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The multiple-point schemes of a finite curvilinear map of codimension one

Steven Kleiman(1), Joseph Lipman(2) and Bernd Ulrich(3)

Abstract. Let X and Y be smooth varieties of dimensions $n-1$ and n over an arbitrary algebraically closed field, $f\colon X\to Y$ a finite map that is birational onto its image. Suppose that f is curvilinear; that is, for all $x\in X$, the Jacobian $\partial f(x)$ has rank at least $n-2$. For $r\geq 1$, consider the subscheme N_r of Y defined by the $(r-1)$th Fitting ideal of the $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$, and set $M_r:=f^{-1}N_r$. In this setting—in fact, in a more general setting—we prove the following statements, which show that M_r and N_r behave like reasonable schemes of source and target r-fold points of f.

If each component of M_r, or equivalently of N_r, has the minimal possible dimension $n-r$, then M_r and N_r are Cohen–Macaulay, and their fundamental cycles satisfy the relation, $f_*[M_r]=r[N_r]$. Now, suppose that each component of M_s, or of N_s, has dimension $n-s$ for $s=1,\dots,r+1$. Then the blowup $\mathrm{Bl}(N_r,N_{r+1})$ is equal to the Hilbert scheme Hilb^r_f, and the blowup $\mathrm{Bl}(M_r,M_{r+1})$ is equal to the universal subscheme Univ^r_f of $\mathrm{Hilb}^r_f\times_Y X$; moreover, Hilb^r_f and Univ^r_f are Gorenstein. In addition, the structure map $h\colon\mathrm{Hilb}^r_f\to Y$ is finite and birational onto its image; and its conductor is equal to the ideal $\mathcal{J}_r$ of N_{r+1} in N_r, and is locally self-linked. Reciprocally, $h_*\mathcal{O}_{\mathrm{Hilb}^r_f}$ is equal to $\mathcal{H}om(\mathcal{J}_r,\mathcal{O}_{N_r})$. Moreover, $h_*[h^{-1}N_{r+1}]=(r+1)[N_{r+1}]$. Similar assertions hold for the structure map $h_1\colon\mathrm{Univ}^r_f\to X$ if $r\geq 2$.

1 Introduction

1.1. Overview

Consider a finite map $f\colon X\to Y$. In the theory of singularities of f, a leading role is played by the loci of source and target r-fold points, $M_r=M_r(f)$ and $N_r=N_r(f)$. They are simple sets: M_r is just the preimage $f^{-1}N_r$, and N_r consists of the (geometric) points y of Y whose fiber $f^{-1}y$ contains a subscheme of degree r (or

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length r). However, M_r and N_r support more refined structures, which reflect the multiplicities of appearance of their points. First of all, they support natural positive cycles, whose classes are given, under suitable hypotheses, by certain multiple-point formulas, which are polynomial expressions in the Chern classes of f. In fact, there are two different, but related, derivations of these formulas: one is based on iteration [19]; the other, on the Hilbert scheme [20]. In this paper, we use the method of iteration to derive results about M_r from corresponding results proved about N_r, to prove results about the Hilbert scheme Hilb_f^r (of degree r-subschemes of the fibers of f), and to derive corresponding results about the universal subscheme Univ_f^r of $\mathrm{Hilb}_f^r \times_Y X$.

Secondly, M_r and N_r support natural scheme structures: N_r is the closed subscheme of Y defined by the Fitting ideal $\mathcal{F}itt_{r-1}^Y(f_*\mathcal{O}_X)$, and M_r is the closed subscheme $f^{-1}N_r$ of X. Under suitable hypotheses, which will be introduced in Article 1.3 below and developed in detail in Sections 2 and 3, these subschemes have many lovely and desirable properties. Work on this matter was carried out by Gruson and Peskine [12] in 1981 and by Mond and Pellikaan [29] in 1988. (Beware: Mond and Pellikaan's M_r is our N_r; moreover, neither they nor Gruson and Peskine really studied our M_r.) In this paper, we aim to carry their work further. In Section 3, we establish some basic properties of the schemes M_r and N_r, and we prove a relation between their fundamental cycles. In Section 4, we relate M_r and N_r to Hilb_f^r and Univ_f^r using some technical commutative algebra developed in Section 5, the final section.

In Section 3, under suitable hypotheses, we prove, that M_r and N_r are 'perfect' subschemes, and that their fundamental cycles satisfy the basic relation,

$$f_*[M_r] = r[N_r]; \tag{1.1.1}$$

see Theorems 3.5 and 3.11. Intuitively, this relation says that a general point of N_r has exactly r preimages. However, the relation takes into account the multiplicity of the point as an r-fold point. In other words, the Fitting ideal gives the "right" nilpotent structure to the schemes M_r and N_r. Of course, off N_{r+1}, the map $M_r \to N_r$ is finite and flat of degree r by a standard property of the Fitting ideal. The subtlety appears when some component of N_r is also a component of N_{r+1}.

For example, under suitable hypotheses, Proposition 3.4 says that N_1 is equal to the scheme-theoretic image of f, and that N_2 is defined in N_1 by the conductor

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of $f_*\mathcal{O}_X$ into $\mathcal{O}_{N_1}$. Therefore, with $r=2$, Relation (1.1.1) recovers the following celebrated result (proved in various forms around 1950 by Apery, by Gorenstein, by Kodaira, by Rosenlicht, and by Samuel): given the local ring of a (closed) point of a curve on a smooth surface, the colength of the conductor in the normalization is equal to twice the colength of the conductor in the given ring; here X is the normalized curve and Y is the smooth surface. However, even with $r=2$, Relation (1.1.1) is more general than that. For instance, it is valid for the birational projection into the plane of any reduced projective curve. Furthermore, it yields the equation $\deg M_2 = 2\deg N_2$ proved indirectly by J. Roberts [31, 2nd par. p. 254] in the case of the birational projection of a smooth projective variety of arbitrary dimension onto a hypersurface.

In Section 4, under suitable hypotheses, we study the Hilbert scheme and the universal subscheme. Notably, we prove Theorem 4.2, which asserts that the structure maps,

$$h\colon \mathrm{Hilb}_f^r \longrightarrow Y \quad \text{and} \quad h_1\colon \mathrm{Univ}_f^r \longrightarrow X,$$

have a number of desirable properties; also, $h^{-1}N_{r+1}$ is a divisor, and

$$h_*[h^{-1}N_{r+1}] = (r+1)[N_{r+1}], \tag{1.1.2}$$

and similar assertions are valid for h_1 and M_{r+1}. We also prove Theorem 4.3, which asserts the equations,

$$\mathrm{Hilb}_f^r = \mathrm{Bl}(N_r, N_{r+1}) \quad \text{and} \quad \mathrm{Univ}_f^r = \mathrm{Bl}(M_r, M_{r+1}). \tag{1.1.3}$$

The first equation is obvious off N_{r+1}; indeed, Hilb_f^r is equal to N_r off N_{r+1}, because $M_r \to N_r$ is finite and flat of degree r there. However, it is not obvious, a priori, even that $h^{-1}N_{r+1}$ is a divisor.

Theorem 4.4 asserts that the ideal $\mathcal{J}_r$ of N_{r+1} in N_r and the direct image $h_*\mathcal{O}_{\mathrm{Hilb}_f^r}$ are reciprocal fractional ideals; that is, the pairing by multiplication,

$$\mathcal{J}_r \times h_*\mathcal{O}_{\mathrm{Hilb}_f^r} \longrightarrow \mathcal{O}_{N_r},$$

is well defined and perfect; in particular, $\mathcal{J}_r$ is the conductor of $h_*\mathcal{O}_{\mathrm{Hilb}_f^r}$ in $\mathcal{O}_{N_r}$. In addition, $\mathcal{J}_r$ is locally a self-linked ideal of $\mathcal{O}_{N_r}$. Furthermore, if $r\geq 2$, then similar assertions hold for h_1. These results about h and h_1 require the development of a lot of supporting commutative algebra; some of it is developed at the end of Section 4, and the rest, including a generalization of Huneke's theory of strong Cohen–Macaulayness, is developed in Section 5.

Intuitively, the first equation of (1.1.3) says that when N_r is blown up along N_{r+1}, then the $(r+1)$-fold points of f on Y are resolved into their constituent r-fold points. Equation (1.1.2) says that the number of constituents is $r+1$, just as there should be since there are $r+1$ different ways in which a group of r points can be chosen from a group of $r+1$ points. Similar statements hold for the r-fold points of f on X. Moreover, the second equation of (1.1.3) formally implies the equation,

$$[M_r] = h_{1*}[\mathrm{Univ}_f^r],$$

which says that $[M_r]$ is equal to the cycle whose class is given by the refined r-fold-point formula of [20, (1.18), p. 107]. Furthermore, the first equation of (1.1.3) implies that the r-fold-point formula is valid when N_s has codimension s for $s = 1, \dots, r$ (assuming the usual hypotheses on f and Y in addition). Thus the present paper clarifies the enumerative significance of the refined r-fold-point formula.

1.2. Applications

The theory in this paper applies, for example, to the enumeration of the secant curves of a given space curve C. Indeed, Gruson and Peskine [12] made their development of the theory to give modern derivations of the nineteenth century formulas for the degree and genus of the curve of trisecant lines of C, and for the number of quadrisecant lines. They used the following setup: Y is the Grassmannian of lines; X is the incidence variety of pairs (P, L) where P is a point of C, and L is a line through P; and $f: X \to Y$ is the projection. Then N_r is the scheme of r-secant lines. So the degree of N_4 is the number of quadrisecants, and it is given by a formula of Cayley. To obtain this formula, Gruson and Peskine used the Grothendieck–Riemann–Roch theorem and Porteous's formula; however, instead, it is possible to use the 4-fold-point formula.

Similarly, by using a stationary multiple-point formula, Colley [8, 5.8, p. 62] recovered Salmon's formula for the number of reincident tangent lines of C. In much the same way, S. Katz [18, 2.5, p. 151] recovered Severi's formula for the number of 8-secant conics to C: he worked out the 8-fold-point formula for the map $f: X \to Y$, where Y is the variety of conics in space, and where X is the incidence variety of pairs (P, L) where P is a point of C and L is a conic through P. Later, Johnsen [17] established the enumerative significance of Severi's formula for curves C of two types: (1) complete intersections of general pairs of surfaces, each of degree at least 15, and (2) general rational curves of degree at least 15. He did so by using Gruson and Peskine's local analysis as a model to show that in each case N_r has codimension r for all r; then he referred to a preliminary version of the present paper, and quoted the discussion given at the end of Article 1.1 above to complete the proof.

Gruson and Peskine obtained the geometric genus of N_3 as follows [12, Theorem 3.6, p. 25]: first they found its arithmetic genus; then they determined that, under blowing-up along N_4, the arithmetic genus drops by the amount of $3\deg N_4$; and finally they proved a necessary and sufficient geometric condition for this blowup to be smooth. To determine the amount of the drop in the genus, they proved an abstract algebraic result [12, 2.6, p. 13] and a related geometric result [12, 2.7, p. 14]. The former applied, a priori, to a certain modification of N_3, and the latter result identified this modification as the blowup of N_3 along N_4. Now, this blowup is equal to Hilb_f^3 by (1.1.3). Correspondingly, their algebraic result, which they proved for an arbitrary r, becomes simply equation (1.1.2). Thus we recover their result. In fact, we derive (1.1.2) from (1.1.1) by induction on r. Initially, the two results coincide as $\mathrm{Hilb}_f^1=X$ and $h=f$; more precisely, (1.1.2) with $r=1$ coincides with (1.1.1) with $r=2$. Gruson and Peskine [12, p. 13] themselves observed that, when $r=1$, they had recovered the old result about the colength of the conductor of a curve on a smooth surface. Thus (1.1.2) and (1.1.1) may be viewed as different generalizations of this old result.

It is of some importance to determine how M_r and N_r vary in a family. Of course, since they are defined by the $(r-1)$th Fitting ideal of the $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$, their formations commute with base change. So the problem is to find conditions guaranteeing that these schemes are flat when X and Y are flat over a given parameter space. Mond and Pellikaan devoted much of their paper [29] to the issue; they considered only N_r, but the situation is similar for M_r. Assuming that the parameter space is smooth, they noted [29, top p. 113] that N_r is flat if it is Cohen–Macaulay and its fibers are equidimensional of constant dimension. Although N_r is defined by a Fitting ideal, the expected codimension r of N_r is not the "generic" value for that Fitting ideal. So a portion of [29] is devoted to re-expressing the ideal of N_r, locally, as the zeroth Fitting ideal of a suitable $\mathcal{O}_Y$-module under suitable hypotheses; see (1.3.1). Then they could conclude that N_r is Cohen–Macaulay.

Similarly, we prove Theorem 3.5 below by re-expressing the ideal of N_r as a zeroth Fitting ideal; this theorem asserts, in particular, that N_r is perfect of grade r. We derive the corresponding result for M_r from this result for N_r in Theorem 3.11. These results yield the flatness of M_r and N_r, by virtue of the local criterion, without any special assumptions on the parameter space.

1.3. Hypotheses

In this paper, we work with a finite map $f\colon X\to Y$ of arbitrary locally Noetherian schemes (whereas Gruson and Peskine [12] worked with algebraic varieties, and Mond and Pellikaan [29] worked with complex analytic spaces, although much in their papers generalizes with little or no change). Thus our results apply not only to individual varieties in arbitrary characteristic, but also to families of varieties, including infinitesimal families and families of mixed characteristic. Moreover, for the most part, there would be little technical advantage in it if we worked over a field, let alone over an algebraically closed field or over a field of characteristic 0.

To develop the theory fully, we need to make a number of hypotheses. Often we need to assume, for an appropriate r, that Y satisfies *Serre's condition* (S_r); that is, every local ring $\mathcal{O}_{Y,y}$ is either Cohen–Macaulay or of depth at least r. In addition, we make six hypotheses on $f: X \to Y$. However, they are not independent. We now discuss these six, one after the other.

The first hypothesis on f is that f be *locally of codimension* 1; in other words, every local ring $\mathcal{O}_{X,x}$ is of dimension 1 less than that of $\mathcal{O}_{Y,fx}$. Without this hypothesis, the N_r need not be Cohen–Macaulay when they should be. To illustrate this point, Mond and Pellikaan [29, bottom p. 110] gave the following example: X is the t-line; Y is 3-space; and $f(t) := (t^3, t^4, t^5)$. Here N_1 is not Cohen–Macaulay at the origin because it is not reduced there. However, this f is not dimensionally generic, as f is singular at the origin; so N_2 has codimension 1 in N_1, whereas its expected codimension is the codimension of f, namely, 2. On the other hand, Joel Roberts (private communication, April 18, 1991, see also [35, Cor. 2.7]) gave the following argument, which shows that the preceding phenomenon is not accidental. Suppose that X and N_1 are both Cohen–Macaulay, and consider the sheaf

$$\mathcal{M}_2 := \mathcal{O}_{f_*\mathcal{O}_X} / \mathcal{O}_{N_1}.$$

Its support is the set N_2 and, at any point x of N_2, its depth is at least $\operatorname{depth}(\mathcal{O}_{X,x}) - 1$, which is equal to $\dim(\mathcal{O}_{X,x}) - 1$. However, the depth of a sheaf is at most the dimension of its support. Hence, at x, the codimension of N_2 in N_1 cannot be 2 or more. However, again, its expected value is the codimension of f.

The second hypothesis is that f be *locally of flat dimension* 1; in other words, every local ring $\mathcal{O}_{X,x}$ is an $\mathcal{O}_{Y,fx}$-module of flat dimension 1. It is equivalent that $f_*\mathcal{O}_X$ be presented, locally, by a square matrix with regular determinant; see [29, 2.1, p. 114] and Lemma 2.3 below. By the same token, it is equivalent that N_1 be a divisor. Now, the second hypothesis implies the first by Corollary 2.5. In practice, often Y is nonsingular; if so, then, by the Auslander–Buchsbaum formula, the second hypothesis obtains if and only if the first does and X is Cohen–Macaulay.

The third hypothesis is that f be *birational* (or of degree 1) onto its image. Suppose that Y satisfies Serre's condition (S_2), and that the first three hypotheses obtain. Then Proposition 3.4 implies that N_1 is equal to the scheme-theoretic image of f; in other words, the Fitting ideal $\mathcal{F}itt_0^Y(f_*\mathcal{O}_X)$ is equal to the annihilator

$\mathcal{A}nn_Y(f_*\mathcal{O}_X)$. Moreover, then N_2 has codimension 2 in Y, and

$$\mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{O}_{N_1}) = \mathcal{F}itt_1^Y(f_*\mathcal{O}_X) = \mathcal{F}itt_0^Y(f_*\mathcal{O}_X/\mathcal{O}_{N_1}).$$

The history of these equations is involved, and was indicated in the discussion of (1.6), (1.7) and (1.8) in [21, p. 202]; since then, Zaare-Nahandi [35] handled a few additional, but special, cases, in which f need not have codimension 1. The first equation above implies that N_2 is defined in N_1 by the conductor; the second equation implies that N_2 is perfect (compare also with J. Roberts [31, Thm. 3.1, p. 258]).

