THE GENERAL CANONICAL CORRELATION DISTRIBUTION

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1. Summary. The general canonical correlation distribution is given as a multiple power series in the true canonical correlations ρ_i. When only one true correlation is not zero, this series is expressible as a generalized hypergeometric function, for the cases both of non-central means and of correlations proper. In the general case of more than one non-zero true correlation the coefficients in the expansion depend on the conditional moments of the sample correlations between the pairs of transformed variables representing the true canonical variables, when the sample canonical correlations between the sample canonical variables are fixed. Methods are given of obtaining these coefficients for both cases, non-central means and correlations proper; and their form up to the fourth order, corresponding to $O(\rho^8)$ in the expansion, listed in Appendix I. The detailed terms making up these coefficients are given, in the case of two non-zero correlations, up to the fourth order, and in the general case, up to the third order, in Appendix II.

2. Introductory remarks; the case of zero roots. In the statistical theory of the relation of one vector variate with another (see Hotelling [1]), the simultaneous distribution of the canonical correlations r_i, which are the roots of a certain determinantal equation, was first obtained in 1939 (Fisher [2], Hsu [3], Roy [4]) in the special but important case when the true roots or correlations ρ_i are zero. Roy [5] has since investigated the case where the true roots are not zero when these non-zero values arise from non-central means. The present investigation is primarily intended to cover the alternative case where non-zero roots arise from the existence of true correlations ρ_i. The method developed is, however, also applicable to the case of non-central means; and it is shown that the general distribution, which for more than one non-zero root becomes very complicated, does not in the case of non-central means agree with the distribution given by Roy [5] except in the case of only one non-zero root.[1]

It will be convenient in this introductory section to sketch (with slight modifications) the method used by Hsu [3] to obtain the solution in the case of zero roots, as some of his intermediate formulae are useful for the present developments. We consider a dependent vector variate with p components, and an independent[2] vector variate with q components. For definiteness we assume

[1] This conclusion has also been reached by T. W. Anderson, who has given a solution of the non-central means problem in the cases of either one or two non-zero roots, (*Annals of Math. Stat.*, Vol. 17 (1946), pp. 409–431).

[2] This classification of a variate as the "dependent variate" or "independent variate" is in the regression sense, and does not necessarily imply statistical dependence or independence.

$p \leq q$, and the sample with $n(\geq p + q)$ degrees of freedom corresponding to the dependent variate is divided in the usual way (see, for example, [6]) into a part with q degrees of freedom corresponding to the independent variate and the remaining part with $n - q$ degrees of freedom. If a_{ij}, b_{ij} denote the sums of squares and products corresponding to this division, then it is known that the joint distribution of a_{ij} and b_{ij}, if the dependent vector variate is normal and actually, in the statistical sense, independent of the second vector variate, is

$$(1)\qquad \frac{|A|^{\frac{1}{2}(q-p-1)}|B|^{\frac{1}{2}(n-q-p-1)}\exp\left[-\frac{1}{2}\sum_{i=1}^{p}(a_{ii}+b_{ii})\right]da\,db}{2^{\frac{1}{2}np}\pi^{\frac{1}{4}p(p-1)}\prod_{i=0}^{p-1}\{\Gamma[\frac{1}{2}(q-i)]\Gamma[\frac{1}{2}(n-q-i)]\}}$$

where $|A|$ denotes the determinant of the matrix $A \equiv \{a_{ij}\}$, and da the product of differentials da_{ij}, and where for convenience the variance matrix of the dependent variate is taken to be the unit matrix.

We make the transformation specified by

$$(2)\qquad \begin{aligned} A &= WDW', \\ A + B &= WW', \end{aligned}$$

where D is a diagonal matrix of the quantities r_i^2 in descending order of magnitude, and $W = \{w_{ij}\}$ is a matrix (with transpose W') uniquely determined by (2) except for an ambiguity of sign for each column; this ambiguity can be eliminated by choosing positive elements in the first row. The Jacobian Δ of the transformation may be shown to be

$$(3)\qquad \Delta = 2^p\,|WW'|^{\frac{1}{2}p+1}\prod_{i=1}^{p}\prod_{j=i+1}^{p}(r_i^2 - r_j^2).$$

By direct substitution, we obtain from (1) the distribution

$$p(a_{ij}, b_{ij}) = p(w_{ij}, r_i^2) = p(w_{ij})p(r_i^2),$$

where $p(x)$ is a general notation[3] for a distribution function in one or more variates x, (including the differential elements); for $p(w_{ij})$ and $p(r_i^2)$ we have

$$(4)\qquad p(w_{ij}) = C_1\,|WW'|^{\frac{1}{2}(n-p)}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{p}w_{ij}^2\right]dw,$$

$$(5)\qquad p(r_i^2) = C_2\prod_{i=1}^{p}\left\{(r_i^2)^{\frac{1}{2}(q-p-1)}(1-r_i^2)^{\frac{1}{2}(n-q-p-1)}\prod_{j=i+1}^{p}(r_i^2-r_j^2)\right\}dr^2,$$

[3] The probability symbol is not of course to be confused with the number p of components in the dependent variate. It should also be noted that for convenience $p(x_i)$ is used to denote the *joint* probability for a set of quantities x_i, whereas $p(x_1)$ or $p(x_2)$ denotes the probability for the specified variate x_1 or x_2 considered separately.

the constants C_1 and C_2 being arranged to give unity on integration of $p(w_{ij})$ or $p(r_i^2)$, i.e. we have

(6) $$C_1 = 2^{p-\frac{1}{2}np}\,\pi^{-\frac{1}{2}p^2}\prod_{i=0}^{p-1}\{\Gamma[\tfrac{1}{2}(p-i)]/\Gamma[\tfrac{1}{2}(n-i)]\},$$

(the w_{ij} varying from $-\infty$ to ∞ except that $w_{1i} \geq 0$), and

(7) $$C_2 = \pi^{\frac{1}{2}p}\prod_{i=0}^{p-1}\{\Gamma[\tfrac{1}{2}(n-i)]/(\Gamma[\tfrac{1}{2}(p-i)]\Gamma[\tfrac{1}{2}(q-i)]\Gamma[\tfrac{1}{2}(n-q-i)])\}.$$

3. Formal determination of the general distribution. The method to be adopted of obtaining the general distribution from the particular case quoted in equation (5) above is the same in principle as the one adopted by Fisher [7] in his derivation of the general distribution of the multiple correlation coefficient. Since the argument is more involved in the present problem, it will be presented first in formal probability terms, before the details of the solution are examined.

We consider a transformation of the components of each vector variate to the true canonical components. Let the observed ordinary correlation coefficients of these mutually independent components for one vector variate with the corresponding components of the second vector variate be denoted by s_i. The true correlations are the true canonical correlations ρ_i. Then we have for the general canonical correlation distribution denoted by[4] $p(r_i \mid \rho_i)$, the expression

$$\begin{aligned} p(r_i \mid \rho_i) &= \int_{s_i} p(r_i, s_i \mid \rho_i) \\ &= \int_{s_i} p(r_i \mid s_i, \rho_i)p(s_i \mid \rho_i) \\ &= \int_{s_i} p(r_i \mid s_i)p(s_1 \mid \rho_1)p(s_2 \mid \rho_2) \cdots p(s_p \mid \rho_p), \end{aligned}$$

the substitution $p(r_i \mid s_i)$ for $p(r_i \mid s_i, \rho_i)$ following from the sufficiency of the independent correlations s_i of the corresponding pairs of canonical components, as statistics for the ρ_i. We now define the function $g(s_1, \rho_1)$ by the relation

$$p(s_1 \mid \rho_1) = p(s_1 \mid \rho_1 = 0)\, g(s_1, \rho_1),$$

whence we have the general solution

(8) $$\begin{aligned} p(r_i \mid \rho_i) &= \int_{s_i} p(r_i \mid s_i)p(s_1 \mid \rho_1 = 0)g(s_1, \rho_1)p(s_2 \mid \rho_2 = 0)g(s_2, \rho_2) \cdots \\ &= \int_{s_i} p(r_i, s_i \mid \rho_i = 0)g(s_1, \rho_1)g(s_2, \rho_2) \cdots \\ &= p(r_i \mid \rho_i = 0) \int_{s_i} p(s_i \mid r_i, \rho_i = 0)g(s_1, \rho_1)g(s_2, \rho_2) \cdots \end{aligned}$$

for $p(r_i \mid \rho_i)$ in terms of the special case $p(r_i \mid \rho_i = 0)$.

