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ON THE WEAK CONVERGENCE OF PROBABILITY MEASURES[1]

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1. Introduction. Let $\mathscr{X}$ be a set carrying a σ-field $\mathscr{A}$ and a family of probability measures $\{P_\theta : \theta \in \Theta\}$. Let L be the smallest L-space which contains all the P_θ, $\theta \in \Theta$. This is the smallest linear space which is complete for the usual total variation norm and contains all the measures smaller than linear combinations of the P_θ. It is a Banach space with a dual M which contains, often properly, the space of equivalence classes of bounded $\mathscr{A}$-measurable functions.

Upon replication of the experiment $\{\mathscr{X}, \mathscr{A}, P_\theta ; \theta \in \Theta\}$ the relevant measures are the product measures $P_\theta \otimes P_\theta$ on the product space $\{\mathscr{X} \times \mathscr{X}, \mathscr{A} \times \mathscr{A}\}$. Several problems about the existence of "consistent tests", "sequential discrimination" and similar subjects lead to the following question:

If μ is a measure in the $w(L, M)$ closure $\bar{S}$ of the set $S = \{P_\theta ; \theta \in \Theta\}$ is the product measure $\mu \otimes \mu$ in the closure of $\{P_\theta \otimes P_\theta ; \theta \in \Theta\}$?

It is an easy consequence of a theorem of Dunford and Pettis (see [1], [2]) that the answer to this question is "yes" if the set $\{P_\theta ; \theta \in \Theta\}$ is $w(L, M)$ relatively compact in L. In particular if there is a *sequence* $\{P_{\theta_n}\}$ which converges to μ then $P_{\theta_n} \otimes P_{\theta_n}$ converges to $\mu \otimes \mu$.

The answer is also "yes" if $\mathscr{X}$ is a countable set. Finally, the answer is "yes" if the set $S = \{P_\theta ; \theta \in \Theta\}$ is convex, since in this case the strong closure of S coincides with its $w(L, M)$ closure. The purpose of the present note is to show that there do exist families $\{P_\theta ; \theta \in \Theta\}$ for which the answer is "no". In fact we shall demonstrate the existence of a *countable* collection $S = \{P_\theta ; \theta \in \Theta\}$ and a measure μ such that $\mu \in \bar{S}$ but such that $\mu \otimes \mu$ is remote from the closure of the convex hull of the set $\{P_\theta \otimes P_\theta ; \theta \in \Theta\}$.

In the usual statistical context one considers not only the products $P_\theta \otimes P_\theta$ but also for each integer n the experiment $\mathscr{E}_n$ consisting of taking n independent identically distributed observations whose distribution is either μ or one of the P_θ ; $\theta \in \Theta$. The available σ-field for $\mathscr{E}_n$ is the product $\mathscr{A}^n$ of n copies of $\mathscr{A}$. The measures are the corresponding products μ^n or $P_\theta^{\,n}$. Let then $\mathscr{P}$ be the set of all probability measures on $\{\mathscr{X}, \mathscr{A}\}$. For each n one can use $\mathscr{A}^n$, or equivalently the space $\mathscr{V}_n$ of equivalence classes of bounded $\mathscr{A}^n$-measurable functions to define a uniform structure $\mathscr{U}_n$ on $\mathscr{P}$. Let also $\mathscr{U}$ be the uniform structure defined by $\bigcup_n \mathscr{V}_n$. Let $\hat{S}$ be the closure of $S = \{P_\theta ; \theta \in \Theta\}$ in $\mathscr{P}$ for the structure $\mathscr{U}$. Let us say that there exist uniformly consistent tests of μ against S if there exist functions $\varphi_n \in \mathscr{V}_n$, $0 \leqq \varphi_n \leqq 1$ such that $E[\varphi_n \mid \mu] \to 1$ while $\sup_\theta \{E[\varphi_n \mid P_\theta] ; \theta \in \Theta\} \to 0$.

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It is a rather special case of Theorem 1 of [3] that there exist uniformly consistent tests of μ against S if and only if $\mu \notin \hat{S}$.

