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Quentin Guignard

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On the ramified class field theory of relative curves

Quentin Guignard

We generalize Deligne's approach to tame geometric class field theory to the case of a relative curve, with arbitrary ramification.

1. Introduction

Let $X \to S$ be a relative curve, i.e., a smooth morphism of schemes of relative dimension 1, with connected geometric fibers, which is Zariski-locally projective over S. Let $Y \hookrightarrow X$ be a relative effective Cartier divisor over S (see Section 4.10), and let U be the complement of Y in X.

The pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module and α is a rigidification of $\mathcal{L}$ along Y, are parametrized by an S-group scheme $\mathrm{Pic}_S(X, Y)$, the relative rigidified Picard scheme (see Proposition 4.8). The Abel–Jacobi morphism

$$\Phi : U \to \mathrm{Pic}_S(X, Y)$$

is the morphism which sends a section x of U to the pair $(\mathcal{O}(x), 1)$, see Proposition 4.14. We prove the following relative version of the main theorem of geometric global class field theory:

Theorem 1.1 (Theorem 5.3). *Let Λ be a finite ring of cardinality invertible on S, and let $\mathcal{F}$ be an étale sheaf of Λ-modules, locally free of rank 1 on U, with ramification bounded by Y* (see Definition 5.2). *There exists a unique (up to isomorphism) multiplicative étale sheaf of Λ-modules $\mathcal{G}$ on $\mathrm{Pic}_S(X, Y)$, locally free of rank 1, such that the pullback of $\mathcal{G}$ by Φ is isomorphic to $\mathcal{F}$.*

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The notion of multiplicative locally free Λ-module of rank 1 is defined in Definition 2.5, and it corresponds to isogenies $G \to \mathrm{Pic}_S(X, Y)$ with constant kernel $\Lambda^\times$. We restrict ourselves in this article to $\Lambda^\times$-torsors, with Λ as in Theorem 1.1, in order to simplify the exposition, since we are able to apply directly our main descent tool in this context, namely Lemma 5.9. However, the latter lemma, and hence Theorem 1.1 can be extended to G-torsors, where G is an arbitrary locally constant finite abelian group on $S_{\text{ét}}$.

The case where S is the spectrum of a perfect field is originally due to Serre [1959] and Lang [1956, §6]. Their proof relies on the Albanese property of Rosenlicht's generalized Jacobians [Rosenlicht 1954]. A similar proof was sketched in a letter of 1974 from Deligne to Serre [Deligne 2001]. However, a more geometric proof was given by Deligne in the tamely ramified case; an account of his proof in the unramified case over a finite field can be found in [Laumon 1990, Section 2]. We generalize the latter approach by Deligne to allow arbitrary ramification and an arbitrary base S. This generalization is inspired by notes by Alain Genestier (unpublished) on arithmetic global class field theory.

Deligne's approach has the advantage over Serre and Lang's to yield an explicit geometric construction of the isogeny over $\mathrm{Pic}_S(X, Y)$ corresponding to a local system of rank 1 over U. This feature of Deligne's approach carries over to ours, and is in fact crucial in order to handle the case of an arbitrary base S.

The author learned during the preparation of this manuscript that Daichi Takeuchi had independently obtained a different proof of Theorem 1.1 in the case where S is the spectrum of a perfect field, also by generalizing Deligne's approach to handle arbitrary ramification. See [Takeuchi 2019].

Notation and conventions. We fix a universe $\mathcal{U}$ [SGA 4_3 1973, I.0]. Throughout this paper, all sets are assumed to belong to $\mathcal{U}$ and we will use the term "topos" as a shorthand for "$\mathcal{U}$-topos" [SGA 4_3 1973, IV.1.1]. The category of sets belonging to $\mathcal{U}$ is simply denoted by Sets.

For any integers e, d we denote by $[\![e, d]\!]$ the set of integers n such that $e \le n \le d$ and by $\mathfrak{S}_d$ the group of bijections of $[\![1, d]\!]$ onto itself.

In this paper, all rings are unital and commutative. For any ring A, we denote by Alg_A the category of A-algebras. For any scheme S, we denote by $\mathrm{Sch}_{/S}$ the category of S-schemes. We denote by $S_{\text{ét}}$ (resp. $S_{\text{Ét}}$) the small étale topos (resp. big étale topos) of a scheme S, i.e., the topos of sheaves of sets for the étale topology [SGA 4_3 1973, VII.1.2] on the category of étale S-schemes (resp. on $\mathrm{Sch}_{/S}$), and by S_{Fppf} the big fppf topos of S, i.e., the topos of sheaves of sets for the fppf topology on $\mathrm{Sch}_{/S}$ [SGA 4_3 1973, VII.4.2]. If $f : X \to S$ is a morphism of schemes, then we denote by (f^{-1}, f_*) the induced morphism of topos from $X_{\text{Ét}}$ to $S_{\text{Ét}}$. The symbol f^* will exclusively denote the pullback functor from $\mathcal{O}_S$-modules to $\mathcal{O}_X$-modules.

2. Preliminaries

2.1. Let E be a topos and let G be an abelian group in E. We denote by GE the category of objects of E endowed with a left action of G. If X is an object of E, we denote by $E_{/X}$ the topos of X-objects in E. If X is considered as an object of GE by endowing it with the trivial left G-action, then we have $(GE)_{/X} = G(E_{/X})$ and this category will be simply denoted by $GE_{/X}$.

Definition 2.2. A *G-torsor* over an object X of E is an object P of $GE_{/X}$ such that $P \to X$ is an epimorphism and the morphism

$$G \times P \to P \times_X P, \quad (g, p) \mapsto (g \cdot p, p)$$

is an isomorphism. We denote by $\mathrm{Tors}(X, G)$ the full subcategory of $GE_{/X}$ whose objects are the G-torsors over X. If $f : Y \to X$ is a morphism in E, we denote by $f^{-1} : \mathrm{Tors}(X, G) \to \mathrm{Tors}(Y, G)$ the functor which associates $f^{-1}P = P \times_{X,f} Y$ to a G-torsor P over X.

The category $\mathrm{Tors}(X, G)$ is monoidal, with product

$$P_1 \otimes P_2 = G_2 \setminus P_1 \times_X P_2,$$

where G_2 is the kernel of the multiplication morphism $G \times G \to G$, and where $G_2 \hookrightarrow G \times G$ acts diagonally on $P_1 \times_X P_2$. The neutral element for this product is the trivial G-torsor over X, namely $G \times X$, and each G-torsor P over X is invertible with respect to $\otimes$, with inverse given by

$$P^{-1} = \underline{\mathrm{Hom}}_{GE_{/X}}(P, G \times X),$$

where $\underline{\mathrm{Hom}}_{GE_{/X}}$ denotes the internal Hom functor in $GE_{/X}$.

Example 2.3. If $G = \Lambda^\times$ for some ring Λ in E, then the monoidal category $\mathrm{Tors}(X, G)$ is equivalent to the groupoid of locally free Λ-modules of rank 1 in $E_{/X}$. The equivalence is given by the functor which sends an object P of $\mathrm{Tors}(X, G)$ to the Λ-module $G \setminus (\Lambda \times P)$, where the action of $G = \Lambda^\times$ on $\Lambda \times P$ is given by the formula $g \cdot (\lambda, p) = (g\lambda, g \cdot p)$. The functor which sends a locally free Λ-module M of rank 1 of $E_{/X}$ to the G-torsor of isomorphisms of Λ-modules from M to Λ defines a quasiinverse to the latter functor.

2.4. Let E be a topos, and let us denote by 1 its terminal object. Let us consider an exact sequence

$$1 \to G \xrightarrow{i} P \xrightarrow{r} Q \to 1$$

of abelian groups in E. The morphism

$$G \times P \to P \times_Q P, \quad (g, p) \mapsto (i(g) + p, p)$$

is an isomorphism, so that P is a G-torsor over Q. Moreover, the multiplication morphism

$$P \times P \to P$$

factors though $G_2 \setminus P \times P$, where $G_2 \hookrightarrow G \times G$ is the kernel of the multiplication morphism of G, acting diagonally on $P \times P$. We thus obtain a morphism

$$p_1^{-1} P \otimes p_2^{-1} P \to m^{-1} P$$

of G-torsors over $Q \times Q$, where p_1 and p_2 are the canonical projections and m is the multiplication morphism of Q.

The following definition is inspired by [Moret-Bailly 1985, I.2.3]:

Definition 2.5. Let G be an abelian group of E and let Q be a commutative semigroup of E (with or without identity). Let $m : Q \times Q \to Q$ be the multiplication morphism of Q. A *multiplicative G-torsor* over Q is a G-torsor $P \to Q$, together with an isomorphism

$$\theta : p_1^{-1}P \otimes p_2^{-1}P \to m^{-1}P$$

of G-torsors over $Q \times Q$, where p_1 and p_2 are the canonical projections, which satisfy the following two properties.

▹ *Symmetry*: If σ is the involution of $Q \times Q$ which switches the two factors, then the isomorphism

$$p_2^{-1}P \otimes p_1^{-1}P \to \sigma^{-1}(p_1^{-1}P \otimes p_2^{-1}P) \xrightarrow{\sigma^{-1}\theta} \sigma^{-1}m^{-1}P \to m^{-1}P$$

is the composition of θ with the canonical isomorphism $p_2^{-1}P \otimes p_1^{-1}P \to p_1^{-1}P \otimes p_2^{-1}P$.

▹ *Associativity*: Let $q_i : Q \times Q \times Q \to Q$ be the projection on the i-th factor, where $i \in [\![1, 3]\!]$, and define $q_{ij} : Q \times Q \times Q \to Q \times Q$ similarly, where $(i, j) \in [\![1, 3]\!]^2$ with $i < j$. If $m_3 : Q \times Q \times Q \to Q$ is the multiplication morphism, then the diagram of G-torsors over $Q \times Q \times Q$

$$\begin{tikzcd}
 & q_1^{-1}P \otimes (mq_{23})^{-1}P \arrow[dr, "(q_1 \times mq_{23})^{-1}\theta"] & \\
q_1^{-1}P \otimes q_2^{-1}P \otimes q_3^{-1}P \arrow[ur, "id \otimes q_{23}^{-1}\theta"] \arrow[dr, "q_{12}^{-1}\theta \otimes id"'] & & m_3^{-1}P \\
 & (mq_{12})^{-1}P \otimes q_3^{-1}P. \arrow[ur, "(mq_{12} \times q_3)^{-1}\theta"'] &
\end{tikzcd}$$

is commutative.

The category of multiplicative G-torsors is fibered in groupoids over the category of commutative semigroups of E. We denote by $\mathrm{Tors}^{\otimes}(Q, G)$ the groupoid of multiplicative G-torsors over a commutative semigroup Q of E.

Remark 2.6. If $G = \Lambda^{\times}$ for some ring Λ in E, we use the term "*multiplicative locally free Λ-module of rank* 1" as a synonym for "multiplicative G-torsor", when we want to emphasize the locally free Λ-module of rank 1 corresponding to a given G-torsor, rather than the G-torsor itself (see Example 2.3).

Proposition 2.7. *Let G be an abelian group in E, let Q be a commutative semigroup in E and let I be an ideal of Q. If the projection morphisms $Q \times I \to Q$ and $I \times I \to I$ onto the first factors are morphisms of descent for the fibered category of multiplicative G-torsors* (see Definition 2.5), *then the restriction functor*

$$\mathrm{Tors}^{\otimes}(Q, G) \to \mathrm{Tors}^{\otimes}(I, G)$$

is fully faithful.

Let $i : I \to Q$ be the canonical injection morphism. Let p_1 and p_2 be the projection morphisms of $Q \times I$ onto its first and second factors respectively, and let $m : Q \times I \to I$ be the multiplication morphism. Let (P, θ) and (P', θ') be multiplicative G-torsors over Q. We have an isomorphism

$$\beta_P : p_1^{-1} P \xrightarrow{(\mathrm{id} \times i)^{-1}\theta} m^{-1} i^{-1} P \otimes p_2^{-1} i^{-1} P^{-1},$$

and similarly for P'. If $\alpha : i^{-1}P \to i^{-1}P'$ is a morphism of multiplicative G-torsors over I, then $\beta_{P'}^{-1}(m^{-1}\alpha \otimes p_2^{-1}\alpha)\beta_P$ is an isomorphism from $p_1^{-1}P$ to $p_1^{-1}P'$, which is compatible with the canonical descent datum for p_1 associated to $p_1^{-1}P$ and $p_1^{-1}P'$. Since p_1 is a morphism of descent for the fibered category of multiplicative G-torsors, there is a unique morphism $\gamma : P \to P'$ of multiplicative G-torsors over Q such that $p_1^{-1}\gamma = \beta_{P'}^{-1}(m^{-1}\alpha \otimes p_2^{-1}\alpha)\beta_P$. The restriction of $p_1^{-1}\gamma$ to $I \times I$ is the pullback of α by the first projection, which is a morphism of descent for the fibered category of multiplicative G-torsors, so that the restriction of γ to I is α.

