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Forcing axioms and the continuum hypothesis. Part II: transcending ω_1-sequences of real numbers

by

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Dedicated to Fennel, Laurel and Stephanie.

1. Introduction

In his seminal analysis of the cardinality of infinite sets, Cantor demonstrated a procedure which, given an ω-sequence of real numbers, produces a new real number which is not in the range of the sequence. He then posed the continuum problem, asking whether the cardinality of the real line was $\aleph_1$ or some larger cardinality. This problem was eventually resolved by the work of Gödel and Cohen who proved that the equality $|\mathbb{R}|=\aleph_1$ was neither refutable nor provable, respectively, within the framework of ZFC. In particular, there is no procedure in ZFC which takes an ω_1-sequence of real numbers and produces a new real number not in the range of the original sequence.

The purpose of this article is to revisit this type of diagonalization and to demonstrate that models of the continuum hypothesis (CH) exhibit a strong form of incompactness. The primary motivation for the results in this paper is to answer a question of Shelah [9, Problem 2.18] concerning which *forcing axioms* are consistent with the CH. The forcing axiom for a class $\mathfrak{C}$ of partial orders is the assertion that if $\mathcal{D}$ is a collection of $\aleph_1$-cofinal subsets of a partial order, then there is an upward directed set G which intersects each element of $\mathcal{D}$. The well-known *Martin's axiom for* $\aleph_1$*-dense sets* ($\mathrm{MA}_{\aleph_1}$) is the forcing axiom for the class of partial orders which satisfy the countable chain condition. Foreman, Magidor and Shelah have isolated the largest class of partial orders $\mathfrak{C}$ for which a forcing axiom is consistent with ZFC. Shelah has established certain sufficient conditions on a class $\mathfrak{C}$ of partial orders in order for the forcing axiom for $\mathfrak{C}$ to be consistent with CH. His question concerns to what extent his result was sharp. The main

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result of the present article, especially when combined with those of [1], show that this is essentially the case.

The solution to Shelah's problem has some features which are of independent interest. Suppose that T is a collection of countable closed subsets of ω_1 which is closed under taking closed initial segments and that T has the following property:

(1) Whenever s and t are two elements of T with the same supremum δ and

$$\lim(s)\cap\lim(t)$$

is unbounded in δ, then $s=t$. (Here $\lim(s)$ is the set of limit points of s.)

As usual, we regard T as a set-theoretic tree by declaring $s\leqslant t$ if s is an initial part of t. Observe that this condition implies that T contains at most one uncountable path: any uncountable path would have a union which is a closed unbounded subset of ω_1 and such subsets of ω_1 must have an uncountable intersection. In fact it can be shown that this condition implies that

$$\{(s,t)\in T^2 : (\mathrm{ht}_T(s)=\mathrm{ht}_T(t))\wedge(s\neq t)\}$$

can be decomposed into countably many antichains (T^2 is *special off the diagonal*). Jensen and Kunen, in unpublished work, have constructed examples of ω_1-Baire trees which are special off the diagonal under the assumption of Jensen's principle $\diamondsuit$, but this article represents the first such construction from CH.

The main result of this article is to prove that if the continuum hypothesis is true, then there is a tree T which satisfies (1) together with the following additional properties:

(2) T has no uncountable path;

(3) for each $t\in T$, there is a closed unbounded set of δ such that $t\cup\{\delta\}$ is in T;

(4) T is proper as a forcing notion and remains so in any outer model with the same set of real numbers in which T has no uncountable path; moreover T is complete with respect to a simple $\aleph_1$-completeness system $\mathbb{D}$, and in fact is $\aleph_1$-completely proper in the sense of [7];

(5) if X is a countable subset of T, then the collection of bounded chains which are contained in X is Borel as a subset of $\mathscr{P}(X)$.

This provides the first example of a consequence of the continuum hypothesis which justifies condition (b) in the next theorem.

THEOREM 1.1. ([8], [4]) *Suppose that $\langle P_\alpha; Q_\alpha : \alpha\in\theta\rangle$ is a countable support iteration of proper forcings which*

(a) *are complete with respect to a simple* 2*-completeness system* $\mathbb{D}$;

(b) *satisfy either of the following conditions*:

(i) *are weakly α-proper for every $\alpha\in\omega_1$*;

(ii) *are proper in every proper forcing extension with the same set of real numbers.*

Then forcing with P_θ does not introduce new real numbers.

Results of Devlin and Shelah [2] have long been known to imply that condition (a) is necessary in this theorem. The degree of necessity of condition (b)—and (i) in particular—has long been a source of mystery, however. Shelah has already shown [8, §XVIII.1] that there is an iteration of forcings in L which satisfies (a) and which introduces a new real at limit stage ω^2. This construction, however, has two serious limitations: it is not possible in the presence of modest large cardinal hypotheses—such as the existence of a measurable cardinal—and it does not refute the consistency of the corresponding *forcing axiom* with the continuum hypothesis. Still, this construction is the starting point for the results in the present article.