For each integer n one can introduce three "closures" $\hat{S}_n$, $\bar{S}_n$, and T_n of the set S in $\mathscr{P}$ as follows: the closure $\hat{S}_n$ for the structure $\mathscr{U}_n$, the closure $\bar{S}_n$ for the weak topology $w(L_n, M_n)$ induced by the adjoint of the L-space spanned by $\{P_\theta{}^n; \theta \in \Theta\}$, and finally the "sequential closure" T_n, that is the smallest set which contains S and contains the limits of $\mathscr{U}_n$ convergent sequences $\{\mu_k\}$, $\mu_k \in T_n$. It is easily verifiable that $\bar{S}_n \subset \hat{S}_n$, and that the two coincide if, for instance, the elements of S are all dominated by the same σ-finite measure. Also, using either the Dunford–Pettis criterion, or the Vitali–Hahn–Saks theorem, one verifies easily that T_n is independent of n so that the following relations always hold:

$$T_1 = T_n \subset \bar{S} = \textstyle\bigcap_n \bar{S}_n \subset \hat{S} = \bigcap_n \hat{S}_n; \ ,$$

also $\bar{S}_n \subset \bar{S}_m$ and $\hat{S}_n \subset \hat{S}_m$ for $m < n$.

The example given below is intended to show that $\hat{S}_2$ may be properly contained in $\hat{S}_1$ *even if S is countable* (in which case $\bar{S}_n = \hat{S}_n$). It is essentially an example showing that if a directed set $\{\mu_\pi\}$ converges for $\mathscr{U}_1$ towards a countably additive measure μ, the product $\mu_\pi \otimes \mu_\pi$ may converge for $\mathscr{U}_2$ towards a functional which is *not* countably additive on the σ-field $\mathscr{A}^2$ or even the field $\mathscr{A} \times \mathscr{A}$. If $\mu_\pi \otimes \mu_\pi$ would converge on $\mathscr{A} \times \mathscr{A}$ to a σ-additive limit, then the convergence would also take place on the whole σ-field $\mathscr{A}^2$ and the same would be true for any directed set $\nu_\pi \otimes \nu_\pi$ with ν_π smaller than a fixed multiple of μ_π.

The example suggests the possibility that $\hat{S}$ may be properly contained in each $\hat{S}_n$ and that T_1 may be a proper subset of $\bar{S}$. Further study of these questions may be of interest since $\bar{S}$ or $\hat{S}$ are rather difficult to identify, but T_1 and $\hat{S}_1$ appear relatively accessible. However, the passage from two to three dimensions appears to involve problems of a different character than the passage from one dimension to two.

2. Some lemmas concerning Lebesgue measure. Let μ be the Lebesgue measure on the Borel subsets of the interval $\mathscr{X} = [0, 1]$. Consider finite partitions $\pi = \{B_j; j = 1, 2, \cdots, n\}$ by elements B_j of the Borel field $\mathscr{A}$. For each such partition π let μ_π be a probability measure of the form $\mu_\pi = \sum_j \mu(B_j)\nu_j$ where ν_j is a probability measure carried by the set B_j. The finite partitions π form a directed set if ordered by refinement. The following easy remark will be used repeatedly below.

LEMMA 1. *Direct the finite partitions of $\mathscr{X}$ by refinement. For each finite partition π let μ_π be a measure constructed as explained above. Then* $\lim_\pi \int f d\mu_\pi = \int f d\mu$ *for every bounded measurable function f.*

NOTE. It is *not* claimed that μ is the $w(L, M)$ limit of the filter μ_π. This may not be true. In fact, if $\mu_\pi = \sum_j \mu(B_j)\,\delta_{x_j}$; where δ_{x_j} is the mass unity carried by a point $x_j \in B_j$ then Lemma 1 holds, but there is a $u \in M$ such that $\langle u, \varphi\rangle = \varphi$ for every positive measure φ dominated by μ but $\langle u, \varphi\rangle = 0$ for φ disjoint from μ. In particular, $\langle u, \mu_\pi\rangle = 0$ if $\mu_\pi = \sum \alpha_j \delta_{x_j}$.

PROOF. Let A be an arbitrary Borel set. There is some finite partition $\pi_0 = \{B_{0,j}\}$ such that A is one of the $B_{0,j}$. If $\pi = \{B_j\}$ refines π_0 then A is a finite sum $A = \bigcup B_j$ of those $B_j \subset A$. This gives

$$\mu_\pi(A) = \sum_j \mu(B_j)\nu_j[B_j \cap A] = \sum_j \{\mu(B_j);\, B_j \subset A\} = \mu(A).$$

It follows that if f is a simple function on $\mathscr{A}$, that is a function of the type $f = \sum_{j=1}^n c_j I_{A_j}$, then $\int f d\mu_\pi = \int f d\mu$ from some π on. Since for every bounded measurable function g and every $\varepsilon > 0$ there is a simple function f such that $\sup_x |g(x) - f(x)| < \varepsilon$, the result follows.

