CHARTS OF THE POWER OF THE F-TEST[1]

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1. Introduction. This paper presents charts of the power of the F-test designed to simplify entry and interpolation. The curves on which the quantity ϕ is constant are given for fixed level of significance α and power β. The coordinates are f_1 and f_2, the number of degrees of freedom in the numerator and denominator, respectively, of the F-statistic. Charts are presented for $\beta = 0.5$, 0.7, 0.8, 0.9 both for $\alpha = 0.01$ and $\alpha = 0.05$ (Figs. 1 to 8). In addition, nomograms are presented for $\alpha = 0.01, 0.05$ (Figs. 9 and 10) which make interpolation in β possible. The latter charts give linear approximations to the curves on which ϕ is constant.

The quantity ϕ is defined as $\sqrt{S_b^{*}/[(f_1 + 1)\sigma^2]}$, where S_b^* is the value of S_b^2 when the observable random variables are replaced by their expectations under the alternative hypothesis considered, and S_b^2 is the sum of squares in the numerator of the F-statistic.

With these charts the following question may be answered: *What experimental setup is required (what combination of f_1 and f_2), in order to obtain a specified power β against a given alternative?*

Tables of the power of the F-test have been given in two forms. Lehmer [2] tabled ϕ for fixed α, β, f_1, and f_2. On the other hand, Tang [4] tabled $P_{II} = 1 - \beta$ for fixed α, ϕ, f_1, and f_2. Essentially the same information as in Tang's tables was given, in graphical form, by Pearson and Hartley [3]. However, neither of these forms is always convenient for the design of experiments where a relation between f_1 and f_2 is desired for fixed α, β for a specified alternative hypothesis.

2. Construction of the charts. The present charts were constructed by interpolation, both numerical and graphical, in the existing tables. For $\beta = 0.5$ and 0.9, Tang's tables were used; while for $\beta = 0.7$ and 0.8, Lehmer's tables were found convenient.

Lehmer remarks that in her tables harmonic interpolation in both f_1 and f_2 is very efficient. For this reason reciprocal scales were used for f_1 and f_2. On this scale the curves of constant ϕ obtained from Lehmer's tables are nearly straight lines (see Figs. 2, 3, 6, and 7). This is especially striking for large f_1 and f_2.

Tang's tables give no entries for $f_1 > 8$. However, formula (13) of Lehmer may be used to compute ϕ for $f_1 = \infty$, while the case $f_2 = \infty$ is covered by the table of Fix [1]. As noted above, replacing the curves by straight lines for large

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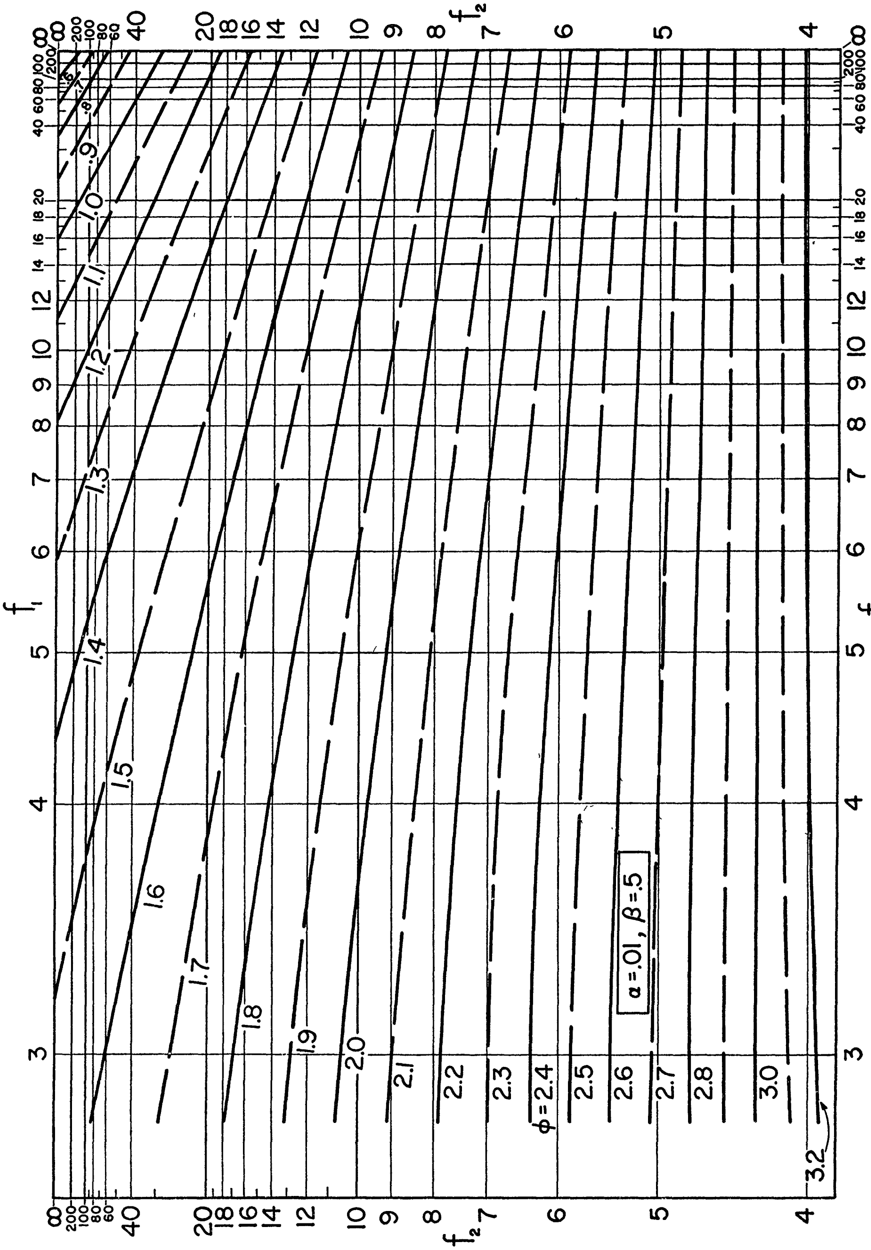

