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WEIGHTED COMPOSITION OPERATORS BETWEEN DIFFERENT WEIGHTED BERGMAN SPACES AND DIFFERENT HARDY SPACES

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Abstract. We characterize bounded and compact weighted composition operators acting between weighted Bergman spaces and between Hardy spaces. Our results use certain integral transforms that generalize the Berezin transform. We also estimate the essential norms of these operators. As applications, we characterize bounded and compact pointwise multiplication operators between weighted Bergman spaces and estimate their essential norms.

1. Introduction

Let D be the open unit disk in the complex plane. Let $\varphi : D \to D$ be an analytic self-map of D and let u be an analytic function on D. The weighted composition operator uC_φ is defined on the space of analytic functions on D by $(uC_\varphi)f(z) = u(z)(f \circ \varphi)(z)$. We are interested in weighted composition operators restricted to Hardy spaces and weighted Bergman spaces. In our previous work [CZ] we characterized bounded and compact weighted composition operators mapping every weighted Bergman space into itself. The main tool was the generalized Berezin transform. We needed a general Poisson transform to find a characterization of boundedness and compactness of these operators from a Hardy space into itself. In this paper we continue this line of investigation and study weighted composition operators from one weighted Bergman space into another weighted Bergman space. We study the same question about boundedness and compactness of these operators acting between different Hardy spaces. We also obtain estimates for the essential norms of uC_φ on these spaces.

Let $dA(z) = (1/\pi)dxdy$ be the normalized Lebesgue measure on D and $dA_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha\, dA(z)$ be the weighted Lebesgue measure, where $-1 < \alpha < \infty$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space

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$L_a^{p,\alpha}$ consists of those functions f analytic on D that satisfy

$$\|f\|_{L_a^{p,\alpha}}^p = \int_D |f(z)|^p \, dA_\alpha(z) < \infty.$$

For $0 < p < \infty$, the Hardy space H^p consists of functions f analytic on D that satisfy

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < \infty.$$

We would like to mention other relevant work in this direction. Composition operators between different Hardy spaces and Bergman spaces were studied by many authors, for example, Goebeler [G], Gorkin and MacCluer [GM], Hammond and MacCluer [HM], Hunziker and Jarchow [HJ], Jarchow [J], Smith [Sm] and Smith and Yang [SY]. Boundedness and compactness of weighted composition operators between Hardy spaces were studied by Contreras and Hernandez-Diaz [CH] using Carleson measures. Our approach uses the generalized Berezin transform and related integral operators to characterize bounded and compact weighted composition operators mapping $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ and H^p into H^q. The generalized Berezin transform also appears in [Li] in characterizations of bounded and compact composition operators acting on the Bergman spaces on strictly pseudoconvex domains in $\mathbb{C}^n$.

As one would expect, our results are different for the $p \leq q$ case and the $q < p$ case. Our results also provide an answer to a question posed by Contreras and Hernandez-Diaz [CH] regarding one of the cases mentioned above in the Hardy space setting.

Our first result concerns bounded weighted composition operators mapping $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ for $p \leq q$. For unweighted spaces, that would mean mapping a larger Bergman space into a smaller one. Our results will be expressed in terms of the integral operator

$$I_{\varphi,\alpha,\beta}(u)(a) = \int_D \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2} \right)^{(2+\alpha)q/p} |u(w)|^q \, dA_\beta(w).$$

Theorem 1. *Let u be an analytic function on D and φ be an analytic self-map of D. Let $0 < p \leq q < \infty$, and $\alpha, \beta > -1$. Then the weighted composition operator uC_φ is bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ if and only if*

$$\sup_{a\in D} I_{\varphi,\alpha,\beta}(u)(a) < \infty. \tag{1}$$

We have the following estimates for the essential norm of uC_φ.

Theorem 2. *Let u be an analytic function on D and φ be an analytic self-map of D. Let $1 < p \leq q < \infty$, and $\alpha, \beta > -1$. Let uC_φ be bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$. Then there is an absolute constant $C \geq 1$ such that*

$$\limsup_{|a|\to 1} I_{\varphi,\alpha,\beta}(u)(a) \leq \|uC_\varphi\|_e^q \leq C \limsup_{|a|\to 1} I_{\varphi,\alpha,\beta}(u)(a).$$

The following corollary is now immediate.

Corollary 1. *Let u be an analytic function on D and φ be an analytic self-map of D. Let $1 < p \le q < \infty$, and $\alpha, \beta > -1$. Let uC_φ be bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$. Then the weighted composition operator uC_φ is compact from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ if and only if*

$$\limsup_{|a|\to 1} I_{\varphi,\alpha,\beta}(u)(a) = 0.$$

Let $\sigma_z(w) = (z-w)/(1-\bar{z}w)$ be a Möbius transformation on D.

Definition. Let φ be an analytic self-map of the unit disk. Let $-1 < \alpha, \beta < \infty$. The weighted φ-Berezin transform of a measurable function h is defined as follows.

$$\begin{aligned} B_{\varphi,\alpha,\beta}(h)(z) &= \int_D |\sigma_z'(\varphi(w))|^{2+\alpha} h(w)\, dA_\beta(w) \\ &= \int_D \frac{(1-|z|^2)^{2+\alpha} h(w)}{|1-\bar{z}\varphi(w)|^{4+2\alpha}}\, dA_\beta(w). \end{aligned}$$

We also write $I_{\varphi,\alpha} = I_{\varphi,\alpha,\alpha}$, $B_{\varphi,\alpha} = B_{\varphi,\alpha,\alpha}$ and $B_\varphi = B_{\varphi,0}$. If $\varphi(z) = z$, $B_{\varphi,\alpha}$ is just the usual weighted Berezin transform B_α.

For the case $q < p$, we have the following characterization of the boundedness of uC_φ.

