Extremal Modular Lattices, McKay Thompson Series, Quadratic Iterations, and New Series for π

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CONTENTS

We give 20 new Ramanujan-type formulae and 20 quadratic approximations to π, parameterized by extremal modular lattices of minimal square norm 2 and 4, at the ten special levels corresponding to square-free orders of the Mathieu group M_{23}. An algorithm for uncovering rational relations between two given power series that we used to discover some of the explicit relations is also given. Explicit relations parameterized by modular lattices between McKay Thompson series for the group $\Gamma_0(\ell)^+$ and $\Gamma_0(2\ell)+p_j$, where p_j range over all odd primes dividing the special level ℓ, are uncovered.

1. INTRODUCTION

1.1 Some Results

In this paper, we prove 20 new Ramanujan type series formulae [Ramanujan 14] and 20 quadratically convergent algorithms for $1/\pi$ that are parameterized uniformly by the extremal strongly modular lattices (analogue of the E_8 and the Leech lattice) of minimal square norm 2 and 4 at the ten special levels ℓ, where $\sum_{d|\ell} d$ divides 24, corresponding to square-free orders of the Mathieu group M_{23}. For the Leech lattice, the new series formula is given by

$$\frac{1}{\pi} = \sqrt{8}\sum_{k=0}^{\infty}\left(\frac{9}{26\sqrt[6]{910}} + k\sqrt[6]{\frac{2610\sqrt{2}+3707}{7280}}\right) \times A_k\left(\frac{5\sqrt{2}-7}{512}\right)^k. \qquad (1\text{–}1)$$

Our formula for the extremal 7-modular Craig's lattice $A_6^{(2)}$ is

$$\frac{1}{\pi} = \sqrt{\frac{8}{7}}\sum_{k=0}^{\infty}(\alpha+\beta k)\,A_k\left(\frac{\sqrt{11+8\sqrt{2}}-\sqrt{2}-3}{8}\right)^k, \qquad (1\text{–}2)$$

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$$\alpha = \frac{(13+9\sqrt{2})\sqrt[6]{-4070183+2878054\sqrt{2}}}{14\sqrt[6]{14}},$$

$$\beta = \frac{1}{2\sqrt[3]{7}}\sqrt[6]{\frac{24717+17479\sqrt{2}-(5075+3591\sqrt{2})\sqrt{11+8\sqrt{2}}}{59216+41872\sqrt{2}-(12536+8864\sqrt{2})\sqrt{11+8\sqrt{2}}}},$$

where the A_k are rationals given by explicitly known polynomial recurrences arising from third-order ordinary differential equations satisfied by some weight 2 modular forms constructed from theta series of the lattices. The recurrences and their initial values are given in Tables 6 and 7. Some similar examples have been given in [Chan et al. 04]. Theorem 3.1 in Section 3 gives uniformly a similar formula for each of the 20 extremal modular lattices (listed in Table 2) where all constants including α and β are given explicitly in terms of invariants attached to the lattices.

The simplest of our new quadratic iterations is probably for the level 2 (note the powers of 2 in the constants) root lattice $D_4 = \{x \in \mathbf{Z^4} : \sum x_i \equiv 0 \mod 2\}$.

Iteration D_4: Let $\kappa_0 = 0, x_0 = (1+\sqrt{2})/2^3$, and set for $n = 1, 2, 3, \ldots$

$$x_n = \frac{1}{2^5}\left(\frac{1+2^3x_{n-1}}{\sqrt{1+2^4x_{n-1}}} - 1\right), \quad (1\text{–}3)$$

$$\kappa_n = \frac{2^n(2^5-2^3)x_n(1+2^4x_n)}{\sqrt{2}(1+2^3x_n)^2(1+2^5x_n)} + \frac{1+2^5x_n}{1+2^3x_n}\kappa_{n-1}, \quad (1\text{–}4)$$

then $\kappa_n \to 1/\pi$ quadratically.

Equivalent iterations, for the 3- and 7-extremal modular lattices of norm 2 (the two-dimensional hexagonal and [2,1;1,4] lattice) were given recently in [Chan et al. 03]. The analogous modular function in the D_4 case is now given by the Hauptmodul

$$X_2(\tau) = \left(\frac{\eta(4\tau)}{\eta(\tau)}\right)^8,$$

of $\Gamma_0(4)$. In fact, the iteration in (1–3) is given by $x_n = X_2(e^{-\pi 2^n/\sqrt{2}})$, and this is equivalent to one case of Elkies' construction of explicit modular towers for the modular curve $\mathbf{H}/\Gamma_0(4)$ [Elkies 01b]. Theorem 1.1 gives uniformly a quadratic iteration for each of the 20 extremal modular lattices, with explicit lattice interpretations for all the occurring terms.

In Section 6.1, we give a heuristic algorithm (Algorithm A) for discovering unknown rational relations between two given power series, which we use to discover the necessary algebraic relations. The typical situation is the application to q expansions of two modular functions (one of which is a Hauptmodul) on some genus zero subgroup of finite area. In this case, a rational relation is guaranteed and once found can be proven by appropriate rearrangement as equality between modular forms and the checking of sufficiently many coefficients, using, for example, the theorem of Sturm [Sturm 87]. We have found Algorithm A to be very useful in several other contexts.

In attempting to prove the relations necessary for Theorem 1.1 and Theorem 3.1, we were led naturally to consider Hauptmoduln on genus zero subgroups arising from the Moonshine [Conway and Norton 79]. Theorem 2.1 and Theorem 2.2 express the McKay Thompson series for $\Gamma_0(\ell)^+$ explicitly as rational functions of the McKay Thompson series for $G_\ell = \Gamma_0(2\ell) + p_j$, where p_j ranges over all odd primes dividing ℓ. It is very pleasing that all the relations (2–6)–(2–11) are parameterized uniformly by the modular lattices.

1.2 20 Special Modular Lattices

Let Λ_1 be the Leech lattice in R^{24} (see [Conway and Sloane 98, Chapter 10] for the explicit construction). The Mathieu group $M_{24} \subset Aut(\Lambda_1) = .O$ acts on Λ_1 via permutation of coordinates. The one point stabilizer subgroup M_{23} has 17 conjugacy classes but only 12 distinct cycle patterns (see column 3 of Table 1). Now let Λ_ℓ be the fixed sublattice of the Leech lattice by a conjugacy class of M_{23} of order ℓ. Clearly, the Λ_ℓ (last column, Table 1) is even integral with minimal norm 4, and dimension $D_\ell = 24\sigma_0(\ell)/\sigma_1(\ell)$ (column 4, Table 2), where $\sigma_k(\ell) = \sum_{d|\ell} d^k$ is the kth power of the divisor function. One can compute the Smith normal form of the Gram matrix of Λ_ℓ (using the GAP command `NormalFormIntMat` [GAP 05]) to see that its elementary divisors $\{d_1, \cdots, d_{D_\ell}\}$ agree exactly with the cycle pattern of the M_{23} conjugacy class. Now, since the glue group $\Lambda_\ell^*/\Lambda_\ell \cong \sum_{j=1}^{D_\ell} Z/d_j$ is exactly given by the elementary divisors (see [Griess 98, page 80 Lemma 8.13]), and since it is well known (and easily verified from Table 1) that the cycle patterns, arranged in order $d_1 \le d_2 \le \cdots \le d_{D_\ell}$, are balanced (see [Koike 84]), i.e., $d_j d_{D_\ell - j+1} = d_{D_\ell} = \ell, 1 \le j \le D_\ell$, we have $\sqrt{\ell}\Lambda_\ell^* \cong \Lambda_\ell$, which implies that the lattices are ℓ modular in the sense of [Quebbemann 95, Quebbemann 97]. In fact, the Λ_ℓ are extremal (densest packings in their genus) and strongly modular (i.e., the theta series are eigenforms on $\Gamma_0(\ell)$ for all the Atkin Lehner involutions) [Nebe 97, Nebe 02, Quebbemann 97, Rains and Sloane 98, Scharlau and Schulze-Pillot 99].

In this paper, we will restrict ourselves to only the ten square-free levels ℓ, which are exactly those ℓ with $\sigma_1(\ell)|24$. There is a second series of strongly modular

Classes	Order	Cycle Pattern	Fixed Sublattice
1	1	1^{24}	Leech
2	2	$1^8.2^8$	BW_{16}
3	3	$1^6.3^6$	K_{12}
4	4	$1^4.2^2.4^4$	$\Lambda_4, \dim = 14$
5	5	$1^4.5^4$	$Q_8(1)$
6	6	$1^2.2^2.3^2.6^2$	$G_2 \otimes F_2$
$7, 8$	7	$1^3.7^3$	$A_6^{(2)}$
9	8	$1^2.2^1.4^1.8^2$	$\Lambda_8, \dim = 6$
$10, 11$	11	$1^2.11^2$	$[4, 1, 0, -2; 1, 4, 2, 0; 0, 2, 4, 1; -2, 0, 1, 4]$
$12, 13$	14	$1^1.2^1.7^1.14^1$	$[4, 1, 0, -1; 1, 4, 1, 0; 0, 1, 4, 1; -1, 0, 1, 4]$
$14, 15$	15	$1^1.3^1.5^1.15^1$	$[4, 2, 2, 1; 2, 4, 1, 2; 2, 1, 6, 3; 1, 2, 3, 6]$
$16, 17$	23	$1^1.23^1$	$[4, 1; 1, 6]$

TABLE 1. Conjugacy classes of M_{23} and their fixed sublattice in Leech.