The fourth hypothesis is that f be *curvilinear*; in other words, the differential corank of f, that is, the corank of the Jacobian map

$$\partial f(x) \colon f^*\Omega^1_Y(x) \longrightarrow \Omega^1_X(x),$$

is at most 1 at every x in X. This hypothesis implies that f is *cyclic*; that is, locally the $\mathcal{O}_Y$-algebra $f_*\mathcal{O}_X$ has a primitive element. This implication was proved in the case that X is smooth by Marar and Mond [26, 2.9, p. 560], and it is proved in full generality in Proposition 2.7 below. As a special case, this proposition contains the usual theorem of the primitive element for a field extension with limited inseparability.

Assume that f is cyclic and that Y satisfies Serre's condition (S_r). If, in addition, N_r locally has codimension r in Y, then N_r is perfect by Theorem 3.5; in fact, if a is a primitive element at $y \in N_r$, then

$$\mathcal{F}itt^Y_{r-1}(f_*\mathcal{O}_X)_y = \mathcal{F}itt^Y_0(\mathcal{M}_r) \tag{1.3.1}$$

where

$$\mathcal{M}_r := (f_*\mathcal{O}_X)_y \Big/ \sum_{i=0}^{r-2} \mathcal{O}_{Y,y}a^i.$$

This relation between Fitting ideals was proved by Mond and Pellikaan [29, 5.2, p. 136] under the additional assumption that N_{r+1} has codimension $r+1$. Briefly put, they proved that the two ideals are equal off N_{r+1}, where the job is simpler because $(f_*\mathcal{O}_X)_y$ is generated by $1, \dots, a^{r-1}$; then they concluded that the two ideals are equal everywhere because the one on the left contains the one on the right, and the latter has no embedded components. However, the relation had already been proved without the assumption on the codimension of N_{r+1} by Gruson and Peskine [12, 1.3, p. 4]; they gave an elementary argument which applies to any finite cyclic extension of an arbitrary commutative ring. The relation plays an essential role in the present paper, entering in the proofs of Lemma 3.6 and Theorem 4.4.

In passing, let us note some other interesting properties of $\mathcal{M}_r$. First,

$$\operatorname{Proj}(\mathcal{S}ym(\mathcal{M}_r)/\mathcal{O}_{N_r}\text{-torsion}) = \operatorname{Bl}(N_r, N_{r+1})$$

if Y satisfies (S_{r+1}), if N_r and N_{r+1} are locally of codimensions r and $r+1$ in Y, and if f is also locally of flat dimension 1. This equation was proved by Gruson and Peskine [12, 2.7, p. 14] for $r=3$, and here in brief is a version of their proof for arbitrary r. There is a natural surjection μ from $\mathcal{M}_r$ onto the sheaf associated to the ideal I in the proof of Theorem 4.4 below. In that proof, it is shown that $\operatorname{Bl}(I)$ is equal to $\operatorname{Bl}(N_r, N_{r+1})$. Now, μ is, obviously, an isomorphism modulo $\mathcal{O}_{N_r}$-torsion; in fact, μ is an isomorphism because the (Fitting) ideals defining N_r and N_{r+1} have the correct grades, namely, r and $r+1$. Thus the equation holds. Second, there is a natural surjection from $\mathcal{S}ym_{r+1}\,\mathcal{M}_r$ onto the ideal $\mathcal{J}_r$ of N_{r+1} in N_r, as Gruson and Peskine [12, 2.2, p. 8] show, and it factors through $\mathcal{S}ym_{r+1}(\mu)$.

The fifth hypothesis is that f be *Gorenstein*; that is, f has finite flat dimension and its dualizing complex $f^!\mathcal{O}_Y$ is quasi-isomorphic to a shifted invertible sheaf. If also the second hypothesis obtains (that is, f is locally of flat dimension 1), then $f_*\mathcal{O}_X$ is presentable locally by a symmetric matrix. This result was proved over the complex numbers by Mond and Pellikaan [29, 2.5, p. 117], and a version of it had been proved earlier by Catanese [7, 3.8, p. 84]. Mond and Pellikaan [29, 4.3, p. 131] went on to prove that, if N_3 has codimension in Y at least 3, then N_3 has codimension exactly 3 and N_3 is Cohen–Macaulay because then $\mathcal{F}itt_2^Y(f_*\mathcal{O}_X)_y$ is locally a symmetric determinantal ideal. Those results will not be recovered in this paper; however, see [22] where there are new proofs, which, unlike the old, work in the present general setting, and there are related new proofs of the characterization (due to Valla and Ferrand) of perfect self-linked ideals of grade 2 in an arbitrary Noetherian local ring as the ideals of maximal minors of suitable n by $n-1$ matrices having symmetric $n-1$ by $n-1$ subblocks.

The sixth hypothesis is that f be a *local complete intersection*; that is, each point of X has a neighborhood that is a complete intersection in some (and so any) smooth Y-scheme (see [2, VIII 1, pp. 466–475]). For example, if X and Y are smooth, then the graph map of f embeds X in $X \times Y$; so f is a local complete intersection. In the presence of the second and fourth hypotheses, the fifth and sixth are equivalent. Indeed, the sixth obviously implies the fifth; the converse is proved in Proposition 2.10 via a version of an old argument of Serre's. (The converse should now be borne in mind when reading [21, bottom p. 200].) Moreover, by Proposition 2.10, the sixth hypothesis, combined with the first, implies the second. The sixth hypothesis is required in global enumerative multiple-point theory to ensure an adequate theory of Chern classes of f and of the pullback operator f^*. Although the sixth hypothesis played no special role in Mond and Pellikaan's paper [29], the hypothesis plays an essential role in the present paper.

The various hypotheses on f are inherited by the iteration map f_1 thanks to Lemma 3.10, and this fact plays a leading role in this paper. The *iteration map* $f_1: X_2 \to X$ is defined as follows: $X_2 := \mathbf{P}(\mathcal{I}(\Delta))$ where $\mathcal{I}(\Delta)$ is the ideal of the diagonal, and f_1 is induced by the second projection. It is remarkable how strong a condition it is for f_1 to satisfy the second hypothesis; indeed, by Proposition 3.12, if f_1 does, if Y satisfies (S_2), and if f satisfies the second and fourth hypotheses, then f satisfies all six. The usefulness of f_1 stems from the equations,

$$M_r(f) = N_{r-1}(f_1) \text{ and } \mathrm{Univ}_f^r = \mathrm{Hilb}_{f_1}^{r-1} \quad \text{for } r \geq 2,$$

which are proved in Lemma 3.9 and Proposition 4.1. These equations permit us to derive general properties of the source multiple-point loci and the universal subschemes from corresponding properties of the target multiple-point loci and the Hilbert schemes, proceeding by induction on r when convenient.

2. Special finite maps

Definition 2.1. A map $f: X \to Y$ of schemes will be said to be *locally of flat dimension s* if X is nonempty and if, for every x in X, the local ring $\mathcal{O}_{X,x}$ is of flat dimension s over $\mathcal{O}_{Y,fx}$.

Proposition 2.2. *Let $f: X \to Y$ be a finite map that locally has a finite presentation. Then f is locally of flat dimension* 1 *if and only if its scheme of target points N_1 is a divisor of Y.*

Proof. The assertion results immediately from the equivalence of (iii) and (v) in the following lemma, where the rings need not be Noetherian.

Lemma 2.3. *Let $\phi: R \to B$ be a nonzero homomorphism of rings. Assume that B has a finite presentation as an R-module. Then the following five conditions are equivalent*:

(i) *The ring B has flat dimension* 1 *over R, and for every prime $\mathbf{p}$ in B, the prime $\mathbf{q} := \phi^{-1}\mathbf{p}$ is nonminimal in R.*

(ii) *For every prime $\mathbf{p}$ in B, the prime $\mathbf{q} := \phi^{-1}\mathbf{p}$ is nonminimal in R, and the localization $B_{\mathbf{p}}$ has flat dimension* at most 1 *over $R_{\mathbf{q}}$.*

(iii) *For every prime $\mathbf{p}$ in B, the localization $B_{\mathbf{p}}$ has flat dimension* exactly 1 *over $R_{\mathbf{q}}$ where $\mathbf{q} := \phi^{-1}\mathbf{p}$.*

(iv) *For every maximal ideal $\mathbf{q}$ of R, the $R_{\mathbf{q}}$-module $B_{\mathbf{q}} := B \otimes R_{\mathbf{q}}$ is presented by a square matrix whose determinant is regular.*

(v) *The zeroth Fitting ideal $\mathcal{F}itt_0^R(B)$ is invertible.*

The preceding (equivalent) conditions imply the following condition, and they are all equivalent if R is Noetherian.

(vi) *The R-module B is perfect of grade* 1.

Furthermore, if (iv) *obtains, then a suitable square matrix may be obtained from any matrix presenting B by omitting suitable columns.*

Proof. Assume (i). Now, for any R-module M,

$$B_{\mathbf{p}} \otimes_B \mathrm{Tor}_i^R(B, M) = \mathrm{Tor}_i^{R_{\mathbf{q}}}(B_{\mathbf{p}}, M_{\mathbf{q}}); \tag{2.3.1}$$

indeed, for any free resolution $E.$ of M,

$$B_{\mathbf{p}} \otimes_B H_i(B \otimes_R E.) = H_i(B_{\mathbf{p}} \otimes_B B \otimes_R E.) = H_i(B_{\mathbf{p}} \otimes_{R_{\mathbf{q}}} R_{\mathbf{q}} \otimes_R E.).$$

Hence $B_{\mathbf{p}}$ has flat dimension at most 1 over $R_{\mathbf{q}}$, because every $R_{\mathbf{q}}$-module N is of the form $M_{\mathbf{q}}$ (for example, take $M := N$). Thus (ii) holds.

Assume (ii). Then the minimal primes $\mathbf{p}'$ in $B_{\mathbf{p}}$ have nonminimal preimages $\mathbf{q}'$ in $R_{\mathbf{q}}$. So, if $\mathbf{q}''$ is a minimal prime contained in $\mathbf{q}'$ and if $a \in (\mathbf{q}' - \mathbf{q}'')$, then a is regular on $R_{\mathbf{q}}/\mathbf{q}''$, but not on $B_{\mathbf{p}'}/\mathbf{q}''B_{\mathbf{p}'}$, since its image in $B_{\mathbf{p}'}/\mathbf{q}''B_{\mathbf{p}'}$ is nilpotent. Hence $B_{\mathbf{p}}$ is not flat over $R_{\mathbf{q}}$. Therefore, $B_{\mathbf{p}}$ has flat dimension exactly 1 over $R_{\mathbf{q}}$. Thus (iii) holds.

Assume (iii). To prove (iv), we may assume that $\mathbf{q} \supset \ker \phi$, because otherwise $B_{\mathbf{q}} = 0$. Then $\mathbf{q}$ is of the form $\phi^{-1}\mathbf{p}$ because ϕ is finite. So (2.3.1) implies that $B_{\mathbf{q}}$ has flat dimension at most 1 over $R_{\mathbf{q}}$. By hypothesis, there is a short exact sequence of $R_{\mathbf{q}}$-modules,

$$0 \longrightarrow E \longrightarrow F \longrightarrow B_{\mathbf{q}} \longrightarrow 0,$$

in which F is free and E is finitely generated. Hence E is flat, and therefore free by, for example, [27, 7.10, p. 51].

Set $I := \mathcal{F}itt_0^R(B)$. Then $Ann(I_{\mathbf{q}})$ vanishes by McCoy's theorem [28, Thm. 51, p. 159] (or by [24, Lem., p. 889]) because E is free. Hence, $\mathbf{q}$ is not a minimal prime of R; indeed, otherwise, $R_{\mathbf{q}}$ would have dimension 0, so $B_{\mathbf{q}}$ would be free because $I_{\mathbf{q}}$ would be equal to $R_{\mathbf{q}}$, but $B_{\mathbf{p}}$ has flat dimension exactly 1. Hence, if $\mathbf{q}'$ is any minimal prime of R contained in $\mathbf{q}$, then $B_{\mathbf{q}} \otimes R_{\mathbf{q}'} = 0$. Therefore, $\mathrm{rk}\, E = \mathrm{rk}\, F$. In other words, $B_{\mathbf{q}}$ is presented by a square matrix $\mathbf{M}$. Now, $\det \mathbf{M}$ generates $I_{\mathbf{q}}$, and $Ann(I_{\mathbf{q}}) = (0)$; so $\det \mathbf{M}$ is regular. Thus (iv) holds.

Obviously (iv) implies (v). The converse is a special case of [25, Lem. 1, p. 423], but may be proved directly as follows. Assume (v). Take any matrix $\mathbf{M}$ presenting $B_{\mathbf{q}}$ over $R_{\mathbf{q}}$; say $\mathbf{M}$ is m by n with $m \le n$. Moreover, we may assume that the zeroth Fitting ideal is generated by the determinant of the submatrix $\mathbf{N}$ formed by the first m columns because the ideal is invertible and $R_{\mathbf{q}}$ is local. Hence

$\det \mathbf{N}$ is regular, and divides the determinant of every m by m submatrix of $\mathbf{M}$. Hence, by Cramer's rule, every column of $\mathbf{M}$ is a linear combination of the first m. Therefore, $\mathbf{N}$ too presents $B_{\mathbf{q}}$. Thus (iv) and the last assertion hold.

Assume (iv). Then, for every prime $\mathbf{q}$ of R, the $R_{\mathbf{q}}$-module $B_{\mathbf{q}}$ has flat dimension at most 1. Hence the R-module B has flat dimension at most 1. If $\mathbf{q}=\phi^{-1}\mathbf{p}$ for some prime $\mathbf{p}$ in B, then $\mathbf{q}R_{\mathbf{q}}$ contains a regular element, namely, the determinant of a matrix presenting $B_{\mathbf{q}}$; indeed, this determinant lies in $Ann_{R_{\mathbf{q}}} B_{\mathbf{q}}$, which is contained in $\mathbf{q}R_{\mathbf{q}}$. Hence, $\mathbf{q}$ is not minimal. Moreover, $\mathrm{Hom}_{R_{\mathbf{q}}}(B_{\mathbf{q}}, R_{\mathbf{q}})=0$; whence, $\mathrm{Hom}_R(B,R)=0$ because B has a finite presentation. Now, if $\mathbf{p}$ is a minimal prime of B, if $\mathbf{q}'$ is a minimal prime contained in $\mathbf{q}=\phi^{-1}\mathbf{p}$ and if $a\in(\mathbf{q}-\mathbf{q}')$, then a is regular on $R/\mathbf{q}'$, but not on $B/\mathbf{q}'B$; hence B is not flat over R. Alternatively, B is not flat over R because its zeroth Fitting ideal is invertible, so nonzero. Thus both (i) and (vi) hold.

Finally, assume that (vi) holds and that R is Noetherian. Then $Ann_R B$ contains a regular element. This element lies in any prime $\mathbf{q}$ of R of the form $\mathbf{q}=\phi^{-1}\mathbf{p}$ where $\mathbf{p}$ is a prime of B. Hence, $\mathbf{q}$ is not minimal. Thus (i) holds, and the proof is complete.

Definition 2.4. Let $f: X\to Y$ be a map of locally Noetherian schemes, and s an integer. Call f *locally of codimension* s if X is nonempty and if, for every x in X,

$$\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,fx} - s.$$

Corollary 2.5. *Let $f: X\to Y$ be a finite map of locally Noetherian schemes. Assume that f is locally of flat dimension* 1.

(1) *Then f is locally of codimension* 1.

(2) *Then the fundamental cycles satisfy the relation, $f_*[X]=[N_1]$.*

Proof. To prove (1), let x be a point of X, and y its image in Y. Let R and B be the corresponding local rings, and C the semi-local ring of $f^{-1}y$. Let '$\widehat{\ }$' denote completion. It suffices to check that $\dim \widehat{B}=\dim \widehat{R}-1$. Now, $\mathcal{F}itt_0^R C$ is invertible by Proposition 2.2; hence, $\mathcal{F}itt_0^{\widehat{R}} \widehat{C}$ is invertible. However, $\widehat{B}$ is a direct summand of $\widehat{C}$. Hence $\mathcal{F}itt_0^{\widehat{R}} \widehat{B}$ is invertible. Therefore, $\dim(\widehat{R}/\mathcal{F}itt_0^{\widehat{R}} \widehat{B})$ is equal to $\dim \widehat{R}-1$. Finally, $\widehat{R}/\mathcal{F}itt_0^{\widehat{R}} \widehat{B}$ and $\widehat{B}$ have the same dimension because $\mathcal{F}itt_0^{\widehat{R}} \widehat{B}$ and $Ann_{\widehat{R}}(\widehat{B})$ have the same radical. Thus (1) holds.