LEMMA 2. *Let $\{B_j\,; j = 0, 1, 2, \cdots, m\}$ be $m+1$ Borel subsets of $\mathscr{X}$ such that $\mu(B_j) > 0$ for each j. Then there exist points $x_j \in B_j$, $j = 0, 1, 2, \cdots, m$ such that*

(1) *all differences $x_j - x_i$ are rational*
(2) *each x_j is a point of density unity of its set B_j.*

PROOF. Let f_j be the indicator of the set of points $x \in B_j$ at which B_j has density unity. Let

$$h(x_1, x_2, \cdots, x_m) \int [\prod_{j=1}^m f_j(x_j + t)] f_0(t)\mu(dt)$$

this is a nonnegative function of the vector variable $z = \{x_1, \cdots, x_m\}$ whose Lebesgue integral on R^m is not zero. Thus h does not vanish identically. We claim that h is a continuous function of z. To prove this select an $\varepsilon > 0$ and continuous functions with compact support, say φ_j, such that $m\int |f_j(u) - \varphi_j(u)|\, \lambda(du) < \varepsilon$ for the Lebesgue measure λ on the whole line. Since $0 \leqq f_j \leqq 1$ one can also assume $0 \leqq \varphi_j \leqq 1$. Write

$$\prod_{j=1}^m f_j - \prod_{j=1}^m \varphi_j = \sum_k g_k[f_k - \varphi_k]$$

with $g_k = (\prod_{j<k} f_j)(\prod_{j>k} \varphi_j)$. This gives $0 \leqq g_k \leqq 1$, and therefore

$$\begin{aligned}&\left|\int\{[\prod f_j(x_j+t)] - [\prod \varphi_j(x_j+t)]\} f_0(t)\mu(dt)\right| \\ &\leqq \sum_k \int g_k \left|f_k(x_k+t) - \varphi_k(x_k+t)\right| f_0(t)\mu(dt) \\ &\leqq \sum_k \int \left|f_k(x_k+t) - \varphi_k(x_k+t)\right| \lambda(dt) < \varepsilon.\end{aligned}$$

The function $\int \prod_j \varphi_j(x_j+t) f_0(t)\mu(dt)$ is obviously a continuous function of $\{x_1, \cdots x_m\}$. Therefore the same is true of the limit h.

Since h is not identically zero and since it is continuous there is a point $\{y_1, \cdots y_m\}$ with rational coordinates such that $h[y_1, \cdots y_m] > 0$. For this point the integral which produces h must be non-zero somewhere. Thus there is some t such that $[\prod_{j=1}^m f_j(y_j + t)] f_0(t) > 0$. Write x_0 for this t and write $x_j = y_j + t$. The differences $x_j - x_i = y_j - y_i$ are rational for all pairs (i, j). Also each x_j is such that $f_j(x_j) > 0$; hence $f_j(x_j) = 1$. This completes the proof of the lemma.

3. Construction of the example. Consider in the Euclidean plane the set of straight lines on which differences of coordinates are rational. More specifically, order the rationals in a sequence $\{r_1, r_2, \cdots, r_n, \cdots\}$. Let D_n be the line defined by

$D_n = \{(x, y);\ y-x = r_n\}$. For a prescribed $\alpha \in (0, 1)$ let U_n be the set of points (x, y) such that $[(y-x)-r_n| < \alpha 2^{-(n+3)}$. Let $G^* = \bigcup_n U_n$ and let G be the intersection $G = G^* \cap C$ of G^* with the square $C = [0, 1] \times [0, 1]$. For the Lebesgue measure $\mu \otimes \mu$ on C the measure of G satisfies the inequality $[\mu \otimes \mu](G) \leqq \sum_n [\mu \otimes \mu][U_n \cap C] \leqq \sum_n \alpha 2^{\frac{1}{2}} 2^{-(n+3)} < \alpha/4$. Furthermore G is an open dense subset of C.

Let $\pi = \{B_j; j = 0, 1, \cdots, m, m+1, \cdots n_\pi\}$ be a Borel partition of $[0, 1]$ such that $\mu(B_j) > 0$ for $j = 0, 1, 2 \cdots, m$ and $\mu(B_j) = 0$ for $j = m+1, \cdots, n_\pi$. Assign to each B_j, $j = 0, 1, 2 \cdots, m$ a point of density $x_j \in B_j$ according to Lemma 2 so that $x_j - x_i$ is rational.

Once the points x_j have been selected one can find some $\varepsilon_\pi > 0$ such that $2^{n_\pi}\varepsilon_\pi < 1$, and such that if $|x-x_i| < 2\varepsilon_\pi$ and $|y-x_j| < 2\varepsilon_\pi$ for some pair (x_i, x_j), then $(x, y) \in G$ and in particular $x \in [0, 1]$.