ℓ	k_1	d_ℓ	D_ℓ	Λ_ℓ^0	Λ_ℓ	T_ℓ	G_ℓ	x_0	Y_0
1	24	8	24	E_8	Leech	$2B$	2	$1/8$	$\frac{-7+5\sqrt{2}}{512}$
2	8	4	16	D_4	BW_{16}	$4C$	4	$\frac{1+\sqrt{2}}{8}$	$\frac{-4+3\sqrt{2}}{128}$
3	6	2	12	$H_3 = A_2$	K_{12}	$6C$	$6+3$	$1/2$	$\frac{-5+3\sqrt{3}}{32}$
5	4	4	8	$QQF.4.a$	$Q_8(1)$	$10B$	$10+5$	$\frac{1+\sqrt{5}}{4}$	$\frac{-3+\sqrt{10}}{8}$
6	2	4	8	$H_3 + 2H_3$	$G_2 \otimes F_4$	$12E$	$12+3$	$\frac{1+\sqrt{3}}{4}$	$\frac{-2+\sqrt{6}}{16}$
7	3	2	6	H_7	$A_6^{(2)}$	$14B$	$14+7$	1	$\frac{-3-\sqrt{2}+\sqrt{11+8\sqrt{2}}}{8}$
11	2	2	4	H_{11}	Λ_{11}	$22B$	$22+11$	y_{11} *	$\frac{-3+\sqrt{11}}{4}$
14	1	4	4	$H_7 + 2H_7$	Λ_{14}	$28C$	$28+7$	$\frac{\sqrt{2}+\sqrt{2+4\sqrt{2}}}{4}$	$\frac{-1+\sqrt{2}}{4}$
15	1	4	4	$H_3 + 5H_3$	Λ_{15}	$30C$	$30+3, 5$	1	$\frac{-2+\sqrt{6}}{4}$
23	1	2	2	H_{23}	$[4, 1; 1, 6]$	$46A$	$46+23$	$\frac{12+\sqrt[3]{(108+12\sqrt{69})^2}}{6\sqrt[3]{108+12\sqrt{69}}}$	$\frac{-2-\sqrt{2}+\sqrt{6+8\sqrt{2}}}{4}$

$* \; y_{11} = \sqrt[3]{\frac{1+2\sqrt{11/108}}{4}} + \sqrt[3]{\frac{1-2\sqrt{11/108}}{4}}, \quad H_\ell = \begin{pmatrix} 2 & 1 \\ 1 & (\ell+1)/2 \end{pmatrix}$

TABLE 2. Data for the extremal modular lattices of minimal norm 2 and 4.

lattices Λ_ℓ^0 (the level ℓ analogue of E_8) of minimal norm 2 and dimension d_ℓ [Rains and Sloane 98, Nebe 02] that are also extremal. The lattices, as well as their dimensions, are listed in columns 3–6 of Table 2. The Gram matrices of the lattices can be explicitly computed as stabilizer lattices using GAP. Surprisingly, we always obtain essentially the same standard Gram matrix as that given in [Harada and Lang 90, Kondo and Tasaka 86, Kondo and Tasaka 87] with small coefficients after applying LLL reduction.

The cycle patterns of the ten special conjugacy classes admit a simple description multiplicatively as $\prod_{d|\ell} d^{24/\sigma_1(\ell)}$, and one can associate with it a weight $D_\ell/2$, level ℓ cusp form

$$\Delta_\ell(\tau) := \prod_{d|\ell} \eta(d\tau)^{k_1}, \quad k_1 = 24/\sigma_1(\ell), \tag{1–5}$$

that, together with $\theta_{\Lambda_\ell}(\tau)$ or $\theta_{\Lambda_\ell^0}(\tau)$, generates the polynomial ring of modular forms of the theta series of strongly modular lattices [Chua and Solé 04, Nebe 02, Quebbemann 95, Quebbemann 97], and [Scharlau and Schulze-Pillot 99]. Note that the integer k_1 gives the number of fixed points of an element of M_{23} of order ℓ and the same special levels also occur in Munkai's work on automorphisms of K_3 surfaces [Munkai 88].

1.3 Quadratic Iterations for π

Now let

$$X_\ell(\tau) = \frac{\Delta_\ell(2\tau)}{\Delta_\ell(\tau)}, \tag{1–6}$$

be the level 2ℓ modular functions. In [Elkies 01a, Elkies 01b], the following relations, quadratic in two variables between $X_\ell(\tau)$ and $Y_\ell(\tau) = X_\ell(2\tau)$, were derived in the context of explicit modular towers:

$$X^2 = Y + 2k_1XY + 2^{D_\ell/2}XY^2, \qquad \text{for } \ell \text{ odd}, \tag{1–7}$$

$$X^2 = Y(1+2k_1X)(1+2k_1Y), \qquad \text{for } \ell \text{ even}. \tag{1–8}$$

A uniform proof of the above relations was given using the fact that $\Delta_\ell(\tau)$ are all Hecke eigenforms in [Chua and Solé 04]. The cubic and septic case ($\ell = 3, 7$) of the above were used to give two new quadratic algorithms for $1/\pi$ in [Chan et al. 03]. We now generalize it to all the 20 special lattices.

Throughout the paper, Λ will denote one of the extremal modular lattices Λ_ℓ or Λ_ℓ^0 of minimal norm 4 or 2 for one of the ten special levels $\ell \in \{1, 2, 3, 5, 6, 7, 11, 14, 15, 23\}$. Also, when the context is clear, we will write $X = X(\tau)$ for the modular functions $X_\ell(\tau)$, and $Y(\tau) = X(2\tau)$. We will also sometimes abuse notation and write $X(e^{2\pi i\tau})$ for $X(\tau)$.

Theorem 1.1. *Let ℓ be one of the ten special levels, Λ be the extremal ℓ modular lattice Λ_ℓ or Λ_ℓ^0, and $\mathcal{R}_\Lambda(X), W_\Lambda(X)$ be the lattice invariants defined explicitly in Section 2.3. The level 2ℓ modular functions $1/X_\ell(\tau) = \Delta_\ell(\tau)/\Delta_\ell(2\tau)$ are McKay Thompson series for the group $\Gamma_0(2\ell) + p_j$, where the p_j range over the odd primes dividing ℓ. Setting $x_n = X_\ell(e^{-\pi 2^n/\sqrt{\ell}})$, we have the following quadratic iterations:*

Iteration Λ: *Let $\kappa_0 = 0$ and $x_0 = X_\ell(e^{-\pi/\sqrt{\ell}})$, which is given explicitly in the second-to-last column of Table 2. Set for $n = 1, 2, 3, \ldots$*

$$x_n = \begin{cases} \frac{-(1+2k_1x_{n-1}) + \sqrt{(1+2k_1x_{n-1})^2 + 2^{D_\ell/2+2}x_{n-1}^3}}{2^{D_\ell/2+1}x_{n-1}}, & \ell \text{ odd}, \\ \frac{1}{4k_1}\left(\sqrt{\frac{1+2k_1x_{n-1}+8k_1x_{n-1}^2}{1+2k_1x_{n-1}}} - 1\right), & \ell \text{ even}, \end{cases} \tag{1–9}$$

and

$$\kappa_n = \frac{2^n x_n}{\sqrt{\ell}}\mathcal{R}'_\Lambda(x_n)\sqrt{W_\Lambda(x_n)} + \mathcal{R}_\Lambda(x_n)\kappa_{n-1}, \tag{1–10}$$

then $\kappa_n \to 1/\pi$ quadratically.

In Section 6.2, we will implement using PARI-GP, the quadratic iterations above as well as the partial sums in Theorem 3.1 to verify the convergence to $1/\pi$. The number of iterations needed for accuracy of 50 decimal digits is given. This gives a check on our formulae. The program code, including the rationality detection Algorithm A, is available on request.

2. MCKAY THOMPSON SERIES RELATED TO MODULAR LATTICES

2.1 McKay Thompson Series as j Functions

Felix Klein's absolute invariant function

$$j(\tau) := \frac{E_4(\tau)^3}{\Delta(\tau)} = \frac{(1+240\sum\sigma_3(n)q^n)^3}{q\prod_{n=1}^\infty(1-q^n)^{24}} = \frac{1}{q} + 744 + c_1q + c_2q^2 + \cdots, \tag{2–1}$$

is a Hauptmodul on the compactification of $\mathbf{H}/PSL_2(Z)$ and contains a large amount of arithmetic information in elliptic modular theory. We note that we can write $j(\tau) = \theta_{\Lambda_1^0}(\tau)^3/\Delta_1(\tau)$ and this leads to our next definition.

Let ℓ be a special level and Λ be Λ_ℓ or Λ_ℓ^0, we define

$$j_\Lambda(\tau) := \frac{\theta_\Lambda(\tau)^{D_\ell/\dim(\Lambda)}}{\Delta_\ell(\tau)}, \tag{2–2}$$

so that

$$j_{\Lambda_\ell^0}(\tau) = \frac{\theta_{\Lambda_\ell^0}(\tau)^{D_\ell/d_\ell}}{\Delta_\ell(\tau)}, \quad j_{\Lambda_\ell}(\tau) = \frac{\theta_{\Lambda_\ell}(\tau)}{\Delta_\ell(\tau)}. \tag{2–3}$$

The modular functions $j_{\Lambda_\ell}(\tau), j_{\Lambda_\ell^0}(\tau)$ are McKay Thompson series for the Monster conjugacy class ℓA in the notation of [Conway and Norton 79], so they are Hauptmoduln on the group $\Gamma_0(\ell)^+$. It is well known that $\theta_{\Lambda_\ell^0}(\tau), \Delta_\ell(\tau)$ generate the space of modular forms containing the theta series of even strongly ℓ modular lattices. Since Λ_ℓ has no vector of squared norm 2, we have

$$\theta_{\Lambda_\ell}(\tau) = \theta_{\Lambda_\ell^0}(\tau)^{D_\ell/d_\ell} - \gamma_\ell\Delta_\ell(\tau), \quad \gamma_\ell = \#\left(\Lambda_\ell^0\right)_2 D_\ell/d_\ell, \tag{2–4}$$

where $\#\left(\Lambda_\ell^0\right)_2$ is the number of roots of Λ_ℓ^0. It follows that

$$j_{\Lambda_\ell^0}(\tau) = j_{\Lambda_\ell}(\tau) + \gamma_\ell. \tag{2–5}$$

Since it is easy to see that

$$j_{\Lambda_\ell}(\tau) = 1/q + k_1 + O(q^2),$$

we have

$$j_{\Lambda_\ell^0}(\tau) = \frac{1}{q} + \left(\frac{24}{\sigma_1(\ell)} + \#\left(\Lambda_\ell^0\right)_2 D_\ell/d_\ell\right) + O(q),$$

and this gives a geometric interpretation, in the special case $\ell = 1$, for the number 744 occurring in Klein's invariant:

$$744 = \dim(\text{Leech}) + (\text{Number of roots of } E_8) \times \dim(\text{Leech})/\dim(E_8).$$

(One can give a similar but somewhat more complicated geometric interpretation of the positive integer $c_1 - 1$, which gives the least dimension of nontrivial irreducible representation of the Monster group in the case $\ell = 1$.)

Now let $X = X_\ell(\tau)$ be as given in (1–2). The functions $T_\ell(\tau) = 1/X_\ell(\tau)$ are also McKay Thompson series of level 2ℓ [Conway and Norton 79] and their associated Monster classes are listed in Table 2. From [Conway and Norton 79], their invariance groups are $G_\ell = \Gamma_0(2\ell) + p_j$ where the p_j range over the odd primes dividing ℓ. We note that $\Gamma_0(2\ell) \subset G_\ell \subset \Gamma_0(2\ell)^+$.

2.2 Relations between McKay Thompson Series

Since $G_\ell \subset \Gamma_0(\ell)^+$, the functions $j_{\Lambda_\ell}(\tau), j_{\Lambda_\ell^0}(\tau)$ are modular functions on G_ℓ and hence must be rational functions of $1/X_\ell$. A computation using Algorithm A gives us Theorem 2.1.