Proposition 2.8. *Let G be an abelian group in E, and let $\rho : M \to Q$ be a morphism of commutative semigroups in E. If ρ (resp. $\rho \times \rho$ and $\rho \times \rho \times \rho$) is a morphism of effective descent (resp. of descent) for the fibered category of G-torsors, then ρ is a morphism of effective descent for the fibered category of multiplicative G-torsors.*

A descent datum of multiplicative G-torsors for ρ yields a descent datum of G-torsors for ρ, hence a G-torsor over Q by hypothesis. Since $\rho \times \rho$ and $\rho \times \rho \times \rho$ are morphisms of descent for the fibered category of G-torsors, the structure of multiplicative G-torsor descends as well. Details are omitted.

Proposition 2.9. *Let G and Q be abelian groups in E. The groupoid* $\mathrm{Tors}^{\otimes}(Q, G)$ *of multiplicative G-torsors over Q is equivalent as a monoidal category to the groupoid of extensions of Q by G in E, with the Baer sum as a monoidal structure.*

We have already seen how to associate a multiplicative G-torsor to an extension of Q by G. This construction is functorial, and the corresponding functor is an equivalence by [Moret-Bailly 1985, I.2.3.10].

Corollary 2.10. *Let G and Q be abelian groups in E. The group of isomorphism classes of multiplicative G-torsors over Q is isomorphic to the group* $\mathrm{Ext}^1(Q, G)$ *of isomorphism classes of extensions of Q by G in E.*

2.11. Let S be a scheme, let X be an S-scheme, and let G be a finite abelian group. Let P be a G-torsor over X in $S_{\text{Ét}}$. Since $P \to X$ is an epimorphism in $S_{\text{Ét}}$, there is an étale cover $(X_i \to X)_{i \in I}$ such that for each $i \in I$, the morphism $X_i \to X$ factors through $P \to X$. In particular, for each $i \in I$ the G-torsor $P \times_X X_i \to X_i$ is isomorphic to the trivial G-torsor $G \times X_i \to X_i$, so that $P \times_X X_i$ is representable by a finite étale X_i-scheme. By étale descent of affine morphisms, we obtain:

Proposition 2.12. *Let G be a finite abelian group, let S be a scheme, and let P be a G-torsor over an S-scheme X in $S_{\text{Ét}}$. Then the étale sheaf $P : \mathrm{Sch}_{/S} \to \mathrm{Sets}$ is representable by a finite étale X-scheme.*

The topos $(S_{\text{Ét}})_{/X}$ coincides with $X_{\text{Ét}}$. The category of G-torsors over X in $S_{\text{Ét}}$ is therefore equivalent to the category of G-torsors over the terminal object in $X_{\text{Ét}}$, and Proposition 2.12 yields:

Corollary 2.13. *Let G be a finite abelian group, let S be a scheme, and let X be an S-scheme. Then the category of G-torsors over X in $S_{\text{Ét}}$ is equivalent to the category of G-torsors over the terminal object in $X_{ét}$.*

2.14. Let S be a scheme, and let G be a finite abelian group. Let Q be a commutative S-group scheme, and let M be a sub-S-semigroup scheme of Q.

Proposition 2.15. *Assume that the morphism*

$$\rho : M \times_S M \to Q, \quad (x, y) \mapsto xy^{-1}$$

is faithfully flat and quasicompact, and that M is flat over S. Then the restriction functor

$$\mathrm{Tors}^{\otimes}(Q, G) \to \mathrm{Tors}^{\otimes}(M, G)$$

is an equivalence of categories.

Let (P, θ) be a multiplicative G-torsor over M. For $i \in [\![1, 4]\!]$, let r_i be the projection of $R = (M \times_S M) \times_{\rho,Q,\rho} (M \times_S M)$ onto its i-th factor. Similarly, for $i, j \in [\![1, 4]\!]$ such that $i < j$, we denote by $r_{ij} : R \to M \times_S M$ the projection on the i-th and j-th factors. We then have a sequence of isomorphisms

$$(r_1^{-1}P \otimes r_2^{-1}P^{-1}) \otimes (r_3^{-1}P \otimes r_4^{-1}P^{-1})^{-1} \longrightarrow r_{14}^{-1}(p_1^{-1}P \otimes p_2^{-1}P) \otimes r_{23}^{-1}(p_1^{-1}P \otimes p_2^{-1}P)^{-1}$$
$$\xrightarrow{r_{14}^{-1}\theta \otimes (r_{23}^{-1}\theta)^{-1}} (mr_{14})^{-1}P \otimes ((mr_{23})^{-1}P)^{-1},$$

of G-torsors over R, where $m : M \times_S M \to M$ is the multiplication of M. Since $mr_{14} = mr_{23}$, the latter G-torsor is canonically trivial. We thus obtain an isomorphism

$$\psi : r_1^{-1}P \otimes r_2^{-1}P^{-1} \to r_3^{-1}P \otimes r_4^{-1}P^{-1},$$

of G-torsors over R. The associativity of θ (see Definition 2.5) implies that ψ is a cocycle, i.e., $(p_1^{-1}P \otimes p_2^{-1}P^{-1}, \psi)$ is a descent datum for ρ. By Proposition 2.12 and since faithfully flat and quasicompact morphisms of schemes are of effective descent for the fibered category of affine morphisms, the conditions of Proposition 2.8 are satisfied, and thus there exists a multiplicative G-torsor P' over Q and an isomorphism $\alpha : \rho^{-1}P' \to p_1^{-1}P \otimes p_2^{-1}P^{-1}$ such that ψ is given by the composition

$$r_1^{-1}P \otimes r_2^{-1}P^{-1} \xrightarrow{r_{12}^{-1}\alpha^{-1}} (\rho r_{12})^{-1}P' = (\rho r_{34})^{-1}P' \xrightarrow{r_{34}^{-1}\alpha} r_3^{-1}P \otimes r_4^{-1}P^{-1}.$$

The association $P \mapsto P'$ then defines a functor from $\mathrm{Tors}^{\otimes}(M, G)$ to $\mathrm{Tors}^{\otimes}(Q, G)$. For any multiplicative G-torsor U over Q, we have an isomorphism $U \to (U \times_Q M)'$ by multiplicativity, which is functorial in U.

We now construct, for any multiplicative G-torsor (P, θ) over M, an isomorphism $P \to P' \times_Q M$ of multiplicative G-torsors which is functorial in P. Let $\nu : M \times_S M \to M \times_S M$ be the morphism which sends a section (x, y) to (xy, y). We have an isomorphism

$$(\rho\nu)^{-1}P' \xrightarrow{\nu^{-1}\alpha} \nu^{-1}(p_1^{-1}P \otimes p_2^{-1}P^{-1}) \to m^{-1}P \otimes p_2^{-1}P^{-1} \xrightarrow{\theta^{-1}} p_1^{-1}P.$$

The diagram

$$\begin{tikzcd}
 & M \times_S M \arrow[dr, "\rho"] & \\
M \times_S M \arrow[ur, "\nu"] \arrow[dr, "p_1"'] & & Q \\
 & M \arrow[ur] &
\end{tikzcd}$$

is commutative; hence $(\rho\nu)^{-1}P'$ is isomorphic to $p_1^{-1}(P' \times_Q M)$. We thus obtain an isomorphism

$$\beta : p_1^{-1}P \to p_1^{-1}(P' \times_Q M),$$

of multiplicative G-torsors. The morphism β is compatible with the canonical descent data for p_1 associated to $p_1^{-1}P$ and $p_1^{-1}(P' \times_Q M)$. Since p_1 is a covering for the fpqc topology, Proposition 2.8 applies, hence there is a unique isomorphism $\gamma : P \to P' \times_Q M$ of multiplicative G-torsors such that $\beta = p_1^{-1}\gamma$. The construction of this isomorphism of multiplicative G-torsors is functorial in P, hence the result.

2.16. Let A be a ring. If M is an A-module, we denote by $\underline{M}$ the functor $B \mapsto M \otimes_A B$ from Alg_A to Sets.

Definition 2.17 [SGA 4_3 1973, XVII 5.5.2.2]. Let M and N be A-modules. A *polynomial map* from M to N is a morphism of functors $\underline{M} \to \underline{N}$. A polynomial map $f : \underline{M} \to \underline{N}$ is *homogeneous of degree d* if for any A-algebra B, any element λ of B and any element m of $\underline{M}(B)$, we have $f(\lambda m) = \lambda^d f(m)$.

For each integer d and any A-module M, let $\mathrm{TS}^d_A(M) = (M^{\otimes_A d})^{\mathfrak{S}_d}$ be the A-module of symmetric tensors of degree d with coefficients in M. If M is a free A-module with basis $(e_i)_{i\in I}$, then we have a decomposition

$$TS^d_A(M) = \Bigg(\bigoplus_{\beta:[\![1,d]\!]\to I} Ae_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)} \Bigg)^{\mathfrak{S}_d} = \bigoplus_{\substack{\alpha:I\to\mathbb{N}\\ \sum_{i\in I}\alpha(i)=d}} Ae_\alpha, \tag{2.17.1}$$

where we have set

$$e_\alpha = \sum_{\substack{\beta:[\![1,d]\!]\to I\\ \forall i, |\beta^{-1}(\{i\})|=\alpha(i)}} e_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)}.$$

In particular $TS^d_A(M)$ is a free A-module, and its formation commutes with base change by any ring morphism $A \to B$.

Proposition 2.18. *Let M be a flat A-module and let $d \geq 0$ be an integer. Then $\mathrm{TS}^d_A(M)$ is a flat module, and for any A-algebra B the canonical homomorphism*

$$\mathrm{TS}^d_A(M) \otimes_A B \to \mathrm{TS}^d_B(M \otimes_A B)$$

is bijective.

Any flat A-module is a filtered colimit of finite free modules. We have already seen that the conclusion of Proposition 2.18 holds whenever M is free, hence the conclusion in general since the functor TS_A^d commutes with filtered colimits.

Proposition 2.19. *Let M be a flat A-module and let $d \geq 0$ be an integer. Let $\gamma_d : \underline{M} \to \underline{\mathrm{TS}_A^d(C)}$ be the functor which sends, for any A-algebra B, an element m of $\underline{M}(B)$ to the element $m^{\otimes d}$ of $\mathrm{TS}_B^d(M \otimes_A B) = \mathrm{TS}_A^d(M) \otimes_A B$* (see Proposition 2.18). *Then, for any homogeneous polynomial map $f : \underline{M} \to \underline{N}$ of degree d, there is a unique A-linear homomorphism $\tilde{f} : \mathrm{TS}_A^d(M) \to N$ such that $f = \tilde{f}\gamma_d$.*

As in Proposition 2.18, we can assume that M is free of finite rank over A. Let $(e_i)_{i\in I}$ be a basis of M. Let us write

$$f\Big(\sum_{i\in I} X_i e_i\Big) = \sum_{\alpha : I \to \mathbb{N}} X^\alpha f_\alpha$$

in $\underline{N}(A[(X_i)_{i\in I}])$ for some elements $(f_\alpha)_\alpha$ of N, where $X^\alpha = \prod_{i\in I} X_i^{\alpha_i}$. Accordingly, we have for any A-algebra B and any element $m = \sum_{i\in I} b_i e_i$ of $\underline{M}(B)$, the formula

$$f(m) = \sum_{\alpha : I \to \mathbb{N}} b^\alpha f_\alpha,$$

where $b^\alpha = \prod_{i\in I} b_i^{\alpha_i}$, by using the naturality of f with the unique morphism of A-algebras $A[(X_i)_{i\in I}] \to B$ which sends X_i to b_i for each i. By applying this to the element $m = \sum_{i\in I} T X_i e_i$ of $\underline{M}(A[T, (X_i)_{i\in I}])$, we obtain

$$f\Big(\sum_{i\in I} T X_i e_i\Big) = \sum_{\alpha : I \to \mathbb{N}} T^{|\alpha|} X^\alpha f_\alpha,$$

where we have set $|\alpha| = \sum_{i\in I} \alpha(i)$. Since f is homogeneous of degree d, the left side of this equation is also equal to

$$T^d f\Big(\sum_{i\in I} X_i e_i\Big) = \sum_{\alpha : I \to \mathbb{N}} T^d X^\alpha f_\alpha.$$

We conclude that $T^d f_\alpha = T^{|\alpha|} f_\alpha$ in $N \otimes_A A[T]$ for any $\alpha : I \to \mathbb{N}$, and thus that $f_\alpha = 0$ whenever $|\alpha|$ differs from d. We therefore have

$$f(m) = \sum_{\substack{\alpha : I \to \mathbb{N} \\ |\alpha| = d}} b^\alpha f_\alpha,$$

for any A-algebra B and any element $m = \sum_{i\in I} b_i e_i$ of $\underline{M}(B)$. Using the decomposition (2.17.1), we also have

$$\gamma_d(m) = \sum_{\beta : [\![1,d]\!] \to I} \otimes_{j=1}^d b_{\beta(j)} e_{\beta(j)} = \sum_{\substack{\alpha : I \to \mathbb{N} \\ |\alpha| = d}} b^\alpha e_\alpha.$$

The conclusion of Proposition 2.19 is achieved by taking $\tilde{f}$ to be the unique morphism of A-modules from $\mathrm{TS}_A^d(M)$ to N which sends e_α to f_α.

