

Research Article

A Novel Optimization Method for Nonconvex Quadratically Constrained Quadratic Programs

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This paper presents a novel optimization method for effectively solving nonconvex quadratically constrained quadratic programs (NQCQP) problem. By applying a novel parametric linearizing approach, the initial NQCQP problem and its subproblems can be transformed into a sequence of parametric linear programs relaxation problems. To enhance the computational efficiency of the presented algorithm, a cutting down approach is combined in the branch and bound algorithm. By computing a series of parametric linear programs problems, the presented algorithm converges to the global optimum point of the NQCQP problem. At last, numerical experiments demonstrate the performance and computational superiority of the presented algorithm.

1. Introduction

The nonconvex quadratically constrained quadratic programs problems have attracted the attention of practitioners and researchers for 30 years. During the past 10 years, curiosity in these problems has been especially intense. In part, this is because the NQCQP problems have a large number of practical applications, for example, pooling problems in petrochemistry [1], modularization of product subassemblies [2], chance-constrained optimization problems, production planning or portfolio optimization [3–5], the fuel mixture problem encountered in oil industry [6], and also placement and layout problems appeared in integrated circuit design (see [7, 8]). In addition, many nonlinear optimization problems can be transformed into the form, for example, special classes of structured stochastic games [9] can be interpreted as quadratic programs problems, the packing problem contained in the unit square can be formulated as concave quadratic constraints quadratic programs problem, {0, 1} variable in 0-1 programming may be also represented by concave quadratic constraints, and minmax location problems [4] also lead to quadratic programs problems with quadratic constraints. Another cause for the strong attention in the NQCQP problems is that, from a research point of

view, the class of problems put forward significant theoretical and computational defiance. This is mainly because these problems are global optimization problem; that is, they are well known to generally own multiple local optimum points that are not globally optimum point. Therefore, it is very essential to put forward a good global optimization method for solving the NQCQP problems.

In this paper, we will investigate the following NQCQP problems:

$$\begin{aligned} \min \quad & G_0(x) = x^T A^0 x + (d^0)^T x \\ \text{s.t.} \quad & G_i(x) = x^T A^i x + (d^i)^T x \leq b_i, \\ & i = 1, \dots, m, \\ & x \in X^0 = \{x \in R^n : l^0 \leq x \leq u^0\}, \end{aligned} \quad (\text{NQCQP})$$

where $A^i = (a_{jk}^i)_{n \times n} \in R^{n \times n}$ ($i = 0, 1, \dots, m$) are all symmetric matrices, $d^0, d^i \in R^n$, $b_i \in R$, $i = 1, \dots, m$, and $l^0 = (l_1^0, \dots, l_n^0)^T$, $u^0 = (u_1^0, \dots, u_n^0)^T$.

In last several decades, some algorithms have been exploited for globally solving the special case or general case of the NQCQP problems. For example, based on a

novel reformulation-linearization/convexification approach, Serali and Tuncbilek [10] proposed a global optimization algorithm for linearly constrained nonconvex quadratic programs problems. Based on outer approximation and branch and bound scheme by solving linear programs subproblems, Al-Khayyal et al. [7] presented an algorithm for computing the approximate global optimal solutions of the NQCQP problems. Based on Lagrangian underestimation method to compute lower bounds and utilize the Interval Newton method to facilitate the convergence of algorithm in the neighborhood of the global optimum point, Van Voorhis [11] developed a branch and bound algorithm for globally solving the NQCQP problems. By partitioning the feasible region into the Cartesian product of two-dimensional triangles and rectangles and by utilizing the convex and concave envelopes of bilinear functions over triangles and rectangles, a simplicial branch and bound algorithm [12] was presented for solving globally the NQCQP problems. Based on semidefinite relaxations and finite KKT-branching method, Burer and Vandebussche [13] presented a finite branch and bound algorithm for globally solving the NQCQP problems. In [14], Zheng et al. presented a decomposition-approximation method for constructing convex relaxations of the NQCQP problems, which can be used to offer a tighter lower bound for solving the problems (NQCQP). Using duality bounds approach, Thoai [15] presented a branch and bound algorithm for solving the NQCQP problems. Based on linear relaxation approximation technique and linearity-based range deleting tactics, Gao et al. [16] presented a rectangle branch and reduce method for the linearly constrained quadratic programs problems; by utilizing linearizing technique and quadratic constraint-based range compressing technique, Gao et al. [17] presented a branch and reduce approach for globally solving the NQCQP problems. By utilizing the special structure of quadratic function and linearization technique, Shen et al. [18] and Shen and Liu [19] proposed two effective global optimization algorithms for computing the NQCQP problems. By making use of linear relaxation approximation technique, Qu et al. [20] and Jiao and Chen [21] proposed two deterministic algorithms for calculating the NQCQP problems. Except for the above reviewed references, several algorithms for solving generalized geometric programming problem presented in [22–30] can be also used to solve the NQCQP problems. For an excellent review of recent advances in global optimization, the reader is referred to Floudas and Gounaris [31].