Properties (3) and (4) of T imply that forcing with T adds an uncountable path through T and does not introduce new real numbers. Thus T can have an uncountable branch in an outer model with the same real numbers. By condition (1), this branch is in a sense *unique*, in that once such an uncountable branch exists, there can never be another. (Of course different forcing extensions will have different cofinal branches through T and in any outer model in which ω_1^V is countable, there are a continuum of paths through T whose union is cofinal in ω_1^V.)

Property (5) of the tree T is notable because it causes a number of related but formally different formulations of *completeness* to be identified [7]. This is significant because it was demonstrated in [1] that in general different notions of completeness can give rise to incompatible iteration theorems and incompatible forcing axioms.

The existence of the tree T can be seen as the obstruction to a diagonalization procedure for ω_1-sequences of reals. For each ω_1-sequence of reals $\mathbf{r}$, there will be an associated sequence $\vec{T}^{\mathbf{r}}$ of trees of length at most ω^2 which satisfy (1). These sequences are such that they have positive length exactly when $\omega_1^{L[\mathbf{r}]}{=}\omega_1^V$, and if ξ is less than the length of the sequence, then

(6) $T^{\mathbf{r}}_\xi$ has an uncountable path if and only if it is not the last entry in the sequence;

(7) if $\xi'{\in}\xi$, then every element of T_ξ is contained in the set of limit points of $E_{\xi'}$, where $E_{\xi'}$ is the union of the uncountable path through $T^{\mathbf{r}}_{\xi'}$;

(8) if $T^{\mathbf{r}}_\xi$ is the last entry in the sequence, then either $L[\mathbf{r}]$ contains a real number not in the range of $\mathbf{r}$, or else $T^{\mathbf{r}}_\xi$ is completely proper in every outer model of $L[\mathbf{r}]$ with the same real numbers;

(9) if the length of the sequence is ω^2 and δ is in the intersection of $\bigcap_{\xi\in\omega^2} E_\xi$, then $\langle E_\xi{\cap}\delta{:}\xi{\in}\omega^2\rangle$ is not in $L[\mathbf{r}]$ (in particular, $\mathbf{r}$ is not an enumeration of $\mathbb{R}$).

Moreover the (partial) function $\mathbf{r}{\mapsto}\vec{T}^{\mathbf{r}}_\xi$ is Σ_1-definable for each $\xi{\in}\omega^2$. Thus in any outer model, the sequence $\vec{T}^{\mathbf{r}}$ may increase in length, but it maintains the entries from the inner model. Observe that if $\mathbf{r}$ is an enumeration of $\mathbb{R}$ in order-type ω_1, then $\vec{T}^{\mathbf{r}}$ has successor length $\eta_{\mathbf{r}}+1$.

Shelah has proved (unpublished) that an iteration of completely proper forcings of length less than ω^2 does not add new real numbers. Thus, if $\mathbf{r}$ is a well ordering of $\mathbb{R}$ in type ω_1 and $\xi\in\omega^2$, then it is possible to go into a forcing extension with the same reals such that $\xi\leqslant\eta_{\mathbf{r}}$. The above remarks show that this result is optimal. Also, for a fixed $\xi\in\omega^2$, the assertion that there is an enumeration $\mathbf{r}$ of $\mathbb{R}$ in type ω_1 with $\eta_{\mathbf{r}}$ at least ξ is expressible by a Σ^2_1-formula. By Woodin's Σ^2_1-absoluteness theorem (see [6]), this means that, in the presence of a measurable Woodin cardinal, CH implies that for each $\xi\in\omega^2$ there is an enumeration $\mathbf{r}$ of $\mathbb{R}$ in type ω_1 such that $\eta_{\mathbf{r}}$ is at least ξ.

While the construction of the sequence of trees is elementary, the reader is assumed to have a solid background in set theory at the level of [5]. Knowledge of proper forcing will be required at some points, although the necessary background will be reviewed for the reader's convenience; further reading can be found in [3], [4] and [8]. Notation is standard and will generally follow that of [5].

2. Background on complete properness

In this section we will review some definitions associated with proper forcing, culminating in the definition of complete properness. This is not necessary to understand the main construction; it is only necessary in order to understand the verification of (4). A *forcing notion* is a partial order with a least element. Elements of a forcing notion are often referred to as *conditions*. If p and q are conditions in a forcing, then $p\leqslant q$ is often read "q extends p" or "q is stronger than p." If two conditions have a common extension, they are *compatible*; otherwise they are *incompatible*.