Theorem 2.1. *$j_{\Lambda_\ell}(\tau), j_{\Lambda_\ell^0}(\tau)$ are rational functions in $X = X_\ell$. More explicitly, we have*

$$j_\Lambda(\tau) = \begin{cases} \frac{p_\Lambda(X)}{X}, & \ell \text{ odd,} \\ \frac{p_\Lambda(X)}{X(1+2k_1X)}, & \ell \text{ even,} \end{cases} \tag{2–6}$$

where the $p_\Lambda(X)$ are explicitly known polynomials given by

$$p_{\Lambda_\ell}(\tau) = \begin{cases} 1 + 2k_1X + 2k_1 2^{D_\ell/2}X^2 + 2^{D_\ell}X^3, & \\ & \ell \text{ odd,} \\ 1 + 4k_1X + (k_1+1)2^{D_\ell/2+1}X^2 + 2^{D_\ell+1}X^3 & \\ \quad +2^{5D_\ell/4}X^4, & \ell \text{ even,} \end{cases} \tag{2–7}$$

and

$$p_{\Lambda_\ell^0}(X) = \begin{cases} p_{\Lambda_\ell}(X) + \gamma_\ell X, & \ell \text{ odd,} \\ p_{\Lambda_\ell}(X) + \gamma_\ell X(1+2k_1X), & \ell \text{ even.} \end{cases} \tag{2–8}$$

Proof: Since $1/X_\ell$ is a Hauptmodul on G_ℓ and $j_\Lambda(\tau)$ are modular on $\Gamma_0(\ell)^+ \supset G_\ell$, we know in advance that $j_\Lambda(\tau)$ is rational in X_ℓ. Applying Algorithm A gives us the explicit form of the rational functions in each individual case. By inspecting the patterns in the numbers, we arrive at the explicit form on the right-hand side of (2–6) and (2–7). We now prove the formula by matching enough coefficients in an appropriate finite-dimensional space of modular forms. For the case where ℓ is odd, both $\theta_{\Lambda_\ell}(\tau)$ and $p_{\Lambda_\ell}(\tau)\Delta_\ell(\tau)/X_\ell(\tau)$ are modular forms of weight $D_\ell/2$ on $\Gamma_0(2\ell)$ with the same quadratic character and integral q expansion. Since we know the index of $\Gamma_0(2\ell)$, we can apply the theorem in [Sturm 87] to conclude equality based on checking finitely many coefficients. Equation (2–8) follows from (2–5). □

We have a similar result for level 2ℓ:

Theorem 2.2. *$j_{\Lambda_\ell}(2\tau), j_{\Lambda_\ell^0}(2\tau)$ are rational functions in $X = X_\ell$. More explicitly, we have*

$$j_\Lambda(2\tau) = \begin{cases} \frac{q_\Lambda(X)}{X^2}, & \ell \text{ odd,} \\ \frac{q_\Lambda(X)}{X^2(1+2k_1X)}, & \ell \text{ even,} \end{cases} \tag{2–9}$$

where the $q_\Lambda(X)$ are explicitly known polynomials given by

$$q_{\Lambda_\ell}(\tau) = \begin{cases} 1 + 2k_1X + 2k_1X^2 + 2^{D_\ell/2}X^3, & \ell \text{ odd,} \\ 1 + 4k_1X + (k_1+1)2^{D_\ell/4+1}X^2 + 2^{D_\ell/2+1}X^3 & \\ \quad +2^{3D_\ell/4}X^4, & \ell \text{ even,} \end{cases} \tag{2–10}$$

and

$$q_{\Lambda_\ell^0}(X) = \begin{cases} q_{\Lambda_\ell}(X) + \gamma_\ell X^2, & \ell \text{ odd,} \\ q_{\Lambda_\ell}(X) + \gamma_\ell X^2(1+2k_1X), & \ell \text{ even.} \end{cases} \tag{2–11}$$

We note that there are simplifications in the cases given in Lemma 2.3.

Lemma 2.3. *For $\ell = 1, 3, 7$,*

$$p_{\Lambda_\ell^0}(X) = (1 + 2^{D_\ell/3}X)^3, \quad q_{\Lambda_\ell^0}(X) = (1 + 2^{D_\ell/6}X)^3. \tag{2–12}$$

Proof: This follows from combining the fact that $2k_1/3 = 2^{D_\ell/6}$ in these cases with (2–7), (2–8), (2–10), and (2–11). □

2.3 Two Lattice Invariants

It follows from (2–7), (2–10), and (2–2) that

$$\frac{\theta_\Lambda(\tau)}{\theta_\Lambda(2\tau)} = \left(\frac{j_\Lambda(\tau)}{Xj_\Lambda(2\tau)}\right)^{\dim(\Lambda)/D_\ell} = \left(\frac{p_\Lambda(X)}{q_\Lambda(X)}\right)^{\dim(\Lambda)/D_\ell}. \tag{2–13}$$

We observe by direct calculation the results in Lemma 2.4.

Lemma 2.4. *$\frac{p_\Lambda}{q_\Lambda}$ are ratios of cubic (ℓ odd) or quartic (ℓ even) irreducibles (over $\mathbf{R}$) in X except for the following* *cases where there are cancellations:*

$$\frac{p_{\Lambda_{15}}(X)}{q_{\Lambda_{15}}(X)} = \frac{1+8X^2}{1+2X^2}, \tag{2–14}$$

$$\frac{p_{\Lambda_\ell^0}(X)}{q_{\Lambda_\ell^0}(X)} = \begin{cases} \left(\frac{1+2^{D_\ell/3}X}{1+2^{D_\ell/6}X}\right)^3, & \text{if } \ell = 1,3,7, \\ \left(\frac{1+32X}{1+8X}\right)^4, & \text{if } \ell = 2 \\ \left(\frac{1+8X}{1+2X}\right)^2, & \text{if } \ell = 5,6, \\ \frac{1+4X+8X^2}{1+2X+2X^2}, & \text{if } \ell = 14. \end{cases} \tag{2–15}$$

As we will be considering the ratio of weight 2 forms, we define our first lattice invariant. Define

$$\mathcal{R}_\Lambda(X) := \left(\frac{p_\Lambda(X)}{q_\Lambda(X)}\right)^{4/D_\ell}. \tag{2–16}$$

$\mathcal{R}_\Lambda(X)$ is not, in general, rational in X but we observe the following:

Lemma 2.5. *$\mathcal{R}_{\Lambda_1^0}(X)^2$ is rational in X and for $\ell > 1$, $\mathcal{R}_{\Lambda_\ell^0}(X)$ is rational in X. We also have*

$$\mathcal{R}_{\Lambda_\ell}(X) = \left(\frac{p_{\Lambda_\ell(X)}q_{\Lambda_\ell^0(X)}}{p_{\Lambda_\ell^0(X)}q_{\Lambda_\ell(X)}}\right)^{4/D_\ell} \mathcal{R}_{\Lambda_\ell^0}(X), \tag{2–17}$$

so that $\mathcal{R}_{\Lambda_\ell}(X)^{D_\ell/4}$ is rational in X.

We will now define our second lattice invariant rational $W_\Lambda(X)$. First set

$$Z_\Lambda(q) := \theta_\Lambda(\tau)^{4/\dim(\Lambda)}, \tag{2–18}$$

which transforms like a weight 2 modular form on $\Gamma_0(\ell)^+$, and note that by (2–13) and (2–16), we have

$$\frac{Z_\Lambda(q)}{Z_\Lambda(q^2)} = \mathcal{R}_\Lambda(X(q)). \tag{2–19}$$

We also have the following lemma:

Lemma 2.6.

$$\left(\frac{q}{Z_{\Lambda_\ell^0}(q)}\frac{dX_\ell}{dq}\right)^2 = X_\ell(q)^2 W_{\Lambda_\ell^0}(X_\ell), \tag{2–20}$$

where $W_{\Lambda_\ell^0}(X)$ is a rational function in X.

Proof: Since X_ℓ is a modular function on G_ℓ, $q\frac{dX_\ell}{dq}$ is a weight 2 form on G_ℓ so that $(q\frac{dX_\ell}{dq})^2$ is weight 4 modular on G_ℓ with the trivial character. Since the $\theta_{\Lambda_\ell^0}(\tau)$ are modular forms of weight $\dim(\Lambda_\ell^0)/2$ on $\Gamma_0(\ell)^+$ with possibly a quadratic character and $\dim(\Lambda_\ell^0)$ divides 8, $Z_{\Lambda_\ell^0}(q)^2 = \theta_{\Lambda_\ell^0}^{8/\dim\Lambda_\ell^0}(\tau)$ is also a weight 4 holomorphic form on $\Gamma_0(\ell)^+ \supset G_\ell$ with the trivial character. The left-hand side of Equation (2–20) must be a rational function of $1/X_\ell$ and as it clearly has a double zero at ∞, (2–20) must hold for some rational W. □

For the extremal modular lattices of minimum norm 4, $Z_{\Lambda_\ell}(q)^2 = \theta_{\Lambda_\ell}^{8/\dim\Lambda_\ell}(\tau)$ is only a fractional $(8/D_\ell)$ power of $\theta_\Lambda(\tau)$ for $\ell = 1,2,3,7$. A similar argument as in the proof of Lemma 2.6 gives the next lemma, where we use Algorithm A to determine the explicit form of W_{Λ_ℓ} and λ_ℓ.

Lemma 2.7.

$$\left(\frac{q}{Z_{\Lambda_\ell}(q)}\frac{dX_\ell}{dq}\right)^2 = X_\ell(q)^2 W_{\Lambda_\ell}(X_\ell), \tag{2–21}$$

where $W_{\Lambda_\ell}(X)^{\lambda_\ell}$ is a rational function in X, and $\lambda_\ell = 3,2,3,3$ for $\ell = 1,2,3,7$ and 1 otherwise.

Also Equations (2–16) and (2–19)–(2–21) clearly imply Lemma 2.8.

Lemma 2.8.

$$W_{\Lambda_\ell}(X) = \left(\frac{p_{\Lambda_\ell^0}}{p_{\Lambda_\ell}}(X)\right)^{8/D_\ell} W_{\Lambda_\ell^0}(X), \tag{2–22}$$

which gives $W_{\Lambda_\ell}(X)$ in term of $W_{\Lambda_\ell^0}(X)$.

We also observe by direct calculations the following general patterns for W_Λ:

Lemma 2.9.

$$W_\Lambda(X) = \begin{cases} \left(\frac{r_\ell(X)}{p_\Lambda(X)}\right)^{8/D_\ell}, & \ell \text{ even}, \\ \left(\frac{r_\ell(X)(1+2k_1X)}{p_\Lambda(X)}\right)^{8/D_\ell}, & \ell \text{ odd}, \end{cases} \tag{2–23}$$

$$\text{where} \begin{cases} r_\ell(X) \text{ is a polynomial of degree } 3, \\ \qquad \text{if } 8|D_\ell, \text{ or } \ell = 1,2,5,6, \\ r_\ell(X)^2 \text{ is a polynomial of degree } 6, \\ \qquad \text{if } 4|D_\ell, \text{ or } \ell = 3,11,14,15, \\ r_\ell(X)^4 \text{ is a polynomial of degree } 12, \\ \qquad \text{if } 2|D_\ell, \text{ or } \ell = 7,23. \end{cases}$$

Explicit forms of the lattice invariants $\mathcal{R}_{\Lambda_\ell^0}(X)$, and $W_{\Lambda_\ell^0}(X)$ are given in Table 3. The invariants of Λ_ℓ can be computed from these using (2–17) and (2–22).