2.20. Let $A \to C$ be a ring morphism such that C is a finitely generated projective A-module of rank d. For any A-algebra B and any element m of $\underline{C}(B)$, we set

$$N_{C/A}(c) = \det_{\underline{A}(B)}(m_c),$$

where m_c is the $\underline{A}(B)$-linear endomorphism of $\underline{C}(B)$ induced by the multiplication by c. This defines a homogeneous polynomial map $N_{C/A} : \underline{C} \to \underline{A}$ of degree d (see Definition 2.17). By Proposition 2.19, there is a unique morphism of A-modules $\varphi : \mathrm{TS}^d_A(C) \to A$ such that $N_{C/A} = \varphi\gamma_d$.

Proposition 2.21 [SGA 4$_3$ 1973, XVII 6.3.1.6]. *The morphism of A-modules $\varphi : \mathrm{TS}^d_A(C) \to A$ is a morphism of A-algebras.*

Let x be an element of C, and let us consider the morphism of A-modules $f : y \to \varphi(\gamma_d(x)y)$ from $\mathrm{TS}^d_A(C)$ to A. For any A-algebra B and any element c of $\underline{C}(B)$, we have

$$f(\gamma_d(c)) = \varphi(\gamma_d(x)\gamma_d(c)) = \varphi(\gamma_d(xc)) = N_{C/A}(xc) = N_{C/A}(x)N_{C/A}(c)$$

by the multiplicativity of determinants, so that $f(\gamma_d(c)) = N_{C/A}(x)\varphi(\gamma_d(c))$. By the uniqueness statement in Proposition 2.19, we obtain $f = N_{C/A}(x)\varphi$, i.e., for all y in $\mathrm{TS}^d_A(C)$ we have

$$\varphi(\gamma_d(x)y) = N_{C/A}(x)\varphi(y). \tag{2.21.1}$$

For any A-algebra B, one can apply this argument to the morphism $B \to \underline{C}(B)$ instead of $A \to C$. Thus (2.21.1) also holds for any element x of $\underline{C}(B)$ and any element y of $\underline{\mathrm{TS}^d_A(C)}(B) = \mathrm{TS}^d_{\underline{A}(B)}(\underline{C}(B))$ (see Proposition 2.18). Now, let y be an element of $\mathrm{TS}^d_A(C)$ and let us consider the morphism of A-modules $g : z \to \varphi(zy)$ from $\mathrm{TS}^d_A(C)$ to A. We have proved that $g\gamma_d = \varphi(y)N_{C/A}$, hence $g = \varphi(y)\varphi$ by Proposition 2.19. Thus φ is a morphism of rings. Since φ is also A-linear, it is a morphism of A-algebras.

2.22. Let S be a scheme.

Definition 2.23 [SGA 1 1971, V.1.7].

- ▹ Let T be an object of a category C endowed with a right action of a group Γ. We say that *the quotient T/Γ exists* in C if the covariant functor

  $$C \to \mathrm{Sets}, \quad U \mapsto \mathrm{Hom}_C(T, U)^\Gamma$$

  is representable by an object of C.
- ▹ Let T be an S-scheme. An action of a finite group Γ on T is *admissible* if there exists an affine Γ-invariant morphism $f : T \to T'$ such that the canonical morphism $\mathcal{O}_{T'} \to f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^\Gamma$.

Proposition 2.24 [SGA 1 1971, V.1.3]. *Let T be an S-scheme endowed with an admissible right action of a finite group Γ. If $f : T \to T'$ is an affine Γ-invariant morphism such that the canonical morphism $\mathcal{O}_{T'} \to f_*\mathcal{O}_T$ induces an isomorphism from $\mathcal{O}_{T'}$ to $(f_*\mathcal{O}_T)^\Gamma$, then the quotient T/Γ exists and is isomorphic to T'.*

Proposition 2.25 [SGA 1 1971, V.1.8]. *Let T be an S-scheme endowed with a right action of a finite group Γ. The action of Γ on T is admissible if and only if T is covered by Γ-invariant affine open subsets.*

Proposition 2.26 [SGA 1 1971, V.1.9]. *Let T be an S-scheme endowed with an admissible right action of a finite group Γ, and let S' be a flat S-scheme. Then, the action of Γ on the S'-scheme $T \times_S S'$ is admissible, and the canonical morphism*

$$(T \times_S S')/\Gamma \to (T/\Gamma) \times_S S'$$

is an isomorphism.

Let X be an S-scheme and let $d \geq 0$ be an integer. The group $\mathfrak{S}_d$ of permutations of $[\![1, d]\!]$ acts on the right on the S-scheme $X^{\times_S d} = X \times_S \cdots \times_S X$ by the formula

$$(x_i)_{i \in [\![1,d]\!]} \cdot \sigma = (x_{\sigma(i)})_{i \in [\![1,d]\!]}.$$

Proposition 2.27. *If X is Zariski-locally quasiprojective over S, then the right action of $\mathfrak{S}_d$ on $X^{\times_S d}$ is admissible. In particular, the quotient $\operatorname{Sym}^d_S(X) = X^{\times_S d}/\mathfrak{S}_d$ exists in the category of S-schemes.*

Since X is Zariski-locally quasiprojective over S, any finite set of points in X with the same image in S is contained in an affine open subset of X. Thus $X^{\times_S d}$ is covered by open subsets of the form $U^{\times_S d}$ where U is an affine open subset of X whose image in S is contained in an affine open subset of S. These particular open subsets are affine and $\mathfrak{S}_d$-invariant, so that the action of $\mathfrak{S}_d$ on $X^{\times_S d}$ is admissible by Proposition 2.25.

Remark 2.28. If $X = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(A)$ are affine, then for any S-scheme T we have

$$\operatorname{Hom}_{\mathrm{Sch}_{/S}}(X^{\times_S d}, T)^{\mathfrak{S}_d} = \operatorname{Hom}_{\mathrm{Alg}_A}(\Gamma(T, \mathcal{O}_T), B^{\otimes_A d})^{\mathfrak{S}_d} = \operatorname{Hom}_{\mathrm{Alg}_A}(\Gamma(T, \mathcal{O}_T), \mathrm{TS}^d_A(B)),$$

see Section 2.16. Thus $\operatorname{Sym}^d_S(X)$ is representable by the S-scheme $\operatorname{Spec}(\mathrm{TS}^d_A(B))$.

Proposition 2.29. *If X is flat and Zariski-locally quasiprojective over S, then $\operatorname{Sym}^d_S(X)$ is flat over S. Moreover, for any S-scheme S', the canonical morphism*

$$\operatorname{Sym}^d_{S'}(X \times_S S') \to \operatorname{Sym}^d_S(X) \times_S S'$$

is an isomorphism.

This follows from Remark 2.28 and from Proposition 2.18.

Proposition 2.30 [SGA 1 1971, IX.5.8]. *Let G be a finite abelian group, let P be a G-torsor over an S-scheme X in $S_{\text{Ét}}$. Assume that P and X are endowed with right actions from a finite group Γ such that the morphism $P \to X$ is Γ-equivariant, and that the following properties hold*:

(a) *The right Γ-action on P commutes with the left G-action.*

(b) *The right Γ-action on X is admissible* (see Definition 2.23), *and the quotient morphism $X \to X/\Gamma$ is finite.*

(c) *For any geometric point $\bar{x}$ of X, the action of the stabilizer $\Gamma_{\bar{x}}$ of $\bar{x}$ in Γ on the fiber $P_{\bar{x}}$ of P at $\bar{x}$ is trivial.*

Then the action of Γ on P is admissible, and P/Γ is a G-torsor over X/Γ in $S_{\text{Ét}}$.

2.31. Let S be a scheme, let X be an S-scheme and let $d \geq 1$ be an integer. Let G be a finite abelian group, and let $P \to X$ be a G-torsor over X in $S_{\text{Ét}}$. By Proposition 2.12, the sheaf P is representable by a finite étale X-scheme.

For each $i \in [\![1, d]\!]$ let $p_i : X^{\times_S d} \to X$ be the projection on i-th factor, and let us consider the G-torsor

$$p_1^{-1}P \otimes \cdots \otimes p_d^{-1}P = G_d \setminus P^{\times_S d}$$

over $X^{\times_S d}$, where $G_d \subseteq G^d$ is the kernel of the multiplication morphism $G^d \to G$. By Proposition 2.12, the object $G_d \setminus P^{\times_S d}$ of $S_{\text{Ét}}$ is representable by an S-scheme which is finite étale over $X^{\times_S d}$. The group $\mathfrak{S}_d$ acts on the right on $G_d \setminus P^{\times_S d}$ by the formula

$$(p_i)_{i \in [\![1,d]\!]} \cdot \sigma = (p_{\sigma(i)})_{i \in [\![1,d]\!]}.$$

This action of $\mathfrak{S}_d$ commutes with the left action of G on $G_d \setminus P^{\times_S d}$.

Proposition 2.32. *If X is Zariski-locally quasiprojective on S, then the right action of $\mathfrak{S}_d$ on $G_d \setminus P^{\times_S d}$ is admissible* (see Definition 2.23), *so that the quotient $P^{[d]}$ of $G_d \setminus P^{\times_S d}$ by $\mathfrak{S}_d$ exists in* $\mathrm{Sch}_{/S}$. *Moreover, the canonical morphism $P^{[d]} \to \mathrm{Sym}_S^d(X)$ is a G-torsor, and the morphism*

$$p_1^{-1}P \otimes \cdots \otimes p_d^{-1}P \to r^{-1}P^{[d]}$$

where $r : X^{\times_S d} \to \mathrm{Sym}_S^d(X)$ is the canonical projection, is an isomorphism of G-torsors over $X^{\times_S d}$.

By Propositions 2.27 and 2.30, it is sufficient to show that if $\bar{x} = (\bar{x}_i)_{i=1}^d$ is a geometric point of $X^{\times_S d}$, then the stabilizer of $\bar{x}$ in $\mathfrak{S}_d$ acts trivially on $(G_d \setminus P^{\times_S d})_{\bar{x}}$. Assume that the finite set $\{\bar{x}_i \mid i \in [\![1, d]\!]\}$ has exactly r distinct elements $\bar{y}_1, \ldots, \bar{y}_r$, where $\bar{y}_j$ appears with multiplicity d_j. Then the stabilizer of $\bar{x}$ in $\mathfrak{S}_d$ is isomorphic to the subgroup $\prod_{j=1}^r \mathfrak{S}_{d_j}$ of $\mathfrak{S}_d$. For each $j \in [\![1, r]\!]$, the G-torsor $P_{\bar{y}_j}$ is trivial, and if e is a section of this torsor then $(e)_{i=1}^{d_j}$ is a section of $G_{d_j} \setminus P_{\bar{y}_j}^{d_j}$ which is $\mathfrak{S}_{d_j}$-invariant. The action of $\mathfrak{S}_{d_j}$ on $G_{d_j} \setminus P_{\bar{y}_j}^{d_j}$ is therefore trivial, so that the action of $\prod_{j=1}^r \mathfrak{S}_{d_j}$ on the G-torsor

$$(G_d \setminus P^{\times_S d})_{\bar{x}} = G_r \setminus \left(\prod_{j=1}^r G_{d_j} \setminus P_{\bar{y}_j}^{d_j} \right)$$

is trivial as well.