FIG. 1. Curves of constant ϕ for the case $\alpha = 0.01, \beta = 0.5$

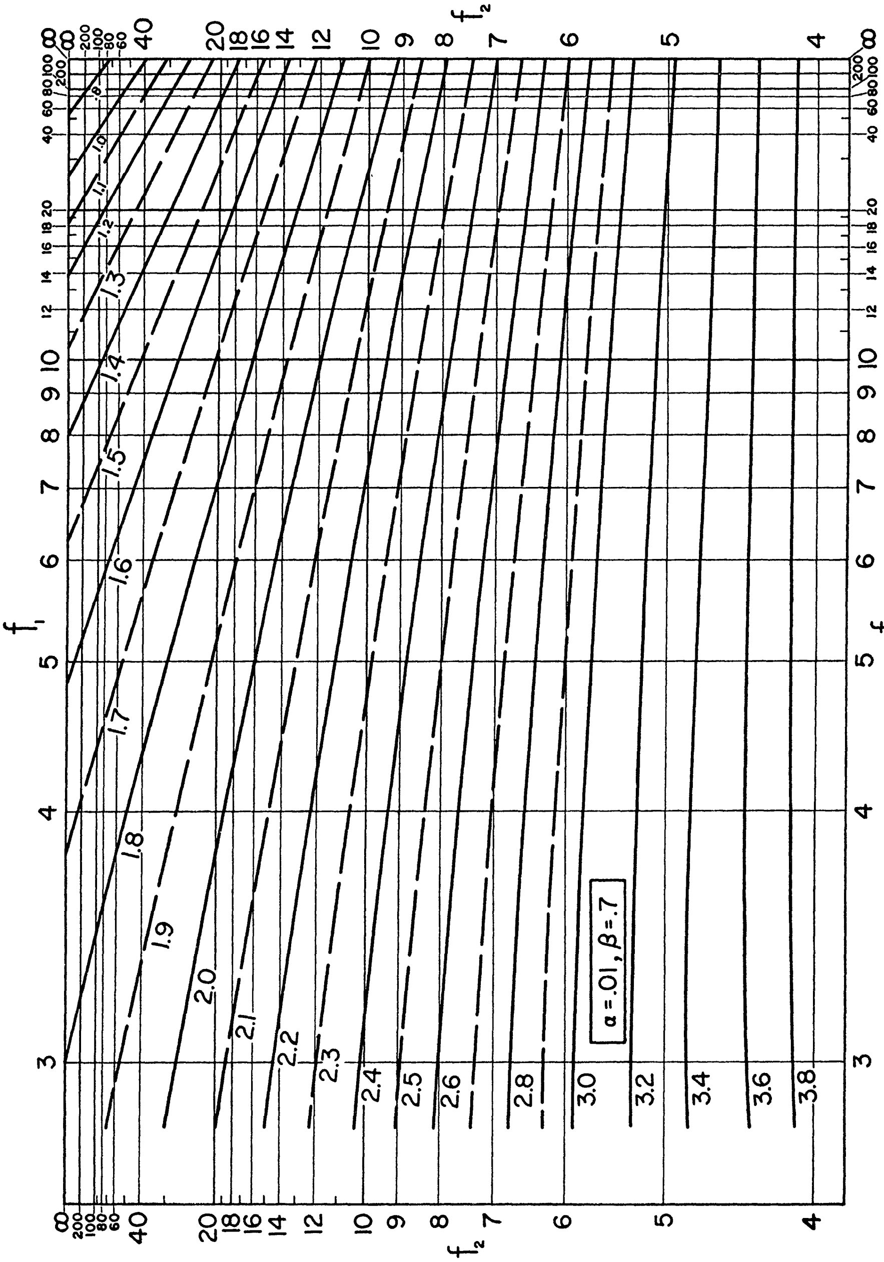

FIG. 2. Curves of constant ϕ for the case $\alpha = 0.01$, $\beta = 0.7$

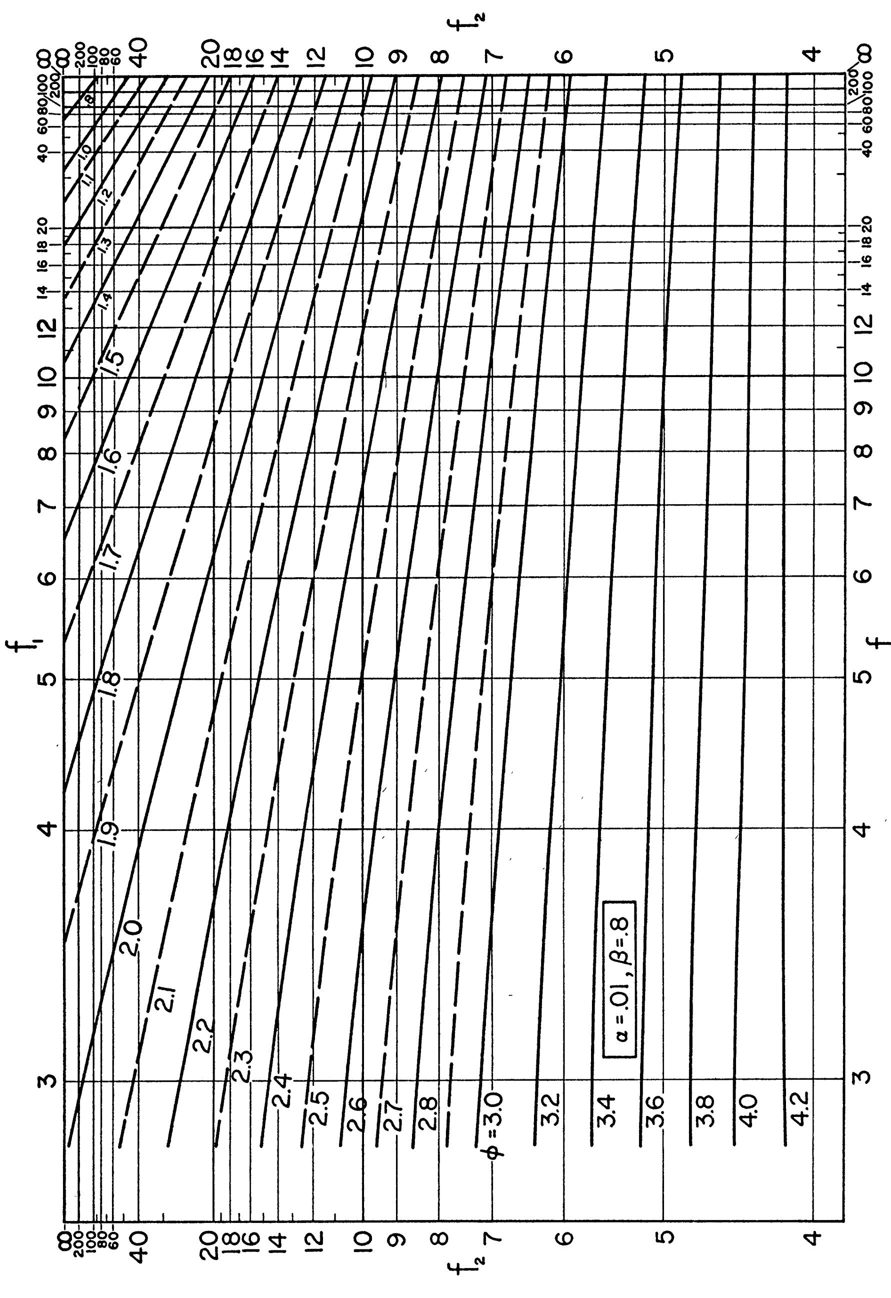