ℓ	Z	$W_{\Lambda_\ell^0}(X)$	$\mathcal{R}_{\Lambda_\ell^0}(X)$	$\alpha_{\Lambda_\ell^0}$	$\beta_{\Lambda_\ell^0}$
1	$\sqrt{\theta_{E8}}$	$\frac{1+64X}{1+256X}$	$\sqrt{\frac{1+256X}{1+16X}}$	$\frac{3\sqrt{10}}{100}$	$\frac{\sqrt{9-4\sqrt{2}}}{2(\sqrt{10}-\sqrt{5})}$
2	θ_{D_4}	$\left(\frac{1+16X}{1+32X}\right)^2$	$\frac{1+32X}{1+8X}$	$\frac{1}{9}$	$\frac{\sqrt{2}}{3}+\frac{1}{2}$
3	$\theta_{A_2+A_2}$	$\frac{1+4X}{1+16X}$	$1/W(X)$	$\frac{\sqrt{2}}{12}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$
5	$\theta_{QQF.4.a}$	$\frac{1+12X+1+16X^2}{(1+8X)^2}$	$\frac{1+8X}{1+2X}$	$\frac{\sqrt{5}}{18}$	$\frac{5\sqrt{2}+2\sqrt{5}}{12}$
6	$\theta_{H_3+2H_3}$	$\left(\frac{1+4X}{1+8X}\right)^2$	$\frac{1+8X}{1+2X}$	$\frac{1}{9}$	$\frac{1}{\sqrt{6}}+\frac{1}{2}$
7	$\theta_{H_7+H_7}$	$\frac{(1+X)(1+8X)(1+5X+8X^2)}{(1+4X)^4}$	$\left(\frac{1+4X}{1+2X}\right)^2$	$\sqrt{7}\frac{\sqrt{11+8\sqrt{2}}}{48+34\sqrt{2}}$	$\sqrt{7}\frac{\sqrt{9+7\sqrt{2}-(1+\sqrt{2})x_7}}{\sqrt{2}(1+\sqrt{2}-x_7)^2}*$
11	$\theta_{H_{11}+H_{11}}$	$\frac{1+4X+8X^2+4X^3}{1+8X+16X^2+16X^3}$	$\frac{1+8X+16X^2+16X^3}{1+4X+8X^2+4X^3}$	$\frac{3\sqrt{2}}{44}$	$\sqrt{\frac{18+5\sqrt{11}}{44}}$
14	$\theta_{H_7+2H_7}$	$\frac{P_{14}(X)}{(1+2X+4X^2)^2(1+4X+8X^2)^2}$ $***$	$\frac{1+4X+8X^2}{1+2X+2X^2}$	$\frac{4\sqrt{7}}{75}$	$\frac{\sqrt{14}+2\sqrt{7}}{10}$
15	$\theta_{H_3+5H_3}$	$\frac{(1+X)(1+4X)(1+4X^2)(1+X+4X^2)}{(1+8X+8X^2+16X^3)^2}$	$\frac{1+8X+8X^2+16X^3}{1+2X+8X^2+4X^3}$	$\frac{\sqrt{10}}{108}$	$\frac{\sqrt{5}(3\sqrt{2}-\sqrt{3})}{18(\sqrt{6}-2)}$
23	$\theta_{H_{23}+H_{23}}$	$\frac{P_{23}(X)}{(1+4X+4X^2+4X^3)^4}$ $****$	$\left(\frac{1+4X+4X^2+4X^3}{1+2X+4X^2+2X^3}\right)^2$	$\sqrt{23}\frac{\sqrt{147+104\sqrt{2}}}{54(24+17\sqrt{2})}$	x_{23} $**$

$*$ $x_7 = \sqrt{11+8\sqrt{2}}$,

$**$ $x_{23} = \dfrac{\sqrt{23}\sqrt{2(671\sqrt{2}+949)-(454+321\sqrt{2})\sqrt{6+8\sqrt{2}}}}{18(6(7+5\sqrt{2})-(10-7\sqrt{2})\sqrt{6+8\sqrt{2}}}$

$***$ $P_{14}(X) = (1+2X)^2(1+X+2X^2)(1+3X+4X^2) \times (1+2X+8X^2)$

$****$ $P_{23}(X) = (1+X+2X^2+X^3)(1+4X+4X^2+8X^3) \times (1+5X+14X^2+25X^3+28X^4+20X^5+8X^6)$

TABLE 3. Data for extremal lattices of minimal norm 2.

2.4 Special Values of $X_\ell(q)$

We derive here the special values

$$x_0 = X_\ell(e^{-\pi/\sqrt{\ell}}),$$
$$X_0 = X_\ell(e^{-2\pi/\sqrt{2\ell}}), \qquad (2–24)$$
$$Y_0 = X_\ell(e^{-4\pi/\sqrt{2\ell}}),$$

needed in Theorems 1.1 and 3.1. First, note that the infinite product expansion for $\eta(\tau)$ implies that for X_ℓ, we have

$$X_\ell(q) = \frac{q\prod_{d|\ell}\prod_{n=1}^{\infty}(1-q^{2dn})^{1/\sigma_1(\ell)}}{\prod_{d|\ell}\prod_{n=1}^{\infty}(1-q^{dn})^{1/\sigma_1(\ell)}}, \qquad (2–25)$$

which implies x_0, X_0, Y_0 are all positive real numbers.

Now, noting that the modular forms $\Delta_\ell(\tau)$ transform like the ℓ modular theta series $\theta_{\Lambda_\ell}(\tau)$ and using (3–2) or, alternatively, the well known eta transformation formula $\eta(-1/\tau) = \sqrt{\frac{\tau}{i}}\eta(\tau)$ [Siegel 54], we have

$$t^{D_\ell/4}\Delta_\ell(e^{-2\pi\sqrt{t/\ell}}) = \Delta_\ell(e^{-2\pi/\sqrt{\ell t}}). \qquad (2–26)$$

Setting $t=2$ in (2–26) gives immediately

$$X_0 = 2^{-D_\ell/4}. \qquad (2–27)$$

Since $X_0 = X(q_0)$ and $Y_0 = X(q_0^2)$ with $q_0 = e^{-2\pi/\sqrt{2\ell}}$, (X_0, Y_0) must satisfy (1–7) and (1–8). Substituting in (2–27) and using the positivity of Y_0 determines Y_0 uniquely as

$$Y_0 = \begin{cases} \dfrac{-(2^{D_\ell/4}+2k_1)+\sqrt{(2^{D_\ell/4}+2k_1)^2+2^{D_\ell/4+2}}}{2^{D_\ell/4+1}}, & \ell \text{ odd}, \\ \dfrac{-(2^{D_\ell/4}+2k_1)+\sqrt{(2^{D_\ell/4}+2k_1)^2+8k_1(1+2^{1-D_\ell/4}k_1)}}{4k_1(2^{D_\ell/4}+2k_1)}, & \ell \text{ even}. \end{cases} \qquad (2–28)$$

For x_0, we have Lemma 2.10.

Lemma 2.10. *x_0 is the unique positive root of the following cubic (ℓ odd) or quartic (ℓ even):*

$$x^3 - 2^{1-D_\ell/2}k_1x - 2^{1-D_\ell/2}, \qquad \ell \text{ odd}, \quad (2–29)$$
$$x^4 - 2^{1-D_\ell/2}k_1x^2 - (2^{-D_\ell/2} + 2^{2-D_\ell}k_1^2)x - 2^{1-D_\ell}k_1, \qquad \ell \text{ even}. \quad (2–30)$$

Proof: We set $\beta_n = \Delta_\ell(e^{-n\pi/\sqrt{\ell}})$ and set $t=4$ in (2–26) to get

$$2^{-D_\ell/2} = \frac{\beta_4}{\beta_1} = \frac{\beta_4\beta_2}{\beta_2\beta_1} = y_0x_0, \qquad (2–31)$$

where $y_0 = X_\ell(q_0^2), x_0 = X_\ell(q_0), q_0 = e^{-\pi/\sqrt{\ell}}$, so that (x_0, y_0) must also satisfy (1–7) and (1–8). These together with (2–31) give (2–29) and (2–30). Now (2–29) and (2–30) clearly have some positive root and by Descartes' rule of sign, the positive root is unique, which must be x_0. $\square$

Explicit values of x_0 computed from (2–29) and (2–30) using MAPLE are given in Table 2. We note, incidentally, by Equation (2–29) that when $\ell = 23$, x_0 is the positive root of $x^3 - x - 1$ so that (see [Serre 02]) it generates over $Q(\sqrt{-23})$ its Hilbert class field, whose Artin L-function is associated with the dihedral weight 1 form $\Delta_{23}(\tau) = \eta(\tau)\eta(23\tau)$.

3. EXTREMAL MODULAR LATTICES AND RAMANUJAN TYPE FORMULAE FOR π

Let Λ be an n-dimensional ℓ modular lattice, $q = e^{2\pi i\tau}$, and let

$$\theta_\Lambda(\tau) := \sum_{x\in\Lambda} q^{\langle x,x\rangle/2}$$

be its theta series. The Jacobi inversion formula [Conway and Sloane 98] states that we have

$$\theta_{\Lambda^*}(\tau) = \sqrt{\det(\Lambda)}\left(\frac{i}{\tau}\right)^{n/2}\theta_\Lambda(-1/\tau). \tag{3–1}$$

By the ℓ modularity [Quebbemann 95], we have $\det(\Lambda) = \ell^{n/2}$ and $\theta_{\Lambda^*}(\tau) = \theta_{\Lambda/\sqrt{\ell}}(\tau)$, so that (3–1) implies

$$\theta_\Lambda(\tau/\ell) = \ell^{n/4}\left(\frac{i}{\tau}\right)^{n/2}\theta_\Lambda(-1/\tau). \tag{3–2}$$

Setting $\tau = i\sqrt{t\ell}$ in (3–2) gives immediately, for any $t > 0$, that

$$t^{n/4}\theta_\Lambda(i\sqrt{t/\ell}) = \theta_\Lambda(i/\sqrt{\ell t}). \tag{3–3}$$

Now, recall by (2–18), we have $Z_\Lambda(q) := \theta_\Lambda(\tau)^{4/n}$, which transforms like a weight 2 modular form, and (3–3) gives

$$tZ_\Lambda(e^{-2\pi\sqrt{\frac{t}{\ell}}}) = Z_\Lambda(e^{-\frac{2\pi}{\sqrt{\ell t}}}). \tag{3–4}$$

Taking the logarithmic derivative of (3–4) gives us

$$\frac{\sqrt{\ell t}}{\pi} = q\frac{Z'_\Lambda}{Z_\Lambda}(q) + tq^t\frac{Z'_\Lambda}{Z_\Lambda}(q^t) \text{ at } q = e^{-\frac{2\pi}{\sqrt{\ell t}}}. \tag{3–5}$$