Consider (2). Since N_1 is a divisor by Proposition 2.2, at the generic point ν of any of its components, the lengths $l_\nu(f_*\mathcal{O}_X)$ and $l_\nu(\mathcal{O}_{N_1})$ are equal by the determinantal length formula, [6, 2.4, 4.3, 4.5] or [3, 2.10, p. 154]; in other words, (2) holds.

Definition 2.6. Let $f: X \to Y$ be a map of schemes, $x \in X$. Call the number,

$$\dim_{k(x)} \Omega^1_f(x),$$

the *differential corank* of f at x. In terms of a base scheme S, this number is simply the corank of the Jacobian map,

$$\partial f(x): f^* \Omega^1_{Y/S}(x) \longrightarrow \Omega^1_{X/S}(x).$$

Call f *curvilinear* if its differential corank is at most 1 at every x in X.

Proposition 2.7. *Let $f: X \to Y$ be a finite map of schemes, $t \geq 1$.*

(1) *Let $x \in X$. Then f has differential corank at most t at x if and only if x has a neighborhood U such that the restriction $f|_U$ factors through an embedding of U into the affine t-space $\mathbf{A}^t_Y$.*

(2) *Let $y \in Y$ be a point whose residue class field is infinite. Then f has differential corank at most t at every point x of $f^{-1}y$ if and only if y has a neighborhood V such that the restriction $f^{-1}V \to V$ factors through a closed embedding of $f^{-1}V$ in the affine t-space $\mathbf{A}^t_V$.*

Proof. The assertions follow immediately from the next lemma.

Lemma 2.8. *Let R be a local ring (which need not be Noetherian). Let B be an R-algebra that is finitely generated as an R-module, $t \geq 1$.*

(1) *Let $\mathbf{m}$ be a maximal ideal of B, let C be an R-subalgebra of B generated by t elements, and set $\mathbf{n} := \mathbf{m} \cap C$. If the canonical map $C_\mathbf{n} \to B_\mathbf{m}$ is surjective, then*

$$\dim_{B/\mathbf{m}} (\Omega^1_{B/R} / \mathbf{m} \Omega^1_{B/R}) \leq t. \tag{2.8.1}$$

(2) *Let $\mathbf{m}$ be a maximal ideal of B such that* (2.8.1) *obtains. Then there exists an R-subalgebra C of B generated by t elements such that, if $\mathbf{n} := \mathbf{m} \cap C$, then the natural map $C_\mathbf{n} \to B_\mathbf{m}$ is bijective.*

(3) *Assume that* (2.8.1) *obtains for every maximal ideal $\mathbf{m}$ of B, and that the residue class field of R is infinite. Then B is generated as an R-algebra by t elements.*

Proof. In (1), say C is generated by $x_1, \ldots, x_t$. Then the images of $dx_1, \ldots, dx_t$ generate $(\Omega^1_{C/R})_\mathbf{n}$. Hence (2.8.1) holds, as asserted.

To prove (2) and (3), we may assume that R is a field. Indeed, let k be the residue field of R. First, consider (3). Suppose that $B \otimes k$ is generated by t elements. Lift them to B, and let C be the resulting subalgebra. Then the inclusion $C \to B$ is surjective by Nakayama's lemma, because $C \otimes k \to B \otimes k$ is surjective by assumption and B is finitely generated as an R-module by hypothesis. Thus, to prove (3), we may replace R and B by k and $B\otimes k$.

Next, consider (2). Suppose that there exists a subalgebra C of B generated by t elements, such that if $\mathbf{n}:=\mathbf{m}\cap C$, then $C_{\mathbf{n}}\otimes k\to B_{\mathbf{m}}\otimes k$ is surjective. If x_1 lies in $\mathbf{n}$, replace it by x_1+1; obviously, doing so does not change C. Now, $B\otimes k$ is equal to the direct product of its localizations at the maximal ideals of B; so there is an element x of B whose image in $B_{\mathbf{m}}\otimes k$ is equal to 1 and whose image in the other localizations is equal to 0. For each i, replace x_i by $x_i x$; doing so may change C, but $C_{\mathbf{n}}\otimes k\to B_{\mathbf{m}}\otimes k$ remains surjective. The following argument, adapted from [11, (18.4.6.1), p. 120], shows that $C_{\mathbf{n}}\to B_{\mathbf{m}}$ is now bijective.

The map $C_{\mathbf{n}}\to B_{\mathbf{m}}$ factors as follows:

$$C_{\mathbf{n}} \xrightarrow{\alpha} B_{\mathbf{n}} \xrightarrow{\beta} B_{\mathbf{m}}.$$

Consider β. It will be bijective if every element of $B-\mathbf{m}$ becomes a unit in $B_{\mathbf{n}}$, so if every maximal ideal $\mathbf{p}$ of $B_{\mathbf{n}}$ contracts to $\mathbf{m}$. Since α is finite, $\alpha^{-1}\mathbf{p}$ is a maximal ideal of $C_{\mathbf{n}}$; hence, $\alpha^{-1}\mathbf{p}=\mathbf{n}C_{\mathbf{n}}$. Therefore, if $\mathbf{q}$ denotes the trace of $\mathbf{p}$ in B, then $\mathbf{q}\cap C=\mathbf{n}$. Since B is a finitely generated C-module, $\mathbf{n}$ is a maximal ideal of C, and so $\mathbf{q}$ is a maximal ideal of B. Now, $x_1\notin\mathbf{q}$ because $x_1\notin\mathbf{n}$ and $x_1\in C$. Since x_1 lies in every maximal ideal of B other than $\mathbf{m}$, necessarily $\mathbf{q}=\mathbf{m}$, as required. Thus β is bijective.

Consider α. It is injective because $C\subseteq B$. Now, α is finite; hence, by Nakayama's lemma, it will be surjective if $\alpha\otimes k$ is. However, $(\beta\alpha)\otimes k$ is surjective by assumption, and β is bijective by the preceding paragraph. Hence α is surjective, so bijective. So $C_{\mathbf{n}}\to B_{\mathbf{m}}$ is bijective. Thus, to prove (2) as well as (3), we may assume that R is a field.

We may also assume that B is local. Indeed, since R is a field, B is equal to the direct product, over its finite set of maximal ideals $\mathbf{m}$, of its localizations: $B=\prod B_{\mathbf{m}}$. Suppose one of the localizations $B_{\mathbf{m}}$ has t generators. Lift them to elements $x_1,\dots,x_t$ of B whose images in the other localizations are equal to 0. Then form C and $\mathbf{n}$ as usual. Clearly $C=B_{\mathbf{m}}$ and $C_{\mathbf{n}}=C$. Hence, to prove (2), we may replace B by $B_{\mathbf{m}}$, and then we have to prove that B is generated as an R-algebra by t elements.

Consider (3). Suppose that each $B_{\mathbf{m}}$ has t generators $x_{\mathbf{m},i}$. Set $x_i:=(x_{\mathbf{m},i})$ in B. Now, in the polynomial ring in one variable $R[\lambda]$, let $f_{\mathbf{m}}(\lambda)$ be a polynomial of minimal degree such that $f_{\mathbf{m}}(x_{\mathbf{m},1})=0$. Suppose that R is infinite. Then we may assume that the $f_{\mathbf{m}}(\lambda)$ are relatively prime; indeed, replacing $x_{\mathbf{m},1}$ by $x_{\mathbf{m},1}+a_{\mathbf{m}}$ where the $a_{\mathbf{m}}$ are suitable elements of R, we may ensure that no two $f_{\mathbf{m}}(\lambda)$ share a root in some algebraic closure of R. Set $f:=\prod f_{\mathbf{m}}$. Then $R[\lambda]/(f)$ is isomorphic to the subalgebra B_1 of B generated by x_1. Hence, by the Chinese remainder theorem, B_1 contains the idempotents of the decomposition $B=\prod B_{\mathbf{m}}$. Therefore, B is generated by the x_i. Thus, to prove (3) as well as (2), we may assume that B is local.

We may also assume $t=1$. Indeed, suppose $\Omega^1_{B/R}/\mathbf{m}\Omega^1_{B/R}$ has dimension at least 2, and let x_1 be an element of B such that the residue class of dx_1 is nonzero. Let B_1 be the subalgebra of B generated by x_1. Then B is a finitely generated B_1-module; so, by the Cohen–Seidenberg theorem, B_1 is local as B is. Moreover, Ω^1_{B/B_1} is equal to the quotient of $\Omega^1_{B/R}$ by the submodule generated by dx_1. Hence, we may assume by induction on t that there exist $t-1$ elements $x_2, \dots, x_t$ of B that generate it as a B_1-algebra. Then $x_1, \dots, x_t$ generate B as an R-algebra.

Suppose that the natural map $R \to B/\mathbf{m}$ is an isomorphism. Then there exists an element x in $\mathbf{m}$ that generates B as an R-algebra. Indeed, $\mathbf{m}/\mathbf{m}^2$ is equal to $\Omega^1_{B/R}/\mathbf{m}\Omega^1_{B/R}$; see, for example, [23, Cor. 6.5(a), p. 96]. So $\dim(\mathbf{m}/\mathbf{m}^2) \le 1$. Let x be an element of $\mathbf{m}$ whose residue class generates $\mathbf{m}/\mathbf{m}^2$ over R; possibly, $x=0$. Then $B=R[x]$. (The argument is standard. Let y be an element of B. For all $n \ge 0$, there is a polynomial $y_n := \sum_{i=0}^n a_i x^i$ with $a_i \in R$ and $y-y_n$ in $\mathbf{m}^{n+1}$; indeed, take a_0 to be the image of y in $R=B/\mathbf{m}$, and given y_n, take a_{n+1} so that $y-y_n$ is equal to $a_{n+1}x^{n+1}$. However, $\mathbf{m}^n=0$ for $n \gg 0$, so $y=y_n$ for $n \gg 0$.)

Suppose that R is infinite. Let R' be an algebraic closure of R, and set $B' := B \otimes_R R'$. Then there exists an element x' of B' that generates it as an R'-algebra by the preceding paragraphs, because B' is a finite R'-algebra whose residue class fields are all equal to R' and because the formation of Ω^1 commutes with base change. To descend the existence of a generator, let $y_1, \dots, y_n$ be a vector space basis of B over R, and let $\tau_1, \dots, \tau_n$ be indeterminates. In the polynomial ring $B[\tau_1, \dots, \tau_n]$, set $u := \sum \tau_i y_i$ and expand the powers u^j for $j=0, \dots, n-1$. Form the matrix $\Phi(\tau_1, \dots, \tau_n)$ such that

$$(1, u, \dots, u^{n-1})^{\mathrm{tr}} = \Phi(\tau_1, \dots, \tau_n)(y_1, \dots, y_n)^{\mathrm{tr}}$$

where 'tr' denotes transpose. Say $x' = \sum \tau'_i y_i$ where $\tau'_i \in R'$. Then the τ'_i do not satisfy the equation $\det \Phi(\tau_1, \dots, \tau_n) = 0$ because x' generates B' as an R'-algebra. Hence, since R is infinite, there exist elements $\tau''_1, \dots, \tau''_n$ in R that do not satisfy this equation. Therefore, $x := \sum \tau''_i y_i$ generates B as an R-algebra.

Finally, suppose that R is finite, say of characteristic p. Take $q := p^e$ so large that $\mathbf{m}^q = 0$ and $z^q = z$ for all $z \in B/\mathbf{m}$. Consider $L := B^q$. Obviously, L is an R-subalgebra. In fact, L is a field: if $x \in B$ and $x^q \ne 0$, then $x \notin \mathbf{m}$ as $\mathbf{m}^q = 0$; hence, there is a $y \in B$ such that $xy=1$; so $x^q y^q = 1$. Now, since $z^q = z$ for all $z \in B/\mathbf{m}$, the natural map $L \to B/\mathbf{m}$ is surjective; so it is an isomorphism because L is a field.

Since $\Omega^1_{B/L}$ is a quotient of $\Omega^1_{B/R}$, by the paragraph before the last, there is an $x \in \mathbf{m}$ such that $B = L[x]$. Since L is finite, its multiplicative group is generated by an element y. Set $z := x + y$, and $B' := R[z]$. Then $z^q = x^q + y^q$, so $z^q = y$. Thus $y \in B'$. Hence $L \subseteq B'$. Moreover, $z - z^q = x + y - y = x$. Thus $x \in B'$. So $L[x]$ lies in B'. Therefore $B = B'$, and the proof is now complete.

Definition 2.9. Let $f: X \to Y$ be a map of locally Noetherian schemes. Following [2, 1.1, p. 466] call f a *local complete intersection* if, locally on X, there is a factorization $f = \pi i$ where $i: X \hookrightarrow P$ is a regular embedding and $\pi: P \to Y$ is smooth. Following [13, p. 144], call f *Gorenstein* if it has finite flat dimension and if in the derived category $f^! \mathcal{O}_Y$ is isomorphic to a (shifted) invertible sheaf.

Proposition 2.10. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes, s an integer. If f is a local complete intersection and is locally of codimension s, then f is Gorenstein and locally of flat dimension s. Moreover, the converse holds if also $s = 1$ and f is curvilinear.*

Proof. Suppose f is a local complete intersection and is locally of codimension s. Then clearly f is Gorenstein; see [13, Cor. 7.3, p. 180], and [13, 3, p. 190]. Now, for every x in X, clearly

$$\operatorname{depth} \mathcal{O}_{X,x} = \operatorname{depth} \mathcal{O}_{Y,fx} - s.$$

Hence, f is locally of flat dimension s by the Auslander–Buchsbaum formula, which applies after completion because f is finite. Thus the direct assertion holds. The converse follows from 2.7(1) and the following lemma.

Lemma 2.11. *Let $R \to B$ be a quasi-finite local homomorphism of Noetherian local rings, and t an integer. Assume*

(i) *that B is of the form S/I where S is a localization at a prime ideal of the polynomial ring in one variable $R[u]$,*

(ii) *that $\operatorname{Ext}^i_S(B, S)$ vanishes for $i \neq t$ and is isomorphic to B for $i = t$, and*

(iii) *that B has flat dimension 1 over R.*

Then $t = 2$ and I is generated by a regular sequence of length 2; moreover, $\dim B = \dim R - 1$.

Proof. Assumptions (i) and (iii) imply that I is flat over R. Say I is generated by n elements with n minimal, and form the exact sequence,

$$(2.11.1) \qquad 0 \longrightarrow F \longrightarrow S^n \xrightarrow{\sigma} I \longrightarrow 0.$$

Then F is flat over R. Moreover, if k denotes the residue field of R, then $F \otimes k$ is free over $S \otimes k$, because $S \otimes k$ is a Principal Ideal Domain. Hence F is a free S-module. So, clearly, $F = S^{n-1}$.

Sequence (2.11.1) therefore yields this free resolution of B over S:

$$0 \longrightarrow S^{n-1} \longrightarrow S^n \longrightarrow S \longrightarrow B \longrightarrow 0. \tag{2.11.2}$$

Hence $t \le 2$. Suppose $t < 2$. Then, because of (ii), dualizing (2.11.2) yields a surjection $v: S^n \twoheadrightarrow S^{n-1}$. However, $v \otimes k = 0$ because n is minimal. Hence $n=1$. Therefore, the map $\sigma: S \to I$ is an isomorphism. Consequently, there is a short exact sequence,

$$0 \longrightarrow \mathrm{Tor}_1^S(B,k) \longrightarrow S\otimes k \longrightarrow S\otimes k \longrightarrow B\otimes k \longrightarrow 0.$$

Since B/R is quasi-finite, $\dim_k B\otimes k$ is finite. Hence the map in the middle is nonzero. Therefore, it is injective because $S\otimes k$ is a domain. So $\mathrm{Tor}_1^S(B,k)=0$. Hence B is flat over R by the local criterion. Thus (iii) is contradicted, and so $t=2$.

Because $t=2$ and because of (ii), dualizing (2.11.2) yields the following exact sequence:

$$0 \longrightarrow S \longrightarrow S^n \longrightarrow S^{n-1} \longrightarrow B \longrightarrow 0.$$

Because S is local, this sequence may be reduced to the sequence,

$$0 \longrightarrow S \longrightarrow S^2 \longrightarrow S \longrightarrow B \longrightarrow 0,$$

by splitting off copies of the trivial exact sequence $0 \to S \to S \to 0$. Therefore, I is generated by two elements, and because $t=2$, they form a regular sequence. Finally, by hypothesis, S is the localization of $R[u]$ at a prime ideal, and this ideal must be maximal because B is quasi-finite over R; hence, $\dim S = \dim R + 1$. Therefore, $\dim B = \dim R - 1$ because I is generated by a regular sequence of length 2.