FIG. 3. Curves of constant ϕ for the case $\alpha = 0.01$, $\beta = 0.8$

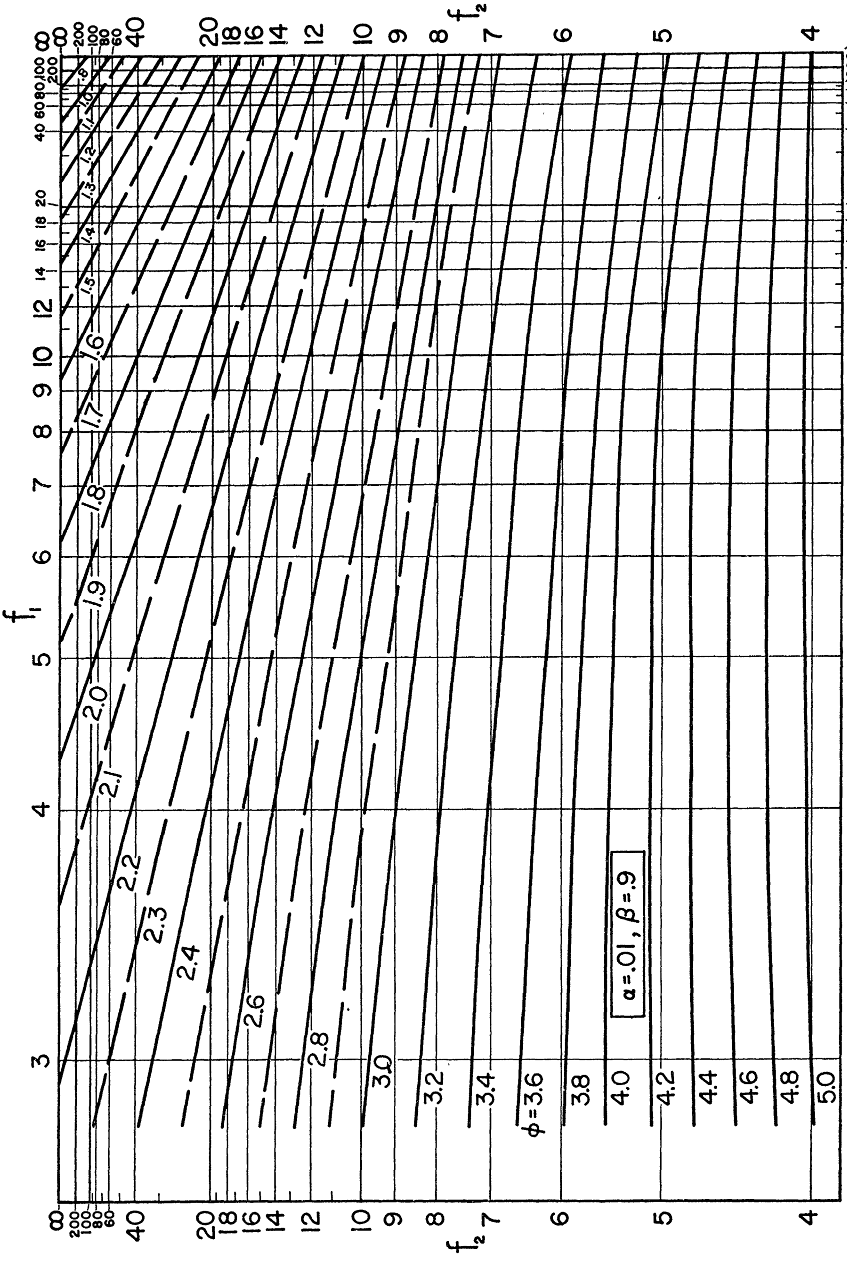

FIG. 4. Curves of constant ϕ for the case $\alpha = 0.01, \beta = 0.9$