Theorem 3. *Let φ be an analytic self-map of the unit disk D and u be an analytic function on D. Let $1 \le q < p < \infty$, and let $-1 < \alpha, \beta < \infty$. Then the following statements are equivalent:*

(i) *uC_φ is bounded from $L_a^{p,\alpha}$ to $L_a^{q,\beta}$;*
(ii) *uC_φ is compact from $L_a^{p,\alpha}$ to $L_a^{q,\beta}$;*
(iii) *$B_{\varphi,\alpha,\beta}(|u|^q) \in L^{p/(p-q),\alpha}$.*

For the weighted composition operators between Hardy spaces, we obtain analogous results using the related integral operator

$$I_{\varphi,-1}(u)(a) = \int_{\partial D} \left(\frac{1-|a|^2}{|1-\bar{a}\varphi(w)|^2}\right)^{q/p} |u(w)|^q\, d\sigma(w),$$

where $a \in D$, ∂D is the unit circle and $d\sigma$ is the normalized arc length measure on ∂D.

Theorem 4. *Let u be an analytic function on D and φ be an analytic self-map of D. Let $0 < p \le q < \infty$. Then the weighted composition operator uC_φ is bounded from H^p into H^q if and only if*

$$\sup_{a\in D} I_{\varphi,-1}(u)(a) < \infty.$$

We also have the following estimates for the essential norm of uC_φ.

Theorem 5. *Let u be an analytic function on D and φ be an analytic self-map of D. Let $1 < p \le q < \infty$. Let uC_φ be bounded from H^p into H^q. Then there is an absolute constant $C \ge 1$ such that*

$$\limsup_{|a|\to 1} I_{\varphi,-1}(u)(a) \le \|uC_\varphi\|_e^q \le C \limsup_{|a|\to 1} I_{\varphi,-1}(u)(a).$$

In particular, uC_φ is compact from H^p into H^q if and only if

$$\limsup_{|a|\to 1} I_{\varphi,-1}(u)(a) = 0.$$

Theorem 6. *Let φ be an analytic self-map of the unit disk D and u be an analytic function on D. Let $1 \le q < p < \infty$. Let uC_φ be bounded from H^p into H^q. Then uC_φ is compact from H^p into H^q if and only if $|\varphi(z)| < 1$ a.e on ∂D.*

Theorems 1–6 are going to be proved in the Sections 2–5, respectively. Throughout the paper, C represents a constant which may vary from line to line.

2. Boundedness between $L_a^{p,\alpha}$ and $L_a^{q,\beta}$ for $p \le q$

In this section we prove Theorem 1. Our main tool is the Carleson measure on the weighted Bergman space. Let μ be a positive Borel measure on D. Let X be a Banach space of analytic functions on D. Let $q > 0$. We say that μ is an (X, q)-Carleson measure if there is a constant $C > 0$ such that, for any $f \in X$,

$$\int_D |f(z)|^q \, d\mu(z) \le C\|f\|_X^q.$$

Let I be an arc in the unit circle ∂D. Let $S(I)$ be the Carleson square defined by

$$S(I) = \{z \in D : 1 - |I| \le |z| < 1, z/|z| \in I\}.$$

The following result is well-known.

Theorem A. *Let μ be a positive Borel measure on D. Let $0 < p \le q < \infty$ and $-1 < \alpha < \infty$. Then the following statements are equivalent:*

(i) *There is a constant $C_1 > 0$ such that, for any $f \in L_a^q$,*

$$\int_D |f(z)|^q \, d\mu(z) \le C_1 \|f\|_{L_a^{p,\alpha}}^q.$$

(ii) *There is a constant $C_2 > 0$ such that, for any arc $I \in \partial D$,*

$$\mu(S(I)) \le C_2 |I|^{(2+\alpha)q/p}.$$

(iii) *There is a constant $C_3 > 0$ such that, for every $a \in D$,*

$$\int_D |\sigma_a'(z)|^{(2+\alpha)q/p}\, d\mu(z) \le C_3.$$

The result was proved by several authors. The equivalence of (i) and (ii) can be found in [H] and [L1], and a proof of the equivalence of (ii) and (iii) can be found in [ASX]. Notice that the best constants C_1, C_2 and C_3 in this theorem are in fact comparable, which means that there is a positive constant M, independent of μ, such that

$$\frac{1}{M}C_1 \le C_2 \le MC_1, \qquad \frac{1}{M}C_1 \le C_3 \le MC_1.$$

To check this fact, one may refer to the proof of Theorem 6.2.2 in [Zhu1, p. 109–110] and [ASX]. We define

$$\|\mu\| = \sup_{I\subset\partial D} \frac{\mu(S(I))}{|I|^{(2+\alpha)q/p}}.$$

Then $\|\mu\|$ and the above constants are comparable.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. By definition, uC_φ is bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ if and only if for any $f \in L_a^{p,\alpha}$,

$$\|(uC_\varphi)f\|^q_{L_a^{q,\beta}} \le C\|f\|^q_{L_a^{p,\alpha}},$$

that is,

$$\int_D |u(z)|^q |f(\varphi(z))|^q\, dA_\beta(z) \le C\|f\|^q_{L_a^{p,\alpha}}. \tag{2}$$

Letting $w = \varphi(z)$ we get

$$\int_D |f(w)|^q\, d\mu_u(w) \le C\|f\|^q_{L_a^{p,\alpha}},$$

where $\mu_u = \nu_u \circ \varphi^{-1}$ and $d\nu_u(z) = |u(z)|^q\, dA_\beta(z)$. But (2) means that $d\mu_u$ is an $(L_a^{p,\alpha}, q)$-Carleson measure. By Theorem A, this is equivalent to

$$\sup_{a\in D} \int_D |\sigma_a'(w)|^{(2+\alpha)q/p}\, d\mu_u(w) < \infty.$$

Changing the variable back to z we get (1). The proof is complete. □

Using the corresponding results on (H^p, q)-Carleson measures for $0 < p \le q < \infty$ analogous to Theorem A (see [D] and [ASX]), the proof of Theorem 4 follows similarly.

3. Essential norm estimates

We need the following two lemmas.

Lemma 1. *Let $0 < r < 1$. Let μ be a positive Borel measure on D. Let*

$$N_r^* = \sup_{|a|\geq r} \int_D |\sigma_a'(z)|^{(2+\alpha)q/p}\, d\mu(z).$$

If μ is an $(L_a^{p,\alpha}, q)$-Carleson measure for $0 < p \leq q < \infty$, then so is $\mu_r = \mu|_{D\setminus D_r}$, where $D_r = \{z \in D : |z| < r\}$. Moreover, $\|\mu_r\| \leq MN_r^*$, where M is an absolute constant.