Proposition 2.33. *If X is flat and Zariski-locally quasiprojective on S, then for any S-scheme S', the canonical morphism*

$$(P \times_S S')^{[d]} \to P^{[d]} \times_S S'$$

is an isomorphism.

By Proposition 2.29, the canonical morphism

$$\mathrm{Sym}^d_{S'}(X \times_S S') \to \mathrm{Sym}^d_S(X) \times_S S'$$

is an isomorphism. Thus the second morphism in the composition

$$(P \times_S S')^{[d]} \to (P^{[d]} \times_S S') \times_{\mathrm{Sym}^d_S(X) \times_S S'} \mathrm{Sym}^d_{S'}(X \times_S S') \to P^{[d]} \times_S S'$$

is an isomorphism, while the first morphism is a morphism of G-torsors, hence an isomorphism.

3. Geometric local class field theory

Let k be a perfect field, and let L be a complete discretely valued extension of k with residue field k. We denote by $\mathcal{O}_L$ its ring of integers, and by $\mathfrak{m}_L$ the maximal ideal of $\mathcal{O}_L$.

3.1. Let us consider the functor

$$\mathbb{O}_L : \mathrm{Alg}_k \to \mathrm{Alg}_{\mathcal{O}_L}, \quad A \mapsto \lim_n A \otimes_k \mathcal{O}_L/\mathfrak{m}_L^n,$$

with values in the category of $\mathcal{O}_L$-algebras.

Proposition 3.2. *The functor $\mathbb{O}_L$ is representable by a k-scheme.*

Indeed, if π is a uniformizer of L, then we have an isomorphism $k((t)) \to L$ which sends t to π, so that the functor $\mathbb{O}_L$ is isomorphic to the functor $A \mapsto A[[t]]$, which is representable by an affine space over k of countable dimension.

Corollary 3.3. *The functor $\mathbb{L} = \mathbb{O}_L \otimes_{\mathcal{O}_L} L$ is representable by an ind-k-scheme.*

We can assume that L is the field of Laurent series $k((t))$. In this case, we have

$$\mathbb{L}(A) = A((t)) = \mathrm{colim}_n\, t^{-n} A[[t]]$$

for any k-algebra A, and for each integer n the functor $A \mapsto t^{-n}A[[t]]$ is representable by a k-scheme, see Proposition 3.2.

Proposition 3.4. *Let G (resp. H) be the functor from Alg_k to the category of groups which associates to a k-algebra A the subgroup $G(A)$ of $A((t))^\times$ consisting of Laurent series of the form $1 + \sum_{r>0} a_r t^{-r}$ where a_r is a nilpotent element of A for each $r > 0$ and vanishes for r large enough (resp. of Laurent series of the form $1 + \sum_{r>0} a_r t^r$ where a_r belongs to A for each $r > 0$). Let $\underline{\mathbb{Z}}$ be the functor which sends a k-algebra A to the group of locally constant functions $\mathrm{Spec}(A) \to \mathbb{Z}$. For any uniformizer π of L, the morphism*

$$\mathbb{G}_{m,k} \times \underline{\mathbb{Z}} \times G \times H \to \mathbb{L}^\times, \quad (a, n, g, h) \mapsto a\pi^n g(\pi)h(\pi),$$

is an isomorphism of group-valued functors.

Let A be a k-algebra. By [Contou-Carrère 2013, 0.8], every invertible element u of $A((t))$ uniquely factors as $u = t^n f(t)h(t)$ where $f(t)$ and $h(t)$ are elements of $A[\![t]\!]^\times$ and $G(A)$ respectively, and $n : \mathrm{Spec}(A) \to \mathbb{Z}$ is a locally constant function. Moreover, there is a unique factorization $f(t) = ag(t)$ where a and $g(t)$ belong to $A^\times$ and $H(A)$ respectively, hence the result.

Corollary 3.5. *The functor $\mathbb{L}^\times$ is representable by an ind-k-scheme. Moreover, its restriction to the category of reduced k-algebras is representable by a reduced k-scheme.*

The groups $\underline{\mathbb{Z}}$ and H from Proposition 3.4 are representable by reduced k-schemes, and so is $\mathbb{G}_{m,k}$. The group G from Proposition 3.4 is the filtered colimit of the functor $n \mapsto G_n$, where G_n is the functor which associates to a k-algebra A the subset $G_n(A)$ of $A((t))^\times$ consisting of Laurent series of the form $1 + \sum_{r=1}^{n} a_r t^{-r}$ where $a_r^n = 0$ for each $r \in [\![1, n]\!]$. For each n, the functor G_n is representable by an affine k-scheme. Thus G is representable by an ind-k-scheme, and so is $\mathbb{L}^\times$ by Proposition 3.4. The last assertion of Corollary 3.5 follows from the fact that $G(A)$ is the trivial group for any reduced k-algebra A.

Corollary 3.6. *Let $d \geq 0$ be an integer. Let $\mathbb{U}_L^{(d)}$ be the subfunctor of $\mathbb{L}^\times$ given by $1 + \mathfrak{m}_L^d \mathbb{O}_L$ if $d \geq 1$ and by $\mathbb{O}_L^\times$ if $d = 0$. Then the functor*

$$\mathbb{L}^\times/\mathbb{U}_L^{(d)} : \mathrm{Alg}_k \to \mathrm{Sets}, \quad A \mapsto \mathbb{L}^\times(A)/\mathbb{U}_L^{(d)}(A),$$

is representable by an ind-k-scheme. Moreover, its restriction to the category of reduced k-algebras is representable by a reduced k-scheme.

According to Proposition 3.4, it is sufficient to show that $(\mathbb{G}_{m,k} \times H)/\mathbb{U}_{k((t))}^{(d)}$ is representable by a reduced k-scheme. The case $d = 0$ is clear, while for $d \geq 1$, we have for any k-algebra A a bijection

$$A^\times \times A^{[\![1,d-1]\!]} \to (\mathbb{G}_{m,k} \times H)(A)/\mathbb{U}_{k((t))}^{(d)}(A), \quad (a_i)_{0 \leq i \leq d-1} \mapsto \sum_{i=0}^{d-1} a_i t^i;$$

hence the result.

3.7. From now on, we consider $\mathrm{Spec}(L)$, $\mathbb{L}^\times$ and $\mathbb{L}^\times/\mathbb{U}_L^{(d)}$ for each integer $d \geq 0$ as objects of the topos $\mathrm{Spec}(k)_{\text{Ét}}$. Let π be an uniformizer of L. We denote by Π the element of $\mathbb{L}(k)$ corresponding to π via the canonical identification $L \simeq \mathbb{L}(k)$. Thus the functor $\mathbb{L}^\times$ is given by

$$\mathbb{L}^\times : A \in \mathrm{Alg}_k \mapsto A((\Pi))^\times.$$

In particular, the Laurent series $(\Pi - \pi)^{-1}\Pi = -\sum_{n \geq 1} \pi^{-n}\Pi^n$ defines an L-point of $\mathbb{L}^\times$. We denote by $\varphi : \mathrm{Spec}(L) \to \mathbb{L}^\times$ the corresponding morphism. We follow here Contou-Carrère's convention; in [Suzuki 2013], the morphism φ corresponds to the point $(\Pi - \pi)\Pi^{-1}$ instead. This is harmless since the inversion is an automorphism of the abelian group $\mathbb{L}^\times$.

Theorem 3.8 [Suzuki 2013, Theorem A(1)]. *Let G be a finite abelian group. The functor*

$$\mathrm{Tors}^{\otimes}(\mathbb{L}^\times, G) \to \mathrm{Tors}(\mathrm{Spec}(L), G), \quad P \to \varphi^{-1}P,$$

is an equivalence of categories (see Definitions 2.2 and 2.5).

In the case where k is algebraically closed, Serre [1961] constructed an equivalence

$$\mathrm{Tors}(\mathrm{Spec}(L), G) \to \mathrm{Tors}^{\otimes}(\mathbb{L}^{\times}, G).$$

Suzuki [2013] shows that the functor from Theorem 3.8 is a quasiinverse to Serre's functor when k is algebraically closed, and extends the result to arbitrary perfect residue fields. In particular, the equivalence from Theorem 3.8 is canonical, even though its definition depends on the choice of π. Suzuki's proof of Theorem 3.8 relies on the Albanese property of the morphism φ, previously established by Contou-Carrère.

Let L^{sep} be a separable closure of L, and let G_L be the Galois group of L^{sep} over L, so that the small étale topos of $\mathrm{Spec}(L)$ is isomorphic to the topos of sets with continuous left G_L-action. By Corollary 2.13, the category of G-torsors over $\mathrm{Spec}(L)$ in $\mathrm{Spec}(k)_{\text{Ét}}$ is isomorphic to the category of G-torsors in the small étale topos $\mathrm{Spec}(L)_{\text{ét}}$. Correspondingly, for each finite abelian group G, the group of isomorphism classes of the category $\mathrm{Tors}(\mathrm{Spec}(L), G)$ is isomorphic to the group of continuous homomorphisms from G_L to G.

We denote by $(G_L^j)_{j\geq -1}$ the ramification filtration of G_L [Serre 1962, IV.3], so that $G_L^{-1} = G_L$ and G_L^0 is the inertia subgroup of G_L, while $G_L^{0+} = \cup_{j>0} G_L^j$ is the wild inertia subgroup of G_L.

Definition 3.9. Let G be a finite abelian group and let $d \geq 0$ be a rational number. A G-torsor over $\mathrm{Spec}(L)$ (in $\mathrm{Spec}(k)_{\text{Ét}}$), corresponding to a continuous homomorphism $\rho : G_L \to G$, is said to have *ramification bounded by d* if $\rho(G_L^d) = \{1\}$. A G-torsor over $\mathrm{Spec}(L)$ with ramification bounded by 0 or 1 is said to be unramified or tamely ramified, respectively.

Proposition 3.10. *Let G be a finite abelian group, let $d \geq 0$ be an integer, and let P be a multiplicative G-torsor P over $\mathbb{L}^{\times}$* (see Definition 2.5). *Assume that k is algebraically closed. Then $\varphi^{-1}P$ has ramification bounded by d* (see Definition 3.9) *if and only if P is the pullback of a multiplicative G-torsor over $\mathbb{L}^{\times}/\mathbb{U}_L^{(d)}$* (see Corollary 3.6).

This follows from [Serre 1961, 3.2 Theorem 1] and from the compatibility of φ^{-1} with Serre's construction [Suzuki 2013, Theorem A(2)].

3.11. Let π and φ be as in Section 3.7. Let K be a closed subextension of k in L, such that $K \to L$ is a finite extension of degree d. Since L is a finite free K-algebra of rank d, we have a canonical morphism of K-schemes

$$\psi : \mathrm{Spec}(K) \to \mathrm{Sym}^d_K(\mathrm{Spec}(L))$$

by Proposition 2.21.

Proposition 3.12. *The composition*

$$\mathrm{Spec}(K) \xrightarrow{\psi} \mathrm{Sym}^d_K(\mathrm{Spec}(L)) \to \mathrm{Sym}^d_k(\mathrm{Spec}(L)) \xrightarrow{\mathrm{Sym}^d_k(\varphi)} \mathrm{Sym}^d_k(\mathbb{L}^{\times}) \to \mathbb{L}^{\times},$$

where the last morphism is given by the multiplication, corresponds to the K-point $P_\pi(\Pi)^{-1}\Pi^d$ of $\mathbb{L}^{\times}$, where the polynomial P_π is the characteristic polynomial of the K-linear endomorphism $x \mapsto \pi x$ of L.

We first describe the morphism ψ. The scheme $\mathrm{Sym}^d_K(\mathrm{Spec}(L))$ is the spectrum of the k-algebra $\mathrm{TS}^d_K(L)$ of symmetric tensors of degree d in L, see Proposition 2.27. The elements $e_i = \pi^{i-1}$ for $i = 1, \ldots, d$ form a K-basis of L, so that we have a decomposition

$$\mathrm{TS}^d_K(L) = \bigoplus_{\substack{\alpha:[\![1,d]\!]\to\mathbb{N} \\ \sum_i \alpha(i)=d}} K e_\alpha,$$

where we have set (see Section 2.16)

$$e_\alpha = \sum_{\substack{\beta:[\![1,d]\!]\to[\![1,d]\!] \\ \forall i, |\beta^{-1}(\{i\})|=\alpha(i)}} e_{\beta(1)} \otimes \cdots \otimes e_{\beta(d)}.$$

Let us write the norm polynomial as

$$N_{L/K}\left(\sum_{i=1}^d x_i e_i\right) = \sum_{\substack{\alpha:[\![1,d]\!]\to\mathbb{N} \\ \sum_i \alpha(i)=d}} f_\alpha x^\alpha,$$

where $x^\alpha = x_1^{\alpha(1)} \cdots x_d^{\alpha(d)}$, and the f_α are uniquely determined elements of K. The morphism $\mathrm{TS}^d_K(L) \to K$ corresponding to ψ is the unique K-linear homomorphism which sends e_α to f_α (see Proposition 2.19 and its proof).