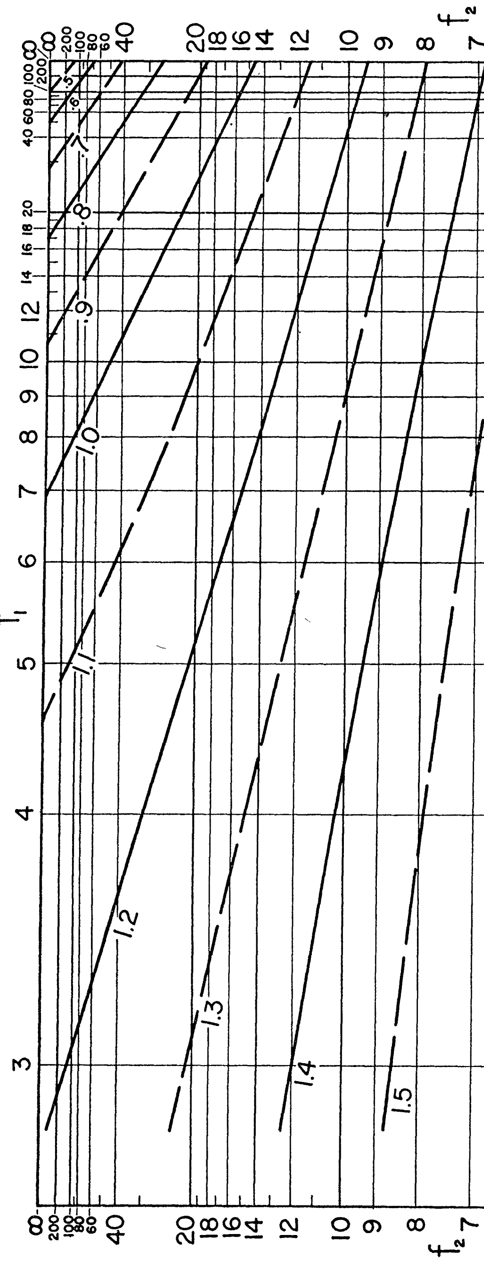

FIG. 5. Curves of constant ϕ for the case $\alpha = 0.05, \beta = 0.5$

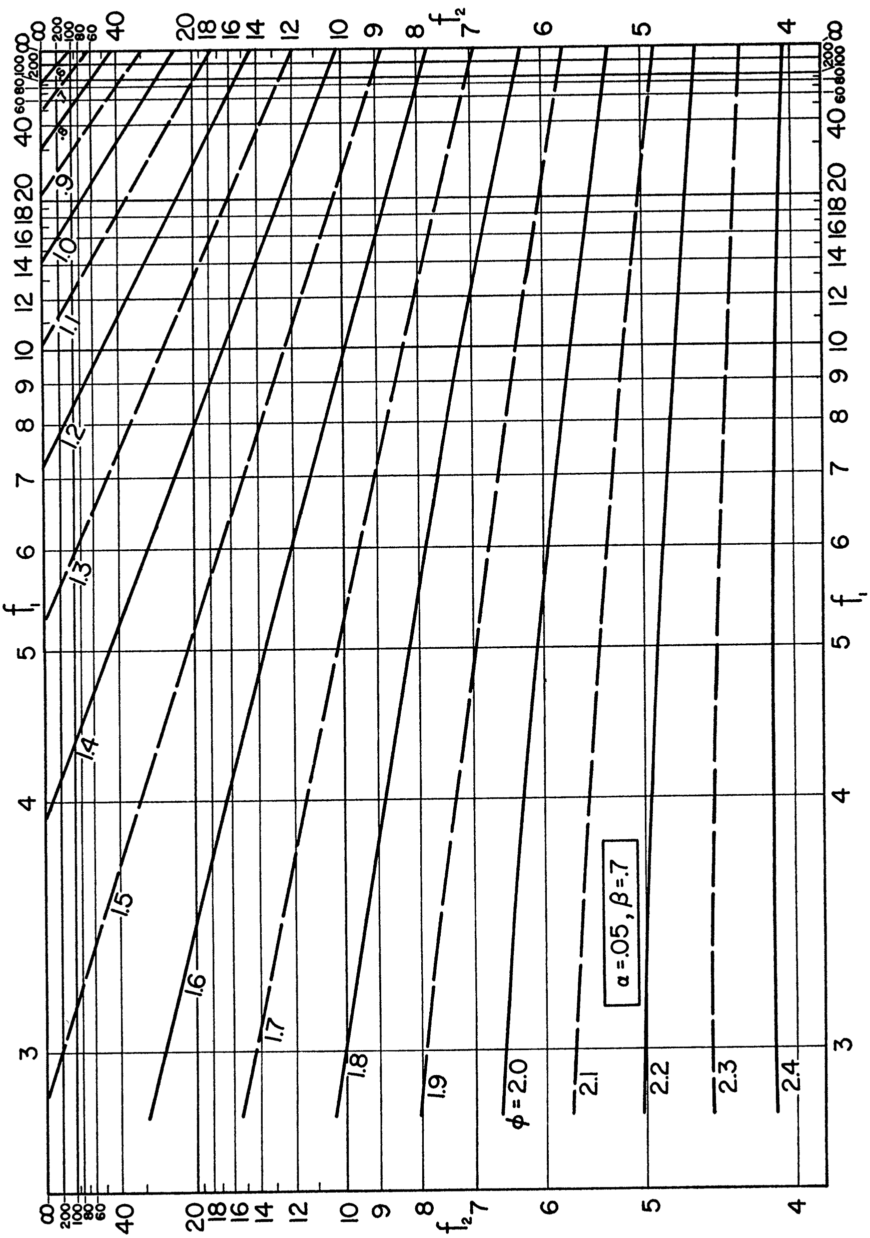

FIG. 6. Curves of constant ϕ for the case $\alpha = 0.05$, $\beta = 0.7$