The proof is the same as the proof of Lemma 1 and Lemma 2 in [CZ], and thus is omitted here.

For $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic on D, let $K_n f(z) = \sum_{k=0}^{n} a_k z^k$ and $R_n = I - K_n$, where $If = f$ is the identity map. Hence $R_n f(z) = \sum_{k=n+1}^{\infty} a_k z^k$. Then we have:

Lemma 2. *If uC_φ is bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ for $0 < p \leq q < \infty$, then*

$$\|uC_\varphi\|_e \leq \liminf_{n\to\infty} \|uC_\varphi R_n\|.$$

Proof. Since $(R_n + K_n)f = f$ and K_n is compact, we have for each n,

$$\|uC_\varphi\|_e \leq \|uC_\varphi R_n + uC_\varphi K_n\|_e \leq \|uC_\varphi R_n\|_e \leq \|uC_\varphi R_n\|.$$

Therefore $\|uC_\varphi\|_e \leq \liminf_{n\to\infty} \|uC_\varphi R_n\|$. □

The proof of Theorem 2 is similar to the proof of Theorem 2 in [CZ], and uses some estimates from [Sh]. We sketch the proof here.

Proof of Theorem 2. First we prove the upper estimate. By Lemma 2,

$$\|uC_\varphi\|_e \leq \liminf_{n\to\infty} \|uC_\varphi R_n\| \leq \liminf_{n\to\infty} \sup_{\|f\|_{L_a^{p,\alpha}}\leq 1} \|(uC_\varphi R_n)f\|_{L_a^{q,\beta}}.$$

However, for any fixed $0 < r < 1$,

$$\begin{aligned}(3)\qquad \|(uC_\varphi R_n)f\|^q_{L_a^{q,\beta}} &= \int_D |u(z)|^q |(R_n f)(\varphi(z))|^q \, dA_\beta(z)\\ &= \int_D |R_n f(w)|^q \, d\mu_u(w)\\ &= \int_{D\setminus D_r} |R_n f(w)|^q \, d\mu_u(w) + \int_{D_r} |R_n f(w)|^q \, d\mu_u(w)\\ &= I_1 + I_2,\end{aligned}$$

where μ_u is the pull-back measure induced by φ defined in Section 2. Since uC_φ is bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$, μ_u is an $(L_a^{p,\alpha}, q)$-Carleson measure.

From the proof of Proposition 3 in [CZ] we see that, for a given $\varepsilon > 0$, and n big enough,

$$|R_n f(w)| \leq \varepsilon \|f\|_{L_a^{p,\alpha}}.$$

Thus

$$I_2 \leq \varepsilon^q \|f\|^q_{L_a^{p,\alpha}} \mu_u(D_r) \leq \varepsilon^q \|f\|^q_{L_a^{p,\alpha}} \|u\|^q_{L_a^{q,\beta}}.$$

Hence, for a fixed r,

$$\sup_{\|f\|_{L_a^{p,\alpha}} \leq 1} I_2 \to 0 \quad \text{as } n \to \infty.$$

On the other hand, if we set $\mu_{u,r} = \mu_u|_{D \setminus D_r}$, then, by Theorem A and Lemma 1,

$$I_1 = \int_{D \setminus D_r} |R_n f(w)|^q \, d\mu_{u,r}(w) \leq K \|\mu_{u,r}\| \|R_n f\|^q_{L_a^{p,\alpha}} \leq KCMN_r^* \|f\|^q_{L_a^{p,\alpha}},$$

where K, C and M are constants independent of u and r, and N_r^* is defined as in Lemma 1. Here we have also used the inequality $\|R_n f\|^q_{L_a^{p,\alpha}} \leq C\|f\|^q_{L_a^{p,\alpha}}$, which can be easily proved by the triangle inequality, and the inequality $\|K_n f\|^q_{L_a^{p,\alpha}} \leq C\|f\|^q_{L_a^{p,\alpha}}$, obtained in [Zhu3] (see Proposition 1 and Corollary 4 there). Taking the supremum in (3) over analytic functions f in the unit ball of $L_a^{p,\alpha}$, and letting $n \to \infty$, we get

$$\liminf_{n\to\infty} \sup_{\|f\|_{L_a^{p,\alpha}} \leq 1} \|(uC_\varphi R_n) f\|^q_{L_a^{p,\alpha}} \leq \liminf_{n\to\infty} KCMN_r^* = KCMN_r^*.$$

Thus $\|uC_\varphi\|_e^q \leq KCMN_r^*$. Letting $r \to 1$ we get

$$\begin{aligned}
\|uC_\varphi\|_e^q &\leq KCM \lim_{r\to 1} N_r^* = KCM \limsup_{|a|\to 1} \int_D |\sigma_a'(w)|^{(2+\alpha)q/p} \, d\mu_u(w) \\
&= KCM \limsup_{|a|\to 1} \int_D |\sigma_a'(\varphi(z))|^{(2+\alpha)q/p} |u(z)|^q \, dA_\beta(z) \\
&= KCM \limsup_{|a|\to 1} I_{\varphi,\alpha,\beta}(u)(a),
\end{aligned}$$

which gives us the desired upper bound.

Now let us prove the lower estimate. Consider the normalized kernel function $k_a(z) = -\sigma_a'(z) = (1-|a|^2)/(1-\bar{a}z)^2$. Let $f_a = k_a^{(2+\alpha)/p}$. Then $\|f_a\|_{L_a^{p,\alpha}} = 1$, and $f_a \to 0$ uniformly on compact subsets of D as $|a| \to 1$. Fix a compact operator $\mathcal{K}$ from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$. Then $\|\mathcal{K} f_a\|_{L_a^{q,\beta}} \to 0$ as $|a| \to 1$.