Next we describe the composition

$$\mathrm{Sym}^d_K(\mathrm{Spec}(L)) \to \mathrm{Sym}^d_k(\mathrm{Spec}(L)) \xrightarrow{\mathrm{Sym}^d_k(\varphi)} \mathrm{Sym}^d_k(\mathbb{L}^\times) \to \mathbb{L}^\times.$$

Its precomposition with the projection $\mathrm{Spec}(L)^{\times_K d} \to \mathrm{Sym}^d_K(\mathrm{Spec}(L))$ corresponds to the element of $L^{\otimes_K d}((\Pi))^\times$ given by the formula

$$\prod_{i=1}^d ((\Pi - 1^{\otimes(i-1)} \otimes \pi \otimes 1^{\otimes(d-i)})^{-1}\Pi) = P(\Pi)^{-1}\Pi^d,$$

where the polynomial $P(\Pi)$ can be computed as follows:

$$P(\Pi) = \prod_{i=1}^d \left(\Pi - 1^{\otimes(i-1)} \otimes \pi \otimes 1^{\otimes(d-i)}\right) = \sum_{r=0}^d (-1)^r \Pi^{d-r} \sum_{\substack{(i_1,\ldots,i_d)\in\{0,1\}^d \\ |\{s|i_s=1\}|=r}} \pi^{i_1} \otimes \cdots \otimes \pi^{i_d} = \sum_{r=0}^d (-1)^r e_{\alpha_r} \Pi^{d-r},$$

where $\alpha_r : [\![1, d]\!] \to \mathbb{N}$ is the map which sends 1 and 2 to $d - r$ and r respectively, and any $i > 2$ to 0. The image of $P(\Pi)$ by ψ in $K[\Pi]$ is the polynomial

$$\sum_{r=0}^d (-1)^r f_{\alpha_r} \Pi^{d-r} = N_{L[\Pi]/K[\Pi]}(\Pi e_1 - e_2).$$

Since $e_1 = 1$ and $e_2 = \pi$, we obtain Proposition 3.12.

Proposition 3.13. *Let G be a finite abelian group, and let Q be a G-torsor over* $\mathrm{Spec}(L)$ *(in* $\mathrm{Spec}(k)_{\text{Ét}}$*) of ramification bounded by d* (see Definition 3.9)*. Then $\psi^{-1}Q^{[d]}$* (see Proposition 2.32) *is tamely ramified on* $\mathrm{Spec}(K)$.

Let K' be the maximal unramified extension of K in a separable closure of K. The formation of $\mathrm{Sym}^d_K(\mathrm{Spec}(L))$ is compatible with any base change by Proposition 2.26 or by Proposition 2.29, and so is the formation of φ. Moreover, a G-torsor over $\mathrm{Spec}(K)$ is tamely ramified if and only if its restriction to $\mathrm{Spec}(K')$ is tamely ramified. By replacing K and L by K' and the components of $K' \otimes_K L$ respectively, we can assume that the residue field k is algebraically closed.

Let P be the multiplicative G-torsor on $\mathbb{L}^\times$ (see Definition 2.5) associated to Q (see Theorem 3.8), so that Q is isomorphic to $\varphi^{-1}P$. Then $\psi^{-1}Q^{[d]}$ is isomorphic to the pullback of P along the composition

$$\mathrm{Spec}(K) \xrightarrow{\psi} \mathrm{Sym}^d_K(\mathrm{Spec}(L)) \to \mathrm{Sym}^d_k(\mathrm{Spec}(L)) \xrightarrow{\mathrm{Sym}^d_k(\varphi)} \mathrm{Sym}^d_k(\mathbb{L}^\times) \to \mathbb{L}^\times$$

considered in Proposition 3.12. By Proposition 3.12, this composition corresponds to the K-point of $\mathbb{L}^\times$ given by $P_\pi(\Pi)^{-1}\Pi^d$, where P_π is the characteristic polynomial of π acting K-linearly by multiplication on L. Let us consider the morphism of pointed sets

$$\begin{aligned} \rho : \mathbb{L}^\times(K) &\to H^1(\mathrm{Spec}(K)_{\text{Ét}}, G) \\ \nu &\to \nu^{-1}P \end{aligned}$$

where an element ν of $\mathbb{L}^\times(K)$ is identified to a morphism $\mathrm{Spec}(K) \to \mathbb{L}^\times$. If ν_1 and ν_2 are elements of $\mathbb{L}^\times(K)$, then using the isomorphism $\theta : p_1^{-1}P \otimes p_2^{-1}P \to m^{-1}P$ from Definition 2.5, we obtain isomorphisms

$$(\nu_1\nu_2)^{-1}P \leftarrow (\nu_1 \times \nu_2)^{-1}m^{-1}P \xleftarrow{(\nu_1\times\nu_2)^{-1}\theta} (\nu_1 \times \nu_2)^{-1}(p_1^{-1}P \otimes p_2^{-1}P) \leftarrow \nu_1^{-1}P \otimes \nu_2^{-1}P.$$

Thus ρ is an homomorphism of abelian groups.

We have to prove that $\rho(\nu)$ is the isomorphism class of a tamely ramified G-torsor over $\mathrm{Spec}(K)$, where $\nu = P_\pi(\Pi)^{-1}\Pi^d$. Since P_π is an Eisenstein polynomial, it can be written as $P_\pi(\Pi) = \Pi^d + cR(\Pi)$, where $c = P_\pi(0)$ is a uniformizer of K, and R is a polynomial of degree $< d$ with coefficients in $\mathcal{O}_K$, such that $R(0) = 1$. Thus we can write

$$\nu = c^{-1}\nu_1\nu_2,$$

where $\nu_1 = R(\Pi)^{-1}\Pi^d$ and $\nu_2 = (1 + c^{-1}\Pi^d R(\Pi)^{-1})^{-1}$, so that $\rho(\nu) = \rho(c)^{-1}\rho(\nu_1)\rho(\nu_2)$.

Since Q has ramification bounded by d (see Definition 3.9), the restriction of ρ to $\mathbb{U}_L^{(d)}(K)$ is trivial (see Proposition 3.10). In particular, $\rho(\nu_2)$ is trivial since ν_2 belongs to $\mathbb{U}_L^{(d)}(K)$.

The element ν_1 belongs to $\mathbb{L}^\times(\mathcal{O}_K)$, so that the morphism $\nu_1 : \mathrm{Spec}(K) \to \mathbb{L}^\times$ factors through $\mathrm{Spec}(\mathcal{O}_K)$. This implies that $\rho(\nu_1)$ is the isomorphism class of an unramified G-torsor over $\mathrm{Spec}(K)$. It remains to prove that $\rho(c)$ is the isomorphism class of a tamely ramified G-torsor over $\mathrm{Spec}(K)$. Since c belongs to $K^\times = \mathbb{G}_{m,k}(K) \subseteq \mathbb{L}^\times(K)$, this is a consequence of the following lemma:

Lemma 3.14. *Let T be a multiplicative G-torsor over the k-group scheme $\mathbb{G}_{m,k}$* (see Definition 2.5). *Then T is tamely ramified at* 0 *and* ∞.

Let G_k be the constant k-group scheme associated to k. By Proposition 2.9, there is a structure of k-group scheme on T and an exact sequence

$$1 \to G_k \to T \to \mathbb{G}_{m,k} \to 1 \tag{3.14.1}$$

in $\mathrm{Spec}(k)_{\text{Ét}}$, such that the structure of G-torsor on T is given by the action of its subgroup G by translations. Since the fppf topology is finer than the étale topology on $\mathrm{Sch}_{/k}$, the sequence (3.14.1) remains exact in the topos $\mathrm{Spec}(k)_{\text{Fppf}}$. In particular, we obtain a class in the group $\mathrm{Ext}^1_{\text{Fppf}}(\mathbb{G}_{m,k}, G_k)$ of extensions of $\mathbb{G}_{m,k}$ by G_k in $\mathrm{Spec}(k)_{\text{Fppf}}$.

Let $n = |G|$. In the topos $\mathrm{Spec}(k)_{\text{Fppf}}$ we have an exact sequence

$$1 \to \mu_{n,k} \to \mathbb{G}_{m,k} \xrightarrow{n} \mathbb{G}_{m,k} \to 1, \tag{3.14.2}$$

where $\mu_{n,k}$ is the k-group scheme of n-th roots of unity. By applying the functor $\mathrm{Hom}(\cdot, G_k)$, we obtain an exact sequence

$$\mathrm{Hom}(\mu_{n,k}, G_k) \xrightarrow{\delta} \mathrm{Ext}^1_{\text{fppf}}(\mathbb{G}_{m,k}, G_k) \xrightarrow{n} \mathrm{Ext}^1_{\text{fppf}}(\mathbb{G}_{m,k}, G_k).$$

Since $n = |G|$, the group $\mathrm{Ext}^1_{\text{Fppf}}(\mathbb{G}_{m,k}, G_k)$ is annihilated by n, so that the homomorphism δ above is surjective. Thus the exact sequence (3.14.1) in $\mathrm{Spec}(k)_{\text{Fppf}}$ is the pushout of (3.14.2) along an homomorphism $\mu_{n,k} \to G_k$. Let n' be the largest divisor of n which is invertible in k. Then the largest étale quotient of $\mu_{n,k}$ is the epimorphism $\mu_{n,k} \to \mu_{n',k}$ given by $x \mapsto x^{n/n'}$. In particular, the homomorphism $\mu_{n,k} \to G_k$ factors through $\mu_{n',k}$, so that (3.14.1) is the pushout of the extension

$$1 \to \mu_{n',k} \to \mathbb{G}_{m,k} \xrightarrow{n'} \mathbb{G}_{m,k} \to 1$$

along an homomorphism $\mu_{n',k} \to G_k$. Since the morphism $\mathbb{G}_{m,k} \xrightarrow{n'} \mathbb{G}_{m,k}$ is tamely ramified above 0 and ∞, so is the morphism $T \to \mathbb{G}_{m,k}$.

4. Rigidified Picard schemes of relative curves

4.1. Let $f : X \to S$ be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus g, which is Zariski-locally projective over S.

Proposition 4.2. *The canonical homomorphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism.*

If S is locally noetherian, then $\mathcal{O}_X$ is cohomologically flat over S in dimension 0 by [EGA III$_2$ 1963, 7.8.6]. This means that for any quasicoherent $\mathcal{O}_S$-module $\mathcal{M}$, the canonical homomorphism $f_*f^*\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{M} \to f_*f^*\mathcal{M}$ is an isomorphism. This implies that the formation of $f_*\mathcal{O}_X$ commutes with arbitrary base change: if $f' : X \times_S S' \to S'$ is the base change of f by a morphism of schemes $S' \to S$, then the canonical morphism $f_*\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to f'_*\mathcal{O}_{X\times_S S'}$ is an isomorphism, see [EGA III$_2$ 1963, 7.7.5.3]. By applying this result to the inclusion $\mathrm{Spec}(\kappa(s)) \to S$ of a point s of S, we obtain that $f_*(\mathcal{O}_X)_s \otimes_{\mathcal{O}_{S,s}} \kappa(s)$ is isomorphic to $H^0(X_s, \mathcal{O}_{X_s}) = \kappa(s)$. Since $f_*(\mathcal{O}_X)$ is a coherent $\mathcal{O}_S$-module, Nakayama's lemma yields that the canonical morphism $\mathcal{O}_S \to f_*(\mathcal{O}_X)$ is an epimorphism. It is also injective since f is faithfully flat, hence the result.

In general one can assume that S is affine and that X is projective over S, in which case there is a noetherian scheme S_0, a morphism $S \to S_0$ and a smooth projective S_0-scheme X_0 with geometrically connected fibers such that X is isomorphic to the S-scheme $X_0 \times_{S_0} S$, see [EGA IV$_3$ 1966, 8.9.1, 8.10.5(xiii); EGA IV$_4$ 1967, 17.7.9]. We have already seen that in this case the canonical homomorphism $\mathcal{O}_{S_0} \to f_*\mathcal{O}_{X_0}$ is an isomorphism, and that the formation of $f_*\mathcal{O}_{X_0}$ commutes with arbitrary base change. In particular, both morphisms in the sequence

$$\mathcal{O}_S \to f_*\mathcal{O}_{X_0} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_S \to f_*\mathcal{O}_X$$

are isomorphisms.