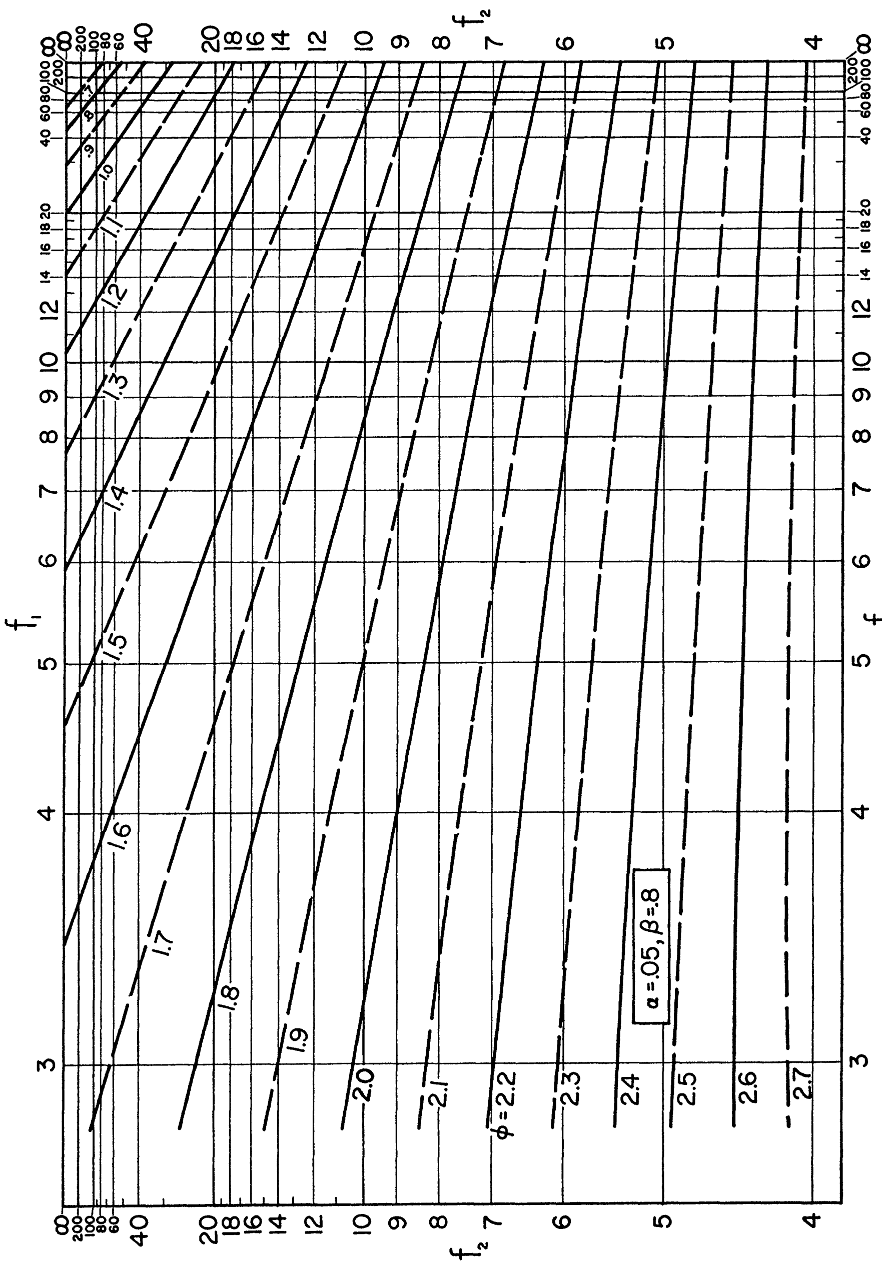

FIG. 7. Curves of constant ϕ for the case $\alpha = 0.05, \beta = 0.8$

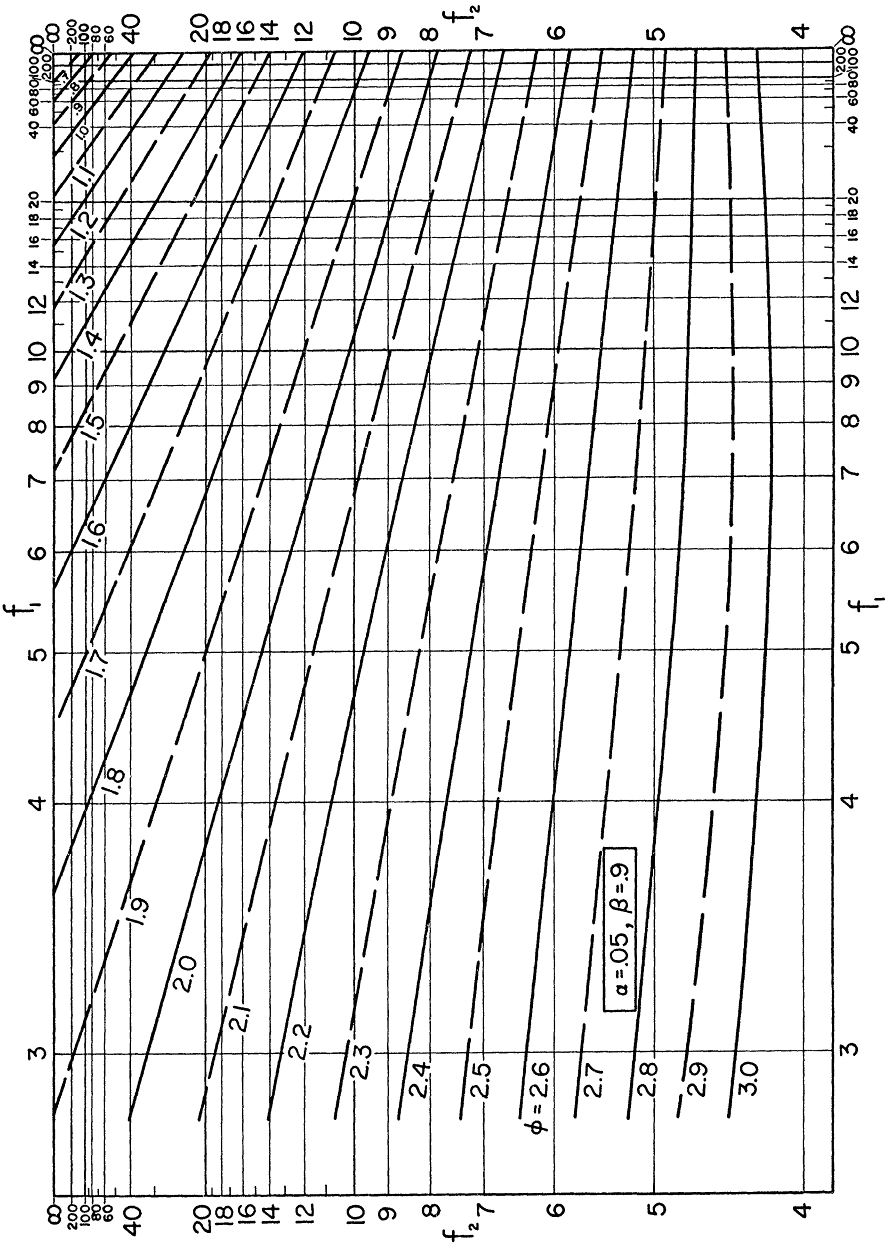

FIG. 8. Curves of constant ϕ for the