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[4] Quantities to the right of the vertical stroke in a probability bracket are *given* quantities on which the probability distribution depends.

Now according as the independent vector variate is considered as (a) a normal variate with which the dependent variate is correlated, (b) a fixed vector in sample space (this includes the non-central means case) Fisher [7] has shown that the distribution of the *multiple* correlation R of a single dependent variate with an independent variate comprising m components is $p(R \mid \rho = 0)g(R, \rho)$, where

$$(9) \quad \begin{aligned} &(a) \qquad g(R, \rho) = F(\tfrac{1}{2} n, \tfrac{1}{2} n; \tfrac{1}{2} m; \rho^2 R^2)\,(1 - \rho^2)^{\frac{1}{2}n}, \\ &(b) \qquad g(R, \rho) = F(\tfrac{1}{2} n; \tfrac{1}{2} m; \tfrac{1}{2} \beta^2 R^2)\, e^{-\frac{1}{2}\beta^2}, \end{aligned}$$

where we replace ρ^2 by a parameter β^2 in case (b), and the notation for hypergeometric functions used is:

$$F(\alpha; \beta; x) = 1 + \frac{\alpha x}{\beta} + \frac{\alpha(\alpha + 1)x^2}{\beta(\beta + 1)2!} + \cdots,$$

$$F(\alpha_1, \alpha_2; \beta; x) = 1 + \frac{\alpha_1 \alpha_2 x}{\beta} + \frac{\alpha_1(\alpha_2 + 1)\alpha_2(\alpha_2 + 1)x^2}{\beta(\beta + 1)\,2!} + \cdots.$$

It follows that we may write $g(s_1, \rho_1)$ above in the form

$$(10) \quad \begin{aligned} &(a) \qquad g(s_1, \rho_1) = F(\tfrac{1}{2} n, \tfrac{1}{2} n; \tfrac{1}{2}; \rho_1^2 s_1^2)\,(1 - \rho_1^2)^{\frac{1}{2}n}, \\ &(b) \qquad g(s_1, \rho_1) = F(\tfrac{1}{2} n; \tfrac{1}{2}; \tfrac{1}{2}\beta_1^2 s_1^2)e^{-\frac{1}{2}\beta_1^2}, \end{aligned}$$

by putting $m = 1$ in (9), (the signs of the s_i are arbitrary, so that we are essentially concerned, as in the multiple correlation distribution, with the squares of the correlations). From these series expansions the integral in (8) consists of terms corresponding to the conditional moments, for any set of positive integers $t_1, t_2, \cdots, t_p$,

$$\begin{aligned} \mu(t_1, t_2, \cdots, t_p) &\equiv E\{(s_1^2)^{t_1}(s_2^2)^{t_2} \cdots (s_p^2)^{t_p} \mid r_i\} \\ &= \int_{s_i} (s_1^2)^{t_1}(s_2^2)^{t_2} \cdots (s_p^2)^{t_p}\, p(s_i \mid r_i, \rho_i = 0). \end{aligned}$$

In the particular case when only $\rho_1 \neq 0$, the moments $\mu(t) \equiv E\{(s_1^2)^t \mid r_i\}$ from the single factor $g(s_1, \rho_1)$ are all that arise, but in the general case it is important to notice that the quantities s_i^2, while statistically independent when unrestricted, are no longer independent for the *conditional* distribution $p(s_i \mid r_i, \rho_i = 0)$.

This completes the formal solution. It remains to evaluate $\mu(t_1, t_2, \cdots, t_p)$.

4. The conditional moment $\mu(t_1, t_2, \cdots, t_p)$. First of all we note from the choice of the components of the dependent vector variate, applying the analysis of section 2 to such components, that the multiple correlation R_i between the ith component and the q components of the independent variate is given by

$$R_i^2 = a_{ii}/(a_{ii} + b_{ii}) = \alpha_{i1}^2 r_1^2 + \alpha_{i2}^2 r_2^2 + \cdots + \alpha_{ip}^2 r_p^2,$$

where

$$\alpha_{ij} = w_{ij}/\sqrt{(w_{i1}^2 + w_{i2}^2 + \cdots + w_{ip}^2)}.$$

To obtain the distribution of the α_{ij} from that of the w_{ij}, we note that the w_{ij} distribution (4) is normal (allowing for convenience w_{1i} to vary from $-\infty$ to ∞) except for the "linkage factor"

$$2^{-\frac{1}{2}p(n-p)} \mid WW' \mid^{\frac{1}{2}(n-p)} \prod_{i=0}^{p-1} \{\Gamma[\tfrac{1}{2}(p-i)/\Gamma[\tfrac{1}{2}(n-i)]\}.$$

Hence if we transform to the variables c_{ii}, θ_{ij} defined by

$$(11)\qquad \begin{aligned} c_{ii} &= w_{i1}^2 + w_{i2}^2 + \cdots + w_{ip}^2, \\ \alpha_{i1} &= \cos\theta_{i1}, \\ \alpha_{i2} &= \sin\theta_{i1}\cos\theta_{i2}, \\ \alpha_{i3} &= \sin\theta_{i1}\sin\theta_{i2}\cos\theta_{i3}, \\ &\ \ \vdots \\ \alpha_{ip} &= \sin\theta_{i1}\sin\theta_{i2}\sin\theta_{i3}\cdots\sin\theta_{i,p-1}, \end{aligned}$$

the sets c_{ii}, θ_{ij} which for normal w_{ij} would all be independent with distributions:

$$(12)\qquad \begin{aligned} &p(c_{ii}) \text{ --- } \chi^2 \text{ distribution with } p \text{ degrees of freedom}, \\ &p(\theta_{ij}) \propto \sin^{p-j-1}\theta_{ij}\, d\theta_{ij}, \\ &(0 \leq \theta_{ij} \leq \pi \text{ for } j = 1, 2, \cdots p-2;\ 0 \leq \theta_{i,p-1} \leq 2\pi), \end{aligned}$$

in general retain their independence for given i, but the linkage factor results in an elevation of the χ^2 distributions to n degrees of freedom, and a linkage factor for the θ_{ij} distributions of

$$(13)\qquad \mid \Lambda \mid^{\frac{1}{2}(n-p)} \prod_{i=1}^{p-1} \left\{\frac{\Gamma[\frac{1}{2}(p-i)]\Gamma[\frac{1}{2}n]}{\Gamma[\frac{1}{2}(n-i)]\Gamma[\frac{1}{2}p]}\right\},$$

where

$$\Lambda \equiv \{\alpha_{i1}\alpha_{j1} + \alpha_{i2}\alpha_{j2} + \cdots + \alpha_{ip}\alpha_{jp}\}.$$