3. The multiple-point schemes

Definition 3.1. A map $f: X \to Y$ of locally Noetherian schemes will be said to be *birational onto its image* if there is an open subset U of Y such that (i) its preimage $f^{-1}U$ is dense in X and (ii) the restriction $f^{-1}U \to U$ is an embedding.

Proposition 3.2. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes.*

(1) *The map f is birational onto its image if and only if the source double-point scheme M_2 is nowhere topologically dense in X. These two equivalent conditions imply that N_2 is nowhere topologically dense in N_1, and all three conditions are equivalent if f is locally of codimension* 1.

(2) *The scheme-theoretic image of f is a closed subscheme of the scheme of target points N_1. The two schemes have the same support, and they are equal off the scheme of target double-points N_2. If they are equal everywhere and if f is locally of flat dimension* 1, *then f is birational onto its image.*

(3) *The map f induces a finite, surjective map $M_r \to N_r$ for $r \ge 1$.*

Proof. Let U be the largest open subset of Y such that $f^{-1}U \to U$ is an embedding. Since f is finite, U consists of all $y \in Y$ at which the comorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. So, by Nakayama's lemma, U consists of the y at which the vector space $(f_*\mathcal{O}_X)(y)$ has dimension at most 1. Hence, $U = Y - N_2$. Therefore, since $M_2 = f^{-1}N_2$, the first assertion of (2) holds. Obviously, if M_2 is nowhere topologically dense in X, then N_2 is nowhere topologically dense in N_1. Finally, if f is locally of codimension 1, then every component of X must map onto a component of N_1; hence, if also N_2 is nowhere topologically dense in N_1, then M_2 is nowhere topologically dense in X. Thus (1) holds.

The scheme-theoretic image of f is defined as the smallest closed subscheme Z of Y through which f factors [10, (6.10.1), p. 324]. Because f is quasi-compact and quasi-separated, Z exists and is associated to the ideal $\mathcal{A}nn_Y(f_*\mathcal{O}_X)$. Since locally on Y there is an integer n such that

$$\mathcal{A}nn_Y(f_*\mathcal{O}_X)^n \subseteq \mathcal{F}itt_0^Y(f_*\mathcal{O}_X) \subseteq \mathcal{A}nn_Y(f_*\mathcal{O}_X),$$

the image Z is a closed subscheme of N_1, and the two schemes have the same support. They are equal off N_2 because there $f_*\mathcal{O}_X$ is a cyclic $\mathcal{O}_Y$-module. If they are equal everywhere, then $\mathcal{A}nn_Y(f_*\mathcal{O}_X) = \mathcal{F}itt_0^Y(f_*\mathcal{O}_X)$. On the other hand, by [5, Thm. 3.1],

$$\mathcal{A}nn_Y(f_*\mathcal{O}_X) = \mathcal{F}itt_0^Y(f_*\mathcal{O}_X) : \mathcal{F}itt_1^Y(f_*\mathcal{O}_X).$$

It follows that the stalk $\mathcal{F}itt_1^Y(f_*\mathcal{O}_X)_x$ is not contained in any associated prime of the stalk $\mathcal{F}itt_0^Y(f_*\mathcal{O}_X)_x$ for any $x \in X$. So N_2 is nowhere topologically dense in N_1. Therefore, f is birational onto its image by (1) and Corollary 2.5(1). Thus (2) holds.

By definition, $M_r = f^{-1}N_r$. Hence f induces a finite map $M_r \to N_r$. It is surjective because $N_r \subseteq N_1$ and because f carries X onto N_1 by (1). Thus (3) holds.

Definition 3.3. Following [11, (5.7.2), p. 103], a locally Noetherian scheme Y will be said to satisfy *Serre's condition* (S_r), if for every $y \in Y$,

$$\operatorname{depth}(\mathcal{O}_{Y,y}) \geq \inf(r, \dim(\mathcal{O}_{Y,y})).$$

Proposition 3.4. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes. Assume that f is locally of flat dimension* 1 *and is birational onto its image. Assume also that Y satisfies* (S_2). *Let Z denote the scheme-theoretic image of f.*

Then $Z = N_1$. Furthermore, N_2 is defined by the adjoint ideal $\mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{O}_Z)$ and M_2 is defined by the conductor $\mathcal{C}_X$. Each component of M_2 has codimension 1 *and maps onto a component of N_2; each component of N_2 has codimension* 2; *and* *the fundamental cycles of these two schemes are related by the equation,*

$$f_*[M_2]=2[N_2].$$

Finally, $\mathcal{O}_{N_2}$ *and* $\mathcal{O}_{f_*M_2}$ *are perfect* $\mathcal{O}_Y$*-modules of grade* 2.

Proof. By Proposition 3.2(2), we have $Z\subseteq N_1$, with equality off N_2, and N_2 is nowhere topologically dense in N_1 by Proposition 3.2(1) and Corollary 2.5(1). Now, N_1 has no embedded components because it is a divisor by Proposition 2.2 and because Y satisfies (S_2). Hence $N_1=Z$, as asserted.

The cyclic $\mathcal{O}_Y$-module with ideal $\mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{O}_Z)$ has a Hilbert–Burch resolution, and

$$\mathcal{A}nn_Y(f_*\mathcal{O}_X/\mathcal{O}_Z)=\mathcal{F}itt_1^Y(f_*\mathcal{O}_X)=\mathcal{F}itt_0^Y(f_*\mathcal{O}_X/\mathcal{O}_Z); \tag{3.4.1}$$

see [21, (3.5), p. 208]. The first equation of (3.4.1) says that the adjoint ideal defines N_2. Since $\mathcal{C}_X$ is the ideal on X induced by the adjoint ideal, therefore $\mathcal{C}_X$ defines M_2. Because of the Hilbert–Burch resolution, (3.4.1) implies that $\mathcal{O}_{N_2}$ is a perfect $\mathcal{O}_Y$-module of grade 2. Hence $\mathcal{O}_{f_*M_2}$ is perfect of grade 2 too because Z is a divisor. Moreover, the determinantal length formula (see [6, 2.4, 4.3, 4.5] or [3, 2.10, p. 154]) gives the following equation, in which ν is the generic point of an arbitrary component of N_2 and l_ν indicates the length of the stalk at ν:

$$l_\nu(f_*\mathcal{O}_X/\mathcal{O}_Z)=l_\nu(\mathcal{O}_Y/\mathcal{F}itt_0^Y(f_*\mathcal{O}_X/\mathcal{O}_Z)).$$

Let $\mathcal{C}_Z$ denote the conductor on Z, namely, the ideal induced by the adjoint ideal. Then, clearly,

$$f_*(\mathcal{O}_X/\mathcal{C}_X)=(f_*\mathcal{O}_X)/\mathcal{C}_Z \quad \text{and} \quad \mathcal{O}_Y/\mathcal{F}itt_0^Y(f_*\mathcal{O}_X/\mathcal{O}_Z)=\mathcal{O}_Z/\mathcal{C}_Z.$$

The preceding two displays yield

$$l_\nu(f_*(\mathcal{O}_X/\mathcal{C}_X))=2l_\nu(\mathcal{O}_Z/\mathcal{C}_Z).$$

Rewritten, the latter equation will become $[N_2]=2f_*[M_2]$, once we prove the assertion about the components of N_2 and M_2.

Let η be the generic point of a component of M_2. Then $\mathcal{O}_{M_2,\eta}$ is an Artin ring, and its residue field is a finite extension of that of $\mathcal{O}_{Y,f\eta}$. So $\mathcal{O}_{M_2,\eta}$ is an $\mathcal{O}_{Y,f\eta}$-module of finite length, and of flat dimension at most 2, hence of projective dimension at most 2. Therefore the intersection theorem of P. Roberts [32] implies that $\mathcal{O}_{Y,f\eta}$ has dimension at most 2. However, every component of N_2 has codimension at least 2. Consequently, the original component of M_2 has codimension 1 by Corollary 2.5(1). Finally, every component of N_2 is the image of a component of M_2 by Proposition 3.2(3). The proof is now complete.

Theorem 3.5. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes, and r an integer, $r \geq 1$. Assume that f is locally of flat dimension 1 and curvilinear. Then each component of N_r has codimension at most r* (that is, the local ring at the generic point has dimension at most r). *Assume further that each component of N_r has codimension r and that Y satisfies Serre's condition* (S_r). *Then $\mathcal{O}_{N_r}$ and $\mathcal{O}_{M_r}$ are perfect $\mathcal{O}_Y$-modules of grade r, each component of M_r has codimension $r-1$ and maps onto a component of N_r, and the fundamental cycles of these two schemes are related by the equation*

$$f_*[M_r] = r[N_r].$$

Proof. The assertion results from Propositions 2.2 and 3.2(3) and the next lemma.

Lemma 3.6. *Let R be a local Noetherian ring, and B an R-algebra that is finitely generated as a module. Assume that, for every maximal ideal $\mathbf{m}$ of B,*

$$\dim_{B/\mathbf{m}}(\Omega^1_{B/R}/\mathbf{m}\Omega^1_{B/R}) \leq 1.$$

Assume that the R-module B is presented by a square matrix whose determinant is regular. Fix $r \geq 1$ and let $F := \mathrm{Fitt}^R_{r-1}(B)$ denote the Fitting ideal, and $\mathrm{ht}\, F$ its height. If $F \neq R$, then $\mathrm{ht}\, F \leq r$. If $\mathrm{ht}\, F = r$ and if R satisfies (S_r), *then both R/F and B/FB are perfect R-modules of grade r; moreover, then, any minimal prime of FB has height $r-1$, and its preimage in R is a minimal prime of F of height r. Finally, if $\dim R = r$ too, then the following length relation obtains*:

$$l_R(B/FB) = r\, l_R(R/F).$$

Proof. By using a standard device, we may assume that R has an infinite residue class field: replace R and B by R' and $B \otimes_R R'$, where R' is the flat local R-algebra obtained by forming the polynomial ring in one variable over R and localizing it at the extension of the maximal ideal of R. Then Lemma 2.8(3) implies that there is an x in B such that $B = \sum_{i=0}^{n-1} Rx^i$ where $n \geq r$. By Lemma 2.3, there is an n by n matrix ψ with entries in R such that the corresponding short sequence is exact:

$$0 \longrightarrow R^n \xrightarrow{\psi} R^n \longrightarrow B \longrightarrow 0$$

where the natural basis element e_i of R^n is mapped to the generator x^{i-1} of B.

Let M be the R-module $B/\sum_{i=0}^{r-2} Rx^i$, and let ϕ be the $n-r+1$ by n matrix consisting of the last $n-r+1$ rows of ψ. Then there is a commutative diagram with exact rows and surjective columns,

$$\begin{array}{ccccccc} R^n & \xrightarrow{\psi} & R^n = \oplus_{i=1}^n Re_i & \longrightarrow & B & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \\ R^n & \xrightarrow{\phi} & R^{n-r+1} = \oplus_{i=r}^n Re_i & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Let $I_{n-r+1}(\phi)$ and $I_{n-r+1}(\psi)$ denote the ideals of $n-r+1$ by $n-r+1$ minors. So

$$F = I_{n-r+1}(\psi) \quad \text{and} \quad \mathrm{Fitt}_0^R(M) = I_{n-r+1}(\phi).$$

Now, Gruson and Peskine [12, Lem. 1.3, p. 4] proved (using the multiplicative structure of B) that $F = \mathrm{Fitt}_0^R(M)$. So F is generated by the maximal minors of ϕ. So, by the classical height result [4, (2.1), p. 10],

$$\mathrm{ht}\, F = \mathrm{ht}\, I_{n-r+1}(\phi) \le n-(n-r+1)+1.$$

Suppose $\mathrm{ht}\, F = r$ and R satisfies (S_r). Then R/F and M are perfect R-modules of grade r by Eagon's theorem [4, (2.16)(c), p. 18].

Let '$^-$' denote reduction modulo $I_{n-r+1}(\phi)$. We will construct a commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \bar{R}^{n-r+1} \otimes_R \bigwedge^{n-r+1} \bar{R}^n & \xrightarrow{\bar{\delta}} & \bar{R}^n & \xrightarrow{\bar{\psi}} & \bar{R}^n & \longrightarrow & B \otimes_R \bar{R} & \longrightarrow & 0 \\ \| & & \| & & \downarrow & & & & \\ \bar{R}^{n-r+1} \otimes_R \bigwedge^{n-r+1} \bar{R}^n & \xrightarrow{\bar{\delta}} & \bar{R}^n & \xrightarrow{\bar{\phi}} & \bar{R}^{n-r+1} & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

in which the right vertical map is surjective. Once constructed, this diagram yields an exact sequence,

$$0 \longrightarrow \bar{R}^{r-1} \longrightarrow B \otimes_R \bar{R} \longrightarrow M \longrightarrow 0. \tag{3.6.1}$$

Now, $B \otimes_R \bar{R}$ is equal to B/FB; hence, B/FB is perfect over R of grade r because $\bar{R}$ and M are.

Let $\mathbf{p}$ be a minimal prime of the B-ideal FB, and let $\mathbf{q}$ be its preimage in R. Then $(B/FB)_{\mathbf{p}}$ is an Artin ring, and its residue field is a finite extension of that of $R_{\mathbf{q}}$. So $(B/FB)_{\mathbf{p}}$ is an $R_{\mathbf{q}}$-module of finite length, and of flat dimension at most r, hence of projective dimension at most r. Therefore the intersection theorem of P. Roberts [32] implies that $R_{\mathbf{q}}$ has dimension at most r. However, $\operatorname{ht} F=r$. Consequently, $\mathbf{q}$ is a minimal prime of F of height r. So, Corollary 2.5(1) implies that $B_{\mathbf{p}}$ has dimension $r-1$, as asserted.

If $\dim R=r$ too, then the determinantal length formula [6, 2.4, 4.3, 4.5] and [3, 2.10, p. 154] yields

$$l_R(M)=l_R(\operatorname{Cok}\phi)=l_R(R/I_{n-r+1}(\phi))=l_R(R/F).$$

Since $R/F=\bar{R}$ and $B\otimes_R\bar{R}=B/FB$, then (3.6.1) yields the asserted length relation.

To define $\bar{\delta}$, use the bases $\{e_r, \dots, e_n\}$ and $\{e_1, \dots, e_n\}$ of R^{n-r+1} and R^n. For $r\leq i\leq n$, for $1\leq j\leq n-r+1$, and for $1\leq k_1<\dots<k_{n-r+1}\leq n$, denote by $d_i^{\mathbf{k}_j}$ the minor (of ϕ) that is obtained from ψ by deleting rows $1, \dots, r-1$, i and taking columns $k_1, \dots, k_{j-1}$, $k_{j+1}, \dots, k_{n-r+1}$. Now, define

$$\delta\colon R^{n-r+1}\otimes_R \bigwedge^{n-r+1} R^n \longrightarrow R^n \quad \text{by}$$

$$\delta(e_i\otimes e_{k_1}\wedge\dots\wedge e_{k_{n-r+1}}) := \sum_{j=1}^{n-r+1} (-1)^j d_i^{\mathbf{k}_j}\cdot e_{k_j}.$$

It is easy to see that the image of the composite map,

$$R^{n-r+1}\otimes_R \bigwedge^{n-r+1} R^n \xrightarrow{\delta} R^n \xrightarrow{\phi} R^{n-r+1},$$

is exactly $I_{n-r+1}(\phi)\cdot R^{n-r+1}$.

On the other hand, since $I_{n-r+1}(\phi)$ has generic grade in R, it follows that an R-resolution of $M=\operatorname{Cok}\phi$ is given by the Buchsbaum–Rim complex [6, 2.4, p. 207],

$$\dots \longrightarrow \bigwedge^{n-r+2} R^n \xrightarrow{\beta} R^n \xrightarrow{\phi} R^{n-r+1} \longrightarrow M \longrightarrow 0$$

where $\bar{\beta}=0$. Since ϕ maps $\operatorname{Im}\delta$ onto $I_{n-r+1}(\phi)\cdot R^{n-r+1}$, therefore the preimage $\phi^{-1}(I_{n-r+1}(\phi)\cdot R^{n-r+1})$ is exactly $\operatorname{Im}\delta+\operatorname{Im}\beta$ and so the sequence

$$\bigwedge^{n-r+2} \bar{R}^n \oplus \left(\bar{R}^{n-r+1}\otimes_R \bigwedge^{n-r+1} \bar{R}^n\right) \xrightarrow{\bar{\beta}\oplus\bar{\delta}} \bar{R}^n \xrightarrow{\bar{\phi}} \bar{R}^{n-r+1} \longrightarrow M\otimes_R\bar{R} \longrightarrow 0$$

is exact. Thus the bottom row of the diagram is exact, since $\bar{\beta}=0$ and $M\otimes_R\bar{R}=M$.

