A PAGE OF MATHEMATICAL AUTOBIOGRAPHY

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INTRODUCTION

As my natural taste has always been to look forward rather than backward this is a task which I did not care to undertake. Now, however, I feel most grateful to my friend Mauricio Peixoto for having coaxed me into accepting it. For it has provided me with my first opportunity to cast an objective glance at my early mathematical work, my algebro-geometric phase. As I see it at last it was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry. But I must not push the metaphor too far.

The time which I mean to cover runs from 1911 to 1924, from my doctorate to my research on fixed points. At the time I was on the faculties of the Universities of Nebraska (two years) and Kansas (eleven years). As was the case for almost all our scientists of that day my mathematical isolation was complete. This circumstance was most valuable in that it enabled me to develop my ideas in complete mathematical calm. Thus I made use most uncritically of early topology à la Poincaré, and even of my own later developments. Fortunately someone at the Académie des Sciences (I always suspected Émile Picard) seems to have discerned "the harpoon for the whale" with pleasant enough consequences for me.

To close personal recollections, let me tell you what made me turn with all possible vigor to topology. From the ρ_0 formula of Picard, applied to a hyperelliptic surface Φ (topologically the product of 4 circles) I had come to believe that the second Betti number $R_2(\Phi)=5$, whereas clearly $R_2(\Phi)=6$. What was wrong? After considerable time it dawned upon me that Picard only dealt with *finite* 2-cycles, the only useful cycles for calculating periods of certain double integrals. Missing link? The cycle at infinity, that is the plane section of the surface at infinity. This drew my attention to cycles carried by an algebraic curve, that is to *algebraic* cycles, and $\cdots$ the harpoon was in!

My general plan is to present the first concepts of algebraic geometry, then follow up with the early algebraic topology of Poincaré plus some of my own results on intersections of cycles. I will then discuss the topology of an algebraic surface. The next step will be a

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summary presentation of the analytical contributions of Picard, Severi and Poincaré leading to my work, application of topology to complex algebraic geometry concluding with a rapid consideration of the effect on the theory of abelian varieties.

This is not however a cold recital of results achieved duly modernized. To do this would be to lose the "autobiographical flavor" of my tale. I have therefore endeavored to place myself back in time to the period described and to describe everything as if I were telling it a half century ago. From the point of view of rigor there is no real loss. Analytically the story is fairly satisfactory and to make it so in the topology all that is needed is to accept the results amply described in my Colloquium Lectures [10].

To place the story into focus I must say something about what we knew and accepted in days gone by. That is I must describe our early background.

In its early phase (Abel, Riemann, Weierstrass), algebraic geometry was just a chapter in analytic function theory. The later development in this direction will be fully described in the following chapters. A new current appeared however (1870) under the powerful influence of Max Noether who really put "geometry" and more "birational geometry" into algebraic geometry. In the classical mémoire of Brill-Noether (Math. Ann., 1874), the foundations of "geometry on an algebraic curve" were laid down centered upon the study of linear series cut out by linear systems of curves upon a fixed curve $f(x, y) = 0$. This produced birational invariance (for example of the genus p) by essentially algebraic methods.

The next step in the same direction was taken by Castelnuovo (1892) and Enriques (1893). They applied analogous methods to the creation of an entirely new theory of algebraic surfaces. Their basic instrument was the study of linear systems of curves on a surface. Many new birationally invariant properties were discovered and an entirely new and beautiful chapter of geometry was opened. In 1902 the Castelnuovo-Enriques team was enriched by the brilliant personality of Severi. More than his associates he was interested in the contacts with the analytic theory developed since 1882 by Émile Picard. The most important contribution of Severi, his theory of the base (see §12) was in fact obtained by utilizing the Picard number ρ (see §11).

The theory of the great Italian geometers was essentially, like Noether's, of algebraic nature. Curiously enough this holds in good part regarding the work of Picard. This was natural since in his time Poincaré's creation of algebraic topology was in its infancy.

Indeed when I arrived on the scene (1915) it was hardly further along.

About 1923 I turned my attention to "fixed points" which took me away from algebraic geometry and into the more rarefied air of topology. I cannot therefore refer even remotely to more recent doings in algebraic geometry. I cannot refrain, however, from mention of the following noteworthy activities:

I. The very significant work of W. V. D. Hodge. I refer more particularly to his remarkable proof that an n-form of V^q which is of the first kind cannot have all periods zero (see Hodge [**13**]).

II. The systematic algebraic attack on algebraic geometry by Oscar Zariski and his school, and beyond that of André Weil and Grothendieck. I do feel however that while we wrote algebraic GEOMETRY they make it ALGEBRAIC geometry with all that it implies.

References. For a considerable time my major reference was the Picard-Simart treatise [2]. In general however except for the writings of Poincaré on topology my Borel series monograph [9] is a central reference. The best all around reference not only to the topics of this report but to closely related material is the excellent Ergebnisse monograph of Zariski [11]. Its bibliography is so comprehensive that I have found it unnecessary to provide an extensive one of my own.

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I. General Remarks on Algebraic Varieties

1. Definition. Function field. It was the general implicit or explicit understanding among algebraic geometers of my day that an algebraic n-variety V^n (n dimensional variety) is the partial or complete irreducible intersection of several complex polynomials or "hypersurfaces" of a projective space S^{n+k}, in which V^n had no singularities (it was homogeneous). Thus V^n was a compact real $2n$-manifold M^{2n} (complex dimension n). It could therefore be considered as its own Riemann manifold as I shall do throughout.

For convenience in analytical operations one customarily represents V^n by a general projection in cartesian S^{n+1}

$$F(x_1, x_2, \cdots, x_n, y) = 0, \tag{1.1}$$

where F is an irreducible complex polynomial of degree m. In this representation, the variety, now called F, occupies no special position relative to the axes.[1] As a consequence (1.1) possesses the simplest singularities. For a curve they consist of double points with distinct tangents, for a surface: double curve with generally distinct tangent planes along this curve.

Incidentally, the recent brilliant reduction of singularities by Hironaka [12] has shown that the varieties as just described are really entirely general.

Returning to our V^n the study of its topology will lean heavily upon the properties of the pencil of hypersurfaces $\{H_y\}$ cut out by the hyperplanes $y=\text{const}$. The particular element of the pencil cut out by $y=c$ is written H_c. As my discourse will be mostly on surfaces I will only describe (later) certain pecularities for varieties.

Function field. Let the complex rational functions $R(x_1, \cdots, x_n, y)$ be identified mod F. As a consequence they constitute an algebraic extension of the complex field K written $K(F)$, called the function field of F.

Let F^* be the nonsingular predecessor of F in S^{n+k} and let $(u_1, \cdots, u_{n+k})$ be cartesian coordinates for S^{n+k}. On F^* they determine elements ξ_h, $h \leqq n+k$ of $K(F)$. The system

$$u_h = \xi_h, \qquad h \leqq n + k$$

is a parametric representation of F^*. F^* is a *model* of $K(F)$.