We may now, having obtained the distribution of the α_{ij}, note their geometrical interpretation. Let us denote the p components of the dependent variate in n-dimensional sample space by the p vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_p$. Let the p orthogonal canonical components corresponding to the *sample* canonical correlations r_i be denoted by the p unit vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_p$. Let the corresponding components for the independent variate be $\mathbf{n}_i$, $\mathbf{y}_i$. The "linkage factor" merely represents the allowance that must be made in the mutual relations of the $\boldsymbol{\xi}$-vectors for the fact that while they must lie in the p-space of the $\mathbf{x}$-vectors, they really belong to the original n-space. We may identify the w_{ij} with the coefficients in the equation

$$\xi_i = w_{i1}\mathbf{x}_1 + w_{i2}\mathbf{x}_2 + \cdots + w_{ip}\mathbf{x}_p \,, \tag{14}$$

where

$$\xi_i^2 \equiv w_{i1}^2 + w_{i2}^2 + \cdots + w_{ip}^2$$

is a χ^2 with n, and not p, degrees of freedom. If we now suppose for convenience ξ_i to be a unit vector, we have in place of (14)

$$\xi_i = \alpha_{i1}\mathbf{x}_1 + \alpha_{i2}\mathbf{x}_2 + \cdots + \alpha_{ip}\mathbf{x}_p \,, \tag{15}$$

with a projection, on the q-space of the $\mathbf{y}$-vectors, of ζ_i, say, where

$$\zeta_i = \alpha_{i1}r_1\mathbf{y}_1 + \alpha_{i2}r_2\mathbf{y}_2 + \cdots + \alpha_{ip}r_p\mathbf{y}_p \,,$$

and hence, as already noted in the algebraic derivation,

$$R_i^2 = (\xi_i \cdot \zeta_i)^2/\zeta_i^2 = \alpha_{i1}^2r_1^2 + \alpha_{i2}^2r_2^2 + \cdots + \alpha_{ip}^2r_p^2 \,,$$

where $(\xi \cdot \zeta)$ denotes a scalar product. The linkage factor (13) indicates that the ξ_i vectors in (15) are not independent in the p-space of the $\mathbf{x}$-vectors, the distribution of their mutual configuration being determined by n-space.

This interpretation enables us to determine the moments of the distribution $p(s_i \mid r_i)$. For if corresponding to (15) we write

$$\mathbf{n}_i = \beta_{i1}\mathbf{y}_1 + \beta_{i2}\mathbf{y}_2 + \cdots + \beta_{iq}\mathbf{y}_q \,, \tag{16}$$

then

$$s_i = \alpha_{i1}\beta_{i1}r_1 + \alpha_{i2}\beta_{i2}r_2 + \cdots + \alpha_{ip}\beta_{ip}r_p \,. \tag{17}$$

If we are considering case (a), the relations of the $\mathbf{n}_i$ to the $\mathbf{y}$-vectors in q-space will be similar to the relations of the ξ_i to the $\mathbf{x}$-vectors in p-space. In case (b), however, the $\mathbf{n}_i$, which represent the true canonical components of a set of q fixed vectors, must remain strictly orthogonal to each other although their relation to the $\mathbf{y}$-vectors can vary. This means that the relations of the $\mathbf{n}_i$ to the $\mathbf{y}$-vectors are determined by a random rotation of a rigid orthogonal set of q vectors in case (b). We may note that if in case (a) we allowed n to tend to infinity, the $\mathbf{n}_i$ would also become rigidly orthogonal, so that the solution in case (b) may conveniently be obtained from case (a) by retaining the same distribution of the α_i, and for the β_i letting $n \to \infty$.

Thus in either case the moments of the s_i can be obtained from (17) in terms of the moments of α_{ij} and β_{ij}, two independent sets of coefficients for which the distribution of each set is known. The above comments suffice theoretically to complete the required solution for $(s_1^2)^{t_1}(s_2^2)^{t_2} \cdots (s_p^2)^{t_p}$ is a function of α_{ij} and β_{ij}; the α_{ij} and the corresponding linkage factor can be expressed in terms of $\sin\theta_{ij}$ and $\cos\theta_{ij}$, and similarly for the β_{ij} in terms of, say, $\sin\phi_{ij}$ and $\cos\phi_{ij}$, and integration carried out over the θ_{ij} and ϕ_{ij}. This method is unfortunately too cumbersome algebraically to be of any practical value except in the case of one non-zero root. This case is considered separately before the general case is discussed further.

5. The case of only one non-zero root. Here we only require $\mu(t)$ and a comparatively simple solution is possible, the linkages within the $\boldsymbol{\xi}_i$ and $\boldsymbol{\eta}_i$ sets being irrelevant. We have in fact, if ψ is the angle between $\boldsymbol{\eta}_1$ and $\boldsymbol{\zeta}_1$, (where $\boldsymbol{\zeta}_1$ was the projection of $\boldsymbol{\xi}_1$ in the q-space), that ψ is a random angle in the q-space, since the α_{ij} and β_{ij} sets are independent. Hence *in this particular case* we may conveniently write $s_1^2 = R_1^2 \cos^2\psi$, which is just the transformation used to obtain the distribution of the multiple correlation R_1^2. Thus we may replace (10) by (9), where $R_1^2 = \alpha_{11}^2 r_1^2 + \alpha_{12}^2 r_2^2 + \cdots + \alpha_{1p}^2 r_p^2$, and

$$(R_1^2)^t = \sum_{u_1+u_2+\cdots=t} \frac{t!\,(r_1^2)^{u_1}(r_2^2)^{u_2}\cdots}{u_1!\,u_2!\cdots} \cdot \cos^{2u_1}\theta_{11}\sin^{2(t-u_1)}\theta_{11}\cos^{2u_2}\theta_{12}\sin^{2(t-u_1-u_2)}\theta_{12}\cdots,$$

where the expected value of the trigonometric term is evaluated as

$$\left\{\frac{\Gamma(u_1+\frac{1}{2})\Gamma(u_2+\frac{1}{2})\cdots}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\cdots}\right\}\frac{\Gamma(\frac{1}{2}p)}{\Gamma(\frac{1}{2}p+t)}. \tag{18}$$

We have now obtained the distribution, $(\rho_2 = \cdots = \rho_p = 0)$,

$$p(r_i \mid \rho_1 \neq 0) = p(r_i \mid \rho_1 = 0) \textstyle\sum_{u_1,u_2,\cdots} C(u_1, u_2, \cdots)(r_1^2)^{u_1}(r_2^2)^{u_2}\cdots,$$

where $p(r_i \mid \rho_i = 0)$ is given by (5); and in case (a)

$$C(u_1, u_2, \cdots) = (1-\rho_1^2)^{\frac{1}{2}n}(\rho_1^2)^t\left[\frac{\Gamma(\frac{1}{2}n+t)}{\Gamma(\frac{1}{2}n)}\right]^2 \cdot \frac{\Gamma(\frac{1}{2}p)\Gamma(\frac{1}{2}q)}{\Gamma(\frac{1}{2}p+t)\Gamma(\frac{1}{2}q+t)}\prod_{j=1}^{p}\left[\frac{\Gamma(u_j+\frac{1}{2})}{\Gamma(\frac{1}{2})u_j!}\right],$$

and in case (b)

$$C(u_1, u_2, \cdots) = e^{-\frac{1}{2}\beta_1^2}(\tfrac{1}{2}\beta_1^2)^t\frac{\Gamma(\frac{1}{2}n+t)\Gamma(\frac{1}{2}p)\Gamma(\frac{1}{2}q)}{\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}p+t)\Gamma(\frac{1}{2}q+t)}\prod_{j=1}^{p}\left[\frac{\Gamma(u_j+\frac{1}{2})}{\Gamma(\frac{1}{2})u_j!}\right],$$

where $u_1 + u_2 + \cdots + u_p$ is denoted by t. $\sum_{u_1,u_2,\cdots}$ denotes summation of all u's from 0 to ∞. The solution in either case contains a generalized hypergeometric function. If we denote the general series

$$\textstyle\sum_{u_1,u_2,\cdots}\left\{\frac{\Gamma(\alpha_1+t)\Gamma(\alpha_2+t)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+t)\Gamma(r_2+t)}\prod_{j=1}^{p}\left[\frac{\Gamma(\beta_j+u_j)x_j^{u_j}}{\Gamma(\beta_j)u_j!}\right]\right\},$$