Therefore,

$$\begin{aligned}\|uC_\varphi - \mathcal{K}\| &\geq \limsup_{|a|\to 1} \|(uC_\varphi - \mathcal{K})f_a\|_{L_a^{q,\beta}} \\ &\geq \limsup_{|a|\to 1} \Big(\|(uC_\varphi)f_a\|_{L_a^{q,\beta}} - \|\mathcal{K}f_a\|_{L_a^{q,\beta}} \Big) \\ &= \limsup_{|a|\to 1} \|(uC_\varphi)f_a\|_{L_a^{q,\beta}}.\end{aligned}$$

Thus

$$\|uC_\varphi\|_e^q \geq \limsup_{|a|\to 1} \|(uC_\varphi)f_a\|^q_{L_a^{q,\beta}} = \limsup_{|a|\to 1} I_{\varphi,\alpha,\beta}(u)(a). \qquad \square$$

The proof of Theorem 5 is similar to that of Theorem 2, using modified versions of Lemma 1 (with $\alpha = -1$) and Lemma 2 for Hardy spaces.

4. Boundedness between $L_a^{p,\alpha}$ and $L_a^{q,\beta}$ for $q < p$

In this section we prove Theorem 3. We need a characterization of the $(L_a^{p,\alpha}, q)$-Carleson measure. The idea of the proof follows that of Theorem 4.4 in [CKY].

Definition. For a positive measure μ on D, $-1 < \alpha < \infty$, and a fixed number $r \in (0,1)$, define

$$\widehat{\mu_{r,\alpha}}(z) = \frac{\mu(D(z,r))}{|D(z,r)|^{1+\alpha/2}}, \qquad B_\alpha(\mu)(z) = \int_D |\sigma_z'(w)|^{2+\alpha}\, d\mu(w),$$

where $D(z,r) = \{w \in D \,|\, |\sigma_z(w)| < r\}$ is the pseudohyperbolic disk with center z and radius r and $|D(z,r)|$ denotes the Lebesgue area measure of $D(z,r)$. Recall that for a measurable function h on D,

$$B_\alpha(h)(z) = \int_D |\sigma_z'(w)|^{2+\alpha} h(w)\, dA_\alpha(w).$$

We need several lemmas.

Lemma 3. *Given $0 < r < 1$, there exists a constant $C = C_r > 0$ such that*

$$g(z) \leq \frac{C_r}{|D(z,r)|} \int_{D(z,r)} g(w)\, dA(w)$$

for all g subharmonic on D, and all $z \in D$.

The proof is the same as that of Proposition 4.3.8 in [Zhu1, p. 62].

Lemma 4. *Let $-1 < \alpha < \infty$. Let μ be a positive measure on D. Given $0 < r < 1$, there exists a constant $C = C_r > 0$ such that*

$$\int_D g(w)\, d\mu(w) \leq C \int_D g(w) \widehat{\mu_{r,\alpha}}(w)\, dA_\alpha(w).$$

for all g subharmonic on D.

Proof. Since $(1-|z|^2)^2 \approx (1-|w|^2)^2 \approx |D(z,r)| \approx |D(w,r)|$ as $z \in D(w,r)$ (see for example [Zhu1, p. 61]), we have

$$\int_{D(w,r)} \frac{d\mu(z)}{|D(z,r)|} \le C\frac{\mu(D(w,r))}{|D(w,r)|} \le C(1-|w|^2)^{\alpha}\widehat{\mu_{r,\alpha}}(w),$$

for some constant C and for all $w \in D$. Therefore, from Lemma 3 and Fubini's Theorem,

$$\begin{aligned}\int_D g(z)\,d\mu(z) &\le C\int_D \frac{1}{|D(z,r)|}\int_{D(z,r)} g(w)\,dA(w)\,d\mu(z)\\ &= C\int_D g(w)\int_{D(w,r)}\frac{d\mu(z)}{|D(z,r)|}\,dA(w)\\ &\le C\int_D g(w)\widehat{\mu_{r,\alpha}}(w)(1-|w|^2)^{\alpha}\,dA(w).\end{aligned}$$

The proof is complete. □

LEMMA 5. *Let $-1 < \alpha < \infty$. Let μ be a positive measure on D. Given $0 < r < 1$, there exists a constant $C = C_r > 0$ such that $B_\alpha(\mu)(z) \le CB_\alpha(\widehat{\mu_{r,\alpha}})(z)$ for any $z \in D$.*

Proof. Setting $g(w) = |\sigma_z'(w)|^{2+\alpha}$ in Lemma 4, we get

$$\begin{aligned}B_\alpha(\mu)(z) &= \int_D |\sigma_z'(w)|^{2+\alpha}\,d\mu(w) \le C\int_D |\sigma_z'(w)|^{2+\alpha}\widehat{\mu_{r,\alpha}}(w)\,dA_\alpha(w)\\ &= CB_\alpha(\widehat{\mu_{r,\alpha}})(z).\end{aligned}$$

The proof is complete. □

LEMMA 6. *Let $-1 < \alpha < \infty$. Then B_α is a bounded operator on $L^{p,\alpha}$ for any $p > 1$.*

Proof. Let $h(z) = (1-|z|^2)^{-1/(pq)}$. By Lemma 4.2.2 in [Zhu1, p. 53], it can be easily checked that

$$\int_D |\sigma_w'(z)|^{2+\alpha}h(z)^q\,dA_\alpha(z) \le Ch(w)^q,$$

and

$$\int_D |\sigma_w'(z)|^{2+\alpha}h(w)^p\,dA_\alpha(w) \le Ch(z)^p.$$

Thus by Schur's Theorem (see, for example, Theorem 3.2.2 in [Zhu1, p. 42]), B_α is bounded on $L^{p,\alpha}$. □

LEMMA 7. *Let $-1 < \alpha < \infty$. Let μ be a positive measure on D. Given $0 < r < 1$, there exists a constant $C = C_r > 0$ such that $\widehat{\mu_{r,\alpha}}(z) \le CB_\alpha(\mu)(z)$ for any $z \in D$.*

Proof. Since $(1-|z|^2)^2 \approx (1-|w|^2)^2 \approx |D(z,r)|$ as $w \in D(z,r)$, we have

$$\begin{aligned} B_\alpha(\mu)(z) &= \int_D |\sigma_z'(w)|^{2+\alpha}\, d\mu(w) = \int_D \frac{(1-|\sigma_z(w)|^2)^{2+\alpha}}{(1-|w|^2)^{2+\alpha}}\, d\mu(w) \\ &\geq (1-r^2)^{2+\alpha} \int_{D(z,r)} \frac{d\mu(w)}{(1-|w|^2)^{2+\alpha}} \geq C_r \frac{\mu(D(z,r))}{|D(z,r)|^{1+\alpha/2}} \\ &= C_r \widehat{\mu_{r,\alpha}}(z). \end{aligned}$$