Let $H(\theta)$ denote the collection of all sets of hereditary cardinality at most θ. If Q is a forcing notion in $H(\theta)$, then a countable elementary submodel M of $H(\theta)$ is *suitable* for Q if it contains the power set of Q. If M is a suitable model for Q and $\bar{q}$ is in Q, then we say that $\bar{q}$ is (M,Q)-*generic* if whenever $\bar{q}\leqslant r$ and $D\subseteq Q$ is a dense subset of Q in M, r is compatible with some element of $D\cap M$. A forcing notion Q is *proper* if whenever M is suitable for Q and q is in $Q\cap M$, there is an extension of q which is (M,Q)-generic.

If M and N are sets, then $\overrightarrow{MN}$ will denote a tuple (M,N,ε), where ε is an elementary embedding ε of $(M,\in)$ into $(N,\in)$ such that the range of ε is a countable element of N. Such a tuple will be referred to as an *arrow*. We will write $M\to N$ to mean $\overrightarrow{MN}$ and also to indicate "$\overrightarrow{MN}$ is an arrow". If $M\to N$ and X denotes an element of M, then X^N will be used to denote $\varepsilon(X)$, where ε is the embedding corresponding to $M\to N$.

If Q is a forcing, M is suitable for Q and $M\to N$, then a filter $G\subseteq Q\cap M$ is $\overrightarrow{MN}$-*prebounded* if, whenever $N\to P$ and the image G' of G under the composite embeddings is in P, P satisfies "G' has a lower bound". If Q is a forcing notion, then a collection of embeddings $M \to N_i$, $i \in I$, will be referred to as a Q-*diagram*. A forcing Q is λ-*completely proper* if, whenever $M \to N_i$, $i \in \gamma$, is a Q-diagram for $\gamma \in 1+\lambda$, there is an (M, Q)-generic filter $G \subseteq Q \cap M$ which is $\overrightarrow{MN}_i$-prebounded for all $i \in \gamma$.

Observe that, as λ increases, λ-complete properness becomes a weaker condition. In [7] it was shown that λ-complete properness implies $\mathbb{D}$-completeness with respect to a simple λ-completeness system $\mathbb{D}$ in the sense of Shelah (see [7] for undefined notions). Moreover, the converse is true for forcing notions Q with the property that, whenever $Q_0 \subseteq Q$ is countable, $\{G \subseteq Q_0 : G \text{ has a lower bound in } Q\}$ is Borel [7].

3. The construction

Assume CH and fix a bijection ind: $H(\omega_1) \to \omega_1$ such that, if x is a countable subset of ω_1, then $\sup(x) \in \mathrm{ind}(x)$. If x and y are in $H(\omega_1)$, we will abuse notation and write $\mathrm{ind}(x, y)$ for $\mathrm{ind}((x, y))$. Fix a Σ_1-definable map $\xi \mapsto \xi^*$ which maps the countable limit ordinals into ω_1 such that, if $\varrho \in \omega_1$, then $\{\xi \in \lim(\omega_1) : \xi^* = \varrho\}$ is uncountable (for instance define ξ^* to be the unique ordinal ϱ such that, for some ς, $\xi = \omega^\varsigma + \omega \cdot \varrho$ and $\omega \cdot \varrho \in \omega^\varsigma$).

If δ is a limit ordinal, let C_δ denote the cofinal subset of δ of order-type ω which minimizes ind; set $C_{\alpha+1} = \{\alpha\}$. Let $e_\beta : \beta \to \omega$ be defined by $e_\beta(\alpha) = \bar{\varrho}_1(\alpha, \beta)$, where $\bar{\varrho}_1$ is defined from $\langle C_\alpha : \alpha \in \omega_1 \rangle$ as in [10]. In what follows, we will only need that the sequence $\langle e_\beta : \beta \in \omega_1 \rangle$ is determined by ind, $e_\beta : \beta \to \omega$ is an injection, and if $\beta \in \beta' \in \omega_1$, then

$$\{\alpha \in \beta : e_\beta(\alpha) \neq e_{\beta'}(\alpha)\}$$

is finite.

Let $E \subseteq \omega_1$ be a club. Define $T = T_E = T_E^{\mathrm{ind}}$ to be all t which are countable closed sets consisting of limit points of E such that the following properties hold:

- if ν is a limit point of t, then $t \cap \nu$ has finite intersection with every ladder in ν which either has index less than $\mathrm{ind}(E \cap \nu)$ or else is C_ν;
- if ν is a limit point of t, then $\min(E \setminus \nu) \in \mathrm{ind}(t \cap \nu)$;
- for all $\alpha \in \beta$, $\mathrm{ind}((t \cap (\beta+1)) \setminus \alpha) \in \min(t \setminus (\beta+1))$.