$$\textstyle\sum_{u_1,u_2,\cdots}\left\{\frac{\Gamma(\alpha+t)}{\Gamma(\alpha)}\frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+t)\Gamma(r_2+t)}\prod_{j=1}^{p}\left[\frac{\Gamma(\beta_j+u_j)x_j^{u_j}}{\Gamma(\beta_j)u_j!}\right]\right\}$$

by

$$F(\alpha_1, \alpha_2; \beta_1, \beta_2, \cdots, \beta_p; r_1, r_2; x_1, x_2, \cdots, x_p),$$

$$F(\alpha; \beta_1, \beta_2, \cdots, \beta_p; r_1, r_2; x_1, x_2, \cdots, x_p)$$

respectively (see [8, p. 300, example 22]), then we have in case (a)

$$p(r_i \mid \rho_1 \neq 0) = p(r_i \mid \rho_i = 0)(1 - \rho_1^2)^{\frac{1}{2}n} \times F(\tfrac{1}{2}n, \tfrac{1}{2}n; \tfrac{1}{2}, \tfrac{1}{2}, \cdots, \tfrac{1}{2}; \tfrac{1}{2}p, \tfrac{1}{2}q; \rho_1^2 r_1^2, \rho_1^2 r_2^2, \cdots, \rho_1^2 r_p^2), \tag{19}$$

and in case (b)

$$p(r_i \mid \beta_1 \neq 0) = p(r_i \mid \rho_i = 0)e^{-\frac{1}{2}\beta_1^2} \times F(\tfrac{1}{2}n; \tfrac{1}{2}, \tfrac{1}{2}, \cdots, \tfrac{1}{2}; \tfrac{1}{2}p, \tfrac{1}{2}q; \tfrac{1}{2}\beta_1^2 r_1^2, \tfrac{1}{2}\beta_1^2 r_2^2, \cdots, \tfrac{1}{2}\beta_1^2 r_p^2). \tag{20}$$

An alternative operational form is obtained by noting that the sum of terms for given $t = u_1 + u_2 + \cdots + u_p$ is generated by means of the coefficient of z^t in

$$\prod_{j=1}^{p} (1 - \rho_1^2 r_j^2 z)^{-\frac{1}{2}},$$

where for definiteness we consider case (a). Hence if we write

$$F(\alpha_1, \alpha_2; r_1, r_2; x) \equiv \sum_t \frac{\Gamma(\alpha_1 + t)\Gamma(\alpha_2 + t)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1 + t)\Gamma(r_2 + t)} x^t,$$

we have

$$F(\tfrac{1}{2}n, \tfrac{1}{2}n; \tfrac{1}{2}, \tfrac{1}{2}, \cdots, \tfrac{1}{2}; \tfrac{1}{2}p, \tfrac{1}{2}q; \rho_1^2 r_1^2, \rho_1^2 r_2^2, \cdots, \rho_1^2 r_p^2) = \Theta F(\tfrac{1}{2}n, \tfrac{1}{2}n; \tfrac{1}{2}p, \tfrac{1}{2}q; z^{-1}) \prod_{j=1}^{p} (1 - \rho_1^2 r_j^2 z)^{-\frac{1}{2}}, \tag{21}$$

where Θ denotes the operation of taking the term independent of z (this might possibly be done by multiplication by z^{-1} and evaluation of a suitable contour integral, but in the use of this formula here the operation Θ has been carried out directly).

It is of some interest to examine a simple case, and, incidentally, to check that

$$\int_{r_i} p(r_i \mid \rho_i) = 1.$$

If we take $p = 2$, $q = 3$, we obtain for $p(r_1^2, r_2^2 \mid \rho_1 = \rho_2 = 0)$ the form

$$\tfrac{1}{4}(n - 2)(n - 3)(n - 4)(1 - r_1^2)^{\frac{1}{2}n-3}(1 - r_2^2)^{\frac{1}{2}n-3}(r_1^2 - r_2^2)\, dr_1^2\, dr_2^2 .$$

Considering the distribution (19) with $p = 2$, $q = 3$, and taking the most elementary case $n = 6$, we obtain on integration of r_2^2 from 0 to r_1^2,

$$p(r_1 \mid \rho_1) = 6(r_1^2)^2\, dr_1^2 (1 - \rho_1^2)^3 \sum_{u_1, u_2} \left[\frac{\Gamma(3 + t)}{\Gamma(3)}\right]^2 \cdot \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} + t)} \frac{\Gamma(\frac{1}{2} + u_1)\Gamma(\frac{1}{2} + u_2)(\rho_1^2 r_1^2)^t}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})u_1!(u_2 + 2)!t!},$$

where $t = u_1 + u_2$. Now from the identity $(1 - x)^{-\frac{1}{2}}(1 - x)^{\frac{3}{2}} = 1 - x$, the coefficient of x^{t+2}, $(t \geq 0)$, is zero in the expansion of the left-hand side. This provides the identity, for all $t \geq 0$,

$$\sum_{u_1} \frac{\Gamma(\frac{1}{2} + u_1)\Gamma(\frac{1}{2} + t - u_1)\frac{1}{2}\cdot\frac{3}{2}}{\Gamma(\frac{1}{2})u_1!\,\Gamma(\frac{1}{2})(u_2 + 2)!} = \frac{\Gamma(\frac{1}{2} + t + 1)}{\Gamma(\frac{1}{2})(t + 1)!} - \frac{\Gamma(\frac{1}{2} + t + 2)}{\Gamma(\frac{1}{2})(t + 2)!}$$
$$= \frac{\frac{1}{2}(t + 3)\Gamma(\frac{3}{2} + t)}{\Gamma(\frac{1}{2})\Gamma(t + 3)},$$

or

$$\text{(22)} \qquad \sum_{u_1} \frac{\Gamma(\frac{1}{2} + u_1)\Gamma(\frac{1}{2} + t - u_1)}{\Gamma(\frac{1}{2})u_1!\,\Gamma(\frac{1}{2})(u_2 + 2)!} = \frac{t + 3}{3}\,\frac{\Gamma(\frac{3}{2} + t)}{\Gamma(\frac{3}{2})\Gamma(t + 3)}.$$

Hence

$$\text{(23)} \qquad \begin{aligned} p(r_1 \mid \rho_1) &= 6(r_1^2)^2\,dr_1^2(1 - \rho_1^2)^3 \sum_t \frac{\Gamma(3 + t)}{[\Gamma(3)]^2}\,\frac{(t + 3)}{3}\,(\rho_1^2 r_1^2)^t \\ &= (r_1^2)^2\,dr_1^2(1 - \rho_1^2)^3 \sum_t \frac{\Gamma(3 + t)(\rho_1^2 r_1^2)^t(t + 3)}{\Gamma(3)t!} \\ &= (1 - \rho_1^2)^3\,dr_1^2\,\partial/\partial r_1^2\{(r_1^2)^3(1 - \rho_1^2 r_1^2)^{-3}\}, \end{aligned}$$

which obviously gives unity on integration of r_1^2 from 0 to 1. In purely algebraic form

$$\text{(24)} \qquad p(r_1 \mid \rho_1) = 3(1 - \rho_1^2)^3(r_1^2)^2\,dr_1^2/(1 - \rho_1^2 r_1^2)^4.$$

Alternatively, making use of formulae (21), we have for the same case $p = 2$, $q = 3$, $n = 6$, the distribution

$$\text{(25)} \quad 6(r_1^2 - r_2^2)\,dr_1^2\,dr_2^2(1 - \rho_1^2)^3\,\Theta F(3, 3; \tfrac{3}{2}; z^{-1})\,(1 - \rho_1^2 r_1^2 z)^{-\frac{1}{2}}\,(1 - \rho_1^2 r_2^2 z)^{-\frac{1}{2}}.$$