The proof is complete. □

THEOREM 7. *Let μ be a positive measure on D. Let $0<q<p<\infty$ and $-1<\alpha<\infty$. Then the following statements are equivalent:*

(i) *μ is an $(L_a^{p,\alpha}, q)$-Carleson measure.*
(ii) *For a fixed $r \in (0,1)$, $\widehat{\mu_{r,\alpha}} \in L^{p/(p-q),\alpha}$.*
(iii) *$B_\alpha(\mu) \in L^{p/(p-q),\alpha}$.*

Proof. The equivalence of (i) and (ii) is given by Luecking [L2] [L4], for the case $\alpha=0$. For $-1<\alpha<\infty$, the result can be similarly proved as in [L4]. We just need to prove (ii) and (iii) are equivalent. However, (iii)⇒(ii) is a direct consequence of Lemma 7. To prove (ii)⇒(iii), let $\widehat{\mu_{r,\alpha}} \in L^{p/(p-q),\alpha}$. By Lemma 6, $B_\alpha(\widehat{\mu_{r,\alpha}}) \in L^{p/(p-q),\alpha}$. By Lemma 5 we get that $B_\alpha(\mu) \in L^{p/(p-q),\alpha}$. The proof is complete. □

Proof of Theorem 3. Let $d\nu_u(z) = |u(z)|^q\, dA_\beta(z)$ and $\mu_u = \nu_u \circ \varphi^{-1}$ be the pull-back measure of ν_u. Then uC_φ is bounded from $L_a^{p,\alpha}$ to $L_a^{q,\beta}$ if and only if for any $f \in L_a^{p,\alpha}$,

$$\int_D |u(z)|^q |f(\varphi(z))|^q\, dA_\beta(z) \leq C\|f\|_{p,\alpha}^q,$$

or

$$\int_D |f(w)|^q\, d\mu_u(w) \leq C\|f\|_{p,\alpha}^q.$$

Thus μ_u is an $(L_a^{p,\alpha}, q)$-Carleson measure. By Theorem 7, this is equivalent to $B_\alpha(\mu_u) \in L^{p/(p-q),\alpha}$. Thus (i) and (iii) are equivalent since $B_\alpha(\mu_u) = B_{\varphi,\alpha,\beta}(|u|^q)$.

The equivalence of (i) and (ii) follows from a general result of Banach space theory. It is known that, for $1 \leq q < p < \infty$, every bounded operator from ℓ^p to ℓ^q is compact (see, for example [LT, p. 31, Theorem I.2.7]). Since the Bergman space $L_a^{p,\alpha}$ is isomorphic to ℓ^p (see, [W, p. 89, Theorem 11]), we get the implication (i)⇒(ii) directly from the above result. On the other hand, it is obvious that (ii) implies (i). □

If $\alpha=\beta=0$, then $B_{\varphi,\alpha,\beta}(h) = B_\varphi(h)$. Thus we have the following consequence.

COROLLARY 2. *Let $1 \le q < p < \infty$. Then uC_φ is compact from L_a^p to L_a^q if and only if $B_\varphi(|u|^q) \in L^{p/(p-q)}$.*

As a byproduct, we show the boundedness of $B_{\varphi,\alpha}$ on $L^{p,\alpha}$ here.

PROPOSITION 1. *For any analytic self-map φ on D and $p > 1$, $B_{\varphi,\alpha}$ is a bounded operator on $L^{p,\alpha}$.*

Proof. Let $h \in L^{p,\alpha}$. Let $d\nu_h = |h(z)|\, dA_\alpha(z)$ and $\mu_h = \nu_h \circ \varphi^{-1}$. Let $q = p - 1$ and $(1/p') + (1/p) = 1$. Noticing that $qp' = p$, we have, for any $f \in L_a^{p,\alpha}$,

$$\begin{aligned}\int_D |h(z)||f(\varphi(z))|^q \, dA_\alpha(z) \\ &\le \left(\int_D |h|^p \, dA_\alpha(z)\right)^{1/p} \left(\int_D |f(\varphi(z))|^p \, dA_\alpha(z)\right)^{1/p'} \\ &= \|h\|_{p,\alpha} \|f \circ \varphi\|_{p,\alpha}^q \le C\|h\|_{p,\alpha}\|f\|_{p,\alpha}^q.\end{aligned}$$

The last inequality is true since the composition operator C_φ is always bounded on $L_a^{p,\alpha}$. Hence μ_h is an $(L_a^{p,\alpha}, q)$-Carleson measure, and by Theorem 7, $B_\alpha(\mu_h) \in L^{p,\alpha}$. Noticing that $|B_{\varphi,\alpha}(h)| \le B_{\varphi,\alpha}(|h|) = B_\alpha(\mu_h)$, we get that $B_{\varphi,\alpha}(h) \in L^{p,\alpha}$. A standard application of the Closed Graph Theorem shows that $B_{\varphi,\alpha}$ is bounded on $L^{p,\alpha}$. □

5. Compactness between H^p and H^q for $q < p$

We first prove the following result, which is of independent interest.

THEOREM 8. *Let φ be an analytic self-map of the unit disk D, and u be an analytic function on D. Let $1 < p < \infty$. Let uC_φ be bounded from H^p into H^1. Then uC_φ is compact from H^p into H^1 if and only if $|\varphi(z)| < 1$ a.e. on ∂D.*

Proof. It is well-known that the sequence $\{z^n\}$ is an H^p-weakly null sequence. Thus the compactness of uC_φ from H^p to H^1 implies that $\|uC_\varphi z^n\|_{H^1} \to 0$ as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \int_0^{2\pi} |u(e^{i\theta})||\varphi(e^{i\theta})|^n \, d\theta = 0.$$

Because uC_φ is bounded from H^p to H^1, it is clear that $u = uC_\varphi 1 \in H^1$. Hence the convergence condition above means that $\{\xi \in \partial D : |\varphi(\xi)| = 1\}$ has measure 0.