Proposition 4.3. *Let $d \geq 2g-1$ be an integer, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module with degree d on each fiber of f. Then, the $\mathcal{O}_S$-module $f_*\mathcal{L}$ is locally free of rank $d-g+1$, the higher direct images $R^j(f_*\mathcal{L})$ vanish for $j > 0$, and the formation of $f_*\mathcal{L}$ commutes with arbitrary base change: if $f' : X' \to S'$ is the base change of f by a morphism $S' \to S$, then the canonical homomorphism $f_*\mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to f'_*(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})$ is an isomorphism.*

We first assume that S is locally noetherian. For each point of s of S and for each integer i, the Riemann–Roch theorem for smooth projective curves implies that the $k(s)$-vector space $H^i(X_s, \mathcal{L}_s)$ is of dimension $d-g+1$ for $i = 0$, and vanishes otherwise. This implies that $R^j f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N})$ vanishes for any integer $j > 0$ and any $\mathcal{O}_S$-module $\mathcal{N}$ by the proof of [EGA III$_2$ 1963, 7.9.8]. Let

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{P} \to 0$$

be an exact sequence of $\mathcal{O}_S$-modules. Since f is flat and since $\mathcal{L}$ is a flat $\mathcal{O}_X$-module, the sequence

$$0 \to \mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N} \to \mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{M} \to \mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{P} \to 0$$

is exact as well. Since $R^1 f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N})$ vanishes, the sequence

$$0 \to f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{N}) \to f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{M}) \to f_*(\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{P}) \to 0$$

is exact. The $\mathcal{O}_X$-module $\mathcal{L}$ is therefore cohomologically flat over S in dimension 0, see [EGA III$_2$ 1963, 7.8.1]. By [EGA III$_2$ 1963, 7.8.4(d)] the $\mathcal{O}_S$-module $f_*\mathcal{L}$ is locally free, and the formation of $f_*\mathcal{L}$ commutes with arbitrary base change. By applying the latter result to the inclusion $\mathrm{Spec}(\kappa(s)) \to S$ of a point s of S and by using that $H^0(X_s, \mathcal{L}_s)$ is of dimension $d-g+1$ over $\kappa(s)$, we obtain that the locally free $\mathcal{O}_X$-module $f_*\mathcal{L}$ is of constant rank $d-g+1$.

In general one can assume that S is affine and that X is projective over S, in which case there is a noetherian scheme S_0, a morphism $S \to S_0$, a smooth projective S_0-scheme X_0, and an invertible $\mathcal{O}_{X_0}$-module $\mathcal{L}_0$ such that X is isomorphic to the S-scheme $X_0 \times_{S_0} S$ and $\mathcal{L}$ is isomorphic to the pullback of $\mathcal{L}_0$ by the canonical projection $X_0 \times_{S_0} S \to X_0$, see [EGA IV$_3$ 1966, 8.9.1, 8.10.5(xiii); EGA IV$_4$ 1967, 17.7.9]. We have seen that the $\mathcal{O}_{S_0}$-module $f_{0*}\mathcal{L}$ is locally free of rank $d-g+1$, and that its formation commutes with arbitrary base change. By performing the base change by the morphism $S \to S_0$, we obtain that $f_*\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of rank $d-g+1$ and that the formation of $f_*\mathcal{L}$ commutes with arbitrary base change.

4.4. Let $f : X \to S$ be as in Section 4.1. The *relative Picard functor* of f is the sheaf of abelian groups $\mathrm{Pic}_S(X) = R^1 f_{\mathrm{Fppf},*}\mathbb{G}_m$ in S_{Fppf}. Alternatively, $\mathrm{Pic}_S(X)$ is the sheaf of abelian groups on S associated to the presheaf which sends an S-scheme T to $\mathrm{Pic}(X \times_S T)$, the abelian group of isomorphism classes of invertible $\mathcal{O}_{X\times_S T}$-modules. For any S-scheme S', we have $(S_{\mathrm{Fppf}})_{/S'} = S'_{\mathrm{Fppf}}$, and we thus have:

Proposition 4.5. *For any S-scheme S', the canonical morphism*

$$\mathrm{Pic}_{S'}(X \times_S S') \to \mathrm{Pic}_S(X) \times_S S'$$

is an isomorphism in S'_{Fppf}.

The elements of $\mathrm{Pic}(X \times_S T)$ which are pulled back from an element of $\mathrm{Pic}(T)$ yield trivial classes in $\mathrm{Pic}_S(X)(T)$, since invertible $\mathcal{O}_T$-modules are locally trivial on T (for the Zariski topology, and thus for the fppf-topology). This yields a sequence

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X \times_S T) \to \mathrm{Pic}_S(X)(T) \to 0, \tag{4.5.1}$$

which is however not necessarily exact. The following is Proposition 4 from [Bosch et al. 1990, 8.1], whose assumptions are satisfied by Proposition 4.2:

Proposition 4.6. *If f has a section, then the sequence (4.5.1) is exact for any S-scheme T.*

By a theorem of Grothendieck [Bosch et al. 1990, 8.2.1] the sheaf $\mathrm{Pic}_S(X)$ is representable by a separated S-scheme. By [Bosch et al. 1990, 9.3.1] the S-scheme $\mathrm{Pic}_S(X)$ is smooth of relative dimension g, and there is a decomposition

$$\mathrm{Pic}_S(X) = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d_S(X),$$

into open and closed subschemes, where $\mathrm{Pic}^d_S(X)$ is the fppf-sheaf associated to the presheaf

$$\begin{aligned} \mathrm{Sch}^{\mathrm{fp}}_{/S} &\to \mathrm{Sets} \\ T &\mapsto \{\mathcal{L} \in \mathrm{Pic}(X \times_S T) \mid \forall \bar{t} \to T, \deg_{X_{\bar{t}}}(\mathcal{L}_{\bar{t}}) = d\}. \end{aligned}$$

Here the condition $\deg_{X_{\bar{t}}}(\mathcal{L}_{\bar{t}}) = d$ runs over all geometric points $\bar{t} \to T$ of T.

4.7. Let $f : X \to S$ be as in Section 4.1, and let $i : Y \hookrightarrow X$ be a closed subscheme of X, which is finite locally free over S of degree $N \geq 1$. A *Y-rigidified line bundle on X* is a pair $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1 and $\alpha : \mathcal{O}_Y \to i^*\mathcal{L}$ is an isomorphism of $\mathcal{O}_Y$-modules. Two Y-rigidified line bundles $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$ are *equivalent* if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ of $\mathcal{O}_X$-modules such that $(i^*\beta)\alpha = \alpha'$. If such an isomorphism β exists, then it is unique. Indeed, any other such isomorphism would take the form $\gamma\beta$ for some global section γ of $\mathcal{O}_X^\times$ such that $i^*\gamma = 1$. Since $f_*\mathcal{O}_X = \mathcal{O}_S$ (see Proposition 4.2), we have $\gamma = f^*\delta$ for some global section δ of $\mathcal{O}_S^\times$. Since the restriction of δ along the finite flat surjective morphism $Y \to S$ is trivial, one must have $\delta = 1$ as well, hence $\gamma = 1$.

Proposition 4.8. *Let $\mathrm{Pic}_S(X, Y)$ be the presheaf of abelian groups on $\mathrm{Sch}^{\mathrm{fp}}_{/S}$ which maps a finitely presented S-scheme T to the set of isomorphism classes of Y_T-rigidified line bundles on X_T. Then, the presheaf $\mathrm{Pic}_S(X, Y)$ is representable by a smooth separated S-scheme of relative dimension $N + g - 1$.*

We first consider the case where $N = 1$:

Lemma 4.9. *The conclusion of Proposition 4.8 holds if $N = 1$.*

Indeed, if $N = 1$ then Y is the image of a section $x : S \to X$ of f. For any finitely presented S-scheme T, we have a morphism

$$\mathrm{Pic}(X \times_S T) \to \mathrm{Pic}_S(X, x)(T), \quad \mathcal{L} \to (\mathcal{L} \otimes (f^*x^*\mathcal{L})^{-1}, \mathrm{id}).$$

The kernel of this homomorphism consists of all invertible $\mathcal{O}_{X\times_S T}$-modules which are given by the pullback of an invertible $\mathcal{O}_T$-module. Moreover, any isomorphism class $(\mathcal{L}, \alpha)$ in $\mathrm{Pic}_S(X, x)(T)$ is the image of $\mathcal{L}$ by this morphism, hence its surjectivity. We conclude by Proposition 4.6 that the canonical projection morphism

$$\mathrm{Pic}_S(X, x) \to \mathrm{Pic}_S(X), \quad (\mathcal{L}, \alpha) \to \mathcal{L},$$

is an isomorphism of presheaves of abelian groups on $\mathrm{Sch}^{\mathrm{fp}}_{/S}$. This yields Lemma 4.9 since $\mathrm{Pic}_S(X)$ is a smooth separated S-scheme of relative dimension g (see Section 4.4).

We now prove Proposition 4.8. Since $X \times_S Y \to Y$ has a section $x = (i \times \mathrm{id}_Y) \circ \Delta_Y$ where $\Delta_Y : Y \to Y \times_S Y$ is the diagonal morphism of Y, we deduce from Lemma 4.9 and its proof that the canonical projection morphism

$$\mathrm{Pic}_Y(X \times_S Y, x) \to \mathrm{Pic}_Y(X \times_S Y) = \mathrm{Pic}_S(X) \times_S Y$$

sending a pair $(\mathcal{L}, \alpha)$ to the class of $\mathcal{L}$ is an isomorphism. Let Z be the Y-scheme $\mathrm{Pic}_Y(X \times_S Y, x)$, and let $(\mathcal{L}_u, \alpha_u)$ be the universal x-rigidified line bundle on $X \times_S Z$. The morphism $Y \times_S Z \to Z$ is finite locally free of rank N, so the pushforward $\mathcal{A}$ of $\mathcal{O}_{Y\times_S Z}$ is a locally free $\mathcal{O}_Z$-algebra of rank N, and the pushforward $\mathcal{M}$ of $i_Z^*\mathcal{L}_u$ is a locally free $\mathcal{O}_Z$-module of rank N. Let $\lambda : \mathcal{M} \to \mathcal{O}_Z$ be the surjective $\mathcal{O}_Z$-linear homomorphism corresponding to $\alpha_u^{-1} : x_Z^*\mathcal{L}_u \to \mathcal{O}_Z$.

Let T be a Y-scheme, and let $(\mathcal{L}, \beta)$ be a Y_T-rigidified line bundle on X_T. The section $x_T : T \to X_T$ uniquely factors through Y_T and we still denote by x_T the corresponding section of Y_T. The pair $(\mathcal{L}, x_T^*\beta)$ is then an x_T-rigidified line bundle on X_T, so that there is a unique morphism $z : T \to Z$ such that $(\mathcal{L}, x_T^*\beta)$ is equivalent to the pullback by z of $(\mathcal{L}_u, \alpha_u)$. Let us assume that $(\mathcal{L}, x_T^*\beta)$ is equal to this pullback. The section β of $i_T^*\mathcal{L}$ over $Y \times_S T$ provides a section $z^*\mathcal{M}$ over T, which we still denote by β, such that $(z^*\lambda)(\beta) = 1$ and $z^*\mathcal{M} = (z^*\mathcal{A})\beta$. Conversely, any such section produces a Y_T-rigidification of $\mathcal{L}$ on X_T. The functor $\mathrm{Pic}_S(X, Y) \times_S Y = \mathrm{Pic}_Y(X \times_S Y, Y \times_S Y)$ is therefore isomorphic to the functor

$$\mathrm{Sch}^{\mathrm{fp}}_{/S} \to \mathrm{Sets}, \quad T \mapsto \{(z, \beta) \mid z \in Z(T), \beta \in \Gamma(T, z^*\mathcal{M}), \lambda(\beta) = 1 \text{ and } \mathcal{M}_T = \mathcal{A}_T\beta\}.$$

This implies that $\mathrm{Pic}_S(X, Y) \times_S Y$ is representable by a relatively affine Z-scheme, smooth of relative dimension $N-1$ over Z. By fppf-descent of affine morphisms of schemes along the fppf-cover $\mathrm{Pic}_S(X) \times_S Y \to \mathrm{Pic}_S(X)$, this implies the representability of $\mathrm{Pic}_S(X, Y)$ by an S-scheme, which is relatively affine and smooth of relative dimension $N-1$ over $\mathrm{Pic}_S(X)$. Since $\mathrm{Pic}_S(X)$ is separated and smooth of relative dimension g over S (see Section 4.1), the S-scheme $\mathrm{Pic}_S(X, Y)$ is separated and smooth of relative dimension $g + N - 1$.