Integrating with respect to r_2^2 from 0 to r_1^2, we obtain

$$\text{(26)} \quad 6\,dr_1^2(1 - \rho_1^2)^3\Theta F(3, 3; \tfrac{3}{2}; z^{-1})\,(1 - \rho_1^2 r_1^2 z)^{-\frac{1}{2}}\left\{\frac{2r_1^2}{\rho_1^2 z} + \frac{4[(1 - \rho_1^2 r_1^2 z)^{\frac{3}{2}} - 1]}{3(\rho_1^2 z)^2}\right\}.$$

Discarding the term for which the irrational expression $(1 - \rho_1^2 r_1^2 z)^{\frac{1}{2}}$ cancels, and hence leaves no terms independent of z, we obtain the distribution $p(r_1 \mid \rho_1)$ given in (23) or (24) by selection of the appropriate terms. We may further integrate directly the expression above with respect to r_1^2, and after discarding again irrelevant terms we obtain

$$\text{(27)} \qquad 6(1 - \rho_1^2)^3\Theta F(3, 3; \tfrac{3}{2}; z^{-1})\left\{-\frac{4}{3(\rho_1^2 z)^2}\,(1 - \rho_1^2 z)^{\frac{3}{2}}\right\},$$

which is readily ascertained to be unity.

6. More than one non-zero root. In the general case the factor multiplying $p(r_i \mid \rho_i = 0)$ is rather remarkable in being symmetrical in both the set r_i and the set ρ_i. As n increases, the convergence of r_1 to ρ_1, r_2 to ρ_2, etc. when the ρ_i are also arranged in descending order of magnitude must result from the restriction $r_1 \geq r_2 \geq \cdots \geq r_p$. The limiting distribution has been discussed by Hsu [9].

In view of the algebraic difficulty of obtaining $\mu(t_1, t_2, \cdots, t_p)$ by direct integration, an unsymmetric method of obtaining the moments was developed. This is fairly tractable in the case of two non-zero roots. The second set w_{2j} of the original variables is transformed by an orthogonal transformation such that the first new variable of the second set is determined by the correlation between w_{1j} and w_{2j}. We may write, for example,

$$w'_{21} = (w_{11}w_{21} + w_{12}w_{22} + \cdots)/(w_{11}^2 + w_{12}^2 + \cdots + w_{1p}^2)^{\frac{1}{2}},$$

$$(28) \qquad w'_{22} = \frac{\left\{\dfrac{-w_{11}(w_{12}^2 + \cdots + w_{1p}^2)w_{21}}{w_{11}^2} + w_{12}w_{22} + \cdots\right\}}{\left\{\dfrac{(w_{11}^2 + \cdots + w_{1p}^2)(w_{12}^2 + \cdots + w_{1p}^2)}{w_{11}^2}\right\}^{\frac{1}{2}}},$$

$$w'_{23} = \frac{\left\{\dfrac{-w_{12}(w_{13}^2 + \cdots + w_{1p}^2)w_{22}}{w_{12}^2} + w_{13}w_{23} + \cdots\right\}}{\left\{\dfrac{(w_{12}^2 + \cdots + w_{1p}^2)(w_{13}^2 + \cdots + w_{1p}^2)}{w_{12}^2}\right\}^{\frac{1}{2}}},$$

$$\vdots$$

which conversely we can at once express as a relation of the w_{2j}, in terms of the w'_{2j}, (since the reciprocal of an orthogonal matrix is simply its transpose). If we write

$$(29) \qquad \begin{aligned} \alpha'_{21} &= w'_{21}/[(w'_{21})^2 + (w'_{22})^2 + \cdots + (w'_{2p})^2]^{\frac{1}{2}}, \\ \alpha'_{22} &= w'_{22}/[(w'_{21})^2 + (w'_{22})^2 + \cdots + (w'_{2p})^2]^{\frac{1}{2}}, \\ &\vdots \end{aligned}$$

and write further

$$a_1 = \cos\theta_{11},\ a_2 = \cos\theta_{12},\ \cdots,\ b_1 = \cos\theta'_{21},$$

$$b_2 = \cos\theta'_{22},\ \cdots,\ \text{where } \alpha'_{21} = \cos\theta'_{21},\ \alpha'_{22} = \sin\theta'_{21}\cos\theta'_{22},\ \cdots$$

we have in particular

$$(30) \qquad \begin{aligned} \alpha_{21} &= a_1b_1 - b_2\sqrt{(1 - a_1^2)}\sqrt{(1 - b_1^2)}, \\ \alpha_{22} &= a_2b_1\sqrt{(1 - a_1^2)} + a_1a_2b_2\sqrt{(1 - b_1^2)} \\ &\qquad - b_3\sqrt{(1 - a_2^2)}\sqrt{(1 - b_1^2)}\sqrt{(1 - b_2^2)}, \end{aligned}$$

where the distribution of the a's and b's is proportional to

$$\{(1 - a_1^2)^{\frac{1}{2}(p-3)}\,da_1\}\{(1 - a_2^2)^{\frac{1}{2}(p-4)}\,da_2\}\cdots\{(1 - b_1^2)^{\frac{1}{2}(n-3)}\,db_1\}\{(1 - b_2^2)^{\frac{1}{2}(p-4)}\,db_2\}\cdots.$$

For the reasons discussed in section 4, it will be noticed that only the distribution of b_1 in the a, b set is affected by the linkage factor. By such methods the expressions

$$\mu(1, 1) \equiv E\{s_1^2 s_2^2 \mid r_i\}, \qquad \mu(2, 1) \equiv E\{s_1^4 s_2^2 \mid r_i\}$$

were fairly readily obtained. If we introduce the notation

$$S_k \equiv \sum_{i=1}^{p} (r_i^2)^k, \qquad S_{kl} \equiv \sum_{i \neq j} (r_i^2)^k (r_j^2)^l, \qquad \text{etc.,}$$

and also symbols for the products of the α and β moments, viz.

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} \equiv E\{\alpha_{11}^2\alpha_{12}^2\}\, E\{\beta_{11}^2\beta_{12}^2\}, \qquad \begin{pmatrix} 2 & \cdot \\ \cdot & 2 \end{pmatrix} \equiv E\{\alpha_{11}^2\alpha_{22}^2\}\, E\{\beta_{11}^2\beta_{22}^2\},$$

etc., we may list the moments $\mu(t_1, t_2, \cdots, t_p)$ as in Appendix I, which gives all moments up to the fourth order in terms of the α and β moments (the numerical coefficients arise from the numbers of ways of forming the two-way partitions). "Half-factors" corresponding to the α moments are listed in Appendix II against their appropriate symbol, the corresponding factors coming from the β moments being obtained in case (a) by writing q for p and in case (b) by writing also[5] $n \to \infty$. Thus in case (a)

$$\begin{aligned} \mu(1, 1) = &\left[\frac{n+2}{np(p+2)}\right]\left[\frac{n+2}{nq(q+2)}\right] S_2 \\ &+ \left\{\left[\frac{np+n-2}{np(p+2)(p-1)}\right]\left[\frac{nq+n-2}{nq(q+2)(q-1)}\right]\right. \\ &\left. + 2\left[\frac{-(n-p)}{np(p+2)(p-1)}\right]\left[\frac{-(n-q)}{nq(q+2)(q-1)}\right]\right\} 2S_{11}, \end{aligned} \tag{32}$$

and in case (b)

$$\begin{aligned} \mu(1, 1) = &\left[\frac{n+2}{np(p+2)}\right]\left[\frac{1}{q(q+2)}\right] S_2 \\ &+ \left\{\frac{(np+n-2)(q+1)+2(n-p)}{np(p+2)(p-1)q(q+2)(q-1)}\right\} 2S_{11}. \end{aligned} \tag{33}$$