Define $\widetilde{T} = \widetilde{T}_E^{\mathrm{ind}} \subseteq T$ by recursion. Begin by declaring $\varnothing \in \widetilde{T}$. Now suppose that we have defined $\widetilde{T} \cap \mathscr{P}(\xi+1)$ for all $\xi \in \delta$. Before defining $\widetilde{T} \cap \mathscr{P}(\delta+1)$, we need to specify a family of logical formulas which will be needed in the definition. We will use $\widetilde{T} \restriction \nu$ to denote $\{t \in \widetilde{T} : \mathrm{ind}(t) \in \nu\}$. Let β be a fixed ordinal less than δ and let $t \in \widetilde{T}$ be such that $t \subseteq \beta$. Consider the following recursively defined formulas about a closed subset x of β:

$\theta_0^\delta(x, t, \beta)$: $\max(t) \in \min(x)$, $t \cup x$ is in $\widetilde{T} \restriction \beta$, and

$$\mathrm{otp}(E \cap \min(x))^* = \mathrm{ind}(t, n) \quad \text{for some } n \in \omega;$$

$\theta_1^\delta(x,t,\beta)$: if D is a dense subset of $\widetilde{T}\restriction\nu$ for some limit ordinal $\nu\in\beta$, $\mathrm{ind}(D)\in\beta$ and

$$\mathrm{otp}(E\cap\min(x))^* = \mathrm{ind}(t, e_\delta(\mathrm{ind}(D))),$$

then $t\cup x$ is in D.

$\theta_2^\delta(x,t,\beta)$: if $y\subseteq\beta$, $e_\delta(\min(y))\in e_\delta(\min(x))$ and

$$\theta_0^\delta\wedge\theta_1^\delta\wedge\theta_2^\delta\wedge\theta_3^\delta(y,t,\beta),$$

then $x\cap y\subseteq\{\min(x)\}$.

$\theta_3^\delta(x,t,\beta)$: if $s,z\subseteq\beta$, $\min(z)=\min(x)$ and

$$\theta_0^\delta\wedge\theta_1^\delta\wedge\theta_2^\delta(z,s,\beta),$$

then $\mathrm{ind}(x)\leqslant\mathrm{ind}(z)$.

The truth of $\theta_i^\delta(x,t,\beta)$ is defined by recursion on the tuple

$$(e_\delta(\min(x)), \mathrm{ind}(x), i)$$

equipped with the lexicographical ordering (δ is fixed). Observe that, since e_δ is injective, there is at most one set D which satisfies the hypotheses of $\theta_1^\delta(x,t,\beta)$. In particular, if $\theta_1^\delta(x,t,\beta)$ holds, then $\theta_1^\delta(x,t,\beta')$ holds for all $\beta\in\beta'\in\delta$.

Now, if t is an element of T with $\sup(t)=\delta$ and δ is not a limit point of t, define t to be in $\widetilde{T}$ if and only if $t\cap\delta$ is in $\widetilde{T}$. If t is an element of T with $\sup(t)=\delta$ and δ is a limit point of t, then we define t to be in $\widetilde{T}$ if $t\cap\alpha+1$ is in $\widetilde{T}$ for all $\alpha\in\delta$ and if, for all but finitely many consecutive pairs $\alpha\in\beta$ in C_δ for which $(\alpha,\beta]\cap t$ is non-empty,

$$\theta_0^\delta\wedge\theta_1^\delta\wedge\theta_2^\delta\wedge\theta_3^\delta(t\cap(\alpha,\beta], t\cap\alpha+1, \beta).$$

LEMMA 3.1. *If E and E' are clubs such that $E'\cap\delta=E\cap\delta$ for some $\delta\in\omega_1$, then*

$$\widetilde{T}_E\cap\mathscr{P}(\delta+1) = \widetilde{T}_{E'}\cap\mathscr{P}(\delta+1).$$

Proof. This follows from the observation that, under the hypotheses of the lemma, $T_E\cap\mathscr{P}(\delta+1)=T_{E'}\cap\mathscr{P}(\delta+1)$ and the fact that $\widetilde{T}_E\cap\mathscr{P}(\delta+1)$ is defined by recursion from $T_E\cap\mathscr{P}(\delta+1)$ and $\widetilde{T}_E\restriction\delta$. □