It is also easy to see that the image of the composite map

$$R^{n-r+1}\otimes_R \bigwedge^{n-r+1} R^n \xrightarrow{\delta} R^n \xrightarrow{\psi} R^n$$

is contained in $I_{n-r+1}(\psi)\cdot R^n$, which is equal to $I_{n-r+1}(\phi)\cdot R^n$. Therefore, the top row of the diagram is a complex. This complex is exact, because every relation on the columns of $\bar{\psi}$ is a relation on the columns of $\bar{\phi}$ and hence contained in the image of $\bar{\delta}$. The proof is now complete.

Corollary 3.7. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes, and r an integer, $r \geq 1$. Assume that f is locally of flat dimension 1, and, if $r \geq 3$, then assume that f is curvilinear. Assume also that Y satisfies (S_{r+1}), that each component of N_r has codimension r, and that each component of N_{r+1} has codimension $r+1$. Then N_r is the scheme-theoretic image of M_r.*

Proof. Since $M_r = f^{-1}N_r$ and f is finite, $f_*\mathcal{O}_{M_r}$ is equal to the restriction of $f_*\mathcal{O}_X$ to N_r. Hence, $f_*\mathcal{O}_{M_r}$ is locally free of rank r on $N_r - N_{r+1}$ by standard linear algebra. So the comorphism $\gamma: \mathcal{O}_{N_r} \to f_*\mathcal{O}_{M_r}$ is injective off N_{r+1}. Now, N_r is perfect by Proposition 2.2 if $r=1$; by Proposition 3.2(2), Corollary 2.5(1), and Proposition 3.4 if $r=2$; and by Theorem 3.5 if $r \geq 3$. Hence $\mathcal{O}_{N_r}$ has no embedded points because Y satisfies (S_{r+1}). Therefore, γ is injective everywhere because each component of N_{r+1} has codimension $r+1$. The proof is now complete.

Definition 3.8. Let $f: X \to Y$ be a finite map of locally Noetherian schemes. Following [19, 4.1, pp. 36–37], [20, (2.10), pp. 112–113], and [21, (3.1)], define the *iteration scheme* X_2 and the *iteration map* $f_1: X_2 \to X$ of f as follows:

$$X_2 := \mathbf{P}(\mathcal{I}(\Delta)) \quad \text{and} \quad f_1: X_2 \xrightarrow{p} X \times_Y X \xrightarrow{p_2} X,$$

where Δ is the diagonal, $\mathcal{I}(\Delta)$ is its ideal, p is the structure map, and p_2 is the second projection. (Thus, X_2 is the residual scheme of Δ.)

Lemma 3.9. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes, and assume that f is curvilinear. Then, for any $r \geq 2$,*

$$M_r(f) = N_{r-1}(f_1).$$

Proof. The structure map $p: X_2 \to X \times_Y X$ is a closed embedding if and only if f is curvilinear. If so, then $p_*\mathcal{O}_{X_2}$ is locally isomorphic to $\mathcal{I}(\Delta)$, and therefore, for any $r \geq 0$,

$$\mathcal{F}itt_r^X(f_{1*}\mathcal{O}_{X_2}) = \mathcal{F}itt_r^X(p_{2*}\mathcal{I}(\Delta)). \tag{3.9.1}$$

These statements are not hard to prove; see [21, (3.2), (3.4)]. In that reference, (3.9.1) is stated only for $r=0$, but the proof works without change for any r.

Since f is an affine map, the operator f_* is exact and commutes with base change. Hence, applying p_{2*} to the natural exact sequence,

$$0 \longrightarrow \mathcal{I}(\Delta) \longrightarrow \mathcal{O}_{X \times X} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0,$$

yields an exact sequence,

$$0 \longrightarrow p_{2*}\mathcal{I}(\Delta) \longrightarrow f^* f_* \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Hence, by standard properties of Fitting ideals,

$$\mathcal{F}itt_r^X(p_{2*}\mathcal{I}(\Delta)) = \mathcal{F}itt_{r+1}^X(f^* f_* \mathcal{O}_X) = \mathcal{F}itt_{r+1}^Y(f_* \mathcal{O}_X)\, \mathcal{O}_X.$$

Therefore, (3.9.1) yields the assertion.

Lemma 3.10. *Let $f: X \to Y$ be a finite and curvilinear map of locally Noetherian schemes.*

(1) *Then $f_1: X_2 \to X$ is finite and curvilinear.*

(2) *Assume that X has no embedded components. Assume either* (i) *that each component of N_2 has codimension at least* 2 *in Y, or* (ii) *that each component of M_2 has codimension at least* 1 *in X, or* (iii) *that f is birational onto its image. Finally, assume that f is a local complete intersection and is locally of codimension* 1. *Then f_1 is also a local complete intersection and locally of codimension* 1.

Proof. Assertion (1) holds because (a) the map $p: X_2 \to X \times_Y X$ is a closed embedding; (b) the projection $p_2: X \times_Y X \to X$ is finite; and (c) $\Omega^1_{p_2} = p_2^* \Omega^1_f$.

Consider (2). Conditions (i) and (ii) are equivalent because f is locally of codimension 1. Conditions (ii) and (iii) are equivalent by Proposition 3.2(1); in particular, (ii) obtains. By (1), $f_1: X_2 \to X$ is finite, and by Proposition 3.2(2), it factors through $N_1(f_1)$. By Lemma 3.9, $N_1(f_1) = M_2$. So, for any $x \in X_2$,

$$\dim \mathcal{O}_{X_2,x} \leq \dim \mathcal{O}_{N_1(f_1), f_1 x} = \dim \mathcal{O}_{M_2, f_1 x} \leq \dim \mathcal{O}_{X, f_1 x} - 1.$$

Since X has no embedded components, and since f is a local complete intersection and locally of codimension 1, it follows that f_1 is also. Indeed, it is not hard to show, see [19, 4.3, p. 39], that X_2 is, locally at any point x, cut out of some smooth X-scheme P, say of relative dimension p, by $p+1$ elements. Since f_1 is finite and locally of codimension 1, a subset of p of the elements must restrict to a system of parameters in the fiber of P through x. Since the fiber is smooth, this system is a regular sequence. Hence, by the local criterion of flatness, the p elements themselves form a regular system and they cut out of P a flat X-scheme Q. Since X has no embedded components, neither does Q because Q is X-flat. Since the remaining element cuts X_2 out of Q, it is regular on Q. Thus the $p+1$ elements form a regular sequence on P. Thus (2) holds.

Theorem 3.11. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes, and r an integer, $r \geq 2$. Assume that f is a local complete intersection, is locally of codimension* 1, *and is curvilinear. Assume also that either* (i) *each component of N_2 has codimension at least* 2 *in Y, or* (ii) *each component of M_2 has codimension at least* 1 *in X, or* (iii) *f is birational onto its image.*

If X has no embedded components, then each component of M_r has codimension at most $r-1$ in X. Furthermore, if each component of M_r has codimension $r-1$ and if Y satisfies Serre's condition (S_r), *then M_r is a perfect subscheme of X.*

Proof. If Y satisfies (S_r), then X satisfies (S_{r-1}) because f is locally of codimension 1, and because, for every x in X, clearly

$$\operatorname{depth} \mathcal{O}_{X,x} = \operatorname{depth} \mathcal{O}_{Y,fx} - 1$$

since f is also a local complete intersection. Hence, in any event, X has no embedded components. Therefore, $f_1: X_2 \to X$ is a local complete intersection and locally of codimension 1 by Lemma 3.10(2). Moreover, f_1 is finite and curvilinear by Lemma 3.10(1). Hence f_1 is locally of flat dimension 1 by Proposition 2.10. Therefore, Lemma 3.9 and Theorem 3.5 yield the assertions.

Proposition 3.12. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes. Assume that f is curvilinear and that Y satisfies* (S_2). *Then the following conditions are equivalent:*

(i) *f is a local complete intersection, is locally of codimension* 1, *and is birational onto its image;*

(ii) *f and f_1 are both locally of flat dimension* 1;

(iii) *f is locally of flat dimension* 1 *and M_2 is a divisor;*

(iv) *f is locally of flat dimension* 1, *is birational onto its image, and is Gorenstein.*

Moreover, in (i) *or* (iv) *or both, the condition that f is birational onto its image may be replaced either by the condition that M_2 is nowhere topologically dense in X or by the condition that N_2 is nowhere topologically dense in N_1.*

Proof. In the course of proving Theorem 3.11, it was shown that (i) implies (ii). Assume (ii). Then $N_1(f_1)$ is a divisor by Proposition 2.2, and $N_1(f_1) = M_2$ by Lemma 3.9. Thus (iii) holds. Next, assume (iii). Then f is birational onto its image by Proposition 3.2(2), and it is locally of codimension 1 by Corollary 2.5(1). Moreover, the ideal of M_2 is equal to the conductor $\mathcal{C}_X$ by Proposition 3.4; so $\mathcal{C}_X$ is invertible. Hence f is Gorenstein by [21, (2.3)]. Thus (iv) holds. Now, (iv) implies (i) by Proposition 2.10. Finally, the last assertion holds by Proposition 3.2(2).

4. The Hilbert scheme

Proposition 4.1. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes, and assume that f is curvilinear. Let $r \geq 2$. Then the universal subscheme* Univ_f^r *of* $\mathrm{Hilb}_f^r \times_Y X$ *is equal to the Hilbert scheme* $\mathrm{Hilb}_{f_1}^{r-1}$ *of the iteration map* $f_1: X_2 \to X$ *defined in Definition* 3.8,

$$\mathrm{Univ}_f^r = \mathrm{Hilb}_{f_1}^{r-1}.$$

Proof. For convenience, set $\mathrm{U}_f^r := \mathrm{Univ}_f^r$ and $\mathrm{H}_f^r := \mathrm{Hilb}_f^r$. It will be shown that both U_f^r and $\mathrm{H}_{f_1}^{r-1}$ have canonical closed embeddings in $\mathrm{H}_f^{r-1} \times_Y X$ and then that the two subschemes are equal. First of all, the structure map $p: X_2 \to X \times_Y X$ is a closed embedding because f is curvilinear. Hence, there is a canonical embedding of $\mathrm{H}_{f_1}^{r-1}$ in $\mathrm{H}_{p_2}^{r-1}$, which is equal to $\mathrm{H}_f^{r-1} \times_Y X$.

Secondly, there is a canonical map $v: V \to \mathrm{U}_f^r$ where V is the residual scheme of U_f^{r-1} in $\mathrm{H}_f^{r-1} \times_Y X$ by [20, (2.9)(1), p. 111]. The map v is an isomorphism by [20, (2.9)(4), p. 111]; indeed, every length-r subscheme z of every fiber $f^{-1}(y)$ is Gorenstein, because $f^{-1}(y)$ is isomorphic, locally at each of its points, to a closed subscheme of the affine line over the field $k(y)$ by Proposition 2.7(2) as f is curvilinear. Moreover, the structure map $V \to \mathrm{H}_f^{r-1} \times_Y X$ is a closed embedding because the ideal of U_f^{r-1} is locally generated by a single element. Indeed, the formation of this ideal commutes with base change through H_f^{r-1} because U_f^{r-1} is flat, and on each fiber of $\mathrm{H}_f^{r-1} \times X$, the ideal is generated by a single element; the latter obtains because the fiber comes via base field extension from a fiber $f^{-1}(y)$, and as noted above, $f^{-1}(y)$ is isomorphic to a closed subscheme of the affine line over the field $k(y)$.

Consider the rth iteration scheme X_r and the corresponding iteration map $f_{r-1}: X_r \to X_{r-1}$ of f. For $r=2$, they are simply the iteration scheme X_2 and the iteration map $f_1: X_2 \to X$, and, for $r \geq 3$, they are defined recursively as the iteration scheme and iteration map of f_{r-2}; see [19, 4.1, pp. 36–37] or [20, (4.4), p. 120]. Since f is curvilinear, there is a canonical finite, flat, and surjective map $u: X_r \to \mathrm{U}_f^r$ by [20, (5.10)(i), p. 128]. Then $u_* \mathcal{O}_{X_r}$ is a locally free $\mathcal{O}_{\mathrm{U}_f^r}$-module, so the comorphism $\mathcal{O}_{\mathrm{U}_f^r} \to u_* \mathcal{O}_{X_r}$ is injective. Hence, U_f^r is equal to the scheme-theoretic image of X_r in $\mathrm{H}_f^{r-1} \times_Y X$.

It is clear from the definition of X_r that it is equal to the $(r-1)$st iteration scheme of f_1. Hence, there is a canonical finite, flat, and surjective map $X_r \to \mathrm{H}_{f_1}^{r-1}$ by [20, (5.10)(i), p. 128]. This map yields a second map from X_r to $\mathrm{H}_f^{r-1} \times_Y X$, and its scheme-theoretic image is equal to $\mathrm{H}_{f_1}^{r-1}$. It may be checked using the universal property of the Hilbert scheme that the two maps from X_r to $\mathrm{H}_f^{r-1} \times_Y X$ are equal. Therefore, U_f^r and $\mathrm{H}_{f_1}^{r-1}$ are equal too.

Theorem 4.2. *Let $f: X \to Y$ be a finite map of locally Noetherian schemes, and let $r \geq 1$. Assume that $f: X \to Y$ is a local complete intersection, locally of codimension 1, and curvilinear. Assume that Y satisfies Serre's condition (S_{r+1}). Finally, assume that each component of N_s has codimension s for $s=1, \dots, r+1$. Let $h: \mathrm{Hilb}_f^r \to Y$ be the structure map. Then h is finite, locally of flat dimension r, locally of codimension r, and Gorenstein. Moreover, $h^{-1}N_{r+1}$ is a divisor, and*

$$h_*[h^{-1}N_{r+1}] = (r+1)[N_{r+1}].$$

Similar assertions hold for the structure map $h_1: \mathrm{Univ}_f^r \to X$ too.

Proof. First of all, h has finite fibers because f does. Hence, h is finite because it is proper. Now, f is locally of flat dimension 1 by Proposition 2.10. Hence, by Theorem 3.5,

$$f_*[M_{r+1}] = (r+1)[N_{r+1}]. \tag{4.2.1}$$

The proof proceeds by induction on r. Suppose $r=1$. Then h is equal to f, and the asserted equation becomes (4.2.1). Now, N_2 is nowhere topologically dense in N_1; hence, f is locally of flat dimension 1, locally of codimension 1, and Gorenstein, and M_2 is a divisor by Proposition 3.12. However, $M_2 = f^{-1}N_2$ essentially by definition. Thus the assertions about h hold. Furthermore, Univ_f^1 is equal to the diagonal subscheme of $X \times_Y X$. Hence the assertions about h_1 hold too when $r=1$.

Suppose $r \geq 2$. Consider the map $f_1: X_2 \to X$ and the diagram

$$\begin{array}{ccccc} X & \xleftarrow{h_1} & \mathrm{Univ}_f^r & = & \mathrm{Hilb}_{f_1}^{r-1} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} & & \\ Y & \xleftarrow{h} & \mathrm{Hilb}_f^r & & \end{array}$$

in which h_1 and u are the natural maps and the equality is that of Proposition 4.1. Since f is a local complete intersection and is locally of codimension 1 and since Y satisfies (S_{r+1}), clearly X satisfies (S_r). In particular, X has no embedded components. So f_1 is finite, curvilinear, a local complete intersection, and locally of codimension 1 by Lemma 3.10. Now, $N_s(f_1) = M_{s+1}$ for $s \geq 1$ by Lemma 3.9, and f induces a finite, surjective map $M_{s+1} \to N_{s+1}$ by Corollary 3.7; hence, $N_s(f_1)$ is of pure codimension s for $s=1, \dots, r$.

The induction hypothesis therefore applies to f_1. Hence, h_1 is locally of flat dimension $r-1$, locally of codimension $r-1$, and Gorenstein; moreover, $h_1^{-1}M_{r+1}$ is a divisor, and

$$h_{1*}[h_1^{-1}M_{r+1}] = r[M_{r+1}].$$

Since f is locally of flat dimension 1, locally of codimension 1, and Gorenstein, therefore fh_1 is locally of flat dimension r, locally of codimension r, and Gorenstein. (With the residue field of an arbitrary point in the image of fh_1 as first argument, the "change of rings" spectral sequence for 'Tor' shows that fh_1 is locally of flat dimension at least r.)

