J. DIFFERENTIAL GEOMETRY
16 (1981) 125–128

NEIGHBORHOOD CLASSIFICATION OF ISOTROPIC EMBEDDINGS

ALAN WEINSTEIN

1. The problem

If M is any manifold, and (P, Ω) is a symplectic manifold, then an *isotropic* embedding of M in P is an embedding $e: M \to P$ such that $e^*\Omega = 0$. (We refer the reader to [1], [3], or [6] for definitions and proofs omitted in this note.) A *neighborhood equivalence* from $e_1: M_1 \to P_1$ to $e_2: M_2 \to P_2$ consists of

(i) a diffeomorphism $g: M_1 \to M_2$,

(ii) open neighborhoods U_i of $e_i(M_i)$ in P_i,

(iii) a symplectomorphism $f: U_1 \to U_2$ such that $f \circ e_1 = e_2 \circ g$.

We write $f: e_1 \to e_2$. The isotropic embeddings and neighborhood equivalences form a category $\mathfrak{E}$.

The *symplectic normal* bundle $SN(e)$ of an isotropic embedding $e: M \to P$ is a symplectic vector bundle over M whose fibre over $m \in M$ is formed as follows. The image $(Te)(T_mM)$ is an isotropic subspace of $T_{e(m)}P$; the symplectic orthogonal space $[(Te)(T_mM)]^\perp$ contains $(Te)(T_mM)$; the quotient of the two, which is symplectic, is the fibre of $SN(e)$. Every neighborhood equivalence $f: e_1 \to e_2$ induces a symplectic bundle isomorphism $SN(f)$ from $SN(e_1)$ to $SN(e_2)$ covering a diffeomorphism from M_1 to M_2; we thus obtain a functor SN from $\mathfrak{E}$ to the category $\mathfrak{S}$ of symplectic vector bundles and bundle isomorphisms covering diffeomorphisms.

It is shown in [6] that the functor SN is surjective in the sense that every bundle isomorphism from $SN(e_1)$ to $SN(e_2)$ is $SN(f)$ for some neighborhood equivalence $f: e_1 \to e_2$; it is also shown that every symplectic vector bundle is isomorphic to $SN(e)$ for some isotropic embedding e. Thus there is a one-to-one correspondence between neighborhood equivalence classes of isotropic embeddings and isomorphism classes of symplectic vector bundles.

The constructions in [6] leave something to be desired: the manifold into which M is embedded with a given symplectic normal bundle E is the

Communicated by R. Bott, December 17, 1979. Research supported by NSF Grant #MCS 77-23579.

Whitney sum $P = T^*M \oplus E$, but the symplectic structure on P is not canonical, so bundle isomorphisms do not appear to lift to neighborhood equivalences. The purpose of this note is to improve the construction in [6] by finding a "symplectic thickening" functor $St: \mathfrak{S} \to \mathfrak{E}$ which is a right inverse to SN in the sense that there is a natural transformation from $ST \circ SN$ to the identity. To do so, we will use the construction in [7] of a phase space for a classical particle in a Yang-Mills field.

The author would like to thank J. Marsden and T. Ratiu for a conversation which stimulated this work.

2. The solution

Let $E \to M$ be a symplectic vector bundle with fibre dimension $2n$. The frame bundle of E is the principal $Sp(2n)$ bundle $B \to M$ whose fibre over m is the manifold of linear symplectomorphisms from $\mathbf{R}^{2n}$ to the fibre of E over m. The bundle associated to $B \to M$ via the usual representation of $Sp(2n)$ on $\mathbf{R}^{2n}$ is just the original vector bundle $E \to M$.

The action of $Sp(2n)$ on $\mathbf{R}^{2n}$ preserves not only the symplectic structure $\Omega = \Sigma_{i=1}^{n} dq_i \wedge dp_i$ but also the 1-form $\omega = \frac{1}{2}\Sigma_{i=1}^{n}(p_i dq_i - q_i dp_i)$ for which $d\omega = -\Omega$. It follows that the action admits an equivariant momentum mapping μ from $\mathbf{R}^{2n}$ to the dual Lie algebra $\mathfrak{sp}(2n)^*$; the mapping μ is quadratic with $\mu^{-1}(0) = \{0\}$.

Given any principal G-bundle over a manifold M, and any symplectic G-manifold Q with an equivariant momentum mapping, the construction described in [7] produces a symplectic manifold P which can be fibred over T^*M with fibre Q. This fibration is associated to the pullback of the principal bundle from M to T^*M. The map $P \to T^*M$ depends on the choice of a connection on the principal bundle, but the symplectic manifold P and the map $P \to M$ do not.