Conversely, suppose $|\varphi(z)| < 1$ a.e. on ∂D. Let $\{f_n\} \subset H^p$ be an arbitrary weakly null sequence. This implies that $\{f_n\}$ converges to 0 uniformly on compact subsets of D. Since uC_φ is bounded from H^p to H^1, it takes a weakly null sequence in H^p into a weakly null sequence in H^1. Hence $u(f_n \circ \varphi) \to 0$

weakly in H^1. Since $|\varphi(z)| < 1$ a.e. on ∂D, it follows that $u(f_n \circ \varphi) \to 0$ a.e. on ∂D. This means that $u(f_n \circ \varphi) \to 0$ in measure. By the Dunford-Pettis Theorem (see [DS]), we have $\|uC_\varphi f_n\|_{H^1} \to 0$. Hence uC_φ is completely continuous and, by the reflexivity of H^p, uC_φ is compact. □

For proving Theorem 6, we first give the following criterion for boundedness of uC_φ from H^p to H^q.

Proposition 2. *Let φ be an analytic self-map of the unit disk D and u be an analytic function on D. Let $1 \le q < p < \infty$. Then uC_φ is bounded from H^p to H^q if and only if*

$$\int_0^{2\pi} \left(\int_{\Gamma(\theta)} \frac{d\mu_u(w)}{1-|w|^2} \right)^{p/(p-q)} d\theta < \infty,$$

where $\mu_u = \nu_u \circ \varphi^{-1}$ and $d\nu_u(z) = |u(z)|^q \, d\sigma(z)$ with $d\sigma(z)$ the normalized measure of ∂D, and $\Gamma(\theta)$ is the Stolz angle at θ, which is defined for real θ as the convex hull of the set $\{e^{i\theta}\} \cup \{z : |z| < \sqrt{1/2}\}$.

Proof. The operator uC_φ being bounded from H^p to H^q means that, for any $f \in H^p$,

$$\int_{\partial D} |u(z) f(\varphi(z))|^q \, d\sigma(z) \le C \|f\|_{H^p}^q.$$

With the change of variable $w = \varphi(z)$ we get

$$\int_D |f(w)|^q \, d\mu_u(w) \le C \|f\|_{H^p}^q,$$

which means that $d\mu_u$ is an (H^p, q)-Carleson measure. By a result of I. V. Videnskii [V] and D. Luecking [L3], this is equivalent to

$$\int_0^{2\pi} \left(\int_{\Gamma(\theta)} \frac{d\mu_u(w)}{1-|w|^2} \right)^{p/(p-q)} d\theta < \infty.$$

The result is proved. □

Remark. From this result we can easily see that if u has no zeros in D, then uC_φ is bounded from H^p to H^q if and only if $u^q C_\varphi$ is bounded from $H^{p/q}$ to H^1. □

Proof of Theorem 6. The necessity follows in the same way as in the previous theorem.

To prove the sufficiency, we follow the argument of Goebeler [G, Theorem 4]. Let us first assume that u is an outer function. Suppose $\{f_n\}$ is a sequence in the unit ball of H^p. For each n, we write $f_n = B_n F_n$, where B_n is inner, F_n is outer. Clearly, both sequences $\{B_n\}$ and $\{F_n\}$ are contained in the unit ball of H^p. The local boundedness of these sequences shows that they are normal families; we can use Montel's Theorem to extract subsequences $\{B_{n_j}\}$ and $\{F_{n_j}\}$ that converge uniformly on compact subsets of D. Put $G_j = F_{n_j}^q$. Then G_j is in the unit ball of $H^{p/q}$. Now recall that uC_φ is bounded from H^p to H^q. From the remark after Proposition 2, this is equivalent to saying that u^qC_φ is bounded from $H^{p/q}$ to H^1. Since $|\varphi(z)| < 1$ a.e. on ∂D by assumption, Theorem 8 applies and it follows that u^qC_φ is compact from $H^{p/q}$ to H^1. Therefore there is a subsequence $\{G_{j_k}\}$ of $\{G_j\}$ such that the sequence $\{u^q(G_{j_k} \circ \varphi)\}$ converges in the norm of H^1. Also, the fact that $|\varphi(z)| < 1$ a.e. on ∂D implies $\{u^q(G_{j_k} \circ \varphi)\}$ converges almost everywhere on ∂D. Vitali's Convergence Theorem implies

$$\lim_{\sigma(E)\to 0} \sup_k \int_E |u|^q |G_{j_k} \circ \varphi| \, d\sigma = 0,$$

where σ denotes the normalized Lebesgue measure on ∂D. As in Goebeler's proof, this implies

$$\lim_{\sigma(E)\to 0} \sup_k \int_E |u|^q |f_{n_{j_k}} \circ \varphi|^q \, d\sigma \le \lim_{\sigma(E)\to 0} \sup_k \int_E |u|^q |G_{j_k} \circ \varphi| \, d\sigma = 0.$$

Again, since $|\varphi(z)| < 1$ a.e. on ∂D, $\{u^q(f_{n_{j_k}} \circ \varphi)\}$ converges almost everywhere on ∂D. Using Vitali's Theorem again, we conclude that $u(f_{n_{j_k}} \circ \varphi)$ converges in H^q. Hence uC_φ is compact from H^p to H^q.

In general, if u is not outer, we can factor $u = B_uF_u$, where B_u is inner and F_u is outer. It is clear that uC_φ is compact from H^p to H^q if and only if F_uC_φ is compact from H^p to H^q. By the proof above, this is equivalent to saying that $|\varphi| < 1$ a.e. on ∂D. □

6. Pointwise multiplication operators

In this section we show how our results lead to the corresponding results about boundedness, compactness and essential norm estimates for the pointwise multiplication operators between weighted Bergman spaces. In this setting, the results are expressed in terms of much simpler expressions than the integral operators $I_{\varphi,\alpha,\beta}$ used for the weighted composition operators. Some of these results were given by the second author in [Zha].

We need the following lemmas.