4.10. Let $f : X \to S$ be as in Section 4.1, and let $i : Y \hookrightarrow X$ be a closed subscheme of X, which is finite locally free over S of degree $N \geq 1$. A *Y-trivial effective Cartier divisor of degree d on X* is a pair $(\mathcal{L}, \sigma)$ such that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank 1 and $\sigma : \mathcal{O}_X \hookrightarrow \mathcal{L}$ is an injective homomorphism such that $i^*\sigma$ is an isomorphism and such that the closed subscheme $V(\sigma)$ of X defined by the vanishing of the ideal $\sigma\mathcal{L}^{-1}$ of $\mathcal{O}_X$ is finite locally free of rank d over S. Two Y-trivial effective divisors $(\mathcal{L}, \sigma)$ and $(\mathcal{L}', \sigma')$ are *equivalent* if there is an isomorphism $\beta : \mathcal{L} \to \mathcal{L}'$ of $\mathcal{O}_X$-modules such that $\beta\sigma = \sigma'$. As in Section 4.7, if such an isomorphism exists then it is unique.

Proposition 4.11. *The map $(\mathcal{L}, \sigma) \mapsto (V(\sigma) \hookrightarrow X)$ is a bijection from the set of equivalence classes of Y-trivial effective Cartiers divisor of degree d on X onto the set of closed subschemes of U which are finite locally free of degree d over S.*

Let $(\mathcal{L}, \sigma)$ be a Y-trivial effective divisor of degree d on X. The ideal $\mathcal{I} = \sigma\mathcal{L}^{-1}$ is an invertible ideal of $\mathcal{O}_X$ such that the vanishing locus $V(\mathcal{I})$ is finite locally free of rank d over S and is contained in U. The pair $(\mathcal{L}, \sigma)$ is equivalent to $(\mathcal{I}^{-1}, 1)$, and $\mathcal{I}$ is uniquely determined by $V(\mathcal{I})$. Conversely for any closed subscheme Z of U which is finite locally free of rank d over S, the scheme Z is proper over S hence closed in X as well, and its defining ideal $\mathcal{I}$ in $\mathcal{O}_{X_T}$ is invertible by [Bosch et al. 1990, 8.2.6(ii)]. The pair $(\mathcal{I}^{-1}, 1)$ is then a Y-trivial effective Cartier divisor of degree d on X.

Proposition 4.12. *Let d be an integer and let $\mathrm{Div}^{d,+}_S(X, Y)$ be the functor which to an S-scheme T associates the set of equivalence classes of Y_T-trivial effective Cartier divisors of degree d on X_T. Then $\mathrm{Div}^{d,+}_S(X, Y)$ is representable by the S-scheme $\mathrm{Sym}^d_S(U)$, the d-th symmetric power of $U = X \setminus Y$ over S* (see Section 2.22)*. In particular $\mathrm{Div}^{d,+}_S(X, Y)$ is smooth of relative dimension d over S.*

By Proposition 4.11, the functor $\mathrm{Div}^{d,+}_S(X, Y)$ is isomorphic to the functor which sends an S-scheme T to the set of closed subschemes of U_T which are finite locally free of rank d over T. In other words, $\mathrm{Div}^{d,+}_S(X, Y)$ is isomorphic to the Hilbert functor of d-points in the S-scheme U.

If x is a T-point of U, we denote $\mathcal{O}(-x)$ the kernel of the homomorphism $\mathcal{O}_{X \times_S T} \to x_*\mathcal{O}_T$, which is an invertible ideal sheaf, and by $\mathcal{O}(x)$ its dual, which is endowed with a section $1_x : \mathcal{O}_{X \times_S T} \hookrightarrow \mathcal{O}(x)$.

The morphism

$$\mathrm{Sym}^d_S(U) \to \mathrm{Div}^{d,+}_S(X, Y), \quad (x_1, \dots, x_d) \to \left(\bigotimes_{i=1}^{d} \mathcal{O}(x_i), \prod_{i=1}^{d} 1_{x_i} \right),$$

is then an isomorphism of fppf-sheaves by [SGA 4_3 1973, XVII.6.3.9], hence Proposition 4.12.

Remark 4.13. Let T be an S-scheme. Let Z be a closed subscheme of U_T which is finite locally free of rank d over T, therefore defining a T-point of $\mathrm{Div}^{d,+}_S(X, Y) = \mathrm{Sym}^d_S(U)$ by Proposition 4.11. By [SGA 4_3 1973, XVII.6.3.9], this T-point is given by the composition

$$T \to \mathrm{Sym}^d_T(Z) \to \mathrm{Sym}^d_T(U_T) \to \mathrm{Sym}^d_S(U),$$

where the first morphism is the canonical morphism from Proposition 2.21.

Proposition 4.14. *Let $d \geq N + 2g - 1$ be an integer, and let $\mathrm{Pic}^d_S(X, Y)$ be the inverse image of $\mathrm{Pic}^d_S(X)$ by the natural morphism $\mathrm{Pic}_S(X, Y) \to \mathrm{Pic}_S(X)$. Then the Abel–Jacobi morphism*

$$\Phi_d : \mathrm{Div}^{d,+}_S(X, Y) \to \mathrm{Pic}^d_S(X, Y), \quad (\mathcal{L}, \sigma) \mapsto (\mathcal{L}, i^*\sigma),$$

is surjective smooth of relative dimension $d - N - g + 1$ and it has geometrically connected fibers.

Let Z be the scheme $\mathrm{Pic}^d_S(X, Y)$, and let $(\mathcal{L}_u, \alpha_u)$ be the universal Y-rigidified line bundle of degree d on X_Z. By [Bosch et al. 1990, 8.2.6(ii)], the closed subscheme Y_Z of X_Z is defined by an invertible ideal sheaf $\mathcal{I}$.

Let $\mathcal{E}$ be the pushforward of $\mathcal{M} = \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{I}$ by the morphism $f_Z : X_Z \to Z$. By Proposition 4.3, the $\mathcal{O}_Z$-module $\mathcal{E}$ is locally free of rank $d - N - g + 1$, and for any morphism $T \to Z$ the canonical homomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_T \to f_{T*}(\mathcal{M} \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_{X_T})$$

is an isomorphism, where $f_T : X_T \to T$ is the base change of f by the morphism $T \to S$. We thus obtain an isomorphism

$$E \to E', \tag{4.14.1}$$

of functors on the category of Z-schemes, where E is the functor $T \mapsto \Gamma(T, \mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$ and E' is the functor $T \mapsto \Gamma(X_T, \mathcal{M} \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_{X_T})$. Let $\mathcal{F}$ be the pushforward of $\mathcal{L}_u$ by the morphism f_Z. By the same argument, we obtain that the $\mathcal{O}_Z$-module $\mathcal{F}$ is locally free of rank $d - g + 1$, and that we have an isomorphism

$$F \to F', \tag{4.14.2}$$

of functors on the category of Z-schemes, where F is the functor $T \mapsto \Gamma(T, \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$ and F' is the functor $T \mapsto \Gamma(X_T, \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_{X_T})$. Let us consider the exact sequence

$$0 \to \mathcal{M} \to \mathcal{L}_u \to \mathcal{L}_u \otimes_{\mathcal{O}_{X_Z}} \mathcal{O}_{Y_Z} \to 0.$$

Since $R^1 f_{Z*}\mathcal{M} = 0$ by Proposition 4.3, we obtain an exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,$$

where $\mathcal{G}$ is a locally free $\mathcal{O}_Z$-module of rank N. Together with (4.14.1) and (4.14.2), this yields an exact sequence

$$0 \to E' \to F' \xrightarrow{b} G \to 0,$$

of Z-group schemes in Z_{fppf}, where G is the functor $T \mapsto \Gamma(T, \mathcal{G}_T \otimes_{\mathcal{O}_Z} \mathcal{O}_T)$. The section α_u of $\mathcal{G}$ over Z corresponds to a morphism $\alpha_u : Z \to G$, and we have an isomorphism

$$\mathrm{Div}_S^{d,+}(X, Y) \to F' \times_{b,G,\alpha_u} Z, \quad (\mathcal{L}, \sigma) \mapsto (\sigma, (\mathcal{L}, i^*\sigma)).$$

Since b is an E'-torsor over G in Z_{fppf}, we obtain that $\mathrm{Div}_S^{d,+}(X, Y)$ is an E'-torsor in Z_{fppf}. Since E' is isomorphic to E by (4.14.1), it is smooth of relative dimension $d - N - g + 1$ over Z with geometrically connected fibers, hence the conclusion of Proposition 4.14.

5. Geometric global class field theory

5.1. Let $f : X \to S$ be a smooth morphism of schemes of relative dimension 1, with connected geometric fibers of genus g, which is Zariski-locally projective over S, and let $i : Y \hookrightarrow X$ be a closed subscheme of X which is finite locally free over S of degree $N \geq 1$. Let $j : U \to X$ be the open complement of Y. Let Λ be a finite ring whose cardinality is invertible on S.

Definition 5.2. A locally free Λ-module $\mathcal{F}$ of rank 1 in $U_{\text{Ét}}$ has *ramification bounded by Y over S* if for any geometric point $\bar{x}$ of Y with image $\bar{s}$ in S, the restriction of $\mathcal{F}$ to $\mathrm{Spec}(\widehat{\mathcal{O}_{X_{\bar{s}},\bar{x}}}) \times_{X_{\bar{s}}} U_{\bar{s}}$ has ramification bounded by the multiplicity of $Y_{\bar{s}}$ at $\bar{x}$ (see Definition 3.9).

Theorem 5.3. *Let $\mathcal{F}$ be a locally free Λ-module of rank* 1 *in $U_{\text{Ét}}$ with ramification bounded by Y over S* (see Definition 5.2). *Then, there is a unique* (*up to isomorphism*) *multiplicative locally free Λ-module $\mathcal{G}$ of rank* 1 *on the S-group scheme* $\mathrm{Pic}_S(X, Y)$ (see Remark 2.6) *such that the pullback of $\mathcal{G}$ by the Abel–Jacobi morphism*

$$U \to \mathrm{Pic}_S(X, Y),$$

which sends x to $(\mathcal{O}(x), 1)$, *is isomorphic to $\mathcal{F}$.*

In Section 5.4, we study the restriction of the locally free Λ-module $\mathcal{F}^{[d]}$ of rank 1 on $\mathrm{Div}_S^{d,+}(X, Y)$ (see Proposition 2.32 and Proposition 4.12) to a geometric fiber of the Abel–Jacobi morphism (see Proposition 4.14)

$$\Phi_d : \mathrm{Div}_S^{d,+}(X, Y) \to \mathrm{Pic}_S^d(X, Y), \quad (\mathcal{L}, \sigma) \mapsto (\mathcal{L}, i^*\sigma).$$

This study will enable us to prove Theorem 5.3 in Section 5.10.

5.4. Let k be an algebraically closed field, let X be a smooth connected projective curve of genus g over k and let $i : Y \to X$ be an effective Cartier divisor of degree N with complement U in X. Let $\mathcal{L}$ be a line bundle of degree $d \geq N + 2g - 1$ on X, and let V be the $(d - N - g + 1)$-dimensional affine space over k associated to the k-vector space $\mathcal{V} = H^0(X, \mathcal{L}(-Y))$, i.e., V is the spectrum of the symmetric algebra of the k-module $\mathrm{Hom}_k(\mathcal{V}, k)$. Let τ be a global section of $\mathcal{L}$ on X such that $i^*\tau : \mathcal{O}_Y \to i^*\mathcal{L}$ is an isomorphism.

Proposition 5.5. *Let Λ be a finite ring of cardinality invertible in k, and let $\mathcal{F}$ be a locally free Λ-module of rank 1 in $U_{\text{Ét}}$, with ramification bounded by Y* (see Definition 5.2). *Then the pullback of $\mathcal{F}^{[d]}$* (see Proposition 2.32) *by the morphism*

$$V \to \mathrm{Div}_k^{d,+}(X, Y),$$

which sends a section s of V to $(\mathcal{L}, \tau - s)$, is a constant étale sheaf.