Any two models F_1^*, F_2^* are birationally equivalent: birationally transformable into one another. The properties that will mainly

[1] That is, F has only those singularities which arise from a general projection on S^{n+1} of a nonsingular $V^n \subset S^{n+k}$.

interest us are those possessing a certain degree of birational invariance (details in §17).

Terminology. Since only algebraic curves, surfaces, varieties will be dealt with, I drop the mention "algebraic" and merely say curve, ...

The symbol $\mathcal{V}^n$ represents a (usually complex) n dimensional vector space.

2. **Differential forms.** Let $\alpha, \beta, \cdots$, denote elements of the function field $K(F)$. I shall refer to various differentials: zero, one, two, $\cdots$ forms $\omega^0, \omega^1, \omega^2, \cdots$, in the sense of Élie Cartan of type

$$\omega^k = \sum \alpha_{i_1,\dots,i_k} d\alpha_{i_1} \cdots d\alpha_{i_k},$$

every α in $K(F)$, as zero, one, two, $\cdots$, forms. They are calculated by the rules of calculus, remembering that the $d\alpha_j$ are skew-symmetric, that is $d\beta d\alpha = -d\alpha d\beta$.

Note that $d\omega^k$ is an ω^{k+1} called *exact* and that if $d\omega^k = 0$ one says that ω^k is *closed*.

Special terms are: ω^k is of the first kind when it is holomorphic everywhere on F; of the second kind when it is holomorphic at any point of F mod some $d\alpha$; of the third kind if neither of the first nor of the second kind.

The evaluation of the number of kinds one or two constitutes one of the main problems to be discussed.

3. **Differential forms on a curve.** Let the curve be

$$f(x, y) = 0 \tag{3.1}$$

and let m be its degree. We refer to it as "the curve f." Under our convention, f has no other singularities than double points with distinct tangents and is identified in a well-known sense with its Riemann surface. Its one-forms are said to be *abelian*. An *adjoint* to f is a polynomial $\phi_n(x, y)$ (n is its degree) vanishing at all double points.

The following are classical properties:

One-forms of the first kind. They are all reducible to the type

$$\frac{\phi_{m-3}dx}{f'_y}. \tag{3.2}$$

They form a $\mathcal{V}^p$, where $2p = R_1$, the first Betti number of the Riemann surface f. Of course the collection $\{\phi_{m-3}\}$ forms likewise a $\mathcal{V}^p$.

One-forms of the second kind. Same type of reduction to (3.2) mod a $d\omega^0$, save that ϕ_{m-3} is replaced by some ϕ_s. Their vector space mod $dK(f)$ is a $\mathcal{V}^{2p}$.

One-forms of the third kind. They have a finite number of logarithmic points with residues whose sum is zero.

Some special properties of one-forms of the first kind. Let

$$\psi = \sum_{h=0}^{r} \alpha_k \psi_h(x, y) = 0 \tag{3.3}$$

be a linear system of polynomials linearly independent mod f and of common degree. Let the general ψ intersect f in a set of points $P_1, \cdots, P_s$ which includes all the variable points and perhaps some fixed points. The collection of all such sets is a *linear series of degree* n and *dimension* r. The series is *complete* when its sets do not belong to an amplified series of the same degree: designation g_n^r (concepts and terminology of Brill and Noether).

(3.4) THEOREM OF ABEL. *Let du be any one-form of the first kind; let $\{P_h\}$ be any element of a g_n^r and let A be a fixed point of f. Then with integration along paths on f:*

$$\sum \int_A^{P_h} du = v$$

is a constant independent of the element $\{P_h\}$ of g_n^r.

Still another classic, a sort of inverse of Abel's theorem is this:

(3.5) THEOREM OF JACOBI. *Let $\{du_h\}$ be a base for the one-forms of the first kind. Then for general values of the constants v_h (exceptions noted) the system*

$$\sum_{k=1}^{p} \int_A^{P_k} du_h = v_h$$

in the p unknowns P_k, $k \leqq p$, has a unique solution.

Periodic properties. Let $\{du_h\}$ be as just stated and let $\{\gamma_\mu^1\}$, $\mu \leqq 2p$ be an integral homology base (see (5.4)) for the module of one-cycles of f. The expression

$$\pi_{h\mu} = \int_{\gamma_\mu^1} du_h$$

is the *period* of $\int du_h$ as to the cycle γ_μ^1. Let the matrix

$$\Pi = [\pi_{h\mu}]; \qquad h, \mu \leqq 2p; \qquad \pi_{h+p,\mu} = \bar{\pi}_{h\mu}, h \leqq p.$$

By means of integration on the Riemann surface f, Riemann has obtained the following comprehensive result (formulation of Scorza):

(3.6) THEOREM OF RIEMANN. *There exists an integral skew-symmetric $2p \times 2p$ matrix M with invariant factors unity such that*

$$i\Pi M \Pi' = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}, \qquad (A^* = \bar{A}') \tag{3.7}$$

is a positive definite hermitian matrix.

Riemann matrices. This is the name given by Scorza to a matrix like Π satisfying a relation (3.7) except that M is merely rational skew-symmetric. The theory of such matrices has been extensively developed by Scorza [6]. He called M: *principal matrix* of Π.

It may very well happen that there is more than one rational skew-symmetric matrix M satisfying a relation (3.7) but without necessarily the positive definite property. These matrices are called singularity matrices. They form a rational vector space whose dimension k is the *singularity index* of the Riemann matrix (Scorza).

II. TOPOLOGY

4. **Results of Poincaré.** Let M^n be a compact orientable n-manifold which admits a cellular subdivision with α_k k-cells (well-known property for varieties). The characteristic is the expression

$$\chi(M^n) = \sum (-1)^k \alpha_k. \tag{4.1}$$

The following two relations were proved by Poincaré:

$$\chi(M^n) = \sum (-1)^k R_k \tag{4.2}$$

$$R_k = R_{n-k} \tag{4.3}$$

where R_k is the kth integral Betti number of M^n: maximum number of linearly independent k-cycles with respect to homology (=with respect to bounding).

5. **Intersections.** In my work on algebraic geometry I freely used the intersection properties described below; they were actually justified and proved topologically invariant a couple of years later in my paper in the 1926 Transactions and much more fully in [**10**].

Let M^n be as before and let γ^p and γ^q be integral p- and q-cycles of M^n. One may define the intersection $\gamma^p \cdot \gamma^q$ and it is a $(p+q-n)$-cycle.