case $\alpha = 0.05$, $\beta = 0.9$

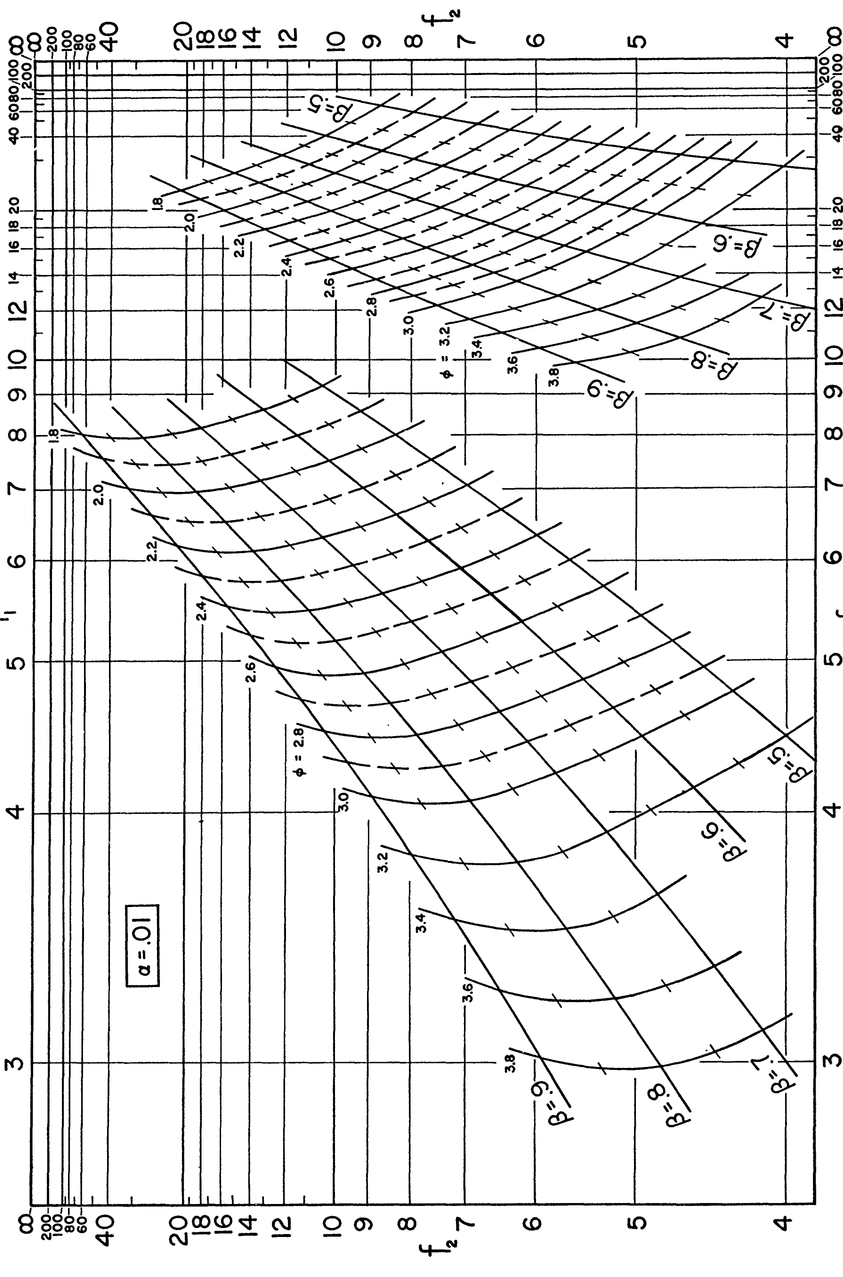

FIG. 9. Nomogram for the case $\alpha = 0.01$

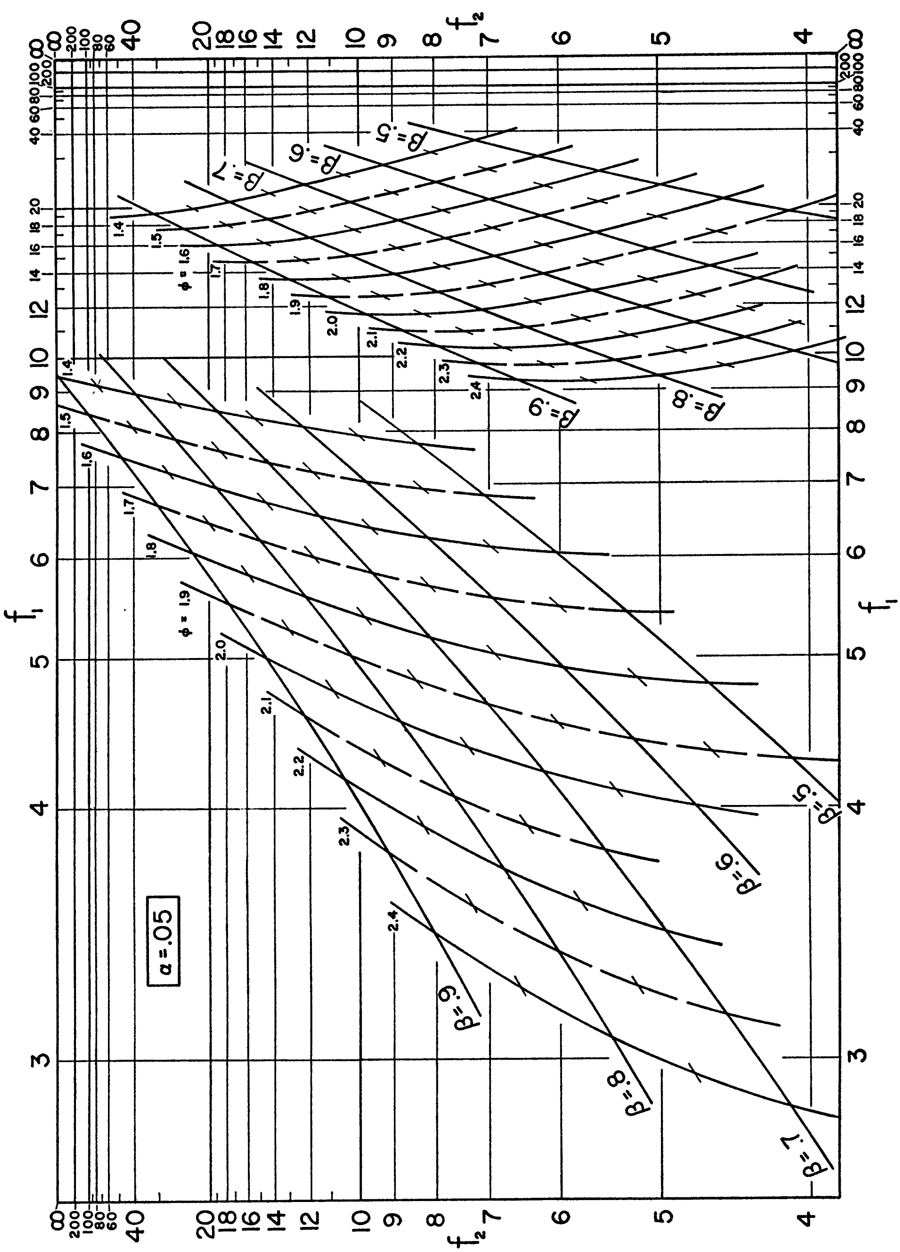

FIG. 10. Nomogram for the case $\alpha = 0.05$

values of f_1 results in very small errors. Therefore, to continue the curves beyond $f_1 = 8$ on the charts obtained from Tang's tables, straight lines were used between $f_1 = 8$ and $f_1 = \infty$.