By means of the transformation (28) it is possible to develop the moments $\mu(t_1, t_2)$ in the case of two non-zero roots, though in obtaining the results quoted in Appendix II, where the formula for $\mu(3, 1)$ and $\mu(2, 2)$ are included, it was found convenient to supplement this method with the devices mentioned in the

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[5] It should be remembered that we have assumed $p \leq q$. If $p > q$, we interchange the dependent and independent vector variates, and hence must interchange p and q in these moment formulae, $p(\leq q)$ now corresponding to the independent variate.

next section. In the case of more than two non-zero roots, it is theoretically possible to carry out a further transformation on the w_{3j} variates, but with the "partial" variates $w_{1j \cdot 2} \equiv w_{1j} - b_{12}w_{2j}$, where

$$b_{12} = (w_{11}w_{21} + w_{12}w_{22} + \cdots)/(w_{11}^2 + w_{12}^2 + \cdots),$$

as coefficients. This enables us to express w_{3j} in terms of new variables, of which the first is related to the partial correlation of w_{3j} with w_{1j} for given w_{2j}, i.e. to the second correlation factor which depends on the "linkage"; and so on. This method is, however, again too cumbersome to be of much use, and a more rapid method of evaluating $\mu(t_1, t_2, \cdots, t_p)$ in general is desirable. This problem has not been entirely solved to the author's satisfaction in this paper, although in the concluding section are mentioned devices which have been found useful, and which enabled the terms for the remaining third-order moment $\mu(1, 1, 1)$ to be completed and added to Appendix II.

7. Relations among the α-moments. Equation (15) defining the α's, the ξ_i being random vectors in the p-space of the x-vectors except for their mutual configuration being determined by the properties of n-space, may be used to provide relations among the α-moments. Thus in addition to the identities

$$\alpha_{i1}^2 + \alpha_{i2}^2 + \cdots + \alpha_{ip}^2 = 1, \qquad (i = 1, 2, \cdots, p), \tag{34}$$

the correlation of any ξ_i with a fixed vector in the p-space, e.g. with x_1 or with $(x_1 + x_2)/\sqrt{2}$, is a random correlation in p-space, whereas the correlation of any ξ_i with any other ξ_i is a random correlation in n-space. The use of these facts is best illustrated by an example and equations sufficient to determine the six α-moments required for $\mu(1, 1, 1)$ will be derived.

For convenience, denote the required mean values of

$$\alpha_{11}^2\alpha_{21}^2\alpha_{31}^2, \ \alpha_{11}^2\alpha_{21}^2\alpha_{32}^2, \ \alpha_{11}^2\alpha_{22}^2\alpha_{33}^2, \ \alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\alpha_{31}^2, \ \alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}\alpha_{33}^2, \ \alpha_{11}\alpha_{12}\alpha_{22}\alpha_{23}\alpha_{31}\alpha_{33}$$

by A, B, C, D, E, F respectively. Multiply the second-order quantities $\alpha_{11}^2\alpha_{21}^2$, $\alpha_{11}^2\alpha_{22}^2$, $\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}$ by expression (34) for $i = 3$; since this expression is identically unity, the consequent mean values are unaltered. This gives the three relations

$$\begin{aligned} &A + (p-1)B = (n+2)/\{np(p+2)\}, \\ &A + 3(p-1)B + (p-1)(p-2)C = 1/p, \\ &A + (p-1)B + 2\,(p-1)(p-2)D \\ &\qquad + (p-1)(p-2)(p-3)E = 1/(np). \end{aligned} \tag{35}$$

The moment A is the mean of the triple product of the squared scalar products of ξ_1, ξ_2 and ξ_3 with x_1. The same value must be realized with any other fixed vector in the p-space, e.g. with either $(x_1 + x_2)/\sqrt{2}$ or with $(x_1 + x_2 + \cdots + x_p)/\sqrt{p}$. This gives two relations

$$\begin{aligned} &A - B - 4\,D = 0 \\ &(p+1)A - 3B - 12D - (p-2)\,(C + 6E + 8F) = 0. \end{aligned} \tag{36}$$

A final linearly independent relation is obtained from the mean triple product of $(\xi_1 \cdot \xi_2)$, $(\xi_1 \cdot \xi_3)$, $(\xi_2 \cdot \xi_3)$, which depends solely on the internal configuration of ξ_1, ξ_2 and ξ_3, and is easily shown (e.g. choose ξ_1 to coincide with one of the original axes of the n-space) to be $1/n^2$. This gives

$$pA + 3p(p-1)D + p(p-1)(p-2)F = 1/n^2. \tag{37}$$

The equations contained in (35), (36) and (37) determine A, B, C, D, E and F.

Similar equations could evidently be constructed for the higher-order moments, e.g. for the terms required for $\mu(2, 1, 1)$ or $\mu(1, 1, 1, 1)$, but the numbers of such terms increase rapidly. From Appendix I it will be seen that there are 24 distinct terms in $\mu(2, 1, 1)$ and 16 in $\mu(1, 1, 1, 1)$.

Appendix I.

$$\mu(1, 1) = S_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 2S_{11} \left\{ \begin{pmatrix} 2 & \cdot \\ \cdot & 2 \end{pmatrix} + 2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

$$\begin{aligned} \mu(2, 1) = {} & S_3 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + S_{21} \left\{ \begin{pmatrix} 4 & \cdot \\ \cdot & 2 \end{pmatrix} + 6 \begin{pmatrix} 2 & 2 \\ 2 & \cdot \end{pmatrix} + 8 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \right\} \\ & + 6S_{111} \left\{ 3 \begin{pmatrix} 2 & 2 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix} + 12 \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & \cdot \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned} \mu(3, 1) = {} & S_4 \begin{pmatrix} 6 \\ 2 \end{pmatrix} + S_{31} \left\{ \begin{pmatrix} 6 & \cdot \\ \cdot & 2 \end{pmatrix} + 15 \begin{pmatrix} 4 & 2 \\ 2 & \cdot \end{pmatrix} + 12 \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix} \right\} \\ & + 2\, S_{22} \left\{ 15 \begin{pmatrix} 4 & 2 \\ \cdot & 2 \end{pmatrix} + 20 \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \right\} \\ & + 2S_{211} \left\{ 15 \begin{pmatrix} 4 & 2 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix} + 45 \begin{pmatrix} 2 & 2 & 2 \\ 2 & \cdot & \cdot \end{pmatrix} + 120 \begin{pmatrix} 3 & 1 & 2 \\ 1 & 1 & \cdot \end{pmatrix} + 30 \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & \cdot \end{pmatrix} \right\} \\ & + 24S_{1111} \left\{ 15 \begin{pmatrix} 2 & 2 & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \end{pmatrix} + 90 \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & \cdot & \cdot \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned} \mu(2, 2) = {} & S_4 \begin{pmatrix} 4 \\ 4 \end{pmatrix} + S_{31} \left\{ 12 \begin{pmatrix} 4 & \cdot \\ 2 & 2 \end{pmatrix} + 16 \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \right\} \\ & + 2S_{22} \left\{ \begin{pmatrix} 4 & \cdot \\ \cdot & 4 \end{pmatrix} + 18 \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + 16 \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \right\} \\ & + 2S_{211} \left\{ 6 \begin{pmatrix} 4 & \cdot & \cdot \\ \cdot & 2 & 2 \end{pmatrix} + 36 \begin{pmatrix} 2 & 2 & \cdot \\ 2 & \cdot & 2 \end{pmatrix} + 72 \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} + 96 \begin{pmatrix} 3 & 1 & \cdot \\ 1 & 1 & 2 \end{pmatrix} \right\} \\ & + 24S_{1111} \left\{ 9 \begin{pmatrix} 2 & 2 & \cdot & \cdot \\ \cdot & \cdot & 2 & 2 \end{pmatrix} + 72 \begin{pmatrix} 1 & 1 & 2 & \cdot \\ 1 & 1 & \cdot & 2 \end{pmatrix} + 24 \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right\} \end{aligned}$$