(5.1) *If γ^p or $\gamma^q \sim 0$ (bounds), then also $\gamma^p \cdot \gamma^q \sim 0$.*

The more important situation arises when $p+q=n$. The intersection (geometric approximation) is then a zero-cycle

$$C^0 = \sum s_j A_j$$

where the s_j are integers. The *intersection number*

$$(\gamma^p, \gamma^{n-p}) = \sum s_j$$

is independent of the approximation. One proves readily

$$(\gamma^p, \gamma^{n-p}) = (-1)^{(n-p)p}(\gamma^{n-p}, \gamma^p). \tag{5.2}$$

A basic result is:

(5.3) THEOREM. *A n.a.s.c. in order that* $\lambda\gamma^p \sim 0$, $\lambda \neq 0$ *is that*

$$(\gamma^p, \gamma^{n-p}) = 0$$

for every γ^{n-p} [9, p. 15], [10, p. 78].

(5.4) HOMOLOGY BASE. The collection $\{\gamma_h^p\}$, $h \leqq R_p$ is a homology base for the p-cycles when the γ_h^p are independent and every γ^p satisfies a relation

$$\lambda\gamma^p \sim \sum s_h\gamma_h^p, \qquad \lambda \neq 0.$$

(5.5) *A n.a.s.c. in order that the* $\{\gamma_h^p\}$, $h \leqq R_p$ *be a homology base for p-cycles is the existence of a set of* R_p *cycles* $\{\gamma_k^{n-p}\}$ *such that the determinant*

$$|(\gamma_h^p, \gamma_k^{n-p})| \neq 0.$$

Then $\{\gamma_k^{n-p}\}$ *is likewise a homology base for* $(n-p)$*-cycles.*

6. The surface F. Orientation. Let P be a point of F and let $u = u' + iu''$, $v = v' + iv''$ be local coordinates for P. Orient F by naming the real coordinates in the order u', u'', v', v''. There results a unique and consistent orientation throughout the surface F. Hence F is an orientable M^4.

Similarly if C is a curve of F and u is a local coordinate at a nonsingular point Q of C. The resulting orientation turns C into a definite two-cycle, still written C.

Let D be a second curve through Q, for which Q is nonsingular and not a point of contact of the two curves. Then Q contributes $+1$ to both the intersection number (C, D) and to the number $[CD]$ of geometric intersections of C and D. This holds also, through certain approximations when Q is a multiple intersection. Hence always

$$(C, D) = [CD]. \tag{6.1}$$

I will return to these questions later.

7. Certain properties of the surface F. Its characteristic. To be a little precise let for a moment F^* denote the nonsingular predecessor of F in projective S^{k+2}. One may always choose a model F^* of the function field $K(F)$ whose hyperplane sections are in general of a fixed genus $p>0$. We pass now to a cartesian representation of degree m:

$$F(x, y, z) = 0 \tag{7.1}$$

which is a general projection of F^* and in particular in general position relative to the axes. The general scheme that follows is due to Picard. Let $\{H_y\}$ be the pencil cut out by the planes $y=\text{const.}$, and let a_h, $h \leqq N$, be the values for which the planes $y=a_h$ are tangent to F. Then the following properties hold:

I. Every H_y, y not an a_k, is of fixed genus p.

II. Every H_y is irreducible.

III. The plane $y=a_k$ has a unique point of contact A_k with F and A_k is a double point of H_{a_k} with distinct tangents. Hence the genus of H_{a_k} is $p-1$.

IV. Among the branch points of the function $z(x)$ taken on H_y exactly two $\to A_k$ as $y \to a_k$.

V. The fixed points $P_1, \cdots, P_m$ of H_y are all distinct.

I denote by S_y the sphere of the complex variable y.

Characteristic. Cover H_y with a cellular decomposition among whose vertices are the fixed points P_h of the curve.

Then if $H_y^* = H_y - \sum P_h$, $\chi(H_y^*) = 2-2p-m$. Decompose also S_y into cells with the a_k as vertices. Were it not for these points, and since a sphere has characteristic two, H_y^* promenading over S_y would generate a set $E = S_y \times H_y^*$ of characteristic

$$\chi(E^*) = 2(2 - 2p - m).$$

Now in comparison with H_y^*, H_{a_k} has lost two one-cycles, and has two points replaced by one. Hence

$$\chi(\overset{*}{H}_{a_k}) = \chi(\overset{*}{H}_y) + 1.$$

Upon remembering to add the missing points P_h we have then

$$\chi(F) = \chi(E^*) + N + m = (N - m - 4p) + 4 = I + 4 \tag{7.2}$$

a formula due to J. W. Alexander (different proof). The number $I = N-m-4p$ is the well-known *invariant of Zeuthen-Segre*.

8. **One-cycles of F.** The first step was taken by Picard who proved this noteworthy result:

(8.1) THEOREM. *Every one-cycle γ^1 of F is $\sim$ a cycle γ^1 contained in an H_y.*

The next important observation made by Picard was that H_y contained a certain number r of one-cycles which are invariant as y varies. That is such a cycle γ^1 situated say in H_a (a not an a_k) has the property that as y describes any closed path from a to a on the sphere S_y the cycle γ^1 returns to a position $\gamma^1 \sim \gamma^1$ in H_y. This draws attention to the nature of the variation $\mathcal{V}\gamma^1$ of any cycle γ^1 under the same conditions.

Draw lacets aa_k on S_y. Owing to (7, IV) as y describes aa_k a certain cycle δ_k^1 of H_y tends to the point of contact A_k of the plane $y = a_k$ and hence is ~ 0 on H_{a_k}. This is the *vanishing cycle* as $y \to a_k$. A simple lacet consideration shows that as y turns once positively around a_k the variation $\mathcal{V}\gamma^1$ of the cycle γ^1 is given by

$$\mathcal{V}\gamma^1 = (\gamma^1, \delta_k^1)\delta_k^1. \tag{8.2}$$

Hence

(8.3) THEOREM. *N.a.s.c. for invariance of the cycle γ^1 is that every*

$$(\gamma^1, \delta_k^1) = 0.$$

A noteworthy generalization is obtained when γ^1 is replaced by a one-chain L uniquely determined in term of y provided that y crosses no lacet.[2] As y turns as above around a_k the variation of L is

$$\mathcal{V}(L) = (L, \delta_k^1)\delta_k^1. \tag{8.4}$$

Noteworthy special cases are

I. L is an oriented arc joining in H_y two fixed points of H_y.

II. Let C be an algebraic curve of F and let $M_1, \cdots, M_n$ be its intersections with H_y. Then L is a set of paths from a P_j to every point M_k in H_y.

(8.5) THEOREM. *The number of invariant cycles of H_y is equal to the Betti number $R_1(F)$ and both are even*: $r = R_1 = 2q$.

[2] In modern terminology, L will be a relative cycle.

This property was first proved in [7], although it was often admitted before. I give here an outline of the proof (not too different from the proof of [7]).[8]

To make the proof clearer I will use the following special notations: Γ a 3-cycle of F; $\{\Gamma_h\}$ base for the Γ's; $\gamma = \Gamma H_y$: (one-cycle of H_y);
$\{\alpha_h\}$, $h \leqq 2p$, base for one-cycles of H_y;
$\{\beta_j\}$, $j \leqq 2p-r$, base for the one cycles of H_y, none invariant; β any linear combination of the β_j;

Matrices such as $[(\beta_h, \Gamma_k)]_F$ will be written $[\beta\Gamma]_F$.