Also logarithmically differentiating (2–19) with respect to q gives

$$q\frac{\mathcal{R}'_\Lambda}{\mathcal{R}_\Lambda}(X(q))X'(q) = q\frac{Z'_\Lambda}{Z_\Lambda}(q) - 2q^2\frac{Z'_\Lambda}{Z_\Lambda}(q^2). \tag{3–6}$$

Fixing $t = 2$ in (3–5) and substituting into (3–6) at $q = q_0 = e^{\frac{-2\pi}{\sqrt{2\ell}}}$, gives us an expression of π in terms of lattice invariants:

$$\frac{\sqrt{2\ell}}{\pi} = \frac{q_0\mathcal{R}'_\Lambda}{\mathcal{R}_\Lambda}(X(q_0))X'(q_0) + 4q_0^2\frac{Z'_\Lambda}{Z_\Lambda}(q_0^2) \text{ at } q_0 = e^{\frac{-2\pi}{\sqrt{2\ell}}}. \tag{3–7}$$

We now compute the right-hand side of (3–7) explicitly for the 20 extremal modular lattices. Lemma 2.6 and 2.7 give

$$q_0X'(q_0) = X(q_0)\sqrt{W_\Lambda(X(q_0))}Z_\Lambda(q_0), \tag{3–8}$$

while (2–19) and (3–8) give

$$q_0\frac{\mathcal{R}'_\Lambda}{\mathcal{R}_\Lambda}(X(q_0))X'(q_0) = X(q_0)\sqrt{W_\Lambda(X(q_0))}\mathcal{R}'_\Lambda(X(q_0))Z_\Lambda(q_0^2). \tag{3–9}$$

Since both $Z_\Lambda(q)$ and $X(q)$ have leading coefficient unity in their q expansions, we can expand

$$Z_\Lambda(q) = \sum_{k=0}^{\infty} A_k X(q)^k. \tag{3–10}$$

Lemmas 2.6 and 2.7 give

$$q\frac{Z'_\Lambda}{Z_\Lambda}(q) = \sum_{k=0}^{\infty} kA_kX(q)^k\sqrt{W_\Lambda(X(q))}. \tag{3–11}$$

Putting (3–9), (3–10), and (3-11) in (3–8) gives

$$\frac{\sqrt{2\ell}}{\pi} = \sum_{k=0}^{\infty}\left(X_0\sqrt{W_\Lambda(X_0)}\mathcal{R}'_\Lambda(X_0) + 4k\sqrt{W_\Lambda(X_0)}\right) \times A_kY_0^k, \tag{3–12}$$

where $X_0 = X_\ell(q_0) = X_\ell(e^{-2\pi/\sqrt{2\ell}})$ and $Y_0 = X(q_0^2) = X_\ell(e^{-2\pi\sqrt{2/\ell}})$. This gives our final formula stated as Theorem 3.1 with the explicit constants

$$\alpha_\Lambda = X_0\sqrt{W_\Lambda(X_0)}\mathcal{R}'_\Lambda(X_0)/4, \quad \beta_\Lambda = \sqrt{W_\Lambda(X_0)}. \tag{3–13}$$

It remains to determine the A_k explicitly. Since Z_Λ is weight 2 modular and X_ℓ is a modular function on G_ℓ, a standard result [Stiller 84] implies Z_Λ must satisfy a third-order ordinary differential equation (ODE) in X_ℓ that translates to a polynomial recurrence defining A_k from its initial terms. We compute the ODE and the recurrences explicitly in Section 5. It will be more interesting to determine the A_k explicitly (solving the recurrences). Computations using the Mathematica package Hyper show that these sequences are not hypergeometric in the sense of [Petkovsek et al. 96]. Note that since the lattices are even integral, when $\dim(\Lambda)$ divides 8, which occurs for 16 of the 20 cases, both Z_Λ and X_ℓ have integral q expansions, so that the A_k are integral. In general, they are only known to be rational, as in the case of $\Lambda_6 = A_6^{(2)}$.

ℓ	Z_{Λ_ℓ}	α_{Λ_ℓ}	β_{Λ_ℓ}
1	$\sqrt[6]{\theta_{\text{Leech}}}$	$\frac{9}{26\sqrt[6]{910}}$	$\sqrt[6]{\frac{2610\sqrt{2}+3707}{7280}}$
2	$\sqrt[4]{\theta_{BW_{16}}}$	$\frac{10}{23\sqrt[4]{69}}$	$\sqrt[4]{27\frac{408\sqrt{2}+577}{23}}/6$
3	$\sqrt[3]{\theta_{K_{12}}}$	$\frac{7}{30}\sqrt[6]{\frac{9}{50}}$	$\sqrt[6]{1250}\frac{\sqrt[3]{9\sqrt{3}+15}}{10}$
5	$\sqrt{\theta_{Q_8(1)}}$	$\frac{\sqrt{6}}{12}$	$\sqrt{3}\frac{\sqrt{7+2\sqrt{10}}}{6}$
6	$\sqrt{\theta_{G_2\otimes F_4}}$	$\frac{4\sqrt{7}}{49}$	$\sqrt{7}\frac{3+\sqrt{6}}{14}$
7	$\sqrt[3]{\theta^2_{A_6^{(2)}}}$	$\frac{(13+9\sqrt{2})\sqrt[6]{-4070183+2878054\sqrt{2}}}{14\sqrt[6]{14}}$	$\frac{1}{2\sqrt[3]{7}}\sqrt[6]{\frac{24717+17479\sqrt{2}-(5075+3591\sqrt{2})\sqrt{11+8\sqrt{2}}}{59216+41872\sqrt{2}-(12536+8864\sqrt{2})\sqrt{11+8\sqrt{2}}}}$
11	$\theta_{\Lambda_{11}}$	$55\frac{\sqrt{2}}{324}$	$\frac{\sqrt{2}(11+5\sqrt{11})}{36}$
14	$\theta_{\Lambda_{14}}$	$\sqrt{7}\frac{16}{169}$	$\sqrt{7}\frac{6+3\sqrt{2}}{26}$
15	$\theta_{\Lambda_{15}}$	$\frac{\sqrt{10}}{12}$	$\frac{(2\sqrt{3}+3\sqrt{2})(\sqrt{35-10\sqrt{6}})}{24}$
23	$\theta_{[4,1;1,6]^2}$	$\sqrt{23}\frac{(536-379\sqrt{2})\sqrt{147+104\sqrt{2}}}{4}$	$\frac{2\sqrt{43654+30866\sqrt{2}-(10442+7383\sqrt{2})\sqrt{6+8\sqrt{2}}}}{984+696\sqrt{2}-(232+164\sqrt{2})\sqrt{6+8\sqrt{2}}}$

TABLE 4. Data for the extremal modular lattices of minimal norm 4.

We have just proved Theorem 3.1.

Theorem 3.1. *Let ℓ be a special level, $\sigma_1(\ell)|24$ and Λ be the extremal ℓ modular lattice Λ_ℓ or Λ_ℓ^0 of minimal norm 4 or 2, then we have*

$$\frac{1}{\pi} = \sqrt{\frac{8}{\ell}}\sum_{k=0}^{\infty}(\alpha_\Lambda + \beta_\Lambda k)A_k Y_0^k, \tag{3–14}$$

where $\alpha_\Lambda, \beta_\Lambda$ are absolute lattice invariants given by (3–13) and computed explicitly in Table 3 and Table 4; $Y_0 = X_\ell(e^{-2\pi\sqrt{2/\ell}})$ are absolute constants depending only on ℓ, given in Table 2; and the A_k are the coefficients in the expansion (3–10), which can be generated explicitly by the polynomial recurrences given in Table 5 and Table 6 with initial conditions given in Table 7.

We note that the procedure actually associates a formula for π for each weight k modular form constructed from a modular lattice of level ℓ. The uniformity of the parametrization here seems to be of particular interest.

4. MODULAR LATTICES AND 20 QUADRATICALLY CONVERGING ALGORITHMS FOR π

Throughout this section, we assume $t > 0$ is a positive real and set $q = e^{-2\pi\sqrt{t/\ell}}$. Note that this means $q \to q^2$ corresponds to $t \to 4t$. Define

$$\mathcal{K}_\Lambda(t) := \frac{1}{\pi Z_\Lambda(q)} - 2t\frac{d}{dt}\left(\frac{1}{\pi Z_\Lambda(q)}\right). \tag{4–1}$$

Lemma 4.1. *The following hold:*

$$\begin{aligned}\mathcal{K}_\Lambda(4t) &= \sqrt{\frac{t}{\ell}}\frac{2}{Z_\Lambda(q^2)}\frac{q\mathcal{R}'_\Lambda}{\mathcal{R}_\Lambda}(X(q))X'(q)\\ &\quad + \mathcal{R}_\Lambda(X(q))\mathcal{K}_\Lambda(t), &(4–2)\\ \mathcal{K}_\Lambda(t) + t\mathcal{K}_\Lambda(1/t) &= 0, &(4–3)\\ \lim_{t\to\infty}\mathcal{K}_\Lambda(t) &= 1/\pi. &(4–4)\end{aligned}$$

Proof: Equation (4–1) gives

$$\mathcal{K}_\Lambda(t) = \frac{1}{\pi Z_\Lambda(q)} - 2\sqrt{\frac{t}{\ell}}q\frac{Z'_\Lambda}{Z^2_\Lambda}(q). \tag{4–5}$$

Substituting into (3–6) gives

$$\begin{aligned}2q\frac{\mathcal{R}'_\Lambda}{\mathcal{R}_\Lambda}(X(q))X'(q) &= \sqrt{\frac{\ell}{t}}\left(\frac{1}{\pi} - Z_\Lambda(q)\mathcal{K}_\Lambda(t)\right)\\ &\quad - 2\sqrt{\frac{\ell}{4t}}\left(\frac{1}{\pi} - Z_\Lambda(q^2)\mathcal{K}_\Lambda(4t)\right),\end{aligned}$$

which is (4–2). Setting $\tau = i\sqrt{\ell t}$ in (3–2) gives

$$tZ_\Lambda(q) = Z_\Lambda(q^{1/t}). \tag{4–6}$$

Replacing t by $1/t$, which means $q \to q^{1/t}$ and $t\frac{d}{dt} \to -t\frac{d}{dt}$, in (4–1) and using (4–6) gives

$$\mathcal{K}_\Lambda(1/t) = \frac{-1}{\pi t Z_\Lambda(q)} + 2\frac{d}{dt}\left(\frac{1}{\pi Z_\Lambda(q)}\right) = -\frac{1}{t}\mathcal{K}_\Lambda(t).$$

Equation (4–4) follows from taking the limit in (4–5) using the fact that $\lim_{t\to\infty} Z_\Lambda(q) = 1$ and the finiteness of $\lim_{t\to\infty} Z'_\Lambda(q)$, since $Z_\Lambda(q)$ is the power of a theta series. $\square$