LEMMA 3.2. *If t and t' are in $\widetilde{T}$ and $\delta\in\omega_1$ is a limit point of* $\lim(t)\cap\lim(t')$, *then*

$$t\cap\delta = t'\cap\delta.$$

Proof. Let $t, t' \in \widetilde{T}$ and δ be a limit point of $\lim(t) \cap \lim(t')$. It is sufficient to show that $t \cap \alpha = t' \cap \alpha$ for cofinally many $\alpha \in \delta$. Suppose that this is not the case. Let $\delta_0 \in \delta$ be arbitrary and $\alpha \in \beta$ be consecutive elements of C_δ such that $\delta_0 \in \alpha$, $(\alpha, \beta] \cap \lim(t) \cap \lim(t')$ is non-empty, and such that

$$\theta_i^\delta(t \cap (\alpha, \beta], t \cap \alpha + 1, \beta) \quad \text{and} \quad \theta_i^\delta(t' \cap (\alpha, \beta], t' \cap \alpha + 1, \beta)$$

hold for $i = 0, 1, 2, 3$. Without loss of generality, we may assume that

$$e_\delta(\mathrm{ind}(t \cap (\alpha, \beta])) \in e_\delta(\mathrm{ind}(t' \cap (\alpha, \beta])).$$

Define $x = t' \cap (\alpha, \beta]$ and $y = t \cap (\alpha, \beta]$. Observe that, since x and y have common limit points, $x \cap y$ is in particular not contained in $\{\min(x), \min(y)\}$. Since $\theta_2^\delta(x, t \cap \alpha, \beta)$ is true, it follows that $\min(x) = \min(y)$ and hence, by $\theta_0^\delta(y, t \cap \alpha + 1, \beta)$ and $\theta_0^\delta(x, t' \cap \alpha + 1, \beta)$, that $t \cap \alpha = t' \cap \alpha$. $\square$

Lemma 3.3. *The tree*

$$\{(s, t) \in T^2 : (\mathrm{ht}_T(s) = \mathrm{ht}_T(t)) \wedge (s \neq t)\}$$

is a union of countably many antichains.

Proof. Let U denote the set of those $(s, t) \in T^2$ such that $\mathrm{ht}_T(s) = \mathrm{ht}_T(t)$, $s \neq t$ and $\max(s) \leqslant \max(t)$. By symmetry it is sufficient to prove that U is a countable union of antichains. For each (s, t) in U, define $\mathrm{Osc}(s, t)$ to be the set of all $\xi \geqslant \min(s \triangle t)$ such that either

- ξ is in s and $\min(t \setminus \xi + 1) \in \min(s \setminus \xi + 1)$, or
- ξ is in t and $\min(s \setminus \xi + 1) \in \min(t \setminus \xi + 1)$.

Let $\mathrm{osc}(s, t)$ denote the order-type of $\mathrm{Osc}(s, t)$, observing that Lemma 3.2 implies that $\mathrm{osc}(s, t) \in \omega^2$ for all (s, t) in U. If $\max(s) \in \max(t)$, define

$$\beta(s, t) = \min\{\beta \in t : \max(s) \in \beta\} \quad \text{and} \quad n(s, t) = e_{\beta(s,t)}(\max(s)),$$

and set $n(s, t) = \omega$ if $\max(s) = \max(t)$. Observe that if (s, t) and (s', t') are in U and $s < s'$ and $t < t'$, then $\mathrm{Osc}(s, t)$ is an initial part of $\mathrm{Osc}(s', t')$ and that no element of $\mathrm{Osc}(s, t)$ is greater than $\max(s)$. Consequently, either $\min(s' \setminus s)$ is in $\mathrm{Osc}(s', t') \setminus \mathrm{Osc}(s, t)$, and hence $\mathrm{osc}(s, t) \neq \mathrm{osc}(s', t')$ or else $\beta(s, t) = \beta(s', t')$ and $n(s, t) \neq n(s', t')$. It follows that, for each $\xi \in \omega^2$ and $k \in \omega + 1$,

$$\{(s, t) \in U : (\mathrm{osc}(s, t) = \xi) \wedge (n(s, t) = k)\}$$

is an antichain. Since $\omega^2 \times (\omega + 1)$ is countable, this finishes the proof. $\square$

LEMMA 3.4. *Suppose that M is a countable elementary submodel of $H((2^{\omega_1})^+)$ with $\widetilde{T}$ in M, and suppose that t_i, $i \in n$, is a sequence of elements of $\widetilde{T}$ such that there is a club $C \subseteq \omega_1$ in M such that $C \cap M \subseteq \bigcup_{i \in n} t_i$. Then $\widetilde{T}$ has an uncountable chain.*

Proof. Let M, t_i, $i \in n$, and C be as in the statement of the lemma. By replacing C with its limit points if necessary, we may assume that $C \cap M \subseteq \bigcup_{i \in n} \lim(t_i)$. Without loss of generality t_i, $i \in n$, are such that if $i \neq j$, then $t_i \cap M \neq t_j \cap M$. By Lemma 3.2, there is a $\zeta \in M \cap \omega_1$ such that if ν is a limit point of $\lim(t_i) \cap \lim(t_j) \cap M$, then $\nu \in \zeta$. Let $i \in n$ be such that $\lim(t_i) \cap C' \cap M$ is non-empty for every club C' in M. Such an i exists since otherwise there would exist clubs C'_i in M for each $i \in n$ such that $C \cap \bigcup_{i \in n} C'_i$ is empty.