Proof that $r = R_1$. $\gamma = \Gamma H_y$ is invariant; conversely γ invariant is a ΓH_y. Moreover $\gamma \sim 0$ in H_y and $\Gamma \sim 0$ in F are equivalent. Hence $\{\gamma_h\}$, $h \leqq R_1$, is a base for invariant cycles and therefore $r = R_1$.

Proof that r is even. Since no β is invariant, $[\beta\delta]$ is of rank $2p-r$. Hence there exist $2p-r$ cycles δ which are independent in H_y. Denote them by $\bar{\delta}_h$, $h \leqq 2p-r$. Since $(\gamma_k \delta_h) = 0$ for every k, the δ_h depend on the β_j in H_y. Hence one may take $\{\gamma_h; \bar{\delta}_k\}$ as base for the one-cycles of H_y. Hence

$$[\gamma\delta] = \begin{bmatrix} [\gamma\gamma] & 0 \\ 0 & [\delta\delta] \end{bmatrix} \tag{8.6}$$

is nonsingular. It follows that $[\gamma\gamma]$ is likewise nonsingular. Since it is skew-symmetric, a well-known theorem of algebra states that r is even.

9. **The two-cycles of** F. From the expression (7.2) of the characteristic we have

$$\chi(F) = I + 4 = R_2 - 2R_1 + 2.$$

Hence

$$R_2 = I + 2R_1 + 2. \tag{9.1}$$

Besides this formula it is of interest to give an analysis of the 2-cycles.

Given a γ^2 one may assume it such that it meets every H_y in at most a finite set of points. Let Q be one of these and let P, Q be a directed path from the fixed point P to the point Q in H_y. Call L the sum of these paths. As y describes $S_y - \sum$ lacets aa_k, L generates a 3-chain C^3 whose boundary ∂C^3 consists of these chains:

(a) As y describes aa_k the vanishing one-cycle δ_k^1 of H_y generates a 2-chain Δ_k whose boundary

[8] The point here is to prove that an invariant cycle, which is also a vanishing cycle, is necessarily zero.

$$\partial\Delta_k = (\overset{1}{\delta_k})_{H_a}.$$

The corresponding contribution to ∂C^3 is $\mu_k\Delta_k$, where (Zariski)

$$\mu_1 = (L, \overset{1}{\delta_1}), \qquad \mu_k = (L + \mu_1\overset{1}{\delta_2} + \cdots + \mu_{k-1}\overset{1}{\delta}_{k-1}, \overset{1}{\delta_k}). \tag{9.2}$$

(b) A part (H_a) of H_a.

(c) $-\gamma^2$ itself.

Hence

$$\partial C^3 = -\gamma^2 + \sum \mu_k\Delta_k + (H_a) \sim 0$$

and so

$$\gamma^2 \sim \sum \mu_k\Delta_k + (H_a). \tag{9.3}$$

Since the right side is a cycle, and $y=a$ is arbitrary we have

$$\sum \mu_k\overset{1}{\delta_k} \sim 0 \qquad \text{in } H_y. \tag{9.4}$$

Conversely when (9.4) holds, (9.3) is a 2-cycle. Thus to obtain R_2 it is merely necessary to compute the number of linearly independent relations (9.4) and add to them one unit for all μ_k zero, that is for the cycle H_a itself. This yields again (9.1).

For purposes of counting certain double integrals Picard required the number of *finite* 2-cycles independent relative to homologies in $F-H_\infty$. This is the number $R_2(F-H)$ and he found effectively

$$R_2(F - H) = R_2 - 1. \tag{9.5}$$

10. **Topology of algebraic varieties.** I have dealt with it at length in both [8] and [9]. Questions of orientation and intersection are easily apprehended from the case of surfaces. I shall only recall here a few properties that are not immediate derivatives from the case of a surface.

The designations V^n, H_y are the same as in Chapter I. The following properties are taken from [9, Chapter V]. The symbol γ^k will represent a k-cycle of V^n.

I. *Every γ^k, $k<n-1$, of H_y is invariant.*

II. *Every γ^k, $k<n$, of V^n is $\sim\gamma^{k\prime}$ in H_y.*

III. *When $k\leqq n-2$, $\gamma^k\sim 0$ in V^n and $\gamma^{k\prime}\sim 0$ in H_y are equivalent relations.*

IV. *Under the same conditions $R_k(V^n)=R_k(H_y)$.*

III. Analysis with Little Topology

This is a rapid résumé of the extensive contributions of Picard, Severi and Poincaré upon which I applied topology (see IV). I will continue to consider the same surface F and all notations of II.

11. **Émile Picard and differentials on a surface.** During the period 1882–1906 Picard developed almost single-handedly the foundations of this theory. His evident purpose was to extend the Abel-Riemann theory and this he accomplished in large measure. Reference: Picard-Simart [2].

Picard studied particularly closed ω^1, that is

$$\omega^1 = \alpha dx + \beta dy, \qquad \partial\alpha/\partial y = \partial\beta/\partial x$$

and ω^2. The choice of closed ω^1 is very appropriate since then $\int\omega^1$ is an element of $K(F)$, and analytic function theory plus topology are fairly readily available.[4]

For closed one-forms the same three kinds as for abelian differentials are distinguished, save that for the third kind logarithmic curves replace logarithmic points.

Significant results are

I. Closed one-forms of the first kind make up a $\mathcal{V}^q$ (Castelnuovo) ($q=\frac{1}{2}R_1$ as I have shown).

II. For the second kind same property save that they form a $\mathcal{V}^{2q}$ mod $dK(F)$. (Picard)

III. Regarding the third kind Picard obtained this noteworthy result: There exists a least number $\rho\geqq 1$ such that any set of $\rho+1$ curves are logarithmic for some closed ω^1 having no other poles.

The 2-forms admit again three kinds: (a) first kind: holomorphic everywhere; (b) second kind: holomorphic to within a $d\omega^1$ about each point; (c) the rest. The third kind is characterized by the possession of periods: *residues* over some 2-cycle γ^2 bounding an arbitrarily small neighborhood of a one-cycle on a curve.

The 2-forms of the first kind were already found by Max Noether. They are of the type

$$\omega^2 = \frac{Q(x, y, z)dxdy}{F_z'}$$

where Q is an adjoint polynomial of degree $m-4$. These ω^2 (or the associated Q) make up a $\mathcal{V}^{p_g}$, where p_g is the *geometric* genus of F, studied at length by Italian geometers.

[4] Strictly speaking, $\int\omega^1$ is in $K(F)$ only if ω^1 has no residues or periods, but since $d\omega^1=0$, $\int\omega^1$ is invariant under a continuous variation in the path of integration.