ℓ	
1	$(4+k)^3A_{k+4}+8(2343+2550k+900k^2+104k^3)A_{k+3}+$ $12288(107+187k+108k^2+20k^3)A_{k+2}+2097152(1+k)(8+19k+14k^2)A_{k+1}+$ $1073741824k^3A_k=0$
2	$(5+k)^3A_{k+5}+8(893+712k+186k^2+16k^3)A_{k+4}+$ $256(519+576k+210k^2+25k^3)A_{k+3}+8192(107+180k+102k^2+19k^3)A_{k+2}+$ $262144(1+k)(5+11k+7k^2)A_{k+1}+8388608k^3A_k=0$
3	$(4+k)^3A_{k+4}+2(570+633k+225k^2+26k^3)A_{k+3}+$ $192(26+46k+27k^2+5k^3)A_{k+2}+512(1+k)(8+19k+14k^2)A_{k+1}+$ $16384k^3A_k=0$
5	$(8+k)^3A_{k+8}+2(8033+3479k+501k^2+24k^3)A_{k+7}+$ $4(49605+25443k+4326k^2+244k^3)A_{k+6}+48(25579+16043k+3326k^2+228k^3)A_{k+5}+$ $384(10394+8330k+2203k^2+192k^3)A_{k+4}+3072(2108+2304k+833k^2+99k^3)A_{k+3}+$ $8192(536+888k+495k^2+91k^3)A_{k+2}+65536(1+k)(11+24k+15k^2)A_{k+1}+$ $524288k^3A_k=0$
6	$(11+k)^3A_{k+11}+2(21317+6472k+654k^2+22k^3)A_{k+10}+$ $4(151764+51639k+5844k^2+220k^3)A_{k+9}+48(106503+41057k+5260k^2+224k^3)A_{k+8}+$ $384(73859+32769k+4826k^2+236k^3)A_{k+7}+768(140870+73463k+12700k^2+728k^3)A_{k+6}+$ $3072(92695+58451k+12208k^2+844k^3)A_{k+5}+24576(20382+16175k+4256k^2+370k^3)A_{k+4}+$ $196608(2756+2927k+1036k^2+121k^3)A_{k+3}+524288(566+897k+480k^2+85k^3)A_{k+2}+$ $4194304(1+k)(11+23k+13k^2)A_{k+1}+33554432k^3A_k=0$
7	$(7+k)^3A_{k+7}+(5450+2763k+465k^2+26k^3)A_{k+6}+$ $(31330+19659k+4062k^2+277k^3)A_{k+5}+4(21128+17070k+4533k^2+395k^3)A_{k+4}+$ $16(6794+7521k+2742k^2+327k^3)A_{k+3}+64(892+1500k+849k^2+157k^3)A_{k+2}+$ $512(1+k)(14+31k+20k^2)A_{k+1}+4096k^3A_k=0$
11	$(12+k)^3A_{k+12}+2(18214+5011k+459k^2+14k^3)A_{k+11}+$ $8(10+k)(4132+845k+43k^2)A_{k+10}+2(859998+292193k+33027k^2+1242k^3)A_{k+9}+$ $32(8+k)(22476+5812k+373k^2)A_{k+8}+32(407314+179369k+26229k^2+1274k^3)A_{k+7}+$ $64(317210+164090k+28131k^2+1599k^3)A_{k+6}+512(41998+26327k+5457k^2+374k^3)A_{k+5}+$ $4096(3624+2874k+753k^2+65k^3)A_{k+4}+512(11896+12723k+4515k^2+526k^3)A_{k+3}+$ $8192(148+238k+129k^2+23k^3)A_{k+2}+8192(1+k)(8+17k+10k^2)A_{k+1}+$ $16384k^3A_k=0$
14	$(20+k)^3A_{k+20}+(190872+30205k+1593k^2+28k^3)A_{k+19}+$ $(2307568+386484k+21567k^2+401k^3)A_{k+18}+2(9265672+1647512k+97575k^2+1925k^3)A_{k+17}+$ $8(13740716+2602656k+164157k^2+3448k^3)A_{k+16}+8(63590440+12880868k+868575k^2+19499k^3)A_{k+15}+$ $16(118494672+25784124k+1867161k^2+45001k^3)A_{k+14}+256(22655978+5323605k+416163k^2+10824k^3)A_{k+13}+$ $64(230140784+58753116k+4988343k^2+140861k^3)A_{k+12}+128(243941080+68145292k+6328941k^2+195419k^3)A_{k+11}+$ $512(108027460+33301444k+3412005k^2+116178k^3)A_{k+10}+512(159353016+54759436k+6252981k^2+237201k^3)A_{k+9}+$ $1024(97084128+37651616k+4852059k^2+207647k^3)A_{k+8}+4096(24089004+10708429k+1582035k^2+77600k^3)A_{k+7}+$ $8192(9523080+4951174k+855927k^2+49123k^3)A_{k+6}+32768(1447900+904897k+188220k^2+12999k^3)A_{k+5}+$ $65536(319776+249944k+65127k^2+5637k^3)A_{k+4}+262144(23164+24101k+8385k^2+970k^3)A_{k+3}+$ $524288(1784+2766k+1443k^2+251k^3)A_{k+2}+2097152(1+k)(20+41k+22k^2)A_{k+1}+$ $8388608k^3A_k=0$
15	$(15+k)^3A_{k+15}+(79136+17203k+1245k^2+30k^3)A_{k+14}+$ $(774194+182955k+14394k^2+377k^3)A_{k+13}+8(538026+138215k+11826k^2+337k^3)A_{k+12}+$ $4(4030340+1127504k+105087k^2+3263k^3)A_{k+11}+16(10+1k^1)(284810+58775k+3031k^2)A_{k+10}+$ $32(3146076+1064848k+120105k^2+4515k^3)A_{k+9}+64(2736492+1036460k+130707k^2+5491k^3)A_{k+8}+$ $512(477348+205825k+29493k^2+1406k^3)A_{k+7}+1024(257580+129518k+21564k^2+1191k^3)A_{k+6}+$ $1024(213100+129640k+25935k^2+1711k^3)A_{k+5}+8192(15788+12124k+3045k^2+250k^3)A_{k+4}+$ $8192(6132+6380k+2187k^2+241k^3)A_{k+3}+16384(652+1020k+537k^2+89k^3)A_{k+2}+$ $65536(1+k)(10+21k+12k^2)A_{k+1}+262144k^3A_k=0$
23	$(21+k)^3A_{k+21}+(172176+26029k+1311k^2+22k^3)A_{k+20}+$ $(1513168+242071k+12900k^2+229k^3)A_{k+19}+(8501534+1439595k+81207k^2+1526k^3)A_{k+18}+$ $2(17329247+3109681k+185913k^2+3703k^3)A_{k+17}+4(27403580+5222560k+331641k^2+7017k^3)A_{k+16}+$ $(279604222+56766531k+3840498k^2+86581k^3)A_{k+15}+4(14+1k^1)(10523000+1533231k+55809k^2)A_{k+14}+$ $4(260430934+60770555k+4725102k^2+122425k^3)A_{k+13}+12(129829640+32750906k+2752229k^2+77057k^3)A_{k+12}+$ $32(61831775+16984233k+1553439k^2+47321k^3)A_{k+11}+32(66647068+20111108k+2019243k^2+67482k^3)A_{k+10}+$ $48(40448994+13555875k+1510014k^2+55933k^3)A_{k+9}+64(23101888+8717168k+1091691k^2+45399k^3)A_{k+8}+$ $64(14486122+6262623k+896910k^2+42569k^3)A_{k+7}+64(7300564+3699228k+619635k^2+34301k^3)A_{k+6}+$ $256(712266+436123k+88110k^2+5861k^3)A_{k+5}+1536(33792+26096k+6648k^2+555k^3)A_{k+4}+$ $512(18982+19719k+6789k^2+762k^3)A_{k+3}+1024(948+1480k+777k^2+133k^3)A_{k+2}+$ $2048(1+k)(14+29k+16k^2)A_{k+1}+4096k^3A_k=0$

TABLE 5. Recurrence relations for $A_k(\Lambda_\ell^0)$.