Let N be a countable elementary submodel of $H(\omega_2)$ in M such that C and ζ are in N and $\delta = N \cap \omega_1$ is in $\lim(t_i)$. Since δ is not in $\lim(t_j)$ for $j \neq i$, there is a $\zeta' \in \delta$ such that $t_j \cap \delta \subseteq \zeta'$ whenever $j \neq i$. Let C' be the set of elements of C which are greater than ζ'. If ν is in $C' \cap N$, then ν is in $\lim(t_i) \setminus \lim(t_j)$ for each $j \in n$ which is different from i. Define b to be the set of $p \in \widetilde{T}$ such that $C' \cap \sup(p)$ is an infinite cofinal subset of $\lim(p)$. Since it is definable from parameters in N, b is in N. Furthermore, b is uncountable since it is not contained in N—it has $t_i \cap (\delta+1)$ as an element. By Lemma 3.2, b is a chain in $\widetilde{T}$ and hence satisfies the conclusion of the lemma. □

LEMMA 3.5. *Suppose that $E \subseteq \omega_1$ is a club and $\widetilde{T}_E$ does not contain an uncountable chain. Then $\widetilde{T}_E$ is completely proper.*

Proof. Let $\widetilde{T}$ denote $\widetilde{T}_E$. Suppose that $M \to N_i$, $i \in \omega$, is a $\widetilde{T}$-diagram and that t_0 is in $\widetilde{T} \cap M$. Define $\delta = M \cap \omega_1$ and let $\eta \in \omega_1$ be an upper bound for $\omega_1^{N_i}$ for each $i \in \omega$. In particular, if $X \subseteq M \cap \omega_1$ is in N_i, then $\mathrm{ind}^{N_i}(X) \in \eta$. Let D_n, $n \in \omega$, enumerate the dense subsets of $\widetilde{T}$ in M and let X_n, $n \in \omega$, enumerate the collection of all cofinal subsets X of δ of order-type ω which are in N_i for some $i \in \omega$. We will construct t_n, α_n and β_n for each $n \in \omega$ by induction, so that the following conditions hold for all $n \in \omega$:

(10) t_{n+1} extends t_n and is in $D_n \cap M$;

(11) $\alpha_n \in \beta_n \in \alpha_{n+1} \in \delta$;

(12) if $i \in n$ then $t_{n+1} \setminus t_n$ is contained in $\beta_n \setminus \alpha_n$ and is disjoint from X_i if $i \in n$;

(13) if $i \in n$ then $e_\delta^{N_i}(\xi) = e_\delta(\xi)$ whenever $\xi \in \beta_n \setminus \alpha_n$;

(14) if $i \in n$, then there are consecutive elements $\bar{\alpha} \in \bar{\beta}$ of $C_\delta^{N_i}$ such that $\bar{\alpha} \in \alpha_n \in \beta_n \in \bar{\beta}$;

(15) there is a limit ordinal ν such that $\max(t_{n+1}) \in \nu \in \beta_{n+1}$ and

$$\mathrm{otp}(E \cap \min(t_{n+1} \setminus t_n))^* = \mathrm{ind}(t_n, e_\delta(\mathrm{ind}(D \restriction \nu)));$$

(16) if $i \in n$, then

$$N_i \models \theta_0^\delta \wedge \theta_1^\delta \wedge \theta_2^\delta \wedge \theta_3^\delta(t_{n+1} \setminus t_n, t_n, \beta_{n+1}).$$

Assuming that this can be accomplished, then define $\bar{t}=\bigcup_{n\in\omega} t_n\cup\{\delta\}$. It follows that, if $i\in\omega$ and $N_i\to\widetilde{N}$ with $\bar{t}\in\widetilde{N}$, then $\bar{t}$ is in $T^{\widetilde{N}}$. If $\bar{t}$ is in $T^{\widetilde{N}}$, then moreover we have arranged that

$$N_i\models\theta_0^\delta\wedge\theta_1^\delta\wedge\theta_2^\delta\wedge\theta_3^\delta(\bar{t}\cap(\alpha,\beta],\bar{t}\cap\alpha+1,\beta)$$

whenever $\alpha\in\beta$ are consecutive elements of $C_\delta^{N_i}$ such that $\max(t_i)\in\alpha$. In particular $\bar{t}$ is in $\widetilde{T}^{\widetilde{N}}$. Thus t_n, $n\in\omega$, is $\overrightarrow{MN}_i$-prebounded in $\widetilde{T}$ for each $i\in\omega$.

Now suppose that t_n is given. Let M' be a countable elementary submodel of $H((2^{\omega_1})^+)$ such that

- M' is in M;
- D_n is in M';
- M' is an increasing union of an $\in$-chain of elementary submodels of $H((2^{\omega_1})^+)$.