Let $\mathfrak{V}^{\rho_0}$ be the vector space of the ω^2 of the second kind mod $d\omega^1$. Picard utilized his topological description of *finite* 2-cycles to arrive at the following formula:

$$\rho_0 = I + 4q - \rho + 2. \tag{11.1}$$

12. **Severi and the theory of the base.** The central idea here is a notion of *algebraic dependence* between curves on the surface F. I must first describe this concept.

Let the nonsingular surface F be in an S^{k+2}. A linear system of hypersurfaces of the space cuts out on F a *linear* system of curves $|C|$. This system is *complete* if its curves are not curves of an amplified linear system.

We owe to the Italian School the following property: Every sufficiently ample complete system $|C|$ is part of a collection $\{C\}$ of ∞^q such systems. The elements $|C|$ of the collection are in an algebraic one-one correspondence with the points of an abelian variety V^q, unique for F and called sometimes the *Picard variety* of F (see § 18).

A system $\{C\}$, ∞^2 at least, without fixed points and with irreducible generic curve is said to be *effective*. Its curves are also called *effective*.

Note the following properties:

(a) An effective system is fully individualized by any one of its curves.

(b) The generic curves of an effective system have the same genus, written $[C]$.

(c) The curves C, D of two effective systems intersect in a set of distinct points whose number is denoted by $[CD]$. In particular we write $[C^2]$ for $[CC]$ and $[C^2]$ is the *degree* of C.[5]

(d) With C, D as before let two curves C, D taken together be individuals of an effective system $\{A\}$. This system is unique and we write

$$A = B + C. \tag{12.1}$$

(e) Any two curves A_1, A_2 of an effective system $\{A\}$ may be joined in $\{A\}$ by a continuous system ∞^1 of curves of $\{A\}$, whose genus, except for those of A_1 and A_2, is fixed and equal to $[A]$.

As an application of (e) let A, B, C be effective and $A = B+C$. Following Enriques, connect A to $B+C$ as indicated in (e). There follows a relation

[5] This degree should not be confused with the degree of C as an algebraic curve in projective space.

$$\chi(A) + [BC] = \chi(B) + \chi(C) - [BC].$$

Hence if we define

$$\phi(A) = \chi(A) + [A^2] = 2 - 2[A] + [A^2],$$

we verify at once that

$$\phi(A) = \phi(B + C) = \phi(B) + \phi(C).$$

That is $\phi(A)$ is *an additive function on effective systems.*

When (12.1) holds between effective systems we set

$$C = A - B$$

and we have

$$\phi(C) = \phi(A - B) = \phi(A) - \phi(B).$$

Note also that as regards the symbols $[BC]$ we may operate as with numbers, that is

$$[(B \pm C)D] = [BD] \pm [CD].$$

Virtual systems: Let $\{A\}$, $\{B\}$ be effective systems. Without imposing any further condition define a virtual system $\{C\} = \{A - B\}$ as the pair of symbols $\{\phi(A) - \phi(B)\}$, $[(A - B)^2]$. This defines automatically $[C]$ and $[C^2]$. It is also clear that they are the same for $A - B$ and $A + D - (B + D)$ whatever D effective. In other words $\{C\}$ depends only upon the difference $A - B$. The symbol $\{C\}$ is called a virtual algebraic system of curves and $[C]$, $[C^2]$ are the related virtual genus and degree.

It may very well happen that while A, B are effective there exist curves C, not necessarily effective such that $B + C$ (B together with C) is a member of $\{A\}$. If so C is considered as a curve of the virtual system $\{C\}$ and has virtual characters $[C]$ and $[C^2]$, not necessarily its actual characters.

If we define $\{0\} = \{A - A\}$, as a virtual curve 0 is unique. One readily finds that $[0] = 1$, $[0^2] = 0$.

To sum up, the totality of effective and virtual curves form a module M_S over the integers: the Severi module. Within M_s a relation

$$\lambda_1 C_1 + \cdots + \lambda_s C_s = 0 \tag{12.2}$$

has a definite meaning. It is a relation of *algebraic dependence* between curves of F in the sense of Severi.

The following remarkable result was proved by Severi:

(12.3) THEOREM OF SEVERI. *The module of curves of F has a base consisting of ρ effective curves $C_1, C_2, \cdots, C_\rho$, where ρ is the Picard number relative to closed ω^1 of the third kind.*

That is any curve C satisfies a relation

$$\lambda C = \lambda_1 C_1 + \cdots + \lambda_\rho C_\rho$$

where λ and the λ_h are integers and $\lambda \neq 0$.

Severi also proved

(12.4) *The base may be chosen minimal, that is such that*

$$\lambda C = \lambda \sum \lambda_h C_h.$$

Moreover there exist effective curves $D_1, D_2, \cdots, D_{\sigma-1}$ such that actually

$$C = \sum \lambda_h C_h + \sum \mu_j D_j.$$

One assumes, as one may that σ is the least possible.

Severi also proved the following criteria:

(12.5) *A n.a.s.c. in order that the curves $C_1, C_2, \cdots, C_s$ be algebraically independent is that, with H a plane section, the matrix*

$$\begin{bmatrix} [C_h & C_k] \\ [C_h & H] \end{bmatrix}$$

be of rank s.

(12.6) *N.a.s.c. in order that $\{C_h\}$, $h \leqq \rho$, be a base is that the determinant $|[C_h C_k]| \neq 0$ and that its order ρ be the highest order for which this holds. Moreover, ρ is the Picard number.*

13. **Poincaré and normal functions.** Through an ingenious application of the theorems of Abel and Jacobi Poincaré arrived at a rapid derivation of some of the major results of Picard and the Italian geometers. I shall mainly deal with the part referring to Severi's theory of the base.

Let me first put in a most convenient form due to Picard and Castelnuovo the ω^1 of the first kind of the curve H_y. A base for them may be chosen of type

$$(13.1) \qquad du_s = \frac{Q_s(x, y, z)dx}{F'_z}, \qquad s \leqq p$$

where Q_s is an adjoint polynomial of degree $m-3$ in x and z. For the first $p-q$ the polynomial is of degree $m-3$ in x, y and z. For $s=p-q+1, \cdots, p$, it is of degree $m-2$ in x, y, z. Actually within this last range one may choose the Q_h so that the du_h only have constant periods relative to the invariant cycles and zero relative to the rest. As for the first $p-q$ they will have zero periods relative to the invariant cycles.

Let C be a curve on F and $M_1, \cdots, M_n$ its intersections with H_y. The sums from a fixed point P_1 of H_y to the M_k

$$\sum_k \int_{P_1}^{M_k} du_s = v_s, \qquad s \leqq p,$$

(integration in H_y) are Poincaré's normal functions.