Lemma 8. *Let $0 < q < \infty$, $-1 < \beta < \infty$ and $1 < s < \infty$. Then there is a constant $C > 0$, depending only on β and s, such that*

$$|u(a)|^q (1 - |a|^2)^{\beta+2-s} \le C \int_D |\sigma_a'(w)|^s |u(w)|^q \, dA_\beta(w).$$

Proof. By subharmonicity we know that for an analytic function g in D and for any fixed r, $0 < r < 1$ (for example, one may choose $r = 1/4$),

$$|g(0)|^q \leq \frac{1}{r^2} \int_{D_r} |g(\zeta)|^q \, dA(\zeta).$$

Replacing g by $u \circ \sigma_a$, we have

$$\begin{aligned} |u(a)|^q &\leq \frac{1}{r^2} \int_{D_r} |u(\sigma_a(\zeta))|^q \, dA(\zeta) \\ &= \frac{1}{r^2} \int_{D(a,r)} |u(w)|^q |\sigma_a'(w)|^2 \, dA(w) \\ &\leq \frac{16}{r^2(1-|a|^2)^2} \int_{D(a,r)} |u(w)|^q \, dA(w). \end{aligned}$$

It is known that for $w \in D(a,r)$, $1 - |w|^2 \sim 1 - |a|^2$ (cf. [Zhu1, p. 61]). So there is a constant C', depending only on β and s, such that

$$\begin{aligned} |u(a)|^q (1-|a|)^{\beta+2-s} &\leq \frac{16(1-|a|^2)^{\beta-s}}{r^2} \int_{D(a,r)} |u(w)|^q \, dA(w) \\ &\leq \frac{16C'}{r^2} \int_{D(a,r)} |u(w)|^q (1-|w|^2)^{\beta-s} \, dA(w) \\ &\leq \frac{16C'}{r^2(1-r^2)^s} \int_{D(a,r)} |u(w)|^q (1-|w|^2)^{\beta-s} (1-|\sigma_a(w)|^2)^s \, dA(w) \\ &= C \int_D |\sigma_a'(w)|^s |u(w)|^q \, dA_\beta(w), \end{aligned}$$

where $C = 16C'/((1+\beta)r^2(1-r^2)^s)$. The proof is complete. □

LEMMA 9. *Let $0 < q < \infty$, $-1 < \beta < \infty$ and $1 < s < \infty$. Then for every $a \in D$,*

$$\int_D |\sigma_a'(w)|^s |u(w)|^q \, dA_\beta(w) \leq \frac{1+\beta}{s-1} \sup_{w \in D} |u(w)|^q (1-|w|^2)^{\beta+2-s}.$$

Proof. Since $s > 1$,

$$\begin{aligned}
&\int_D |\sigma_a'(w)|^s |u(w)|^q \, dA_\beta(w) \\
&\quad \le (1+\beta) \sup_{w\in D} |u(w)|^q (1-|w|^2)^{\beta+2-s} \int_D |\sigma_a'(w)|^s (1-|w|^2)^{s-2} \, dA(w) \\
&\quad = (1+\beta) \sup_{w\in D} |u(w)|^q (1-|w|^2)^{\beta+2-s} \\
&\qquad\qquad \times \int_D (1-|\sigma_a(w)|^2)^s (1-|w|^2)^{-2} \, dA(w) \\
&\quad = (1+\beta) \sup_{w\in D} |u(w)|^q (1-|w|^2)^{\beta+2-s} \int_D (1-|z|^2)^{s-2} \, dA(z) \\
&\quad = \frac{1+\beta}{s-1} \sup_{w\in D} |u(w)|^q (1-|w|^2)^{\beta+2-s}.
\end{aligned}$$

The proof is complete. □

Let M_u denote the pointwise multiplication operators. Then $M_u(f) = uf$, and M_u is the weighted composition operator uC_φ with $\varphi = \mathrm{id}$, the identity map of D.

We have the following result.

THEOREM 9. *Let u be an analytic function on D. Let $0 < p \le q < \infty$, and $\alpha, \beta > -1$. Then the pointwise multiplication operator M_u is bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ if and only if*

$$\sup_{a\in D} |u(a)|(1-|a|^2)^\gamma < \infty, \tag{4}$$

where $\gamma = (\beta+2)/q - (\alpha+2)/p$.

Proof. By Theorem 1, we know that M_u is bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ if and only if

$$\sup_{a\in D} I_{\mathrm{id},\alpha,\beta}(u)(a) = \int_D |\sigma_a'(w)|^{(2+\alpha)q/p} |u(w)|^q \, dA_\beta(w) < \infty.$$

The result clearly follows from Lemma 8 and Lemma 9 (with $s = (\alpha+2)q/p$). □

Remark. Let $\alpha > 0$, and let the Bloch type space B^α be the space of analytic functions f on D such that $\sup_{z\in D} |f'(z)|(1-|z|^2)^\alpha < \infty$. It is known that, as $\alpha > 1$, $f \in B^\alpha$ if and only if $\sup_{z\in D} |f(z)|(1-|z|^2)^{\alpha-1} < \infty$ (see [Zhu2]). Therefore, if $\gamma = (\beta+2)/q - (\alpha+2)/p > 0$, condition (4) means that $u \in B^{1+\gamma}$. If $\gamma = 0$ or $\gamma < 0$, condition (4) is clearly the same as $u \in H^\infty$, or $u \equiv 0$, respectively. Theorem 9 was first proved by the second author in [Zha, Theorem 1 (i), (ii) and (iii)]. □

The following result is new.

THEOREM 10. *Let u be an analytic function on D. Let $1 < p \le q < \infty$, and $\alpha, \beta > -1$. Let M_u be bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$. Then there is an absolute constant $C \ge 1$ such that*

$$\limsup_{|a|\to 1} |u(a)|(1-|a|^2)^\gamma \le \|M_u\|_e \le C \limsup_{|a|\to 1} |u(a)|(1-|a|^2)^\gamma,$$

where $\gamma = (\beta+2)/q - (\alpha+2)/p$. Consequently, M_u is compact from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ if and only if

$$\limsup_{|a|\to 1} |u(a)|(1-|a|^2)^\gamma = 0. \tag{5}$$

Proof. By Theorem 2,

$$\|M_u\|_e \ge \limsup_{|a|\to 1} (I_{\mathrm{id},\alpha,\beta}(u)(a))^{1/q}.$$

Applying Lemma 8 with $s = (\alpha+2)q/p$, we obtain a constant C such that

$$C(I_{\mathrm{id},\alpha,\beta}(u)(a))^{1/q} \ge |u(a)|(1-|a|^2)^\gamma,$$

which gives the lower estimate.