ℓ	
1	$(10+k)^3A_{k+10}+8(9+k)(2190+477k+26k^2)A_{k+9}+$ $192(1536035+588367k+74772k^2+3156k^3)A_{k+8}+1536(31585155+13748086k+1980084k^2+94568k^3)A_{k+7}+$ $786432(31536825+16483723k+2857899k^2+164041k^3)A_{k+6}+8053063680(459310+288881k+60183k^2+4134k^3)A_{k+5}+$ $824633720832(696710+561007k+150354k^2+13369k^3)A_{k+4}+3377699720527872(3+k)(5240+3999k+766k^2)A_{k+3}+$ $864691128455135232(2+k)(1000+1149k+339k^2)A_{k+2}+590295810358705651712(1+k)(20+43k+26k^2)A_{k+1}+$ $302231454903657293676544k^3A_k=0$
2	$(13+k)^3A_{k+13}+8(12+k)(2053+339k+14k^2)A_{k+12}+$ $64(373106+102901k+9438k^2+288k^3)A_{k+11}+512(2999195+911722k+92040k^2+3088k^3)A_{k+10}+$ $8192(10614573+3628672k+411840k^2+15520k^3)A_{k+9}+262144(14290955+5537782k+712296k^2+30384k^3)A_{k+8}+$ $8388608(14537155+6494098k+963612k^2+47424k^3)A_{k+7}+268435456(11466771+6036481k+1057212k^2+61512k^3)A_{k+6}+$ $4294967296(12770005+8123777k+1722240k^2+121536k^3)A_{k+5}+68719476736(8872092+7077103k+1884480k^2+167360k^3)A_{k+4}+$ $1099511627776(3+k)(1131009+825216k+152576k^2)A_{k+3}+52776558133248(2+k)(96597+104704k+29184k^2)A_{k+2}+$ $281474976710656(1+k)(24577+51200k+28672k^2)A_{k+1}+18446744073709551616k^3A_k=0$
3	$(10+k)^3A_{k+10}+2(9+k)(1704+369k+20k^2)A_{k+9}+$ $24(57508+22199k+2835k^2+120k^3)A_{k+8}+96(280266+122029k+17553k^2+836k^3)A_{k+7}+$ $384(1275546+680213k+119877k^2+6962k^3)A_{k+6}+6144(938150+589201k+122463k^2+8376k^3)A_{k+5}+$ $98304(424180+355505k+99045k^2+9128k^3)A_{k+4}+1572864(3+k)(51976+37563k+6848k^2)A_{k+3}+$ $25165824(2+k)(4547+7221k+2784k^2)A_{k+2}+134217728(1+k)(5654+9313k+1664k^2)A_{k+1}+$ $274877906944k^3A_k=0$
5	$(12+k)^3A_{k+12}+4(11+k)(1244+223k+10k^2)A_{k+11}+$ $32(31795+9581k+960k^2+32k^3)A_{k+10}+24(538223+182082k+20420k^2+760k^3)A_{k+9}+$ $128(8+1k^1)(113729+29680k+1900k^2)A_{k+8}+1024(761323+340508k+50418k^2+2468k^3)A_{k+7}+$ $4096(912939+482553k+84564k^2+4900k^3)A_{k+6}+131072(96040+61711k+13185k^2+934k^3)A_{k+5}+$ $1048576(26212+21218k+5730k^2+515k^3)A_{k+4}+6291456(3+1k^1)(1752+1315k+250k^2)A_{k+3}+$ $33554432(2+k)(256+287k+83k^2)A_{k+2}+268435456(1+k)(8+17k+10k^2)A_{k+1}+$ $1073741824k^3A_k=0$
6	$(15+k)^3A_{k+15}+2(14+k)(3179+451k+16k^2)A_{k+14}+$ $8(192208+44493k+3429k^2+88k^3)A_{k+13}+32(596649+150832k+12672k^2+354k^3)A_{k+12}+$ $128(1425705+396790k+36648k^2+1124k^3)A_{k+11}+1024(1356927+419492k+43002k^2+1462k^3)A_{k+10}+$ $4096(2074113+719435k+82728k^2+3152k^3)A_{k+9}+16384(2544566+1002061k+130896k^2+5664k^3)A_{k+8}+$ $65536(2479337+1124993k+169560k^2+8472k^3)A_{k+7}+262144(1874814+998587k+177048k^2+10424k^3)A_{k+6}+$ $1048576(1059077+678657k+145152k^2+10336k^3)A_{k+5}+4194304(419900+335743k+89856k^2+8032k^3)A_{k+4}+$ $50331648(3+k)(11627+8448k+1568k^2)A_{k+3}+67108864(2+k)(6559+7040k+1952k^2)A_{k+2}+$ $268435456(1+k)(449+928k+512k^2)A_{k+1}+68719476736k^3A_k=0$
7	$(13+k)^3A_{k+13}+(12+k)(4719+777k+32k^2)A_{k+12}+$ $(755784+206085k+18702k^2+565k^3)A_{k+11}+2(3368472+1021398k+102801k^2+3437k^3)A_{k+10}+$ $4(10683164+3644433k+411948k^2+15439k^3)A_{k+9}+24(8352880+3248814k+418417k^2+17837k^3)A_{k+8}+$ $192(3620288+1629790k+243089k^2+11993k^3)A_{k+7}+256(6859324+3644340k+642753k^2+37539k^3)A_{k+6}+$ $2048(1518698+976959k+209211k^2+14874k^3)A_{k+5}+8192(439336+354822k+95691k^2+8597k^3)A_{k+4}+$ $65536(3+k)(12168+9066k+1715k^2)A_{k+3}+262144(2+k)(1378+1531k+439k^2)A_{k+2}+$ $2097152(1+k)(26+55k+32k^2)A_{k+1}+16777216k^3A_k=0$
11	$(15+k)^3A_{k+15}+2(14+k)(2395+339k+12k^2)A_{k+14}+$ $8(82122+18902k+1449k^2+37k^3)A_{k+13}+2(2113914+531297k+44397k^2+1234k^3)A_{k+12}+$ $8(2452862+679287k+62439k^2+1906k^3)A_{k+11}+32(10+1k^1)(215083+44735k+2290k^2)A_{k+10}+$ $96(1938026+670593k+76885k^2+2920k^3)A_{k+9}+256(1522738+598929k+78087k^2+3371k^3)A_{k+8}+$ $1024(613808+278403k+41910k^2+2090k^3)A_{k+7}+1536(498110+265377k+47021k^2+2764k^3)A_{k+6}+$ $2048(331130+212559k+45495k^2+3238k^3)A_{k+5}+16384(25112+20166k+5415k^2+485k^3)A_{k+4}+$ $8192(3+k)(6210+4569k+856k^2)A_{k+3}+327680(2+k)(42+46k+13k^2)A_{k+2}+$ $131072(1+k)(10+21k+12k^2)A_{k+1}+262144k^3A_k=0$
14	$(19+k)^3A_{k+19}+(18+k)(6535+723k+20k^2)A_{k+18}+$ $(1129926+199189k+11700k^2+229k^3)A_{k+17}+2(3802612+714964k+44757k^2+933k^3)A_{k+16}+$ $12(3271426+659551k+44232k^2+987k^3)A_{k+15}+8(20317160+4415786k+318957k^2+7659k^3)A_{k+14}+$ $64(8646435+2037482k+159435k^2+4144k^3)A_{k+13}+64(24543560+6309652k+538341k^2+15245k^3)A_{k+12}+$ $128(29234276+8256685k+773724k^2+24051k^3)A_{k+11}+256(29254400+9150306k+949707k^2+32687k^3)A_{k+10}+$ $1024(12249633+4283607k+497313k^2+19147k^3)A_{k+9}+2048(8509012+3365027k+442233k^2+19283k^3)A_{k+8}+$ $4096(4829717+2191236k+330852k^2+16591k^3)A_{k+7}+8192(2188658+1160733k+205251k^2+12073k^3)A_{k+6}+$ $16384(763989+485945k+103296k^2+7320k^3)A_{k+5}+32768(193836+153587k+40770k^2+3618k^3)A_{k+4}+$ $196608(3+k)(3583+2566k+470k^2)A_{k+3}+131072(2+k)(1419+1502k+410k^2)A_{k+2}+$ $262144(1+k)(73+148k+80k^2)A_{k+1}+4194304k^3A_k=0$
15	$(15+k)^3A_{k+15}+(14+k)(2395+339k+12k^2)A_{k+14}+$ $(194198+44985k+3468k^2+89k^3)A_{k+13}+2(432090+109763k+9252k^2+259k^3)A_{k+12}+$ $4(733262+206039k+19155k^2+590k^3)A_{k+11}+8(10+1k^1)(102151+21745k+1130k^2)A_{k+10}+$ $16(1168536+411650k+47889k^2+1839k^3)A_{k+9}+32(1102128+441130k+58383k^2+2549k^3)A_{k+8}+$ $128(429810+198313k+30312k^2+1529k^3)A_{k+7}+256(270552+146426k+26361k^2+1569k^3)A_{k+6}+$ $1024(67390+43807k+9510k^2+685k^3)A_{k+5}+2048(25208+20434k+5565k^2+505k^3)A_{k+4}+$ $8192(3+k)(1070+789k+151k^2)A_{k+3}+16384(2+k)(230+251k+71k^2)A_{k+2}+$ $65536(1+k)(10+21k+12k^2)A_{k+1}+262144k^3A_k=0$
23	$(21+k)^3A_{k+21}+(20+k)(6461+643k+16k^2)A_{k+20}+$ $(919188+144805k+7602k^2+133k^3)A_{k+19}+(4479326+747135k+41505k^2+768k^3)A_{k+18}+$ $2(8293867+1471290k+86862k^2+1707k^3)A_{k+17}+4(12315916+2333675k+147048k^2+3082k^3)A_{k+16}+$ $(121041462+24607467k+1662330k^2+37325k^3)A_{k+15}+2(14+1k^1)(8954090+1322657k+48323k^2)A_{k+14}+$ $8(55357100+13144051k+1035933k^2+27102k^3)A_{k+13}+4(167814632+43422402k+3728601k^2+106229k^3)A_{k+12}+$ $16(54683648+15521817k+1462149k^2+45686k^3)A_{k+11}+16(61196316+19203848k+2000595k^2+69127k^3)A_{k+10}+$ $16(58481606+20479617k+2382402k^2+91943k^3)A_{k+9}+32(23629616+9341286k+1227945k^2+53579k^3)A_{k+8}+$ $64(7948990+3599777k+542808k^2+27193k^3)A_{k+7}+64(4348036+2299548k+405639k^2+23805k^3)A_{k+6}+$ $256(465972+295564k+62655k^2+4427k^3)A_{k+5}+512(73736+58326k+15453k^2+1368k^3)A_{k+4}+$ $1536(3+k)(1722+1235k+226k^2)A_{k+3}+1024(2+k)(434+463k+127k^2)A_{k+2}+$ $2048(1+k)(14+29k+16k^2)A_{k+1}+4096k^3A_k=0$

TABLE 6. Recurrence relations for $A_k(\Lambda_\ell)$.

ℓ	$A_k(\Lambda_\ell^0)$
1	$1, 120, -9000, 1133760$
2	$1, 24, -168, 1728, -20520$
3	$1, 12, -36, 192$
5	$1, 6, -6, 12, -30, 12, 1140, -17640$
6	$1, 6, -6, 36, -198, 780, -2340, 5880, -16326, 67644, -328332$
7	$1, 4, 0, -8, 48, -288, 1848$
11	$1, 4, -4, 12, -36, 112, -388, 1488, -6084, 25840, -112816, 504336$
14	$1, 2, 4, -8, 16, -32, 56, -96, 208, -480, 736, -128, -1480, -2352, 27280, -63552, -10736, 361504, -458688, -1917568$
15	$1, 6, -6, 12, -30, 84, -228, 648, -2046, 6780, -22644, 76632, -264372, 925464, -3272760$
23	$1, 4, 0, -4, 8, -8, 8, -24, 72, -152, 264, -496, 1208, -3344, 9128, -24432, 67272, -195440, 591720, -1830928, 5748104$
ℓ	$A_k(\Lambda_\ell)$
1	$1, 0, 32760, 1223040, -2743289640, -197636624640, 328290842093760, 36028811422356480, -4530790638390884868 0,$ -6736675123454415659520
2	$1, 0, 1080, -1920, -1737000, 5817600, 4373135040, -22362301440, -12930728023080, 88334562101760, 41701008005305920,$ $-357099557883171840, -1418497986320662 49280$
3	$1, 0, 252, -1680, -46116, 639072, 17704512, -404441856, -6231046500, 238053966528$
5	$1, 0, 60, -360, 660, 3360, 38760, -1137120, 8605620, -6477120, -189479280, -2820997920$
6	$1, 0, 36, -48, -612, 1440, 24096, -101376, -895716, 5364288, 42154704, -336673728, -1936393920, 21086825472, 88371039744$
7	$1, 0, 28, -392/3, 448, -4928/3, 95480/9, -256592/3, 5333552/9, -288363712/81, 195108704/9, -12244267840/81, 276105536168/243$
11	$1, 0, 12, -36, 108, -384, 1596, -7200, 33660, -160512, 777744, -3823152, 19033212, -95786016, 486488496$
14	$1, 0, 8, -8, 0, 0, 56, -208, 336, -64, -608, -448, 7224, -15008, -6992, 81760, -52720, -515456, 1409408$
15	$1, 0, 6, 0, -6, -24, 132, -384, 1170, -4176, 15108, -53328, 188796, -678000, 2456184$
23	$1, 0, 4, -4, 8, -24, 72, -216, 648, -1944, 5896, -18160, 56600, -177840, 562632, -1791696, 5739816, -18484176,$ $59799880, -194266864, 633477224$

TABLE 7. Initial values for $A_k(\Lambda)$.

We can now prove Theorem 1.1.