Let L be the set of integration paths and with δ_k^1 as in (8) let

$$\Omega_{ks} = \int_{\delta_k^1} du_s. \tag{13.2}$$

Then with the μ_k as in (9.2) we find

$$\begin{aligned} v_h &= \sum \frac{\mu_k}{2\pi i} \int_a^{a_k} \frac{\Omega_{kh}(Y)dY}{Y-y} \qquad h = 1, 2, \cdots, p-q \\ v_{p-q+j} &= \alpha_j(\text{constant}) \qquad j = 1, 2, \cdots, q. \end{aligned} \tag{13.3}$$

REMARK. The only condition imposed upon the points M_k is that they be rationally defined together on H_y. They may represent for example the following special cases: (a) any sum of multiples of the fixed points P_h of H_y, in particular they may represent just μP_h; (b) if C is reducible say $C = C_1 + C_2$ with M_{1h} and M_{2h} as respective intersections one might have any set $t_1 \sum M_{1h} + t_2 \sum M_{2h}$, and similarly for several reducible curves; (c) any combination of the preceding two cases. In what follows, "curve" must be understood to include all these special cases.

As usual when dealing with abelian sums the v_s are only determined mod periods of the related u_s.

(13.4) THEOREM OF POINCARÉ. *N.a.s.c. to have a set of v_s given by* (13.3) *represent a curve by means of Jacobi's inversion theorem are*

$$\sum_h \mu_h \Omega_{hs}(a) = 0, \qquad s = 1, 2, \cdots, p. \tag{13.4a}$$

(13.4b) *Let $P(x, y, z)$ be any linear combination of the $P_s(x, y, z)$ divisible by $y-a$ and let*

$$du = \frac{P(x, y, z)dx}{F'_z}, \qquad \Omega_k(y) = \int_{\delta_k^1} du.$$

Then one must have

$$\sum \mu_k \int_a^{a_k} \Omega_k(y)dy = 0. \tag{13.4c}$$

(13.5) *Comparison with Severi's results.* Let the collection $\{\mu_s\}$ of the μ_s occurring in any set of normal functions be designated by μ. The collection $\{\mu\}$ is a module U. Let U_0 be the submodule of all the elements μ^0 corresponding to the $\sum t_h P_h$, t_h an integer. The quotient $U_1 = U/U_0$ is the factor-module corresponding to all the curves which are not a plane section or more generally a $\sum t_j P_j$. The U_1 module has a base made up of $\rho - 1$ algebraically independent curves and a minimum base consisting of $\rho + \sigma - 2$ curves. By adding the $\mu(H)$ one has respectively ρ and $\rho + \sigma - 1$ for base and minimum base.

The quotient module $U_1 = U/U_0$ is the module of all μ of curves none a plane section H. The module $U_1 + H = M_p$ is the *Poincaré* module and it is isomorphic with the Severi module M_s.

(13.6) REMARK. In order to get rapidly to the "heart of the matter" I have assumed at the outset that in (13.1) the polynomials Q_{p-q+j} were of degree $m-2$ in x, y, z. This was based upon rather deep results of Picard and Castelnuovo. Poincaré however merely assumed that the degree of Q_{p-q+j} was $m - 2 + \nu_j$. As a consequence in (13.3) the constants α_j must be replaced by polynomials $\alpha_j(y)$ of degree ν_j. Then Poincaré shows that on the strength of the theorems of Abel and Jacobi every $\nu_j = 0$ hence the $\alpha_j(y)$ must be constants and one has in fact the form (13.3).

Notice also that from the form of the Q_{m-3+j} one may find another adjoint polynomial R_{m-3+j} of degree $m-3$ in y, z and $m-2$ in x, y, z such that

$$dw_j = \frac{Q_{m-3+j}dx + R_{m-3+j}dy}{F_z'}$$

is a closed ω^1 of the first kind. The set $\{dw_j\}$ is then shown to be a base for such differentials. This proves rapidly that their "independent number" is q. Finally since the α_j are arbitrary constants the form of (13.3) shows implicitly that a complete (maximal) algebraic system of curves consists of ∞^q linear systems in one-one correspondence with the points of an abelian q dimensional variety (see IV, §17).

In outline this shows how normal functions enabled Poincaré to obtain with ease a number of the major results of Picard and the Italian geometers.

IV. Analysis with Topology

14. **On the Betti number R_1.** In II I recalled my proof that R_1 is even and $R_1=2q$, the number of invariant cycles of the curve H_y. This gave incidentally a direct topological proof that the number of independent one-cycles in any curve of a sufficiently general system was fixed and equal to R_1. It showed also that the irregularity q of a surface, in the sense of Castelnuovo and Enriques was actually a topological character. As I will show in §17, a topological proof that q is an "absolute invariant" is immediate. Notice also that the distribution of complete algebraic systems in ∞^q linear systems, referred to in (13.6) is also shown to have topological character.

(14.1) Let $\{du_k\}$, $k\leqq q$, be a base for the closed ω^1 of the first kind of F. On H_y they coincide with the u_{p-q+h} of §13. Let $\pi_{k\mu}$, $\mu\leqq 2q$, be the periods of u_k relative to a homology base $\{\gamma_\mu^1\}$, $\mu\leqq 2q$, for the one-cycles of F. From the fact that the periods of the differentials of the first kind of H_y form a Riemann matrix, we infer:

(14.2) Theorem. *The matrix π of the periods of the u_k and their conjugates $\bar{u}_k$ as to the γ_μ^2 is a Riemann $2q\times 2q$ matrix.*

15. **On algebraic two-cycles.** A collection of mutually homologous 2-cycles is a *homology class*. In this manner algebraic cycles yield algebraic homology classes. Through addition they generate a module M_L. Thus in relation to the collection of curves on a surface F there are three definite modules: M_S (Severi module), M_P (Poincaré module) and M_L (Lefschetz module).

(15.1) Theorem. *The three modules M_S, M_P and M_L are identical.*

This property will be a final consequence of an extensive argument.

Returning to Poincaré's normal functions (III, §13) a glance at his two conditions for a set of normal functions to represent an algebraic curve reveals immediately that Poincaré's first condition simply means that

$$\begin{cases}\gamma^2 = \sum \mu_k\Delta_k + (H_a)\\ \sum \mu_k\overset{1}{\delta_k} \sim 0 \qquad \text{in } H_y\end{cases} \tag{15.2}$$

is a cycle. As to the second condition it says merely that if

$$\omega^2 = \frac{Q(x,y,z)dxdy}{F_z'}$$

is of the first kind, that is if Q is adjoint of order $m-4$, then

$$\int_{\gamma 2} \omega^2 = 0.$$

Hence Poincaré's conditions are equivalent to the following result:

(15.3) THEOREM. *A n.a.s.c. for a cycle γ^2 to be algebraic is that the period of every 2-form of the first kind relative to γ^2 be zero.*

(15.4) REMARK. Among all the "algebraic" curves there were included all the sums $\sum m_j P_j$, where the P_j are the fixed points of H_y. It is evident that for these special "2-cycles" $\int \omega^2$ is zero.

