see for example Section 7.2 in [**I**]—and Corollary 1.2 in [**MS**]. Note that the argument in [**I**] can be adapted to our setting using Remark 3.7 and the estimates from Section 3 on the number of surgeries.

Corollary 5.3 (Weak Convergence)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in \mathcal{C}(R, \alpha)$ for some R, α. Let $\bar{u}$ be the weak solution generated by Ω, and denote by u_i the solution of the flow with surgeries starting from $\partial\Omega$ and with parameters H_1^i, H_2^i, H_3^i. For all $0 \leq t \leq \bar{T}$ we have*

$$\mu_t^i \to \bar{\mu}_t, \quad \tilde{\mu}_t^i \to \bar{\mu}_t, \quad \partial\Sigma_t^i \to \bar{\Gamma}_t, \quad \partial\tilde{\Sigma}_t^i \to \bar{\Gamma}_t$$

as measures and varifolds, respectively (see Definition 5.1*).*

Proof. We have already established the first two convergence statements. The remaining two claims require Allard's compactness theorem; see for example Section 1.9 in [**I**]. Using (3.6), Hölder's inequality, and (5.2), we can find a uniform constant $C(n, \varepsilon, \alpha) > 0$ depending only on n, ε, α such that

$$\int_{\mathcal{M}_t^i} H \, d\mu_t^i \leq C(n, \varepsilon, \alpha) R^{n-1} \tag{5.3}$$

for all $0 \leq t \leq \bar{T}$. Applying (5.2), Allard's compactness theorem, and the first two convergence statements, we can therefore find subsequences $\{\partial\Sigma_t^{i_j}\}_{j\geq 1}$, $\{\partial\tilde{\Sigma}_t^{i_j}\}_{j\geq 1}$ such that $\partial\Sigma_t^{i_j} \to \bar{\Gamma}_t$ and $\partial\tilde{\Sigma}_t^{i_j} \to \bar{\Gamma}_t$ as varifolds. This is independent of the choice of subsequence. q.e.d.

Regularity estimates. The estimate (5.3) and Allard's compactness theorem combine to establish that the n-rectifiable sets $\bar{\Gamma}_t$ carry a generalised mean curvature vector $\bar{H}$ for all $0 \leq t \leq \bar{T}$. We can now use Corollary 5.3 to pass the integral estimates from Section 3 to limits. The next result states that for all $0 \leq t \leq \bar{T}$ we have $\bar{H} \in L^p(\bar{\Gamma}_t)$ for all $p < n - 1$.

Lemma 5.4 (L^p Estimate for Weak Solution)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in \mathcal{C}(R, \alpha)$ for some R, α. Let $\bar{u}$ be the weak solution generated by Ω, and denote by u_i the solution of the flow* *with surgeries starting from $\partial\Omega$. For all $0 < \varepsilon \le 1$ there exists a constant $C_\varepsilon = C_\varepsilon(n, \alpha) > 0$ depending only on n, ε, α such that*

$$\int_{\bar{\Gamma}_t} |\bar{H}|^{n-1-\varepsilon}\, d\bar{\mu}_t \le C_\varepsilon R^{1+\varepsilon} \tag{5.4}$$

for all $0 \le t \le \bar{T}$, where $\bar{H}$ denotes the generalised mean curvature vector associated with $\bar{\Gamma}_t$.

Proof. Using Corollary 5.3 and Allard's compactness theorem, we obtain $\mu_t^i \llcorner \vec{H}_i \to \bar{\mu}_t \llcorner \bar{H}$ as vector-valued Radon measures, where $\vec{H}_i$ denotes the mean curvature vector $-H_i\nu_i$ associated with $\mathcal{M}_t^i$. We hereafter suppress the subscript i. Since $|\cdot|^p$ is convex, we can therefore apply Theorem 4.4.2 in [**Hu**] to obtain the lower semi-continuity property

$$\int_{\bar{\Gamma}_t} |\bar{H}|^p\, d\bar{\mu}_t \le \liminf_{i\to\infty} \int_{\mathcal{M}_t^i} H^p\, d\mu_t^i \tag{5.5}$$

for all $0 \le t \le \bar{T}$. The assertion then follows from (3.6). q.e.d.

Using Fatou's lemma and Theorem 3.6, we can in fact find a uniform constant $C(n, \varepsilon, \alpha) > 0$ depending only on n, ε, α such that

$$\int_0^{\bar{T}} \int_{\bar{\Gamma}_t} |\bar{H}|^{n+1-\varepsilon}\, d\bar{\mu}_t\, dt \le C(n, \varepsilon, \alpha) R^{1+\varepsilon}. \tag{5.6}$$

Note that the exponent here is greater than the dimension n. Finally, we point out that (5.5) can be turned into an equality using White's estimate on the size of the singular set of $\bar{u}$—see [**W**]—and Brakke's local regularity theorem; see [**B**, **I**]. The proof of the following result closely follows Lemma 6.1 in [**S**] and Lemma 3.3 in [**MS**].

Corollary 5.5 (L^p Convergence)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in \mathcal{C}(R, \alpha)$ for some R, α. Let $\bar{u}$ be the weak solution generated by Ω, and denote by u_i the solution of the flow with surgeries starting from $\partial\Omega$. We have*

$$\int_{\mathcal{M}_t^i} H^p\, d\mu_t^i \to \int_{\bar{\Gamma}_t} |\bar{H}|^p\, d\bar{\mu}_t$$

for all $0 \le t \le \bar{T}$ and for all $0 < p < n-1$, where $\bar{H}$ denotes the generalised mean curvature vector associated with $\bar{\Gamma}_t$.

Proof. For any $\xi > 0$, we can use White's theorem on the size of the singular set of $\bar{u}$ to find a neighborhood U of the singular set such that

$$\limsup_{i\to\infty} \int_{\mathcal{M}_t^i \cap U} H^p\, d\mu_t^i \le C(n, p, \alpha) R^{n-p} \xi$$

for all $0 < p < n-1$ and for all $0 \le t \le \bar{T}$. Here we have used Theorem 3.6 and Hölder's inequality. In addition, we can follow the proof of Lemma 3.3 in [**MS**]—with minor modifications—and apply

Brakke's local regularity theorem to conclude that $\mathcal{M}_t^i \to \Gamma_t$ smoothly away from the singular set. This completes the proof. q.e.d.

We point out that convergence for almost every $0 \leq t \leq \bar{T}$—and with respect to an appropriate subsequence—follows from (5.6), Allard's regularity theorem, Theorem 3.6, and a standard application of Rellich's theorem (see for example Section 6 of [**S**] or Section 5 of [**HI**]), without the use of White's estimate or Brakke's local regularity theorem.

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