We now obtain an upper estimate. Again by Theorem 2 we know

$$\begin{aligned}\|M_u\|_e^q &\le C \limsup_{|a|\to 1} (I_{\mathrm{id},\alpha,\beta}(u)(a))\\ &= C \limsup_{|a|\to 1} \int_D |\sigma_a'(w)|^{(\alpha+2)q/p} |u(w)|^q \, dA_\beta(w).\end{aligned}$$

For any fixed $0 < r < 1$, we write the above integral as $I_1 + I_2$, where

$$I_1 = \int_{D\setminus D_r} |\sigma_a'(w)|^{(\alpha+2)q/p} |u(w)|^q \, dA_\beta(w),$$

and

$$I_2 = \int_{D_r} |\sigma_a'(w)|^{(\alpha+2)q/p} |u(w)|^q \, dA_\beta(w).$$

Then

$$
\begin{aligned}
I_1 &\le (1+\beta) \sup_{w\in D\setminus D_r} |u(w)|^q (1-|w|^2)^{\beta+2-(\alpha+2)q/p} \\
&\qquad\qquad \times \int_{D\setminus D_r} \frac{(1-|\sigma_a(w)|^2)^{(\alpha+2)q/p}}{(1-|w|^2)^2}\, dA(w) \\
&\le (1+\beta) \sup_{w\in D\setminus D_r} |u(w)|^q (1-|w|^2)^{\beta+2-(\alpha+2)q/p} \\
&\qquad\qquad \times \int_{D} (1-|z|^2)^{(\alpha+2)q/p-2}\, dA(z) \\
&\le C \sup_{w\in D\setminus D_r} (|u(w)|(1-|w|^2)^{\gamma})^q,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\le (1+\beta) \sup_{w\in D} |u(w)|^q (1-|w|^2)^{\beta+2-(\alpha+2)q/p} \\
&\qquad\qquad \times \int_{D_r} \frac{(1-|\sigma_a(w)|^2)^{(\alpha+2)q/p}}{(1-|w|^2)^2}\, dA(w) \\
&\le (1+\beta) \sup_{w\in D} |u(w)|^q (1-|w|^2)^{\beta+2-(\alpha+2)q/p} \\
&\qquad\qquad \times \int_{D(a,r)} (1-|z|^2)^{(\alpha+2)q/p-2}\, dA(z) \\
&\le C(1-|a|^2)^{(\alpha+2)q/p-2} \sup_{w\in D} (|u(w)|(1-|w|^2)^{\gamma})^q |D(a,r)| \\
&\le C(1-|a|^2)^{(\alpha+2)q/p} \sup_{w\in D} (|u(w)|(1-|w|^2)^{\gamma})^q,
\end{aligned}
$$

where $|D(a,r)|$ is the normalized area measure of $D(a,r)$. Here we used the fact that $(1-|z|^2)^2 \sim (1-|a|^2)^2 \sim |D(a,r)|$ for a fixed r and for any $z \in D(a,r)$. Since M_u is bounded from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$, by Theorem 9, we know that $\sup_{w\in D}(|u(w)|(1-|w|^2)^{\gamma})^q < \infty$. Notice that $(\alpha+2)q/p > 1$ and hence $I_2 \to 0$ as $|a| \to 1$. Therefore

$$
\|M_u\|_e \le \limsup_{|a|\to 1} I_1^{1/q} \le C \sup_{w\in D\setminus D_r} |u(w)|(1-|w|^2)^{\gamma}
$$

for any fixed $0 < r < 1$, which implies $\|M_u\|_e \le \limsup_{|w|\to 1} |u(w)|(1-|w|^2)^{\gamma}$. The proof is complete. □

Remark. Let $\alpha > 0$, and let B_0^{α} be the space of analytic functions f on D such that $\lim_{|z|\to 1} |f'(z)|(1-|z|^2)^{\alpha} = 0$. It is known that, as $\alpha > 1$, $f \in B_0^{\alpha}$ if and only if $\lim_{|z|\to 1} |f(z)|(1-|z|^2)^{\alpha-1} = 0$ (see [Zhu2]). Therefore, as $\gamma = (\beta+2)/q - (\alpha+2)/p > 0$, condition (5) means that $u \in B_0^{1+\gamma}$. As $\gamma \le 0$, condition (5) implies $u \equiv 0$. Thus, as $\gamma \le 0$, M_u is compact from $L_a^{p,\alpha}$ into $L_a^{q,\beta}$ if and only if $u \equiv 0$. □

For the case $0 < q < p < \infty$, Attele [A] characterized analytic multipliers from L^p_a into L^q_a, and the second author [Zha] extended this result to the weighted cases. We show here how these results follow from our Theorem 3.

THEOREM 11. *Let u be an analytic function on D. Let $1 \leq q < p < \infty$, and $\alpha, \beta > -1$. Then the following statements are equivalent:*

(i) *M_u is bounded from $L^{p,\alpha}_a$ to $L^{q,\beta}_a$;*
(ii) *M_u is compact from $L^{p,\alpha}_a$ to $L^{q,\beta}_a$;*
(iii) *$u \in L^{s,\delta}_a$, where $1/s = 1/q - 1/p$ and $\delta/s = \beta/q - \alpha/p$.*

Proof. By Theorem 3, (i) and (ii) are equivalent, and both are equivalent to the condition

$$B_{\mathrm{id},\alpha,\beta}(|u|^q) \in L^{p/(p-q),\alpha}. \tag{6}$$

Suppose (6) holds. From Lemma 8 it follows

$$B_{\mathrm{id},\alpha,\beta}(|u|^q)(a) = \int_D |\sigma_a'(w)|^{2+\alpha}|u(w)|^q\, dA_\beta(w) \geq C^{-1}|u(a)|^q(1-|a|^2)^{\beta-\alpha}.$$

We conclude that $|u(a)|^q(1-|a|^2)^{\beta-\alpha} \in L^{p/(p-q),\alpha}$, which is the same as $u \in L^{s,\delta}_a$.

Conversely, if $u \in L^{s,\delta}_a$, then by Hölder's inequality we easily get M_u is bounded from $L^{p,\alpha}_a$ to $L^{q,\beta}_a$. The proof is complete. □

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