(15.5) COROLLARY. *Severi's number σ is merely the order of the torsion group of the two-cycles (or equally of the torsion group of the one-cycles).*

For if γ^2 is a torsion 2-cycle we have $\lambda\gamma^2 \sim 0$, $\lambda \neq 0$, and hence

$$\int_{\gamma 2} \omega^2 = 0$$

for every ω^2 of the first kind.

(15.6) THEOREM. *The number ρ is the Betti number of algebraic cycles.*

This is a consequence of the following property:

(15.7) *Let $C_1, \cdots, C_s$ be a set of curves and let $\overline{C}_h$ be the cycle of C_h. Then*

P_a: *algebraic independence of the C_h*

P_h: *homology independence of the $\overline{C}_h$*

are equivalent properties.

From obvious considerations P_a implies P_h. Conversely let P_h hold. We must show that $\overline{C} \sim 0$ implies $\lambda C = 0$, $\lambda \neq 0$. Here I follow Albanese's rapid argument. Let $C = A - B$, A and B effective. Since $\overline{A} \sim \overline{B}$ and $[CD] = (\overline{C}, \overline{D})$ we have

$$[A^2] = [AB] = [B^2]; \qquad [AH] = [BH]$$

where H is a plane section. Hence Severi's independence criterion is violated between A and B. Consequently $\lambda A = \mu B$, $\lambda\mu \neq 0$. From $[AH] = [BH]$ follows $\lambda = \mu$ and therefore $\lambda(A - B) = 0 = \lambda C$, $\lambda \neq 0$. This proves (15.7).

It follows that $M_S = M_L$ and as $M_P = M_S$, (15.1) is proved.

Notice that we may now give the following *very simple* definition of virtual curve: it is merely an algebraic 2-cycle. Simplicity is even augmentable by replacing everywhere the symbol = of algebraic dependence (=) by the homology symbol $\sim$.

16. **On 2-forms of the second kind.** The basic result is the proof of the formula

$$\rho_0 = R_2 - \rho. \tag{16.1}$$

I shall just indicate an outline of my proof. I shall also show that the process outlined obtains incidentally Picard's fundamental result for ρ concerning logarithmic curves of a closed ω^1 of third kind. The steps follow closely an analogous outline in my monograph [9].

For convenience I call ω^1 and ω^2 *regular* when

$$\omega^1 = \frac{Pdy - Qdx}{\phi(y)F_z'}, \qquad \omega^2 = \frac{Pdxdy}{\phi(y)F_z'},$$

where P and Q are adjoint polynomials and $\phi(y)$ is a polynomial.

If $\omega^2 = d\omega^1$, ω_2 is said to be *improper*. Thus ρ_0 is the dimension of the vector space of the ω^2 of the second kind mod those which are improper.

By *reduction* of ω^2 I understand the subtraction of an improper ω^2.

I. *The periods and residues of a normal 2-form are arbitrary.*

II. *One may reduce any ω^2 of the second kind to the regular type.*

III. *A regular ω^2 such that $\int\omega^2$ has neither residues nor periods is reducible to a regular $d\omega^1$.*

Except for the presence of the polynomial $\phi(y)$ the proofs of the preceding propositions are very close to those of Picard. It is true that allowing $\phi(y)$ in regular ω^1 and ω^2 considerably simplifies every step (see [9, Note I]).

IV. *Let C be a curve of order s. We may choose coordinates such that C does not pass through any of the fixed points P_j of H_y, nor through the points of contact of the planes $y = a_k$. One may form an $\omega^1 = Rdx$, $R \in K(F)$ possessing on H_y the s-logarithmic points of CH_y with logarithmic period $2\pi i$ and say P_1 with logarithmic period $-2\pi is$. One may even select R so that $(\partial R/\partial y)dx$ has no periods. From this follows that there is an $S(x, y, z) \in K(F)$ such that*

$$\omega^2 = d(Rdx + Sdy)$$

is regular.

Take now $C_1, C_2, \cdots, C_t$ and the axes so chosen that they all behave like C. Let ω_h^2 be analagous to ω^2 for C_h.

Owing to III *it is now readily shown that n.a.s.c. in order that some linear combination*

$$\omega^2 = \sum \alpha_h \omega_h^1$$

be without periods is that the C_s *and* H *be logarithmic curves of a closed* ω^1.

Since R_2 is finite there is a least $\rho - 1$ such that for $s = \rho$ the curve C_h, H are logarithmic curves of a closed ω^1 of the third kind. Hence

(16.2) *Picard's fundamental property for* ρ *is a consequence of the finiteness of the Betti number* R_2.

V. *To proceed one may form* $\rho - 1$ *linearly independent* ω^2 *which are improper. Since the total number of distinct periods is equal to* $R_2(F-H) = R_2 - 1$ *we have then* $\rho_0 = R_2 - \rho$, *as asserted.*

(16.3) *On Picard's treatment of* ρ_0 *and* ρ. Owing to lack of topological technique Picard proved directly that ρ_0 was finite by showing through strong algebraic operations that if

$$\omega^2 = \frac{Q(x, y, z)dxdy}{F'_z}$$

where Q is adjoint, is of the second kind, the degree of Q was bounded.

Although Picard did not observe it, his later treatment of ω^2 of the second kind contained implicitly (argument of 16.2) the proof that ρ had the property by which he defined it relative to closed ω^1 of the third kind.

17. Absolute and relative birational invariance. Take again a general n-variety

$$F(x_1, \cdots, x_n, y) = 0. \tag{17.1}$$

Let $\{\xi_0, \cdots, \xi_r\}$ be a homogeneous base for the function field $K(F)$. Then the system

$$\tau y_h = \xi_h, \qquad h = 0, 1, 2, \cdots, r > n$$

represents a model F_1 of F in the projective space S^r, with homogeneous coordinates y_h. If $\{\eta_0, \cdots, \eta_s\}$, $s > n$, is a second homogeneous base for $K(F)$, the system

$$\sigma z_k = \eta_k, \qquad k = 0, 1, \cdots, s$$

represents a second model F_2 of F in a projective space S^s. Since $\{\xi_h\}$ and $\{\eta_k\}$ are homogeneous bases for $K(F)$ F_1 and F_2 are birationally transformable into one another. The simple example of two elliptic curves of degrees 3 and 4 show however that the corresponding structures need not be homeomorphic. The difficulty is caused by the presence of singularities. A standard device for curves enables one to "forget" singularities and restore homeomorphism. No such device is known for a V^n, $n>1$.