The morphism

$$V \to \mathrm{Div}_k^{d,+}(X, Y),$$

which sends a point σ of V to $(\mathcal{L}, \tau - \sigma)$, is an isomorphism from V to the fiber of Φ_d over the k-point $(\mathcal{L}, i^*\tau)$, see Proposition 4.14. Proposition 5.5 thus implies:

Corollary 5.6. *Let $\mathcal{F}$ be as in Proposition 5.5. Then the locally free Λ-module $\mathcal{F}^{[d]}$ on $\mathrm{Div}_k^{d,+}(X, Y)_{\text{Ét}}$ is constant on the fiber at $(\mathcal{L}, i^*\tau)$ of the morphism*

$$\Phi_d : \mathrm{Div}_k^{d,+}(X, Y) \to \mathrm{Pic}_k^d(X, Y)$$

from Proposition 4.14 .

We now prove Proposition 5.5. To this end, we consider the morphism

$$\psi : \mathbb{A}_V^1 \to \mathrm{Div}_k^{d,+}(X, Y),$$

which sends a pair (t, σ), where t and σ are points of $\mathbb{A}_k^1$ and V respectively, to the point $(\mathcal{L}, \tau - t\sigma)$ of $\mathrm{Div}_k^{d,+}(X, Y)$. Let $\mathcal{F}$ be as in Proposition 5.5, and let $\mathcal{G}$ be the pullback by ψ of $\mathcal{F}^{[d]}$ (see Proposition 2.32). Denoting by $\iota_t : V \to \mathbb{A}_V^1$ the section corresponding to an element t of $k = \mathbb{A}_k^1(k)$, we must prove that the sheaf $\iota_1^{-1}\mathcal{G}$ is constant. The sheaf $\iota_0^{-1}\mathcal{G}$ is constant, since $\psi\iota_0$ is a constant morphism, hence it is sufficient to prove that $\iota_1^{-1}\mathcal{G}$ and $\iota_0^{-1}\mathcal{G}$ are isomorphic. The latter fact follows from the following lemma:

Lemma 5.7. *The locally free Λ-module $\mathcal{G}$ is the pullback of an étale sheaf on V by the projection $\pi : \mathbb{A}_V^1 \to V$.*

We now prove Lemma 5.7. We start by proving that $\mathcal{G}$ is constant on each geometric fiber of the projection π. Since the formation of ψ and $\mathcal{G}$ is compatible with the base change along any field extension of k, it is sufficient to show that $\mathcal{G}$ is constant on each fiber of the projection $\mathbb{A}_V^1 \to V$ at a k-point σ of V. If $\sigma = 0$, then the restriction of ψ to the fiber of π above σ is constant, hence $\mathcal{G}$ is constant on this fiber.

We now assume that σ is nonzero. Since σ vanishes on the nonempty divisor Y and τ does not, the sections σ and τ are k-linearly independent in $H^0(X,\mathcal{L})$. Let D be the greatest divisor on X such that $D \leq \operatorname{div}(\sigma)$ and $D \leq \operatorname{div}(\tau)$. Since the divisor of τ is contained in U, so is D. We can then write $\sigma = \tilde{\sigma} 1_D$ and $\tau = \tilde{\tau} 1_D$, where 1_D is the canonical section of $\mathcal{O}(D)$ and $\tilde{\sigma}, \tilde{\tau}$ are global sections of $\mathcal{L}(-D)$ on X without common zeroes. Thus $f = [\tilde{\tau} : \tilde{\sigma}]$ is a well defined nonconstant morphism from X to $\mathbb{P}^1_k$. Thus, if W is the closed subscheme of $X \times_k \mathbb{A}^1_k$ defined by the vanishing of $\tau - t\sigma$, where t is the coordinate on $\mathbb{A}^1_k$, then we have

$$W = D \times_k \mathbb{A}^1_k \cup (\mathrm{Graph}(f) \cap X \times_k \mathbb{A}^1_k) \hookrightarrow U \times_k \mathbb{A}^1_k.$$

Moreover, the projection $W \to \mathbb{A}^1_k$ is finite flat of degree d, and the restriction of ψ to the fiber at σ factors as

$$\mathbb{A}^1_k \xrightarrow{\varphi} \mathrm{Sym}^d_{\mathbb{A}^1_k}(W) \to \mathrm{Sym}^d_{\mathbb{A}^1_k}(U \times_k \mathbb{A}^1_k) \to \mathrm{Sym}^d_k(U) \to \mathrm{Div}^{d,+}_k(X,Y),$$

where the first morphism φ is obtained from Proposition 2.21, and the last morphism is the isomorphism from Proposition 4.12. Moreover, the pullback of $\mathcal{F}^{[d]}$ to $\mathrm{Sym}^d_{\mathbb{A}^1_k}(W)$ coincides with $(p_1^{-1}\mathcal{F})^{[d]}$, where $p_1 : W \to U$ is the first projection. In particular, the sheaf $\mathcal{G}$ is isomorphic to $\varphi^{-1}(p_1^{-1}\mathcal{F})^{[d]}$.

Set $K = k((t^{-1}))$ and let $\eta = \mathrm{Spec}(K) \to \mathbb{A}^1_k$ be the corresponding punctured formal neighborhood of ∞. Consider the commutative diagram

$$\begin{tikzcd} \mathbb{A}^1_k \arrow[r, "\varphi"] & \mathrm{Sym}^d_{\mathbb{A}^1_k}(W) \\ \eta \arrow[u] \arrow[r] & \mathrm{Sym}^d_\eta(W \times_{\mathbb{A}^1_k} \eta). \arrow[u] \end{tikzcd}$$

We can then write

$$W \times_{\mathbb{A}^1_k} \eta = D \times_k \eta \cup \mathrm{Graph}(f) \times_{\mathbb{P}^1_k} \eta = D \times_k \eta \cup X \times_{f,\mathbb{P}^1_k} \eta.$$

The divisors $D \times_k \eta$ and $X \times_{f,\mathbb{P}^1_k} \eta$ of $X \times_k \eta$ are disjoint, since the former lies over closed points of X, while the latter lies over the generic point of X. We thus have a decomposition

$$W \times_{\mathbb{A}^1_k} \eta = D \times_k \eta \amalg X \times_{f,\mathbb{P}^1_k} \eta = \coprod_i \mathrm{Spec}(L_i)$$

where L_i is either of the form $K[T]/(T^{d_i})$ if $\mathrm{Spec}(L_i)$ is a connected component of $D \times_k \eta$, or a field extension of degree d_i of K if $\mathrm{Spec}(L_i)$ is a connected component of $X \times_{f,\mathbb{P}^1_k} \eta$. In the former case, the restriction of $p_1^{-1}\mathcal{F}$ to $\mathrm{Spec}(L_i)$ is constant, while in the latter case, we have the further information that the restriction of $p_1^{-1}\mathcal{F}$ to $\mathrm{Spec}(L_i)$ has ramification bounded by d_i (see Definition 3.9), since the ramification index of f at a point x above ∞ is greater than or equal to the multiplicity of Y at x, and $\mathcal{F}$ has ramification bounded by Y by assumption. Moreover, we have $\sum_i d_i = d$, and the morphism

$\eta \to \mathrm{Sym}^d_\eta(W \times_{\mathbb{A}^1_k} \eta)$ factors through the canonical morphism

$$\prod_i \mathrm{Sym}^{d_i}_\eta(\mathrm{Spec}(L_i)) \to \mathrm{Sym}^d_\eta(W \times_{\mathbb{A}^1_k} \eta).$$

By Proposition 3.13, we obtain that the restriction of $\mathcal{G}$ to η is tamely ramified. Since the tame fundamental group of $\mathbb{A}^1_k$ is trivial, we conclude that $\mathcal{G}$ is a constant étale Λ-module on the fiber of π at σ. The conclusion of Lemma 5.7 then follows from a descent result, namely Lemma 5.9 below.

Remark 5.8. While the proof of Proposition 3.13, which constitutes the core of the proof of Lemma 5.7 above, uses geometric local class field theory, it should be noticed that its statement does not refer to it. This explains why no form of local-global compatibility is required in the proof of Lemma 5.7.

Lemma 5.9. *Let $g : T' \to T$ be a quasicompact smooth compactifiable morphism of schemes of relative dimension δ with geometrically connected fibers, and let $\mathcal{G}$ be an étale sheaf of Λ-modules on $T'_{\text{ét}}$ which is constant on each geometric fiber of g. Then $\mathcal{G}$ is isomorphic to the pullback by g of an étale sheaf of Λ-modules on $T_{\text{ét}}$.*

By [SGA 4$_3$ 1973, XVIII 3.2.5] the functor $Rg_!$ on the derived category of Λ-modules on T admits the functor $g^! : K \mapsto g^*K(\delta)[2\delta]$ as a right adjoint. Let us apply the functor $\mathcal{H}^0$ to the adjunction morphism $\mathcal{G} \to g^! Rg_! \mathcal{G}$. The morphism

$$\mathcal{G} \to \mathcal{H}^0(g^! Rg_! \mathcal{G}) = g^* R^{2\delta} g_! \mathcal{G}(\delta)$$

is an isomorphism, as can be seen by checking the stalks at geometric points with the proper base change theorem.

5.10. We now prove Theorem 5.3. Let $\mathcal{F}$ be a locally free Λ-module of rank 1 over $U_{\text{Ét}}$. The family $(F^{[d]})_{d\geq 0}$ of locally free Λ-modules of rank 1 yields a multiplicative étale Λ-module of rank 1 over the S-semigroup scheme

$$\mathrm{Div}^+_S(X, Y) = \coprod_{d\geq 0} \mathrm{Div}^{d,+}_S(X, Y).$$

For each integer $d \geq N + 2g$, Corollary 5.6 implies that the locally free Λ-module $\mathcal{F}^{[d]}$ of rank 1 on $\mathrm{Div}^{d,+}_S(X, Y)$ (see Propositions Proposition 2.32 and 4.12) is constant on the geometric fibers of the smooth surjective morphism (see Proposition 4.14)

$$\Phi_d : \mathrm{Div}^{d,+}_S(X, Y) \to \mathrm{Pic}^d_S(X, Y), \quad (\mathcal{L}, \sigma) \mapsto (\mathcal{L}, i^*\sigma).$$

This morphism satisfies the conditions of Lemma 5.9 by Proposition 4.14. We can therefore apply Lemma 5.9, and we obtain a locally free Λ-module $\mathcal{G}_d$ of rank 1 over $\mathrm{Pic}^d_S(X, Y)$ such that $\Phi_d^{-1}\mathcal{G}_d$ is isomorphic to $\mathcal{F}^{[d]}$. By Proposition 2.8, the family $(\mathcal{G}_d)_{d\geq N+2g}$ yields a multiplicative locally free Λ-module of rank 1 on the S-semigroup scheme

$$M = \coprod_{d\geq N+2g} \mathrm{Pic}^d_S(X, Y).$$

Since the morphism

$$\rho : M \times_S M \to \mathrm{Pic}_S(X, Y), \quad (x, y) \mapsto xy^{-1},$$

is faithfully flat and quasicompact, we can apply Proposition 2.15, which yields a multiplicative locally free Λ-module $\mathcal{G}$ of rank 1 over $\mathrm{Pic}_S(X, Y)$ whose restriction to $\mathrm{Pic}^d_S(X, Y)$ coincides with $\mathcal{G}_d$ for $d \geq N + 2g$. The families $(\mathcal{F}^{[d]})_{d\geq 0}$ and $(\Phi_d^{-1}\mathcal{G}_d)_{d\geq 0}$ yield multiplicative locally free Λ-modules of rank 1 on the S-semigroup scheme $\mathrm{Div}^+_S(X, Y) = \coprod_{d\geq 0} \mathrm{Div}^{d,+}_S(X, Y)$, whose restrictions to the ideal

$$I = \coprod_{d\geq N+2g} \mathrm{Div}^{d,+}_S(X, Y)$$

of $\mathrm{Div}^+_S(X, Y)$ are isomorphic. We obtain by Proposition 2.7 an isomorphism from $\mathcal{F}^{[d]}$ to $\Phi_d^{-1}\mathcal{G}_d$ for each $d \geq 0$. In particular, the locally free Λ-module $\Phi_1^{-1}\mathcal{G}_1$ of rank 1 is isomorphic to $\mathcal{F}$.

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quentin.guignard@ens.fr Institut des Hautes Etudes Scientifiques, Bures-Sur-Yvette, France

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