Applying this construction with $G = Sp(2n)$ and $Q = \mathbf{R}^{2n}$, we obtain a symplectic manifold P which can be fibred over T^*M with fibre $\mathbf{R}^{2n}$. This fibration is just the pullback of E to T^*M, which is the same thing as the Whitney sum $T^*M \oplus E$.

Now we must find as natural isotropic embedding from M to P. The idea is to construct a natural "zero section" from T^*M to P, even though the map $P \to T^*M$ is not well-defined. To do so, we must look at the explicit construction of P.

According to [7], we must take the product symplectic manifold $T^*B \times \mathbf{R}^{2n}$, with its $Sp(2n)$ action, and "reduce at $0 \in \mathfrak{sp}(2n)^*$, following the procedure of [4]. Specifically, we consider the momentum mapping λ: $T^*B \to \mathfrak{sp}(2n)^*$ which is dual to the usual mappings $\mathfrak{sp}(2n) \to T_bB$ onto the tangent spaces along the fibres of the principal bundle. Next, we take the submanifold $\Sigma = \{(\beta, v) \in T^*B \times \mathbf{R}^{2n} | \lambda(\beta) = \mu(v)\}$. Finally, P is the orbit space $\Sigma/Sp(2n)$.

To get a map $P \to T^*M$, we would need an $Sp(2n)$-equivariant projection from T^*B to T^*M, which is essentially a connection on $B \to M$. But let us restrict our attention to $\lambda^{-1}(0)$, which consists of those cotangent vectors to B which annihilate the fibres of $B \to M$. This set $\lambda^{-1}(0)$ is naturally isomorphic to the pullback of T^*M to B. Now Σ contains as a submanifold $\lambda^{-1}(0) \times \mu^{-1}(0) = \lambda^{-1}(0) \times \{0\}$, which gives in P a submanifold $[\lambda^{-1}(0) \times \{0\}]/Sp(2n) \approx \lambda^{-1}(0)/Sp(2n)$, which may be identified with T^*M itself. Thus the zero section $M \to T^*M$ gives an embedding e: $M \to P$.

Finally, one may check by using local trivializations of E that $\lambda^{-1}(0)/Sp(2n)$ is a symplectic submanifold of P and that the tangent bundle to P along $e(M)$ splits symplectically as the Whitney sum $T^*M \oplus E$. It follows that e is an isotropic embedding and that there is a natural isomorphism $n(E)$ from $SN(e)$ to E. Thus if we set $ST(E) = e$, we find that n is a natural transformation from $SN \circ ST$ to the identity.

3. A remark

With the benefit of hindsight, we may see that the construction just described could have been "predicted" from Guillemin's symbol calculus [2] for isotropic submanifolds of cotangent bundles. The quantization of P (see [3], [7], and the "dictionary" in [5]) consists of the sections of the bundle over M which is associated with the frame bundle of E, and whose typical fibre is a quantization of $\mathbf{R}^{2n}$. (We will ignore half-densities and half-forms in this remark.) A quantization of $\mathbf{R}^{2n}$ is given by the space of rapidly decreasing smooth functions on $\mathbf{R}^n$, with the metaplectic representation. Thus, at least if E admits a metaplectic structure, the quantization of P is just a space of symplectic spinors as used in [2].

References

[1] R. Abraham & J. Marsden, *Foundations of mechanics*, 2nd edition, Benjamin/Cummings, Reading, MA, 1978.

[2], V. Guillemin, *Symplectic spinors and partial differential equations*, Géom. Symplectique et Phys. Math. Colloq. Interna-C.N.R.S. **237** (1975) 217–252.

[3] V. W. Guillemin & S. Sternberg, *Geometric asymptotics*, Amer. Math. Soc. 1976.

[4] J. Marsden & A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Mathematical Phys. **5** (1974) 121–130.

[5] ______, Review of *Geometric asymptotics* by V. Guillemin & S. Sternberg and *Symplectic geometry and Fourier analysis* by N. Wallach, Bull. Amer. Math. Soc. **1** (1979) 545–553.

[6] A. Weinstein, *Lectures on symplectic manifolds*, Regional Conf. Math. Vol. 29, Amer. Math. Soc. 1977.

[7] ______, *A universal phase space for particles in Yang-Mills fields*, Lett. Math. Phys. **2** (1978) 417–420.

UNIVERSITY OF CALIFORNIA, BERKELEY