For simplicity let me limit the argument to surfaces. I have really considered a surface as a nonsingular model in some projective space. Let F_1, F_2 be two such distinct models and suppose that the *field* $K(F)$ *is not that of a ruled surface.* Then according to Castelnuovo and Enriques a birational transformation T: $F_1 \rightarrow F_2$ may take a finite number δ_{12} of *exceptional* points of F into *disjoint* nonsingular rational curves. There exists an analogous δ_{21} for T^{-1}. Let a point P of F_1 be sent by T into a curve C of F_2. Since C is rational and nonsingular it is topologically a sphere. Hence its characteristic $\chi(C)=2$. Hence the gain in $\chi(F_1)$ through δ_{12} exceptional points is δ_{12}. Therefore

$$\chi(F_1) + \delta_{12} = \chi(F_2) + \delta_{21}. \tag{17.2}$$

Now a character, numerical or other of F is said to be an *absolute* invariant if it is unchanged under all transformations such as T. A *relative invariant* is one that may change under certain transformations T.

Let me examine some of the characters that have been introduced.

It is readily shown that under T both R_2 and ρ are increased by the same amount $\delta_{12}-\delta_{21}$. Hence both are relative invariants and $\rho_0=R_2-\rho$ is an absolute invariant.

Since

$$\chi(F) = R_2 - 2R_1 + 2$$

and both χ and R_2 vary in the same way, χ is a relative invariant and R_1 is an absolute invariant.

Therefore:

(17.3) *The dimensions of the spaces of closed ω^1 of the first and second kinds and of ω^2 of the second kind are absolute invariants.*

18. **Application to abelian varieties.** Let Π and M be a Riemann matrix and its principal matrix (see §3).

Introduce the following vectors:

$$u = (u_1, \cdots, u_{2p}), \qquad u_{p+j} = \bar{u}_j$$

$$\pi_\mu = (\pi_{1\mu}, \cdots, \pi_{2p,\mu}), \qquad \mu = 1, 2, \cdots, 2p; \qquad \pi_{p+j,\mu} = \bar{\pi}_{j\mu}.$$

Through the hyperplanes

$$u = \sum s_\mu \pi_\mu,$$

s_μ an integer, real $2p$-space is partitioned in a familiar way into paralellotopes. A suitable fundamental domain D is

$$u = \sum t_\mu \pi_\mu, \qquad 0 \leqq t_\mu < 1.$$

The identification of congruent boundary points turns this domain into a $2p$-ring R^{2p} (product of $2p$ circles):

Corresponding to Π and M there may be defined a whole family of functions θ of various orders. Each such function ϕ is a holomorphic function in the domain D. Those of a given order, say n, are characterized by the property that $\phi_n(u+\pi_\mu)=\phi_n(u)$ times a fixed linear exponential function of u. I have shown that one may find an n such that if $\{\theta_n^j(u+\alpha)\}$, α a fixed p-vector, $j=0, 1, \cdots, r$, is a finite linear base for the $\theta_n(u+\alpha)$ then the system

$$kx_j = \theta_n^j(u + \alpha)$$

represents a nonsingular p-variety V^p in projective S^r, and this V^p is in analytic homeomorphism with the ring R^{2p}. This is an abelian p-variety (see [8])

The topological relation $V^p \leftrightarrow R^{2p}$ assigns an exceptionally simple topology to V^p. Let the edges of D oriented from the origin out be designated by $1, 2, \cdots, 2p$. Any i_h defines a one-cycle represented by (i_h); any two edges i_h, i_k define a 2-cycle represented by (i_h, i_k), etc. The (i_h, i_k), $i_h < i_k$ form a base for the 2-cycles of V^p, etc.

I am mainly concerned with the 2-cycles. In view of $(\nu, \mu) = -(\mu, \nu)$ a general 2-cycle is represented by a homology

$$\gamma^2 \sim \sum m_{\mu\nu}(\mu, \nu), \qquad m_{\mu\nu} = - m_{\nu\mu}.$$

On the other hand

$$\overset{2}{\omega}_{jk} = du_j du_k; \qquad j, k \leqq p; \qquad j < k$$

is a closed 2-form of the first kind of V^p and $\{du_h du_k\}$ is a base for all such forms.

(18.2) REMARKS. *On a general n-variety V^n, $n \geqq 2$.* Considerations of the same type as in §12 may be extended automatically to algebraic dependence of hypersurfaces of V^n (its V^{n-1}), and also to their $(2n-2)$-cycles. Algebraic and homology dependence give rise to a number $\rho(V^n)$. I single out especially the following proposition from [9, p. 104] (Corollary):

(18.3) THEOREM. *Let Φ be a fixed surface of V^n (general intersection of hyperplane sections of V^n) and let $C_1, \cdots, C_s$ be hypersurfaces which cut Φ in curves C_h^*, $h \leqq s$. Then the following relations are all equivalent: relations of algebraic dependence between the C_h in V^n, the same between the C_h^* in Φ; relations of homology between the standard oriented cycles $\overline{C}_h$, C_h in V^n; the same for the C_h^* in Φ.*

Returning now to the abelian variety V^p let Φ, C, C^* be this time the same as above but for V^p. Now the ω_{hk}^2 taken on Φ become ω^2 of the first kind for Φ. If $\{C_s\}$, $s \leqq \rho(V^p)$, is a base as to $=$, or equivalently as to $\sim$ and algebraic $(2p-2)$-cycles of V^p, then the same holds for the curves C_s^* in Φ. Hence by theorem (15.2):

$$\int_{C_s^*} du_j du_l = 0; \qquad j, l \leqq p. \tag{18.4}$$

On the other hand since the (μ, ν) are cycles in Φ we have in Φ

$$C_s^* \sim \sum m_{\mu\nu}^s(\mu, \nu), \qquad m_{\mu\nu} = -m_{\nu\mu}.$$

Hence (18.4) yields

$$\sum m_{\mu\nu}^s \pi_{j\mu} \pi_{k\nu} = 0; \qquad j, k \leqq p. \tag{18.5}$$

This really means that the ρ matrices $[m_{\mu\nu}^s]$ are linearly independent singularity matrices for the Riemann matrix Π. If the singularity index of Π is k, then one must have

$$\rho \leqq k. \tag{18.6}$$

The possible inequality is due to the fact that an algebraic 2-cycle of Φ must satisfy a relation such as (18.4) not merely with respect to the closed ω^2 of the first kind of V^p in Φ, but also with respect to all ω^2 of the first kind of Φ, and one cannot exclude the possible existence of such ω^2 other than the closed taken on Φ. However, the following two properties hold:

(a) There is a base for the forms M made up of principal forms.

(b) Each principal M gives rise to a particular system of functions ϕ à la θ. These functions are said to be *intermediary*.

(c) If $\{M_j\}$, $j \leqq k$, is a base for the matrices M, and ϕ_j is an intermediary function relative to M_j then $\phi_j = 0$ represents a hypersurface of V^p and these hypersurfaces are algebraically independent.

It follows that $\rho \geqq k$ and therefore

$$\rho = k. \tag{18.7}$$

This is the result that I was looking for.

Actually the relations between the hypersurfaces *as* cycles and their Severi independence are the same as for their sections with the surface Φ. That is,

(18.8) THEOREM. *For hypersurfaces of V^p algebraic dependence and homology in V^p are equivalent relations.*

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