Since for practical purposes the curves for constant ϕ may be replaced by straight lines, it is sufficient to provide two points for each value of ϕ with fixed α and β. Thus, on the nomograms the curves of constant ϕ may be reconstructed by connecting corresponding points in the two grids. For example, if the curve for $\phi = 1.6$ with $\alpha = 0.05$ and $\beta = 0.8$ is desired, it may be obtained by connecting the two intersections of the curves for $\phi = 1.6$ with the curves for $\beta = 0.8$ on Fig. 10. When this connecting line is extended, it intersects the vertical line $f_1 = 4$ at the horizontal line $f_2 = 75$. Also this extended line intersects $f_1 = 5$ at $f_2 = 30$, $f_1 = 60$ at $f_2 = 10$, etc., where the f_2 values are always rounded to the next larger integer. Thus, the power $\beta = 0.8$ can be achieved when $\phi = 1.6$ with any of these pairs of values of f_1 and f_2.

The nomograms have the disadvantage of restricting the range of values of ϕ.

On the nomograms, the curves for $\beta = 0.6$ were added by linear interpolation along the curves for ϕ. Intersections for $\beta = 0.55, 0.65, 0.75, 0.85$ were obtained in the same way.

3. Interpolation. For values of ϕ intermediate to those given in Figs. 1 to 8 linear interpolation along the normals to the curves may be used. On the nomograms linear interpolation along the β curves may be used for intermediate values of ϕ, and vice versa.

4. Example. As an illustration of the use of the charts, we consider the design of an experiment to test for possible effects of geographic locality on electrodermal resistance in 10-year-old children. We shall test children from $k = 6$ cities. Let the hypothesis to be tested at the 5 per cent significance level be that the locality effects are zero. Suppose we want a reasonable chance β of detecting that the locality effects are not zero when they are really δ_i, $i = 1, \cdots, k$, where $\sum_1^k \delta_i = 0$. In particular, suppose that when $\sum \delta_i^2/\sigma^2 = 2$, that is, when the sum of squares of locality effects in units of the standard deviation σ of a single measurement is 2, we want the probability that we conclude that the locality effects are not zero to be at least $\beta = 0.8$. What number n of children must be tested in each city to achieve this power?

In this case, $f_1 = k - 1 = 5$ and $f_2 = k(n - 1) = 6(n - 1)$. Furthermore,

$$\phi = \sqrt{S_b^*/[(f_1 + 1)\sigma^2]} = \sqrt{n \sum \delta_i^2/(k\sigma^2)}.$$

A procedure for determining n is the following:

(a) We assume a trial value of n. When one of Figs. 1 to 8 is to be used, we may obtain this trial value by reading the value of ϕ for the curve meeting $f_2 = \infty$ at our value of f_1 and then solving for n in the relation $\phi = \sqrt{n \sum \delta_i^2/(k\sigma^2)}$ using the next larger integer. (In this case we read $\phi = 1.46$. Solving for n we obtain $n = \phi^2 k\sigma^2/\sum \delta_i^2 = 6(1.46)^2/2 = 6.39$. Thus, we use $n = 7$ as our first trial value.)

(b) We fix $\sum \delta_i^2/\sigma^2$ at the value for which it is desired that the power be β. (In this case $\sum \delta_i^2/\sigma^2 = 2$.)

(c) We compute ϕ and f_2. (In this case $\phi = \sqrt{7(2)/6} = 1.527$ and $f_2 = 6(6) = 36$.)

(d) Turning to the chart appropriate to our α and β, we find the intersection of the curve for the value of ϕ in (c) with the line for the value of f_1. (In this case we use Fig. 7 and find the intersection of the curve for $\phi = 1.527$ with the line $f_1 = 5$. This is at $f_2 = 60$.)

(e) We repeat steps (a) through (d) until we have two consecutive values of n such that for one the value of f_2 obtained in (e) is larger than that obtained in (d) and for the other it is smaller. The larger of these two consecutive values of n is the required value.

The following table summarizes the results of this procedure for our example:

Trial n	$\phi = \sqrt{n(2)/6}$	$f_2 = 6(n-1)$	f_2 from Chart
7	1.527	36	60
8	1.633	42	23

Thus, we require $n = 8$.

Suppose we require a more stringent design. For example, suppose that with α, k, and $\sum \delta_i^2/\sigma^2$ as before we wish $\beta \geqq 0.85$. Since interpolation in β is necessary, Fig. 10 must be used. Otherwise the procedure is the same. In this case the above table becomes

Trial n	$\phi = \sqrt{n(2)/6}$	$f_2 = 6(n-1)$	f_2 from Chart
10	1.825	54	17
9	1.732	48	25
8	1.633	42	55

Here we obtained the line for trial value $n = 10$ by connecting with a ruler the interpolated point for $\beta = 0.85$, $\phi = 1.825$ on the left grid of Fig. 10 with the interpolated point on the right grid. Reading horizontally from the intersection of this line extended to the line $f_1 = 5$, we found $f_2 = 17$.

Since for trial value $n = 9$ the computed f_2 is larger than f_2 from the chart, while for trial value $n = 8$ the computed f_2 is smaller than f_2 from the chart, we require $n = 9$.

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