$$\mu(1,1,1) = S_3\begin{pmatrix}2\\2\\2\end{pmatrix} + S_{21}\left\{3\begin{pmatrix}2 & \cdot\\2 & \cdot\\\cdot & 2\end{pmatrix} + 12\begin{pmatrix}1 & 1\\1 & 1\\2 & \cdot\end{pmatrix}\right\}$$

$$+ 6S_{111}\left\{\begin{pmatrix}2 & \cdot & \cdot\\\cdot & 2 & \cdot\\\cdot & \cdot & 2\end{pmatrix} + 6\begin{pmatrix}1 & 1 & \cdot\\1 & 1 & \cdot\\\cdot & \cdot & 2\end{pmatrix} + 8\begin{pmatrix}1 & 1 & \cdot\\\cdot & 1 & 1\\1 & \cdot & 1\end{pmatrix}\right\}$$

$$\mu(2,1,1) = S_4\begin{pmatrix}4\\2\\2\end{pmatrix} + S_{31}\left\{2\begin{pmatrix}4 & \cdot\\2 & \cdot\\\cdot & 2\end{pmatrix} + 6\begin{pmatrix}2 & 2\\2 & \cdot\\2 & \cdot\end{pmatrix}\right.$$

$$\left. + 16\begin{pmatrix}3 & 1\\1 & 1\\2 & \cdot\end{pmatrix} + 4\begin{pmatrix}4 & \cdot\\1 & 1\\1 & 1\end{pmatrix}\right\} + 2S_{22}\left\{\begin{pmatrix}4 & \cdot\\\cdot & 2\\\cdot & 2\end{pmatrix} + 6\begin{pmatrix}2 & 2\\2 & \cdot\\\cdot & 2\end{pmatrix}\right.$$

$$\left. + 16\begin{pmatrix}3 & 1\\1 & 1\\\cdot & 2\end{pmatrix} + 12\begin{pmatrix}2 & 2\\1 & 1\\1 & 1\end{pmatrix}\right\} + 2S_{211}\left\{\begin{pmatrix}4 & \cdot & \cdot\\\cdot & 2 & \cdot\\\cdot & \cdot & 2\end{pmatrix} + 12\begin{pmatrix}2 & 2 & \cdot\\2 & \cdot & \cdot\\\cdot & \cdot & 2\end{pmatrix}\right.$$

$$+ 3\begin{pmatrix}\cdot & 2 & 2\\2 & \cdot & \cdot\\2 & \cdot & \cdot\end{pmatrix} + 16\begin{pmatrix}3 & 1 & \cdot\\1 & 1 & \cdot\\\cdot & \cdot & 2\end{pmatrix} + 2\begin{pmatrix}4 & \cdot & \cdot\\\cdot & 1 & 1\\\cdot & 1 & 1\end{pmatrix} + 24\begin{pmatrix}2 & \cdot & 2\\1 & 1 & \cdot\\1 & 1 & \cdot\end{pmatrix}$$

$$\left. + 48\begin{pmatrix}1 & 1 & 2\\1 & 1 & \cdot\\2 & \cdot & \cdot\end{pmatrix} + 48\begin{pmatrix}2 & 1 & 1\\1 & 1 & \cdot\\1 & \cdot & 1\end{pmatrix} + 24\begin{pmatrix}2 & 1 & 1\\\cdot & 1 & 1\\2 & \cdot & \cdot\end{pmatrix} + 32\begin{pmatrix}3 & 1 & \cdot\\\cdot & 1 & 1\\1 & \cdot & 1\end{pmatrix}\right\}$$

$$+ 24S_{1111}\left\{3\begin{pmatrix}2 & 2 & \cdot & \cdot\\\cdot & \cdot & 2 & \cdot\\\cdot & \cdot & \cdot & 2\end{pmatrix} + 6\begin{pmatrix}2 & 2 & \cdot & \cdot\\\cdot & \cdot & 1 & 1\\\cdot & \cdot & 1 & 1\end{pmatrix} + 24\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & \cdot & \cdot\\\cdot & \cdot & 1 & 1\end{pmatrix}\right.$$

$$\left. + 24\begin{pmatrix}2 & 1 & 1 & \cdot\\\cdot & 1 & 1 & \cdot\\\cdot & \cdot & \cdot & 2\end{pmatrix} + 48\begin{pmatrix}1 & 1 & \cdot & 2\\\cdot & 1 & 1 & \cdot\\1 & \cdot & 1 & \cdot\end{pmatrix}\right\}$$

$$\mu(1,1,1,1) = S_4\begin{bmatrix}2\\2\\2\\2\end{bmatrix} + S_{31}\left\{4\begin{bmatrix}2 & \cdot\\2 & \cdot\\2 & \cdot\\\cdot & 2\end{bmatrix} + 24\begin{bmatrix}1 & 1\\1 & 1\\2 & \cdot\\2 & \cdot\end{bmatrix}\right\}$$

$$+ 2S_{22}\left\{3\begin{bmatrix}2 & \cdot\\2 & \cdot\\\cdot & 2\\\cdot & 2\end{bmatrix} + 24\begin{bmatrix}2 & \cdot\\\cdot & 2\\1 & 1\\1 & 1\end{bmatrix} + 8\begin{bmatrix}1 & 1\\1 & 1\\1 & 1\\1 & 1\end{bmatrix}\right\}$$

$$+ 2S_{211}\left\{6\begin{bmatrix}2 & \cdot & \cdot\\2 & \cdot & \cdot\\\cdot & 2 & \cdot\\\cdot & \cdot & 2\end{bmatrix} + 48\begin{bmatrix}2 & \cdot & \cdot\\1 & 1 & \cdot\\1 & 1 & \cdot\\\cdot & \cdot & 2\end{bmatrix} + 48\begin{bmatrix}1 & \cdot & 1\\1 & 1 & \cdot\\1 & 1 & \cdot\\1 & \cdot & 1\end{bmatrix}\right.$$

$$+ 96\begin{bmatrix}1 & 1 & \cdot\\ \cdot & 1 & 1\\ 1 & \cdot & 1\\ 2 & \cdot & \cdot\end{bmatrix} + 12\begin{bmatrix}2 & \cdot & \cdot\\ \cdot & 1 & 1\\ \cdot & 1 & 1\\ 2 & \cdot & \cdot\end{bmatrix}\Bigg\}\Bigg\}$$

$$+ 24S_{1111}\left\{\begin{bmatrix}2 & \cdot & \cdot & \cdot\\ \cdot & 2 & \cdot & \cdot\\ \cdot & \cdot & 2 & \cdot\\ \cdot & \cdot & \cdot & 2\end{bmatrix} + 12\begin{bmatrix}1 & 1 & \cdot & \cdot\\ 1 & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & 1\\ \cdot & \cdot & 1 & 1\end{bmatrix} + 32\begin{bmatrix}1 & 1 & \cdot & \cdot\\ \cdot & 1 & 1 & \cdot\\ 1 & \cdot & 1 & \cdot\\ \cdot & \cdot & \cdot & 2\end{bmatrix}\right.$$

$$\left. + 12\begin{bmatrix}2 & \cdot & \cdot & \cdot\\ \cdot & 1 & 1 & \cdot\\ \cdot & 1 & 1 & \cdot\\ \cdot & \cdot & \cdot & 2\end{bmatrix} + 48\begin{bmatrix}1 & 1 & \cdot & \cdot\\ \cdot & 1 & 1 & \cdot\\ \cdot & \cdot & 1 & 1\\ 1 & \cdot & \cdot & 1\end{bmatrix}\right\}\Bigg\}.$$