Since fh_1 is locally of flat dimension r and locally of codimension r, so is h because $fh_1=hu$ and because u is flat and finite. Also because u is finite, the dualizing complexes of hu and h are related by the formula,

$$u_*\omega_{hu} = \mathcal{H}om(u_*\mathcal{O}_{\mathrm{Univ}_f^r}, \omega_h).$$

Since $u_*\mathcal{O}_{\mathrm{Univ}_f^r}$ is locally free and since hu is Gorenstein (being equal to fh_1), it follows that h is Gorenstein. Finally, since $h_1^{-1}M_{r+1}$ is a divisor, so is $h^{-1}N_{r+1}$ because $fh_1=hu$ and because u is flat and finite. Since u is of degree r,

$$u_*[h_1^{-1}M_{r+1}] = r[h^{-1}N_{r+1}].$$

Since $f_*h_{1*}=h_*u_*$, therefore

$$h_*[h^{-1}N_{r+1}] = f_*[M_{r+1}].$$

Consequently, the asserted equation follows from (4.2.1). Thus, the theorem is proved.

Theorem 4.3. *Under the conditions of Theorem* 4.2, *the Hilbert scheme* Hilb_f^r *is the blowup* $\mathrm{Bl}(N_r, N_{r+1})$, *and the universal subscheme* Univ_f^r *of* $\mathrm{Hilb}_f^r \times_Y X$ *is the blowup* $\mathrm{Bl}(M_r, M_{r+1})$; *that is,*

$$\mathrm{Hilb}_f^r = \mathrm{Bl}(N_r, N_{r+1}) \quad \textit{and} \quad \mathrm{Univ}_f^r = \mathrm{Bl}(M_r, M_{r+1}).$$

Proof. First of all, the structure map $h\colon \mathrm{Hilb}_f^r \to Y$ factors through a map

$$\beta\colon \mathrm{Hilb}_f^r \longrightarrow \mathrm{Bl}(N_r, N_{r+1}),$$

which restricts to an isomorphism off $h^{-1}N_{r+1}$. Indeed, a map $g\colon G\to Y$ factors through $N_r - N_{r+1}$ if and only if $g^*f_*\mathcal{O}_X$ is locally free of rank r by [30, (∗), p. 56]. Hence, h induces an isomorphism,

$$(\mathrm{Hilb}_f^r - h^{-1}N_{r+1}) \xrightarrow{\sim} (N_r - N_{r+1}).$$

Therefore, the ideal of $h^{-1}N_r$ in Hilb_f^r vanishes off $h^{-1}N_{r+1}$. So the ideal vanishes everywhere because $h^{-1}N_{r+1}$ is a divisor by Theorem 4.2. Consequently, h factors through N_r. Therefore, since $h^{-1}N_{r+1}$ is a divisor, the universal property of the blowup implies that h factors through a map β, as claimed.

For convenience, set $B{:=}\mathrm{Bl}(N_r, N_{r+1})$, denote the exceptional divisor by E, and set $U{:=}B-E$. To construct an inverse γ to β, it suffices to construct a length-r subscheme Z of $X\times B/B$ whose restriction over U is equal to $X\times U$. Indeed, such a Z defines a map $\gamma\colon B\to\mathrm{Hilb}^r_f$ such that $\beta\gamma$ is equal to the identity off E and $\gamma\beta$ is equal to the identity off $h^{-1}N_{r+1}$. Since E and $h^{-1}N_{r+1}$ are both divisors and since B and Hilb^r_f are both separated over N_r, each composition is equal to the identity everywhere. (Indeed, each is equal to the identity on a closed subscheme of the source because its target is separated; this subscheme is equal to the source because it contains an open subscheme that includes every associated point of the source.)

Let $\iota\colon U\to B$ denote the inclusion, let f_B and f_U denote the base extensions of f, and let $\mathcal{E}$ denote the image of $f_{B*}\mathcal{O}_{X\times B}$ in $\iota_*(f_{U*}\mathcal{O}_{X\times U})$. Since $\mathcal{E}$ is the image of an $f_{B*}\mathcal{O}_{X\times B}$-map, $\mathcal{E}$ is an $f_{B*}\mathcal{O}_{X\times B}$-module. Hence $\mathcal{E}$ is equal to the direct image of the structure sheaf of a subscheme Z of $X\times B/B$. This Z has the desired properties, because $\mathcal{E}$ is locally free of rank r, as will now be proved.

The question is local on B. Now, each point of B has a neighborhood V on which $f_{B*}\mathcal{O}_{X\times B}$ has a free quotient $\mathcal{F}$ of rank r by Lemma 4.7(3) applied to any matrix $\mathbf{X}$ presenting $f_{B*}\mathcal{O}_{X\times B}$ over the local ring R of the point and applied with any minor generating the $(r+1)$st Fitting ideal as Δ_{i}. On $U\cap V$, the canonical surjection from $f_{B*}\mathcal{O}_{X\times B}|_V$ to $\mathcal{F}$ is an isomorphism because the source is locally free of rank r. Hence there is an induced map $u\colon\mathcal{F}\to\mathcal{E}|_V$, which is an isomorphism on $U\cap V$. Since E is a divisor, $\mathcal{F}$ has no associated point off U. Hence, u is injective on all of V. On the other hand, u is surjective because $\mathcal{E}$ is a quotient of $f_{B*}\mathcal{O}_{X\times B}$. Thus $\mathcal{E}$ is locally free, and the first assertion is proved.

The second assertion follows from the first applied to $f_1\colon X_2\to X$ because of Proposition 4.1 and because f_1 satisfies the corresponding hypotheses; the claim about f_1 was established in the proof of Theorem 4.2.

Theorem 4.4. *Under the conditions of Theorem* 4.2, *the map* $h\colon\mathrm{Hilb}^r_f\to N_r$ *is finite and birational, its conductor is equal to the ideal* $\mathcal{J}_r$ *of* N_{r+1} *in* N_r, *and reciprocally,* $h_*\mathcal{O}_{\mathrm{Hilb}^r_f}$ *is equal to* $\mathcal{H}om(\mathcal{J}_r,\mathcal{O}_{N_r})$. *Moreover,* $\mathcal{J}_r$ *is locally a self-linked ideal of* $\mathcal{O}_{N_r}$; *in fact, locally there exist sections* t *of* $\mathcal{J}_r$ *such that* $\mathcal{J}_r\mathcal{O}_{\mathrm{Hilb}^r_f}$ *is equal to* $t\mathcal{O}_{\mathrm{Hilb}^r_f}$, *and* $\mathcal{J}_r=(t\mathcal{O}_{N_r})\colon\mathcal{J}_r$ *for any such* t. *Furthermore, if* $r\geq 2$, *then similar assertions hold for the structure map* $h_1\colon\mathrm{Univ}^r_f\to M_r$.

Proof. First of all, the assertions about h_1 follow formally from those about h; see the third paragraph of the proof of Theorem 4.2. Now, h is finite and birational by Theorems 4.2 and 4.3, and $h^{-1}\mathcal{J}_r$ is invertible by Theorem 4.2. Hence, locally,

$h^{-1}\mathcal{J}_r$ is generated by a single section of $\mathcal{J}_r$. The remaining three assertions are local on Y; so we may assume that Y is the spectrum of a local ring R.

By using the standard device of making a suitable (faithfully) flat change of base, we may assume that R has an infinite residue class field; namely, we may, clearly, replace R by the flat local R-algebra obtained by forming the polynomial ring in one variable over R and localizing it at the extension of the maximal ideal of R. Then $f_*\mathcal{O}_X$ can be presented by a square matrix $\mathbf{X}$ that satisfies the hypotheses of Theorem 5.9 below; indeed, the condition on grade $I_i(\mathbf{X})$ follows from the hypotheses, and the condition $I_i(\mathbf{X})=I_i(\mathbf{X}_i)$ follows by the reasoning in the first two paragraphs of the proof of Lemma 3.6. Hence Theorem 5.9 implies that, in the local ring A of N_r, there are an A-regular element Δ and an ideal I containing Δ such that $IJ=\Delta J$ and $J=(\Delta){:}I$ where J is the ideal in A of N_{r+1}. Hence, Lemma 4.5 will yield the remaining three assertions after we prove that the Hilbert scheme Hilb_f^r and the two blowups $\mathrm{Bl}(I)$ and $\mathrm{Bl}(J)$ are all equal.

The isomorphism γ in the proof of Theorem 4.3 clearly factors as follows:

$$\gamma\colon \mathrm{Bl}(J) \xrightarrow{\eta} \mathrm{Bl}(I) \xrightarrow{\theta} \mathrm{Hilb}_f^r$$

where η is the map given by Lemma 4.5(5) and θ is given by a construction similar to that of γ, but based on the fact that I is generated by elements of the form given in Theorem 5.9. The composition $\eta\gamma^{-1}\theta$ is equal to the identity off the exceptional divisor of $\mathrm{Bl}(I)$; so it is equal to the identity everywhere, because $\mathrm{Bl}(I)$ is separated over N_r. Therefore, the maps η and θ are isomorphisms, and the proof is complete.

Lemma 4.5. *Let A be a ring, Δ an A-regular element, I an ideal containing Δ, and J an ideal containing I. Let K be the total quotient ring of A. Set $B:=A[I/\Delta]$ and $C:=\{x\in K \mid xB\subset A\}$.*

(1) *If $J=(\Delta){:}I$, then $C\subset J$.*

(2) *If $IJ=\Delta J$, then $JB=J$ and $J\subset C$.*

(3) *If $IJ=\Delta J$ and if JB is invertible, then $B=\{x\in K \mid xJ\subset J\}$. If in addition $J=(\Delta){:}I$, then $B=\{x\in K \mid xJ\subset A\}$.*

(4) *If $IJ=\Delta J$ and if J is finitely generated, then* $\mathrm{Spec}(B)=\mathrm{Bl}(I)$.

(5) *If $IJ=\Delta J$, then there is an A-map $\eta\colon \mathrm{Bl}(J)\to\mathrm{Bl}(I)$.*

(6) *If $J=C$ and if $J=tB$ for some t, then t is an A-regular element of A, and $J=tA{:}J$.*

Proof. (1) Let $x\in C$. Then $x=x\cdot 1$, so $x\in A$. Moreover, $x(I/\Delta)\subset A$, so $xI\subset\Delta A$. Hence $x\in(\Delta){:}I$, but $(\Delta){:}I=J$.

(2) By hypothesis, $IJ=\Delta J$. So $J(I/\Delta)=J$. Hence, $J(I/\Delta)^n=J$ for any $n\geq 1$. Therefore, $JB=J$. Consequently, $J\subset C$.

(3) Let $x \in K$, and suppose $xJ \subset J$. Then $xJB \subset JB$. Hence $x \in B$ because JB is invertible. Conversely, if $x \in B$, then $xJ \subset J$ by (2).

Let $y \in K$, and suppose $yJ \subset A$. Then $yJB \subset A$ by (2). So $yJ \subset C$ by definition of C. If in addition $J=(\Delta):I$, then $yJ \subset J$ by (1), and so $y \in B$ by the preceding paragraph.

(4) First, consider any local A-algebra D such that ID is invertible. Say $ID = dD$ and $\Delta = ed$. Then $IJD = \Delta JD$. So $JD = eJD$ because d is regular on D. Hence e is a unit by Nakayama's lemma because J is finitely generated. Hence $ID = \Delta D$. Therefore, the map $A \to D$ factors through B.

Obviously, $\mathrm{Spec}(B)$ is a principal open subscheme of $\mathrm{Bl}(I)$. Let $x \in \mathrm{Bl}(I)$ and set $D := \mathcal{O}_x$. By the preceding observation, there is an A-map from $\mathrm{Spec}(D)$ to $\mathrm{Spec}(B)$. This map agrees with the canonical map of $\mathrm{Spec}(D)$ into $\mathrm{Bl}(I)$ because $\mathrm{Bl}(I)$ is separated and the two maps agree off the closed subscheme $V(ID)$, which is a divisor. Hence, $x \in \mathrm{Spec}(B)$.

(5) Since $IJ = \Delta J$ and since $J\mathcal{O}_{\mathrm{Bl}(J)}$ is invertible, $I\mathcal{O}_{\mathrm{Bl}(J)}$ is generated by Δ. Moreover, Δ is regular on $\mathcal{O}_{\mathrm{Bl}(J)}$ because it is regular on the complement of the exceptional divisor. Thus $I\mathcal{O}_{\mathrm{Bl}(J)}$ is invertible. Hence the asserted map η exists.

(6) Since $t \in J$, also $t \in A$. Since $\Delta = bt$ for some $b \in B$ and since Δ is A-regular, so is t. Now, $J^2 = Jt$ because $J = tB$; hence, $J \subseteq tA{:}J$. Finally, suppose $x \in tA{:}J$. Then $xtB \subset At$. Hence $x \in C$, but $C = J$. Thus $J \supseteq tA{:}J$, and the proof is complete.

Lemma 4.6. *Let R be a ring, and $\mathbf{X}$ an m by n matrix. Fix $p \geq 1$, and set $A := R/I_{p+1}(\mathbf{X})$ and $J := I_p(\mathbf{X})A$ where $I_q(\mathbf{X})$ denotes the ideal of q by q minors. Denote the image in J of the minor of $\mathbf{X}$ formed using rows $i_1, \dots, i_p$ and columns $k_1, \dots, k_p$ by $d_{\mathbf{i}}^{\mathbf{k}}$.*

(1) *Let $\mathbf{R}_i$ denote the ith row of $\mathbf{X}$. Then, for any $\mathbf{i}$ and $\mathbf{k}$,*

$$d_{\mathbf{i}}^{\mathbf{k}} \mathbf{R}_i = \sum_{j=1}^{p} (-1)^{j+p} d_{\mathbf{i}i_j}^{\mathbf{k}} \mathbf{R}_{i_j}$$

where $\mathbf{i}i_j$ is the sequence $i_1, \dots, i_p, i$ without its jth element.

(2) (Sylvester's relation) *Then $d_{\mathbf{i}}^{\mathbf{k}} d_{\mathbf{j}}^{\mathbf{l}} = d_{\mathbf{j}}^{\mathbf{k}} d_{\mathbf{i}}^{\mathbf{l}}$ for any $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and $\mathbf{l}$.*

Proof. We may assume that $\mathbf{X}$ is a matrix of indeterminates and that R is obtained by adjoining them to the integers. Then A is a domain.

To prove (1), form a $p+1$ by n matrix $\mathbf{Y}$ using rows $\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_p}$, $\mathbf{R}_i$. For each k, form a $p+1$ by $p+1$ matrix $\mathbf{Y}^{(k)}$ by taking out of $\mathbf{Y}$ columns $k_1, \dots, k_p$ and column k. Finally, expand the determinant of $\mathbf{Y}^{(k)}$ along the last column to get the asserted equation.

To prove (2), denote the p by n submatrix of $\mathbf{X}$ consisting of rows $i_1, \dots, i_p$ by $\mathbf{X_i}$. Set $d{:=}d^{\mathbf{l}}_{\mathbf{j}}$. Then, (1) implies that there is a p by p matrix $\mathbf{M}$ such that $d\mathbf{X_i}=\mathbf{M}\mathbf{X_j}$; here $\mathbf{M}$ depends on $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{l}$, but not on $\mathbf{k}$. Hence,

$$d^p d^{\mathbf{k}}_{\mathbf{i}} d^{\mathbf{l}}_{\mathbf{j}} = |\mathbf{M}| d^{\mathbf{k}}_{\mathbf{j}} d^{\mathbf{l}}_{\mathbf{j}} = d^p d^{\mathbf{k}}_{\mathbf{j}} d^{\mathbf{l}}_{\mathbf{i}}.$$

Since $d \neq 0$ and A is a domain, the assertion follows.

Lemma 4.7. *Preserve the conditions of Lemma 4.6. Let $\mathbf{k}$ range over all sequences $1 \le k_1 < \dots < k_p \le n$. Given elements $a^{\mathbf{k}}$ of A, set*

$$\Delta_{\mathbf{i}} := \sum_{\mathbf{k}} a^{\mathbf{k}} d^{\mathbf{k}}_{\mathbf{i}}.$$

Let I be the ideal generated by the various $\Delta_{\mathbf{i}}$.