Proof of Theorem 1.1: Setting $t = 4^{n-1}$, $\kappa_n = \mathcal{K}_\Lambda(4^n)$, and $x_n = X(e^{-2^n\pi/\sqrt{\ell}})$ in (4–2), using (3–9) gives

$$\kappa_n = \frac{2^n}{\sqrt{\ell}} x_n \sqrt{W_\Lambda(x_n)} \mathcal{R}_\Lambda'(x_n) + \mathcal{R}_\Lambda(x_n)\kappa_{n-1}. \qquad (4\text{–}7)$$

By (4–3), $\kappa_0 = \mathcal{K}_\Lambda(1) = 0$ and by (4–4), $\kappa_n \to 1/\pi$ as $n \to \infty$. To see that the convergence is quadratic, we note from (4–2), (4–5), and (3–6), we have

$$\frac{\mathcal{K}_\Lambda(4t) - 1/\pi}{\mathcal{R}_\Lambda(X(q))\mathcal{K}_\Lambda(t) - 1/\pi} = \frac{\sqrt{\frac{t}{\ell}}\frac{2}{Z_\Lambda(q^2)}\left(q\frac{Z_\Lambda'}{Z_\Lambda}(q) - 2q^2\frac{Z_\Lambda'}{Z_\Lambda}(q^2)\right)}{\left(\frac{\mathcal{R}_\Lambda(X(q))}{\pi Z_\Lambda(q)} - 1/\pi\right) - 2q\mathcal{R}_\Lambda(X(q))\sqrt{\frac{t}{\ell}}\frac{Z_\Lambda'}{Z_\Lambda}(q)} + 1. \qquad (4\text{–}8)$$

Since, as $t \to \infty$, $q \to 0$, and $Z_\Lambda(q), Z_\Lambda(q^2), \mathcal{R}_\Lambda(X(q))$ all $\to 1$, we have from (4–8) that

$$\lim_{t\to\infty} \frac{\mathcal{K}_\Lambda(4t) - 1/\pi}{\mathcal{K}_\Lambda(t) - 1/\pi} = 0, \qquad (4\text{–}9)$$

which proves the quadratic convergence. □

5. DIFFERENTIAL EQUATIONS AND POLYNOMIAL RECURRENCES

Since each of the $Z_\Lambda(q)$ transforms like a weight 2 modular form on $\Gamma_0(\ell)^+ \supset G_\ell$ and X_ℓ is a modular function on G_ℓ, Z_Λ should satisfy a third-order differential equation in X, with a priori algebraic coefficients, for an example see Stiller [Stiller 84]. In our case, since $1/X_\ell$ are actually Hauptmoduln, the coefficients of the differential equations are rational. This implies immediately that the coefficients A_k in (3–10) satisfy polynomial recurrences. For the 20 special lattices, we compute, here, the coefficients of the differential equations and the recurrences explicitly. We define

$$\begin{aligned} G_1(X) &:= q\frac{d}{dq}\log X = \sqrt{W_\Lambda(X)}Z_\Lambda(X), \\ G_2(X) &:= q\frac{d}{dq}\log Z_\Lambda(X), \end{aligned} \qquad (5\text{–}1)$$

and also the modular functions

$$\begin{aligned} p_1(X) &:= \frac{q\frac{dG_1}{dq} - G_1G_2}{G_1^2} = \frac{XW_\Lambda'(X)}{2W_\Lambda(X)}, \\ p_2(X) &:= -\frac{q\frac{dG_2}{dq} - G_2^2/2}{G_1^2}. \end{aligned} \qquad (5\text{–}2)$$

These are clearly rational in X since $G_1(X)$ and $G_2(X)$ are powers of rationals in X. From Theorem 1 of [Yang 04], $Z_\Lambda(X)$ must satisfy the explicit third-order differential equation

$$D_X^3 Z_\Lambda + 3p_1 D_X^2 Z_\Lambda + (2p_1^2 + Xp_1' + 2p_2) D_X Z_\Lambda + (2p_1p_2 + Xp_2')Z_\Lambda = 0, \tag{5–3}$$

where D_X denotes $X\frac{d}{dX}$. By clearing denominators, we get an equation with polynomial coefficients.

5.1 Polynomial Recurrences from the Differential Equations

The ordinary differential equation (5-3) is clearly equivalent to polynomial recurrences for the coefficients A_k of $Z_\Lambda = \sum_{k=0}^{\infty} A_k X^k$ in (3–10). Writing t for X for simplicity, we consider a general power series $F(t) = \sum_k A_k t^k$. For any polynomial $p(y)$, we have $p(D_t)F = \sum_k p(k)A_k t^k$. Now, for a general nth-order ODE with polynomial coefficients,

$$\sum_{j=0}^{n} p_{n-j}(t) D_t^j F = p_0(t)D_t^n F + p_1(t) D_t^{n-1} F + \cdots p_n(t)F = 0. \tag{5–4}$$

Let $p(t, y) := \sum_{j=0}^{n} p_{n-j}(t) y^j$ and rewrite it as $p(t, y) = \sum_{j=0}^{m} h_{m-j}(y)t^j$, where m is the maximal degree of the $p_j(t)$. The ODE (5–4) is then simply

$$p(t, D_t)F = 0, \tag{5–5}$$

but we also have

$$p(t, D_t)F = \sum_{j=0}^{m} t^j h_{m-j}(D_t) \sum_k A_k t^k = \sum_k A_k \sum_{j=0}^{m} t^j h_{m-j}(k)t^k. \tag{5–6}$$

So the ODE leads to the following for each k:

$$h_m(k)A_k t^k + h_{m-1}(k)A_k t^{k+1} + h_{m-2}(k)A_k t^{k+2} + \cdots h_0(k)A_k t^{k+m} = 0. \tag{5–7}$$

Hence taking coefficients of terms t^{m+k}, we have an explicit recurrence

$$h_0(k)A_k + h_1(k+1)A_{k+1} + \cdots h_m(k+m)A_{k+m} = 0. \tag{5–8}$$

In this way, we obtain from (5–3) explicit polynomial recurrences for our A_k for each of the 20 modular lattices, as given in Tables 5 and 6 and their initial values in Table 7. The coefficients of the ODE are polynomials with large coefficients and will not be given. The initial terms are computed by explicit q expansion of the theta series for Z_Λ.

6. NUMERICAL COMPUTATIONS AND METHODS

6.1 Algorithm A: Discovering Rational Relations between Power Series

We indicate here our procedure for discovering the two invariants $W_\Lambda(X)$ and $\mathcal{R}_\Lambda(X)$ in Theorems 2.1 and 2.2, (2–6)–(2–11), and Lemmas 2.6 and 2.7, (2–20)–(2–21), since our proofs, as well as the explicit forms of the ODE and recurrences of the A_k, depend on explicit knowledge of them. We first note that in each case, what we need is to discover a unknown rational relation $f = R(g)$, given the series expansion up to a certain order

$$\begin{aligned} f(q) &= a_0 + a_1 q + a_2 q^2 + \cdots O(q^N), \\ g(q) &= b_1 q + b_2 q^2 + \cdots O(q^N), \quad b_1 \neq 0. \end{aligned} \tag{6–1}$$

Setting $t = g(q)$, it is easy to solve for the coefficients p_j term by term to get an expansion of f in terms of g:

$$f = p_0 + p_1 t + p_2 t^2 + \cdots p_N t^N. \tag{6–2}$$

So what we need is a procedure to discover when a power series, given the initial terms as in (6–2), actually comes from a rational function. So suppose that there is a rational relation of degree k, say

$$p_0 + p_1 t + p_2 t^2 + \ldots = \frac{a_0 + a_1 t + \ldots a_k t^k}{1 + b_1 t + b_2 t^2 + \ldots b_k t^k}. \tag{6–3}$$

Clearly, this means we have

$$a_0 = p_0, \quad p_j + p_{j-1}b_1 + p_{j-2}b_2 + \cdots p_1 b_{j-1} + p_0 b_j = \begin{cases} a_j, & if \;\; 1 \le j \le k, \\ 0, & if \;\; j > k, \end{cases} \tag{6–4}$$

which implies, in particular, that we have, for each $s \geq 0$, the $k \times k$ linear system

$$\begin{pmatrix} p_{k+s} & p_{k-1+s} & \cdots & p_{1+s} \\ p_{k+1+s} & p_{k+s} & \cdots & p_{2+s} \\ \vdots & \vdots & \vdots & \vdots \\ p_{2k-1+s} & p_{2k-2+s} & \cdots & p_{k+s} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix} = - \begin{pmatrix} p_{k+1+s} \\ p_{k+2+s} \\ \vdots \\ p_{2k+s} \end{pmatrix}. \tag{6–5}$$

Since (6–5) must have the same solutions for different s, it gives us a way to detect or reject if a rational relation of order k exists by comparing solutions for a few s.

We fix a small s_0, say $s_0 = 4$, and check for each $k = 1, 2, 3, \ldots, k_0$, that the matrix on the left-hand side of (6–5) is nonsingular and solve it for each $0 \leq s \leq s_0$, for the vector $(b_1, b_2, ..., b_k)^T$. If we get the same vector for each of the s, we are certain that the series segment is rational. The a_j can then be read off by the first k equations in (6–4), which can then be used to verify up to the given accuracy. If some solutions disagree, we go on to the next k. The procedure thus allows us to either discover a possible rational relation or conclude that no such rational relation exists up to our checking order k_0. In practice, however, any naturally occurring (in any case explicitly comprehensible) relations of interest are unlikely to have a very high order, say more than 20, so the procedure is very effective. Note that this is only a heuristic for discovering a relation; other considerations, e.g., invoking the theory of modular forms as in the case of Theorem 2.1, are needed in order to conclude equality of the functions from where the power series come.

The algorithm is implemented using PARI-GP. Setting $N = 80$ suffices in all cases we considered and led us to the explicit formulae for $W_\Lambda(X)$ and $\mathcal{R}_\Lambda(X)$. The more explicit form of the relations (2–7) and (2–10) in terms of lattice invariants are deduced simply by observing the patterns in the numbers. Recently, we learned that M. Somos has alternative ways for finding rational relations, but we include the procedure here as it may be useful for some readers.

ℓ	1	2	3	5	6	7	11	14	15	23
n_{Λ_ℓ}	5	5	6	6	6	6	7	7	7	7
m_{Λ_ℓ}	40	55	64	74	75	75	75	75	75	75
$n_{\Lambda_\ell^0}$	6	6	7	7	7	7	7	8	8	8
$m_{\Lambda_\ell^0}$	34	34	47	72	54	75	75	75	75	75

TABLE 8. Number of iterations needed for accuracy to 50 decimal places.

6.2 Numerical Checks

In PARI-GP, we computed explicitly the quadratic iterations (1–9), (1–10), as well as the partial sums (3–14) for each of the 20 modular lattices which give convergence to $1/\pi$. The number of iterations $n_\Lambda(m_\Lambda)$ needed to obtain accuracy to 50 decimal places for $1/\pi$ for the quadratic iterations (series partial sums) for each of the 20 lattices are tabulated in Table 8. Note the quadratic convergence in the case of the quadratic iterations.

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