Appendix II.

$$\begin{pmatrix}2\\2\end{pmatrix}\frac{n+2}{np(p+2)},\quad \begin{pmatrix}2 & \cdot\\ \cdot & 2\end{pmatrix}\frac{np+n-2}{np(p+2)(p-1)},\quad \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\frac{-(n-p)}{np(p+2)(p-1)},$$

$$\begin{pmatrix}4\\2\end{pmatrix}\frac{3(n+4)}{np(n+2)(p+4)},\qquad \begin{pmatrix}4 & \cdot\\ \cdot & 2\end{pmatrix}\frac{3(np+3n-4)}{np(p+2)(p+4)(p-1)},$$

$$\begin{pmatrix}2 & 2\\ 2 & \cdot\end{pmatrix}\frac{np+n+2p-4}{np(p+2)(p+4)(p-1)},\quad \begin{pmatrix}2 & 2 & \cdot\\ \cdot & \cdot & 2\end{pmatrix}\frac{np+3n-4}{np(p+2)(p+4)(p-1)},$$

$$\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}\frac{-3(n-p)}{np(p+2)(p+4)(p-1)},\quad \begin{pmatrix}1 & 1 & 2\\ 1 & 1 & \cdot\end{pmatrix}\frac{-(n-p)}{np(p+2)(p+4)(p-1)},$$

$$\begin{pmatrix}6\\2\end{pmatrix}\frac{15(n+6)}{np(p+2)(p+4)(p+6)},\quad \begin{pmatrix}6 & \cdot\\ \cdot & 2\end{pmatrix}\frac{15(np+5n-6)}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}4 & 2\\ 2 & \cdot\end{pmatrix}\frac{3(np+n+4p-6)}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}4 & 2\\ \cdot & 2\end{pmatrix}\frac{3(np+3n+2p-6)}{np(p+2)(p+4)(p+6)(p-1)}.$$

$$\begin{pmatrix}4 & 2 & \cdot\\ \cdot & \cdot & 2\end{pmatrix}\frac{3(np+5n-6)}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}2 & 2 & 2\\ 2 & \cdot & \cdot\end{pmatrix}\frac{np+3n+2p-6}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}2 & 2 & 2 & \cdot\\ \cdot & \cdot & \cdot & 2\end{pmatrix}\frac{np+5n-6}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}5 & 1\\ 1 & 1\end{pmatrix}\frac{-15(n-p)}{np(p+2)p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}3 & 3\\ 1 & 1\end{pmatrix} \frac{-9(n-p)}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}3 & 1 & 2\\ 1 & 1 & \cdot\end{pmatrix} \frac{-3(n-p)}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}1 & 1 & 4\\ 1 & 1 & \cdot\end{pmatrix} \frac{-3(n-p)}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}1 & 1 & 2 & 2\\ 1 & 1 & \cdot & \cdot\end{pmatrix} \frac{-(n-p)}{np(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}4\\ 4\end{pmatrix} \frac{9(n+4)(n+6)}{n(n+2)p(p+2)(p+4)(p+6)},$$

$$\begin{pmatrix}4 & \cdot\\ \cdot & 4\end{pmatrix} \frac{9\{n^2(p+3)(p+5) + 2n(p+1)(p+3) - 8(2p+3)\}}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}2 & 2\\ 4 & \cdot\end{pmatrix} \frac{3\{n^2(p+3) + 6n(p+1) + 8(p-3)\}}{n(n+2)p(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix} \frac{n^2(p^2+4p+15) + 6n(p+1)(p-3) + 4(5p^2+2p-6)}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}4 & \cdot & \cdot\\ \cdot & 2 & 2\end{pmatrix} \frac{3\{n^2(p+3)(p+5) + 2n(p+1)(p+3) - 8(2p+3)\}}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}2 & 2 & \cdot\\ 2 & \cdot & 2\end{pmatrix} \frac{n^2(p+3)^2 + 2n(p+1)(2p+3) + 4(p^2-4p-6)}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}2 & 2 & \cdot & \cdot\\ \cdot & \cdot & 2 & 2\end{pmatrix} \frac{n^2(p+3)(p+5) + 2n(p+1)(p+3) - 8(2p+3)}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix} \frac{-9(n-p)(n+4)}{n(n+2)p(p+2)(p+4)(p+6)(p-1)},$$

$$\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix} \frac{-9(n-p)(np+3n+2p)}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}1 & 1 & 2\\ 1 & 1 & 2\end{pmatrix} \frac{-(n-p)(np-3n+8p+12)}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}3 & 1 & \cdot\\ 1 & 1 & 2\end{pmatrix} \frac{-3(n-p)(np+3n+2p)}{n(n+2)p(p-2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}1 & 1 & 2 & \cdot\\ 1 & 1 & \cdot & 2\end{pmatrix} \frac{-(n-p)(np+3n+2p)}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix} \frac{3(n-p)(n-p-2)}{n(n+2)p(p+2)(p+4)(p+6)(p-1)(p+1)},$$

$$\begin{pmatrix}2\\2\\2\end{pmatrix}\frac{(n+2)(n+4)}{n^2p(p+2)(p+4)}, \quad \begin{pmatrix}2 & \cdot\\2 & \cdot\\ \cdot & 2\end{pmatrix}\frac{(n+2)(np+3n-4)}{n^2p(p+2)(p+4)(p-1)},$$

$$\begin{pmatrix}2 & \cdot & \cdot\\ \cdot & 2 & \cdot\\ \cdot & \cdot & 2\end{pmatrix}\frac{n^2(p^2+3p-2)-6n(p+2)+16}{n^2p(p+2)(p+4)(p-1)(p-2)},$$

$$\begin{pmatrix}1 & 1\\1 & 1\\2 & \cdot\end{pmatrix}\frac{-(n-p)(n+2)}{n^2p(p+2)(p+4)(p-1)},$$

$$\begin{pmatrix}1 & 1 & \cdot\\1 & 1 & \cdot\\ \cdot & \cdot & 2\end{pmatrix}\frac{-(n-p)(np+2n-4)}{n^2p(p+2)(p+4)(p-1)(p-2)},$$

$$\begin{pmatrix}1 & 1 & \cdot\\ \cdot & 1 & 1\\1 & \cdot & 1\end{pmatrix}\frac{(n-p)(2n-p)}{n^2p(p+2)(p+4)(p-1)(p-2)}.$$

REFERENCES

[1] H. Hotelling, "Relations between two sets of variates", *Biometrika*, Vol. 28 (1936), pp. 321–377.

[2] R. A. Fisher, "The sampling distribution of some statistics obtained from non-linear equations", *Ann. Eugen.*, Vol. 9 (1939), pp. 238–249.

[3] P. L. Hsu, "On the distribution of roots of certain determinantal equations", *Ann. Eugen.*, Vol. 9 (1939), pp. 250–258.

[4] S .N. Roy, "*P*-statistics or some generalizations in analysis of variance appropriate to multivariate problems", *Sankhyā*, Vol. 4 (1939), pp. 381–396.

[5] S. N. Roy, "Analysis of variance for multivariate normal populations. The sampling distribution of the requisite p-statistics on the null and non-null hypothesis", *Sankhyā*, Vol. 6 (1942), pp. 35–50.

[6] M. S. Bartlett, "The vector representation of a sample", *Proc. Camb. Phil. Soc.*, Vol. 30 (1934), pp. 327–360.

[7] R. A. Fisher, "The general sampling distribution of the multiple correlation coefficient", *Proc. Roy. Soc.*, Vol. A 121 (1928), pp. 654–673.

[8] E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge Univ. Press, 4th ed., 1935.

[9] P. L. Hsu, "On the limiting distribution of the canonical correlations", *Biometrika*, Vol. 32 (1941), pp. 38–45.