Set $\beta_{n+1}=M'\cap\omega_1$ and let $F\subseteq M'$ be a finite set such that, if $i\in n$, then

$$C_\delta^{N_i}\cap M'\subseteq F,\quad X_i\cap M'\subseteq F\quad\text{and}\quad\{\xi\in M'\cap\omega_1:e_\delta^{N_i}(\xi)\neq e_\delta(\xi)\}\subseteq F.$$

Fix an elementary submodel M'' of $H((2^{\omega_1})^+)$ such that $F\subseteq M''\in M'$. Set $\nu=M''\cap\omega_1$ and $\zeta=\operatorname{ind}(D_n\cap M'')$, and let $\alpha_{n+1}\in\nu$ be such that $\max(F)\in\alpha_{n+1}$. Observe that, by elementarity, ζ is in M' since it is definable from ν, D_n and ind. Furthermore, $\nu\leqslant\zeta$ since otherwise $D_n\cap M''$ would be in M''. Observe that $e_\delta^{N_i}(\zeta)=e_\delta(\zeta)$ for all $i\in n$. Let ξ be a limit point of E such that $\alpha_{n+1}\in\xi\in\nu$ and

$$\operatorname{otp}(E\cap\xi)^*=\operatorname{ind}(t_n,e_\delta(\zeta)).$$

Let y_i, $i\in l$, list the closed subsets y of δ such that

- for some $j\in n$, we have $y\subseteq(\bar{\alpha},\bar{\beta}]$, where $\bar{\alpha}\in\bar{\beta}$ are the consecutive elements of $C_\delta^{N_j}$ with $\bar{\alpha}\leqslant\alpha_{n+1}\in\beta_{n+1}\leqslant\bar{\beta}$;
- $e_\delta^{N_j}(\min(y))\in e_\delta^{N_j}(\xi)$;
- $\theta_0^\delta\wedge\theta_1^\delta\wedge\theta_2^\delta\wedge\theta_3^\delta(y,t_n,\bar{\beta})$.

Observe that θ_3^δ ensures that there are only finitely many such y's. Furthermore, Lemma 3.4 and our hypothesis implies that $\bigcup_{i\in l} y_i$ does not contain a set of the form $C\cap M''$, where C is a club in M''. Thus there is a countable elementary submodel M''' of $H(\omega_2)$ in M'' such that E, $\widetilde{T}$, D_n, ξ and ind are in M''' and $M'''\cap\omega_1$ is not in $\bigcup_{i\in l} y_i$. Let η be a limit point of E which is in M''' such that ξ and $\max(\bigcup_{i\in l} y_i)$ are less than η. Since D_n is dense, elementarity of M''' ensures that there is an extension of $t_n\cup\{\xi,\eta\}$ which is in $D_n\cap M'''$. Such an extension $\tilde{t}$ necessarily satisfies that $\tilde{t}\setminus t_n$ is disjoint from $y_i\setminus\{\xi\}$ for all $i\in l$. We have therefore arranged that $\theta_0^\delta\wedge\theta_1^\delta\wedge\theta_2^\delta(\tilde{t}\setminus t_n,t_n,\beta_{n+1})$ holds. Let t_{n+1} the element of $\widetilde{T}$ be such that

$$\min(t_{n+1}\setminus t_n)=\xi,\quad\theta_0^\delta\wedge\theta_1^\delta\wedge\theta_2^\delta(t_{n+1}\setminus t_n,t_n,\beta_{n+1})\text{ holds,}$$

and $\mathrm{ind}(t_{n+1}\setminus t_n)$ is minimized. It follows that $\theta_3^\delta(t_{n+1}\setminus t_n, t_n, \beta_{n+1})$ holds. This finishes the inductive construction and, therefore, the proof. $\square$

Theorem 3.6. *Assume* CH. *Then there is a club E such that $\widetilde{T}_E$ has no uncountable branch. In particular*, CH *implies the negation of the forcing axiom for the class of completely proper forcing notions.*