(1) *Let $\mathbf{X}^{\mathbf{l}}$ be the submatrix of $\mathbf{X}$ consisting of columns $l_1, \dots, l_p$. Then $d^{\mathbf{l}}_{\mathbf{j}} I = \Delta_{\mathbf{j}} I_p(\mathbf{X}^{\mathbf{l}})$.*

(2) *Let $\mathbf{X_j}$ be the submatrix of $\mathbf{X}$ consisting of rows $j_1, \dots, j_p$. Then $\Delta_{\mathbf{i}} I_p(\mathbf{X_j}) \subseteq \Delta_{\mathbf{j}} J$. Furthermore, if $J = I_p(\mathbf{X_j})A$, then $IJ = \Delta_{\mathbf{j}} J$.*

(3) *Suppose that $\Delta_{\mathbf{i}}$ is regular on A and generates I. Then every row of $\mathbf{X}$ is a linear combination of rows $i_1, \dots, i_p$ modulo $I_{p+1}(\mathbf{X})$.*

Proof. Sylvester's relation Lemma 4.6(2) yields

$$\Delta_{\mathbf{i}} d^{\mathbf{l}}_{\mathbf{j}} = \Delta_{\mathbf{j}} d^{\mathbf{l}}_{\mathbf{i}}. \tag{4.7.1}$$

Varying $\mathbf{i}$ in (4.7.1) yields (1). On the other hand, varying $\mathbf{l}$ in (4.7.1) yields $\Delta_{\mathbf{i}} I_p(\mathbf{X_j}) \subseteq \Delta_{\mathbf{j}} J$. If $I_p(\mathbf{X_j})A = J$, then $IJ \subseteq \Delta_{\mathbf{j}} J$; hence, $IJ = \Delta_{\mathbf{j}} J$ because $\Delta_{\mathbf{j}} \in I$. Thus (2) holds. Finally, Lemma 4.6(1) yields

$$\Delta_{\mathbf{i}} \mathbf{R}_i = \sum_{j=1}^{p} (-1)^{j+p} \Delta_{\mathbf{i}i_j} \mathbf{R}_{i_j}.$$

By hypothesis, $\Delta_{\mathbf{i}}$ is regular on A and divides each $\Delta_{\mathbf{i}i_j}$. Hence (3) holds.

5. Strongly perfect ideals

Definition 5.1. Let B be a Noetherian ring, A a factor ring of B, and I an A-ideal of grade g such that $\text{grade}_B A/I = s$. Call I *strongly perfect* over B if there exists a generating set $f_1, \dots, f_n$ of I such that, for $0 \le i \le n-g$, the Koszul homology modules $H_i(f_1, \dots, f_n; A)$ are perfect B-modules of grade s.

Remark 5.2. The notion of strong perfection generalizes Huneke's notion of strong Cohen–Macaulayness [16, p. 739]. Indeed, let I be an ideal of a local Cohen–Macaulay ring A, and write $\hat{A}$ as a factor ring of a regular local ring B. Then $\hat{I}$ is strongly perfect over B if and only if I is strongly Cohen–Macaulay.

Some general results about strong perfection will now be proved. The corresponding results about strong Cohen–Macaulayness were proved by Huneke in [15] and [16].

Lemma 5.3. *Let B be a Noetherian ring, A a factor ring of B, and I an A-ideal. Set $s{:=}\mathrm{grade}_B\, A/I$ Let $f_1, \dots, f_n$ be an arbitrary generating set of I.*

(1) *The ideal I is strongly perfect over B if and only if, for every i, the flat dimension over B of $H_i(f_1, \dots, f_n; A)$ is at most s.*

(2) *If I is strongly perfect over B, then the condition in Definition* 5.1 *is satisfied for $f_1, \dots, f_n$.*

Proof. To prove (1), recall that I annihilates $H_i(f_1, \dots, f_n; A)$ for all i and that $H_i(f_1, \dots, f_n; A) \neq 0$ if and only if $0 \leq i \leq n-g$ where $g{:=}\mathrm{grade}\, I$. Hence, I is strongly perfect if the flat dimension of all the Koszul homology is at most s. Moreover, the converse holds if the condition in Definition 5.1 is satisfied for $f_1, \dots, f_n$; so the full converse follows from (2).

To prove (2), it suffices to compare a generating set $f_1, \dots, f_n$ with one of the form $f_1, \dots, f_n, f$. However, there is a natural isomorphism,

$$H_i(f_1, \dots, f_n, f; A) = H_i(f_1, \dots, f_n; A) \oplus H_{i-1}(f_1, \dots, f_n; A),$$

and the assertion follows from the portion of (1) already proved.

Lemma 5.4. *Let B be a Noetherian ring, A a factor ring of B, and I an A-ideal. Let $f_1, \dots, f_n$ be a generating set of I.*

(1) *Let $\Delta_1, \dots, \Delta_m$ be an A-regular sequence contained in I, and let ' $'$ ' indicate the image in $A'{:=}A/(\Delta_1, \dots, \Delta_m)$. Then I' is strongly perfect over B if and only if I is so.*

(2) *Let $a_1, \dots, a_r$ be a sequence of elements in B that is regular on B, on A, and on A/I. Set $\bar{B}{:=}B/(a_1, \dots, a_r)B$ and $\bar{A}{:=}A/(a_1, \dots, a_r)A$. Let '$^-$' indicate the image in $\bar{B}$ and in $\bar{A}$. If I is strongly perfect over B, then there are natural isomorphisms,*

$$H_i(\bar{f}_1, \dots, \bar{f}_n; A) = H_i(f_1, \dots, f_n; A) \otimes_A \bar{A},$$

and $\bar{I}$ is strongly perfect over $\bar{B}$.

Proof. To prove (1), we may assume that $m{=}1$. Since $\Delta_1 H_j(f_1, \dots, f_n; A)$ vanishes, the exact sequence,

$$0 \longrightarrow A \xrightarrow{\Delta_1} A \longrightarrow A' \longrightarrow 0,$$

induces exact sequences

$$0 \longrightarrow H_i(f_1, \dots, f_n; A) \longrightarrow H_i(f_1', \dots, f_n'; A') \longrightarrow H_{i-1}(f_1, \dots, f_n; A) \longrightarrow 0.$$

The assertion now follows by induction on i from Lemma 5.3(1).

To prove (2), we may assume that $r=1$. Set $s:=\operatorname{grade}_B A/I$. Let $\mathbf{p}$ be an associated prime of the B-module $H_i(f_1, \dots, f_n; A)$. Since I is strongly perfect over B, it follows that $\operatorname{depth} B_{\mathbf{p}}=s$; hence, since $\mathbf{p}$ is in the support of A/I, it follows that $\mathbf{p}$ is associated to A/I; for both these conclusions, see [4, (16.17), p. 209] for example. Set $a:=a_1$. Then, therefore, a is regular on $H_i(f_1, \dots, f_n; A)$. Now, the exact sequence,

$$0 \longrightarrow A \xrightarrow{a} A \longrightarrow \bar{A} \longrightarrow 0,$$

induces exact sequences,

$$0 \longrightarrow H_i(f_1, \dots, f_n; A) \xrightarrow{a} H_i(f_1, \dots, f_n; A) \longrightarrow H_i(\bar{f}_1, \dots, \bar{f}_n; \bar{A}) \longrightarrow 0.$$

Hence, they yield the asserted natural isomorphisms. Furthermore, $\bar{I}$ is strongly perfect over $\bar{B}$ because $\operatorname{grade}_{\bar{B}} \bar{A}/\bar{I} \geq s$.

Proposition 5.5. *Let B be a Noetherian ring, A a factor ring of B, and I an A-ideal that is strongly perfect over B. Let $\Delta_1, \dots, \Delta_m$ be an A-regular sequence contained in I, and let $a_1, \dots, a_r$ be a sequence of elements in B that is regular on B, on A, and on A/I. Let '$^-$' denote images in $\bar{A}:=A/(a_1, \dots, a_r)A$, and assume that $\bar{\Delta}_1, \dots, \bar{\Delta}_m$ form an $\bar{A}$-regular sequence. Then, in $\bar{A}$,*

$$\overline{(\Delta_1, \dots, \Delta_m)} : \bar{I} = \overline{(\Delta_1, \dots, \Delta_m) : I}.$$

Proof. It suffices to verify the asserted equality locally at every associated prime ideal of the ideal on the right; so we may assume that all the rings in question are local. Then, since $a_1, \dots, a_r, \Delta_1, \dots, \Delta_m$ form an A-regular sequence, $\Delta_1, \dots, \Delta_m, a_1, \dots, a_r$ do as well, and hence the sequence $a_1, \dots, a_r$ is regular on $A/(\Delta_1, \dots, \Delta_m)$. Hence, using Lemma 5.4(1), we may reduce to the case $m=0$. Now, let $f_1, \dots, f_n$ be a generating set of I with $n \geq 1$. It follows from the definition of the Koszul complex that there are natural identifications,

$$0 : I = H_n(f_1, \dots, f_n; A) \quad \text{and} \quad \bar{0} : \bar{I} = H_n(\bar{f}_1, \dots, \bar{f}_n; \bar{A}).$$

Hence the assertion follows from the first assertion of Lemma 5.4(2).

Lemma 5.6. *Let B be a Noetherian ring, A a factor ring of B, and I an A-ideal that is strongly perfect over B with* $\operatorname{grade}_B A/I=s$. *Set $J=0{:}I$. Assume that $I+J\neq A$, that* $\operatorname{grade}(I+J)\geq 1$, *and that* $\operatorname{grade}_B A/(I+J)\geq s+1$. *Finally, let '‾' denote images in $\bar{A}:=A/J$. Then $\bar{I}$ is a strongly perfect over B with* $\operatorname{grade}_B \bar{A}/\bar{I}=s+1$.

Proof. Obviously, $J\neq 0$. Hence $\operatorname{grade} I=0$ because $J=0{:}I$. Therefore, $I\cap J=0$ because $\operatorname{grade}(I+J)\geq 1$. Let $f_1,\dots,f_n$ be a generating set of I. Then, by [16, 1.4, p. 744], for each i, there is an exact sequence,

$$0\longrightarrow \bigoplus J\longrightarrow H_i(f_1,\dots,f_n;A)\longrightarrow H_i(\bar{f}_1,\dots,\bar{f}_n;\bar{A})\longrightarrow 0,$$

where the first term is a direct sum of copies of J. Now, $J=0{:}I=H_n(f_1,\dots,f_n;A)$. And, the $H_i(f_1,\dots,f_n;A)$ have flat dimension at most s by Lemma 5.3(1). Hence the $H_i(\bar{f}_1,\dots,\bar{f}_n;\bar{A})$ have flat dimension at most $s+1$. Hence Lemma 5.3(1) yields the assertion.

Proposition 5.7. *Let R be a Noetherian ring. Let $\mathbf{X}$ be a $p+1$ by n matrix of variables with $n\geq p+1\geq 2$, let $\mathbf{Y}$ be the $p+1$ by p matrix consisting of the first p columns of $\mathbf{X}$, and set*

$$B:=R[\mathbf{X}],\quad A:=B/I_{p+1}(\mathbf{X}),\quad I:=I_p(\mathbf{Y})A$$

where $I_{p+1}(\mathbf{X})$ and $I_p(\mathbf{Y})$ are the ideals of minors of the indicated sizes. Then I is an A-ideal of grade 1 that is strongly perfect over B.

Proof. Induct on n. Suppose $n=p+1$. Then $I=I_p(\mathbf{Y})/I_{p+1}(\mathbf{X})$ where $I_{p+1}(\mathbf{X})$ is generated by a single B-regular element. On the other hand, Avramov and Herzog [1, (2.1)(a), p. 252] proved that $I_p(\mathbf{Y})$ is a strongly perfect B-ideal of grade 2. Hence, Lemma 5.4(1) implies that I is an A-ideal of grade 1 that is strongly perfect over B.

Suppose $n\geq p+2$ and that the assertion holds for $n-1$. Let $\mathbf{X}'$ be the matrix consisting of the first $n-1$ columns of $\mathbf{X}$, set

$$A':=B/I_{p+1}(\mathbf{X}')B,\quad I':=I_p(\mathbf{Y})A',\quad J':=I_{p+1}(\mathbf{X})A',$$

and let $\Delta'\in A'$ be the image of the $p+1$ by $p+1$ minor of $\mathbf{X}$ made of columns $1,\dots,p,n$. Then I' is an A'-ideal of grade 1 that is strongly perfect over B by induction, because the properties in question are stable under the flat base extension from $R[\mathbf{X}']$ to B. Since, moreover, A' is a perfect B-module of grade $n-p-1$, it follows (from [4, (16.18), p. 209] for example) that

$$s:=\operatorname{grade}_B A'/I'=\operatorname{grade}_B A'+\operatorname{grade}_{A'} I'=n-p.$$

First, we show that Δ' is A'-regular. To this end, let $\mathbf{q}'$ be an associated prime of A', and let $\mathbf{q}$ be the trace of $\mathbf{q}'$ in R. Since A' is R-flat and $A'/\mathbf{q}A'$ is a domain by [4, (2.10), p. 14], it follows that $\mathbf{q}'=\mathbf{q}A'$. Therefore, $\Delta'\notin\mathbf{q}'$.

Next, we verify that $\operatorname{grade}_B A'/(I'+J')\geq s+1$ and $\operatorname{grade} I'+J'\geq 2$. Suppose that the grade of $I_p(\mathbf{Y})+I_{p+1}(\mathbf{X})$ were equal to that of $I_{p+1}(\mathbf{X})$, which is $n-p$. Since the grade of an ideal is the minimum of $\operatorname{depth} B_{\mathbf{q}}$ as $\mathbf{q}$ ranges over all primes containing the ideal, there would be some $\mathbf{q}$ containing $I_p(\mathbf{Y})+I_{p+1}(\mathbf{X})$ with $\operatorname{depth} B_{\mathbf{q}}=n-p$. Since $\mathbf{q}$ also contains $I_{p+1}(\mathbf{X})$, and that ideal is perfect of grade $n-p$, it follows that $\mathbf{q}$ would be an associated prime of $I_{p+1}(\mathbf{X})$. However, an argument like the one above shows that the B-ideal $I_p(\mathbf{Y})$ is not contained in any associated prime of the B-ideal $I_{p+1}(\mathbf{X})$. Thus

$$\operatorname{grade}(I_p(\mathbf{Y})+I_{p+1}(\mathbf{X}))>\operatorname{grade} I_{p+1}(\mathbf{X})=n-p.$$

Therefore, $\operatorname{grade}_B A'/(I'+J')\geq n-p+1=s+1$. Furthermore, since A' is perfect over B of grade $n-p-1$, it follows (from [4, (16.18), p. 209] for example) that

$$\operatorname{grade} I'+J'\geq \operatorname{grade}_B A'/(I'+J')-\operatorname{grade}_B A'\geq 2.$$

We also have $I'J'\subset(\Delta')$; see [16, proof of 4.1, p. 754]. Indeed, let $d_1,\dots,d_{p+1}$ denote the maximal minors of $\mathbf{Y}$, with alternating signs. Then in $A'/(\Delta')$,

$$(d_1,\dots,d_{p+1})\mathbf{X}=0,$$

and hence $I_{p+1}(\mathbf{X})$ annihilates each of the d_i in $A'/(\Delta')$. Therefore, $J'\subseteq(\Delta'){:}I'$, and equality will hold if it holds locally at every associated prime $\mathbf{p}$ of the B-module A'/J'. However, since A'/J' is a perfect B-module, $\operatorname{depth} B_{\mathbf{p}}$ is equal to $\operatorname{grade} A'/J'$ (by [4, (16.17), p. 209] for example). Hence $I'_{\mathbf{p}}=A'_{\mathbf{p}}$ because

$$\operatorname{grade} A'/J'<\operatorname{grade} A'/(I'+J').$$

Therefore, $J'=(\Delta'){:}I'$ holds locally at $\mathbf{p}$, so globally.

Note that $I=(I'+J')/J'\subset A=A'/J'$. Factoring out (Δ') and using Lemmas 5.4(1) and 5.6, we now conclude that I is strongly perfect over B with

$$\operatorname{grade}_B A/I=s+1=n-p+1.$$

But then, by [4, (16.18), p. 209] for example,

$$\operatorname{grade} I=\operatorname{grade}_B A/I-\operatorname{grade}_B A=1.$$

Lemma 5.8. *Let B be a Noetherian ring, A a factor ring of B, and I an A-ideal of grade 1 that is strongly perfect over B. Assume that A is perfect over B, and let J be a proper A-ideal such that $J \cong I$. Then J is an A-ideal of grade 1 that is strongly perfect over B.*

Proof. Set $s := \operatorname{grade}_B A/I$. Since A and A/I are perfect B-modules with $\operatorname{grade} I = 1$, it follows (from [4, (16.18), p. 209] for example) that $\operatorname{grade}_B A = s-1$; hence $\operatorname{grade}_B A/J \geq s$. Furthermore, $aI = bJ$ for some non-zero divisors a and b in A. Say $I = (f_1, \dots, f_n)$ and $J = (h_1, \dots, h_n)$ with $af_j = bh_j$.