Finally, since each x_i is a point of density of its B_j there is an ε such that

$$\mu\{B_j \cap [x_i - h, x_i + h]\} \geqq (1-\alpha_\pi)\, \mu[_i x - h, x_i + h]$$

for all $h \in (0, \varepsilon)$ and for α_π smaller than say $[n_\pi 2^{n_\pi}]^{-1}$. One can assume that ε_π has been taken smaller than this ε.

Construct measures m_π as follows. Let g_i be the indicator of the set $B_i \cap [x_i - \varepsilon_\pi, x_i + \varepsilon_\pi]$. Let $g_i = g_i[\int g_i\, d\mu]^{-1}$. Let m_π be the measure whose density with respect to the Lebesgue measure is $dm_\pi/d\mu = \sum \mu(B_j) g_i$.

LEMMA 3. *Let the measures m_π be constructed as just explained. Then*

(1) $[m_\pi \otimes m_\pi]G = 1$.

(2) *As the partitions π are refined the measures m_π converge to the Lebesgue measure μ for the topology $w(L, M)$ induced by the dual M of the space of all finite signed measures on* $[0, 1]$.

PROOF. The first statement follows from the construction. For the second statement note that by Lemma 1 the measures m_π converge to μ in the sense that $\int f\, dm_\pi \to \int f\, d\mu$ for all bounded Borel functions f. However, since all the m_π are absolutely continuous with respect to μ this is equivalent to $w(L, M)$ convergence.

As a final modification let us reconsider the functions g_i described above. This g_i is a probability density with respect to μ on a set $B_j \cap [x_i - \varepsilon_\pi, x_i + \varepsilon_\pi]$. Let $h_i(x) = (2\varepsilon_\pi)^{-1}$ for $x \in [x_i - \varepsilon_\pi, x_i + \varepsilon_\pi]$ and let $h_i(x) = 0$ otherwise. By construction

$$\begin{aligned}\int |h_i(x) - g_i(x)|\, \mu(dx) &= 2 \int |h_i(x) - g_i(x)|^+\, \mu(dx) \\ &\leqq (2\varepsilon_\pi)^{-1} \mu\{[x_i - \varepsilon_\pi, x_i + \varepsilon_\pi] \cap B_j^c\} \\ &\leqq (2\varepsilon_\pi)^{-1} \alpha_\pi\, \mu[x_i^- - \varepsilon_\pi, x_i + \varepsilon_\pi] \leqq \alpha_\pi.\end{aligned}$$

Let λ_π be the measure whose density with respect to μ is $\sum_j \mu(B_j) h_j(x)$. By construction $\|m_\pi - \lambda_\pi\| \leqq n_\pi \alpha_\pi \leqq 2^{-n_\pi}$. Now for each i one can find some interval $[\alpha_i, \beta_i] \subset [x_i - \varepsilon_\pi, x_i + \varepsilon_\pi]$ having rational end points and such that if f_i is the uniform probability density on $[\alpha_i, \beta_i]$ then $\int |f_i - h_i|\, d\mu \leqq (n_\pi 2^{n_\pi})^{-1}$. Select rational numbers $\mu_i \geqq 0$ such that $\sum |\mu_i - \mu(B_j)| < \varepsilon_\pi$ and $\sum \mu_i = 1$. Let ν_π be the measure whose density with respect to μ is $\sum_i \mu_i f_i$.

The properties of the measures ν_π are summarized in the following statement.

THEOREM 1. *Given any $\alpha > 0$ one can find an open set G of the square* [0, 1] × [0, 1] *and a* countable *set S of probability measures ν on* [0, 1] *with the following properties.*

(1) $[\mu \otimes \mu](G) < \alpha$.

(2) $[\nu \otimes \nu](G) = 1$ *for all* $\nu \in S$.

(3) *All the measures ν are absolutely continuous with respect to the Lebesgue measure μ on* [0, 1].

(4) *The measure $\nu \in S$ can be indexed by a directed set $\{\pi\}$ in such a way that*

$$\lim_\pi \int f \, d\mu_\pi = \int f \, d\mu$$

for every bounded measurable function f defined on [0, 1]. *Equivalently μ belongs to the $w(L, M)$ closure $\bar{S}$ of S.*

To prove this, note that $\lim_\pi \|\nu_\pi - \lambda_\pi\| = 0$ and $\lim_\pi \|m_\pi - \lambda_\pi\| = 0$ by construction. Thus the result follows from Lemma 3.

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