Proof. Suppose for contradiction that CH holds and $\widetilde{T}_E$ contains an uncountable branch for every club $E\subseteq\omega_1$. Let $A\subseteq\omega_1$ be such that ind is in $L[A]$. In particular, $\mathbb{R}\subseteq L[A]$ and $\omega_1=\omega_1^{L[A]}$. Inductively construct clubs E_ξ, $\xi\in\omega^2$, as follows. Since $L[A]$ satisfies $\diamondsuit$, there is a function $h\colon\omega_1\to\omega_1$ such that, if E is a club in $L[A]$, then for some limit point δ of E, there is a ladder $X\subseteq\delta$ with $X\cap E$ infinite and $\mathrm{ind}(X)\in h(\delta)$. Let E_0 be the $<_{L[A]}$-least club in $L[A]$ such that $h(\delta)\in\min(E_0\setminus(\delta+1))$ for all δ. Given E_ξ, let $E_{\xi+1}$ be the union of the branch through $\widetilde{T}_{E_\xi}$. If E_ξ, $\xi\in\eta$, has been defined for a limit η less than ω^2, define $E_\eta=\bigcap_{\xi\in\eta}E_\xi$. Observe that E_2 is not in $L[A]$ since, whenever δ is a limit point of E_1,

$$h(\delta)\in\min(E_0\setminus\delta+1)\in\mathrm{ind}(E_1\cap\delta),$$

and hence E_2 has the property that whenever δ is a limit point of E_2, $E_2\cap\delta$ is disjoint from X whenever X is a ladder in δ with index less than $h(\delta)$. Observe that, if $\xi\in\eta$, then E_η is contained in the limit points of E_ξ. Also, if δ is a limit point of E_η, then

$$\min(E_\xi\setminus\delta+1)\in\mathrm{ind}(E_\eta\cap\delta)\in\min(E_\eta\setminus\delta+1).$$

In particular, if $\delta\in\bigcap_{\xi\in\omega^2}E_\xi$, then, for all $k\in\omega$,

$$\sup_{i\in\omega}\mathrm{ind}(E_{\omega\cdot k+i}\cap\delta)\in E_\xi$$

whenever $\xi\in\omega\cdot(k+1)$.

Let δ be the least element of $\bigcap_{\xi\in\omega^2}E_\xi$. Observe that $\langle E_\xi\cap\delta:\xi\in\omega^2\rangle$ is in $L[A]$. We will obtain a contradiction once we show that $\langle E_\xi:\xi\in\omega^2\rangle$ is in $L[A]$. Working in $L[A]$, define ν_α and $t_\alpha(\xi)$ for each $\alpha\in\omega_1$ and $\xi\in\omega^2$ by simultaneous recursion as follows. The sets $t_\alpha(\xi)$ will satisfy that they are $E_\xi\cap\nu_\alpha$ and the ordinals ν_α will each be elements of $\bigcap_{\xi\in\omega^2}E_\xi$. Set $\nu_0=\delta$ and $t_0(\xi)=E_\xi\cap\delta$. Given ν_α and $t_\alpha(\xi)$, $\xi\in\omega^2$, define

$$\nu_{\alpha+1,k}=\sup_{\xi\in\omega\cdot k}\mathrm{ind}(t_\alpha(\xi))\quad\text{and}\quad\nu_{\alpha+1}=\sup_{k\in\omega}\nu_{\alpha+1,k}.$$

Next we define $t_{\alpha+1}(\xi)$, $\xi\in\omega^2$, by recursion on ξ. We set $t_{\alpha+1}(0)=E_0\cap\nu_{\alpha+1}$. Given $t_{\alpha+1}(\xi)$, define $t_{\alpha+1}(\xi+1)$ to be the unique element t of $\widetilde{T}_{E_\xi}\cap\mathscr{P}(\nu_{\alpha+1}+1)$ such that $\sup t=\nu_{\alpha+1}$ and $\nu_{\alpha+1,k}\in t$ for all k (here we are employing Lemmas 3.1 and 3.2). If $\eta\in\omega^2$ is a limit ordinal, set

$$t_{\alpha+1}(\eta)=\bigcap_{\xi\in\eta}t_{\alpha+1}(\xi).$$

If ν_α has been defined for all $\alpha\in\beta$, set $\nu_\beta=\sup_{\alpha\in\beta}\nu_\alpha$ and $t_\beta(\xi)=\bigcup_{\alpha\in\beta}t_\alpha(\xi)$.

It is now easily seen that $t_\alpha(\xi)=E_\xi\cap\nu_\alpha$, and therefore that E_ξ is in $L[A]$ for all $\xi\in\omega^2$. This is a contradiction, however, since E_2 is not in $L[A]$. $\square$

We will finish by remarking that if $\mathbf{r}$ is an ω_1-sequence of reals, then the sequence $\vec{T}^{\mathbf{r}}$ described in the introduction is defined as follows. If $\omega_1^{L[\mathbf{r}]}<\omega_1$, then define $\vec{T}^{\mathbf{r}}$ to be the empty sequence. Otherwise, let ind be the $<_{L[\mathbf{r}]}$-least bijection between $H(\omega_1)\cap L[\mathbf{r}]$ and ω_1. Let $A\subseteq\omega_1$ be such that $L[A]=L[\mathbf{r}]$ and define $T^{\mathbf{r}}_\xi=T^{\mathrm{ind}}_{E_\xi}$ as detailed in the proof of Theorem 3.6.

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