Let B_i and Z_i denote the modules of boundaries and cycles in the Koszul complex $(K_\cdot, \partial_\cdot)$. For every i, there is a commutative diagram

$$\begin{array}{ccc} K_i(f_1, \dots, f_n; A) & \xrightarrow{\partial_i(f_1, \dots, f_n)} & K_{i-1}(f_1, \dots, f_n; A) \\ \| & & \downarrow \mu_a \\ K_i(af_1, \dots, af_n; A) & \xrightarrow{\partial_i(af_1, \dots, af_n)} & K_{i-1}(af_1, \dots, af_n; A) \end{array}$$

where μ_a denotes multiplication by a. Since μ_a is injective, this diagram yields an identification,

$$Z_i(f_1, \dots, f_n; A) = Z_i(af_1, \dots, af_n; A).$$

Hence, the isomorphism theorem yields a natural isomorphism,

$$B_{i-1}(f_1, \dots, f_n; A) = B_{i-1}(af_1, \dots, af_n; A).$$

Thus there are natural isomorphisms (compare [15, 1.10 pf., p. 1050]):

$$\begin{aligned} Z_i(f_1, \dots, f_n; A) &= Z_i(af_1, \dots, af_n; A) \\ &= Z_i(bh_1, \dots, bh_n; A) = Z_i(h_1, \dots, h_n; A). \end{aligned}$$

Likewise, $B_i(f_1, \dots, f_n; A) = B_i(h_1, \dots, h_n; A)$.

The B-module A has flat dimension $s-1$ because it is perfect of grade $s-1$. The B-module $H_i(f_1, \dots, f_n; A)$ has flat dimension at most s by Lemma 5.3(1). It follows by induction on i that $Z_i(f_1, \dots, f_n; A)$ and $B_i(f_1, \dots, f_n; A)$ have flat dimension at most $s-1$ because their quotient is $H_i(f_1, \dots, f_n; A)$ and because $Z_i(f_1, \dots, f_n; A)$ is a first syzygy module of $B_{i-1}(f_1, \dots, f_n; A)$. Therefore, the above isomorphisms yield that $H_i(h_1, \dots, h_n; A)$ has flat dimension at most s. But $s \leq \operatorname{grade}_B A/J$. Hence J is strongly perfect over B by Lemma 5.3(1), and the proof is complete.

Theorem 5.9. *Let R be a Noetherian ring. Let $\mathbf{X}$ be an m by n matrix with $n\geq m\geq 2$ and with entries in R. For $1\leq i\leq m$, let $\mathbf{X}_i$ denote the submatrix of $\mathbf{X}$ consisting of the last i rows. Fix $p\geq 1$, and assume that the ideals of minors satisfy these conditions*:

$$\operatorname{grade} I_i(\mathbf{X})=n-i+1 \quad \textit{and} \quad I_i(\mathbf{X})=I_i(\mathbf{X}_i) \quad \textit{for } i=p,p+1.$$

Set $A:=R/I_{p+1}(\mathbf{X})$ and $J:=I_p(\mathbf{X})A$. Denote the image in J of the minor of $\mathbf{X}$ with rows $i_1<\dots<i_p$ and columns $k_1<\dots<k_p$ by $d_{\mathbf{i}}^{\mathbf{k}}$.

(1) *Let $\mathbf{p}$ be the sequence $m-p+1,\dots,m$. Then there exists an A-regular element Δ of the form $\Delta=\sum_{\mathbf{k}} a^{\mathbf{k}}d_{\mathbf{p}}^{\mathbf{k}}$.*

(2) *Given an A-regular element Δ as in* (1), *set $\Delta_{\mathbf{i}}:=\sum_{\mathbf{k}} a^{\mathbf{k}}d_{\mathbf{i}}^{\mathbf{k}}$ and let I be the subideal of J generated by the various $\Delta_{\mathbf{i}}$. Then $IJ=\Delta J$ and $J=(\Delta):I$.*

Proof. Obviously, $A=R/I_{p+1}(\mathbf{X}_{p+1})$. Since $I_{p+1}(\mathbf{X}_{p+1})$ has generic grade, A is a perfect R-module; so A is grade unmixed (by [4, (16.17), p. 209] for example). Moreover, $\operatorname{grade} I_p(\mathbf{X}_p)>\operatorname{grade} I_{p+1}(\mathbf{X})$. Therefore, (1) holds.

Consider (2). Obviously, Lemma 4.7(2) yields $IJ=\Delta J$. So $J\subseteq(\Delta):I$. To prove the opposite inclusion, we may replace $\mathbf{X}$ by $\mathbf{X}_{p+1}$. Indeed, A and J are obviously unchanged. Let I' be the ideal generated by the $\Delta_{\mathbf{i}}$ with $m-p\leq i_1$, and suppose $J\supseteq(\Delta):I'$. Now, $(\Delta):I'\supseteq(\Delta):I$ since $I'\subseteq I$. Hence $J=(\Delta):I$. Thus we may assume $p=m-1$.

Since $J\subseteq(\Delta):I$, equality will hold if it holds locally at every associated prime $\mathbf{q}$ of J. Therefore, localizing at $\mathbf{q}$, we may assume that R is local with (Δ), I, and J contained in the maximal ideal of A.

The equation $J=(\Delta):I$ will now be proved in the "generic" case and then specialized. Let $\mathbf{m}$ be the maximal ideal of R, let $\widetilde{\mathbf{X}}$ be an m by n matrix of indeterminates over R, and let $\widetilde{B}$ denote the localization of the polynomial ring $R[\widetilde{\mathbf{X}}]$ at the ideal $(\mathbf{m},\widetilde{\mathbf{X}}-\mathbf{X})$. Let '$\widetilde{\ }$' indicate the corresponding objects defined using $\widetilde{B}$ and $\widetilde{\mathbf{X}}$ instead of R and $\mathbf{X}$, except for $\tilde{J}$, which will now denote $I_p(\widetilde{\mathbf{X}}_p)\tilde{A}$. Let $\mathbf{a}$ be the $\widetilde{B}$-regular sequence consisting of the mn entries of the difference matrix $\widetilde{\mathbf{X}}-\mathbf{X}$. Then $\tilde{A}/(\mathbf{a})$ is equal to A. Furthermore, since $\tilde{A}$ is a perfect $\widetilde{B}$-module and since $\operatorname{grade}_R A$ is equal to $\operatorname{grade}_{\widetilde{B}}\tilde{A}$, it follows that $\mathbf{a}$ is $\tilde{A}$-regular. Hence $\mathbf{a},\tilde{\Delta}$ is $\tilde{A}$-regular, and so $\tilde{\Delta},\mathbf{a}$ is $\tilde{A}$-regular. In particular, $\tilde{\Delta}$ is $\tilde{A}$-regular.

Obviously, Lemma 4.7(2) yields $\tilde{I}\tilde{J}\subseteq(\tilde{\Delta})$. Hence, $\tilde{J}\subseteq(\tilde{\Delta}):\tilde{I}$, and equality will hold if it holds locally at every associated prime $\tilde{\mathbf{q}}$ of $\tilde{J}$. The trace $\mathbf{q}$ of $\tilde{\mathbf{q}}$ is an associated prime of R, and $\tilde{\mathbf{q}}/(\tilde{J}+\mathbf{q}\tilde{A})$ is an associated prime of $\tilde{A}/(\tilde{J}+\mathbf{q}\tilde{A})$; indeed, $\tilde{A}/\tilde{J}$ is equal to $\widetilde{B}/I_p(\widetilde{\mathbf{X}}_p)$ because $I_p(\widetilde{\mathbf{X}}_p)$ contains $I_{p+1}(\widetilde{\mathbf{X}})$ as $m=p+1$, and $\widetilde{B}/I_p(\widetilde{\mathbf{X}}_p)$ is (well-known to be) flat over R. Now, $\tilde{A}/(\tilde{J}+\mathbf{q}\tilde{A})$ is a domain because it is equal to $\widetilde{B}/(I_p(\widetilde{\mathbf{X}}_p)+\mathbf{q}\widetilde{B})$ and the latter is a domain because $R/\mathbf{q}$ is a domain by [4, (2.10), p. 14]. Therefore, $\widetilde{\mathbf{q}}=\tilde{J}+\mathbf{q}\tilde{A}$.

Suppose $\tilde{I}\subseteq\widetilde{\mathbf{q}}$. Then $\tilde{\Delta}_{\mathbf{i}}\in\widetilde{\mathbf{q}}$ for every $\mathbf{i}$. Take $\mathbf{i}$ to be the sequence $1,\dots,p$, and pass momentarily modulo the ideal generated by the last row of $\widetilde{\mathbf{X}}$. Then $\tilde{J}$ vanishes, whence $\widetilde{\mathbf{q}}$ is equal to $\mathbf{q}\tilde{A}$. Hence, since the $a^{\mathbf{k}}$ are the coefficients in the definition of $\tilde{\Delta}_{\mathbf{i}}$, they must lie in $\mathbf{q}$. Returning to the previous setup, conclude that $\tilde{\Delta}\in\mathbf{q}\tilde{A}$. Now, $\tilde{A}/\mathbf{q}\tilde{A}$ is equal to $\widetilde{B}/(I_{p+1}(\widetilde{\mathbf{X}})+\mathbf{q}\widetilde{B})$, which is a domain. So $\mathbf{q}\tilde{A}$ is a prime. So it is an associated prime because $\mathbf{q}$ is. Hence $\tilde{\Delta}$ is a zero divisor on $\tilde{A}$, contrary to the conclusion drawn above. Now, $\tilde{I}\not\subseteq\widetilde{\mathbf{q}}$; so $(\tilde{\Delta}):\tilde{I}=(\tilde{\Delta})$ locally at $\widetilde{\mathbf{q}}$. However, $(\tilde{\Delta})\subseteq\tilde{J}$. Thus $\tilde{J}=(\tilde{\Delta}):\tilde{I}$.

To prove that this equation specializes, we first prove that the ideal $\tilde{I}$ has grade 1 and is strongly perfect over $\widetilde{B}$. Now, Lemma 4.7(1) yields

$$\tilde{d}^{\mathbf{l}}_{\mathbf{p}}\tilde{I}=\tilde{\Delta}_{\mathbf{p}}I_p(\widetilde{\mathbf{X}}^{\mathbf{l}})\tilde{A}$$

for any $\mathbf{l}$. This equation yields an isomorphism of $\tilde{A}$-modules between $\tilde{I}$ and $I_p(\widetilde{\mathbf{X}}^{\mathbf{l}})\tilde{A}$ because $\tilde{d}^{\mathbf{l}}_{\mathbf{p}}$ and $\tilde{\Delta}_{\mathbf{p}}$ are regular on $\tilde{A}$. Furthermore, Proposition 5.7 implies that $I_p(\widetilde{\mathbf{X}}^{\mathbf{l}})\tilde{A}$ is either the unit ideal or else an $\tilde{A}$-ideal of grade 1 that is strongly perfect over $\widetilde{B}$. Hence, Lemma 5.8 yields that $\tilde{I}$ is an $\tilde{A}$-ideal of grade 1 that is strongly perfect over $\widetilde{B}$.

Finally, $\operatorname{grade}_{\widetilde{B}}\tilde{A}/\tilde{I}\le\operatorname{grade}_R A/I$ because $\tilde{I}$ has grade 1; hence, because $\tilde{A}/\tilde{I}$ is perfect, $\mathbf{a}$ is regular on $\tilde{A}/\tilde{I}$. Proposition 5.5 now implies that the equation $\tilde{J}=(\tilde{\Delta}):\tilde{I}$ specializes to $J=(\Delta):I$. Thus (2) is proved.

Remark 5.10. If we assume in Theorem 5.9 that $\operatorname{grade}(d^{\mathbf{k}}_{\mathbf{p}})=1$ for some $\mathbf{k}$, then in (2) we can take $\Delta=d^{\mathbf{k}}_{\mathbf{p}}$ and $I=I_p(\mathbf{X}^{\mathbf{k}})A$ where $\mathbf{X}^{\mathbf{k}}$ is the m by p submatrix of $\mathbf{X}$ consisting of columns $k_1<\dots<k_p$. Moreover, the proof becomes slightly shorter.

Remark 5.11. Lemmas 5.4(1) and 5.8 yield answers to some unpublished questions asked by Avramov and Huneke. Let B be a Noetherian ring, and A a factor ring that is a perfect B-module. The lemmas imply that, given two A-ideals in the same even linkage class, one ideal is strongly perfect over B if and only if the other is too; in particular, every B-ideal in the linkage class of a complete intersection is strongly perfect over B. Huneke [15, Thm. 1.11, p. 1051] proved the corresponding result for strongly Cohen–Macaulay ideals in a Gorenstein local ring.

To prove the general case, obviously it suffices to prove the following assertion. Let K be a proper A-ideal, let $x_1,\dots,x_m$ and $y_1,\dots,y_m$ be A-regular sequences contained in K, and set

$$I:=(x_1,\dots,x_m):K \quad\text{and}\quad J:=(y_1,\dots,y_m):K.$$

Then I is the unit ideal or is strongly perfect over B if and only if J is one or the other.

Induct on m. If $m=0$, then $I=J$ and the assertion is trivial. Suppose $m=1$. If $I=A$, then $K=(x)$, and so $(x^2){:}K=(x)$. Now, (x) is a proper ideal; moreover, it is strongly perfect because A is perfect. Hence, we may replace x by x^2, and so assume that I is proper. Similarly, we may assume that J is proper. Now, in the total quotient ring of A, consider the fractional ideal xK^{-1}. It lies in A because $x \in K$. Hence $xK^{-1}=(x){:}K$. Similarly, $yK^{-1}=(y){:}K$. Therefore, I and J are isomorphic. Consequently, the assertion follows from Lemma 5.8.

Suppose $m>1$. Then we can modify y_1 modulo $y_2, \dots, y_m$ so that $x_1, \dots, x_{m-1}$, y_1 form an A-regular sequence and y_1 is still A-regular (see [15, proof of Thm. 1.11, p. 1051]). Set

$$L := (x_1, \dots, x_{m-1}, y_1) : K, \quad \bar{A} := A/(x_1), \quad \text{and} \quad A' := A/(y_1),$$

and let '$^-$' indicate the image in $\bar{A}$ and '$'$' that in A'. Then I is the unit ideal or is strongly perfect over B if and only if $\bar{I}$ is so by Lemma 5.4(1), if and only if $\bar{L}$ is so by the induction hypothesis, if and only if L' is so by Lemma 5.4(1) applied twice. Now, L' is so if and only if J' is so by the induction hypothesis because the ideals $(x_1, \dots, x_{m-1})$ and $(y_2, \dots, y_m)$ are still generated by A'-regular sequences of length $m-1$ although the given generators need not form A'-regular sequences. Finally, J' is so if and only if J is so, by Lemma 5.4(1) again. Thus I is the unit ideal or is strongly perfect over B if and only if J is so, as asserted.

It follows that certain powers of certain ideals I of B have finite projective dimension; more precisely, if I has grade m and is in the linkage class of a complete intersection, then I^i has projective dimension at most $m+i-2$ in the range $1 \le i \le k$ provided that, for every prime $\mathbf{p}$ containing I with depth $R_{\mathbf{p}} \le m+k-2$, the number of generators of $I_{\mathbf{p}}$ is at most depth $R_{\mathbf{p}}$. Indeed, given a generating set $f_1, \dots, f_n$ of I, set

$$H_j := H_j(f_1, \dots, f_n; B) \quad \text{and} \quad S_j := \operatorname{Sym}_j(B^n).$$

Consider the component $\mathcal{M}_i$ of degree i of the 'approximation complex' of Simis and Vasconcelos [33, p. 351]:

$$\mathcal{M}_i : 0 \longrightarrow H_i \otimes S_0 \longrightarrow \dots \longrightarrow H_j \otimes S_{i-j} \longrightarrow \dots \longrightarrow H_0 \otimes S_i \longrightarrow 0.$$

These component complexes are acyclic in the range $0 \le i \le k-1$ by the acyclicity lemma because the B-modules H_j are either zero or perfect of grade m and because of the assumption on the number of generators of each $I_{\mathbf{p}}$; see the proof of Theorem 4.2 in [33, p. 353]. Hence, by the proof of Theorem 4.6 in Herzog, Simis, and

Vasconcelos [14, p. 105],

$$H_0(\mathcal{M}_i) = I^i/I^{i+1} \quad \text{for } 0 \leq i \leq k-1.$$

Hence, I^i/I^{i+1} has projective dimension at most $m+i$ for $0 \leq i \leq k-1$ again because the H_j are either zero or perfect of grade m. Therefore, I^{i+1} has projective dimension at most $m+i-1$ for $0 \leq i \leq k-1$, as asserted.

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Steven Kleiman
Department of Mathematics
2–278 MIT
Cambridge, MA 02139
U.S.A.

Joseph Lipman
Department of Mathematics
Purdue University
West Lafayette, IN 47907
U.S.A.

Bernd Ulrich
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027
U.S.A.