In this paper, by combing a parametric linearizing approach with a cutting down approach, we will present a novel global optimization method for solving the NQCQP problems. The main characteristics of the presented approach are expounded as follows. Firstly, a novel linear relaxation approximation technique, that is, a parametric linearizing approach, is constructed for inconsistently transforming the NQCQP problems into a sequence of parametric linear programs relaxation problems, and by subsequently subdividing the initial hyperrectangle the optimal point of parametric linear programs relaxation problems can infinitely approach the global optimum point of the problem (NQCQP). Secondly, the proposed parametric linear

programs relaxation problems are putted into a branch and bound framework without adding any new variables and constraint functions and which can be easily calculated by any effective linear programs algorithm procedure. Thirdly, a cutting down approach is exploited to eliminate a large part of the currently investigated subhyperrectangle which does not contain the global optimum point of the problem (NQCQP). Combing the parametric linear programs relaxation problem with the cutting down approach in a branch and bound procedure, a new optimization method is displayed for globally solving the NQCQP problems. At last, numerical results indicate that the proposed method can be employed to obtain the global optimum point of the problem (NQCQP).

The remainder of this paper is composed as follows. The next section describes a novel parametric linearizing approach and the parametric linear programs relaxation of the problem (NQCQP) is constructed. In Section 3 a cutting down approach is presented. Section 4 combines the cutting down approach within a branch and bound scheme; an optimization algorithm and its global convergence are described. In Section 5 some test examples and their results are reported to demonstrate the feasibility and superiority of the presented algorithm. At last, some concluding remarks are described.

2. Parametric Linear Relaxation

The principal composition in the configuration of a branch and bound algorithm for globally solving the problem (NQCQP) is the computation of lower bounds of the problem (NQCQP) and its divided subproblems. The lower bounds of the global minimum values of the problem (NQCQP) and its divided subproblems can be calculated by solving a series of parametric linear programs relaxation problem (PLPRP). In order to construct the PLPRP, the proposed approach is to replace each quadratic function $G_i(x)$ by a parametric linear function.

Let $X = (X_j)_{n \times 1} \subseteq X^0$ with $X_j = [l_j, u_j]$ ($j = 1, \dots, n$). For each $i = 0, 1, \dots, m$, define $x(\theta_i) = l + \theta_i(u - l)$, where $\theta_i \in \{0, 1\}$, $l^T = \{l_1, l_2, \dots, l_n\}$, $u^T = \{u_1, u_2, \dots, u_n\}$, $x(0) = l$, and $x(1) = u$. For convenience in expression, for all $x \in X \subseteq X^0$, for each $i = 0, 1, \dots, m$, some symbols are given as follows:

$$\underline{Z}_i = 2 \times \begin{pmatrix} \sum_{k=1, a_{1k}^i > 0}^n a_{1k}^i l_k + \sum_{k=1, a_{1k}^i < 0}^n a_{1k}^i u_k \\ \sum_{k=1, a_{2k}^i > 0}^n a_{2k}^i l_k + \sum_{k=1, a_{2k}^i < 0}^n a_{2k}^i u_k \\ \vdots \\ \sum_{k=1, a_{nk}^i > 0}^n a_{nk}^i l_k + \sum_{k=1, a_{nk}^i < 0}^n a_{nk}^i u_k \end{pmatrix},$$

$$\bar{Z}_i = 2 \times \begin{pmatrix} \sum_{k=1, a_{1k}^i > 0}^n a_{1k}^i u_k + \sum_{k=1, a_{1k}^i < 0}^n a_{1k}^i l_k \\ \sum_{k=1, a_{2k}^i > 0}^n a_{2k}^i u_k + \sum_{k=1, a_{2k}^i < 0}^n a_{2k}^i l_k \\ \vdots \\ \sum_{k=1, a_{nk}^i > 0}^n a_{nk}^i u_k + \sum_{k=1, a_{nk}^i < 0}^n a_{nk}^i l_k \end{pmatrix},$$

$$f_i(x) = x^T A^i x, \tag{1}$$

$$Z_i(\theta_i) = \underline{Z}_i + \theta_i (\bar{Z}_i - \underline{Z}_i),$$

$$Z_i(1 - \theta_i) = \underline{Z}_i + (1 - \theta_i) (\bar{Z}_i - \underline{Z}_i),$$

$$f_i^l(x, X, \theta_i) = Z_i(\theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) - Z_i(\theta_i)^T x(\theta_i),$$

$$f_i^u(x, X, \theta_i) = Z_i(1 - \theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) - Z_i(1 - \theta_i)^T x(\theta_i).$$

Obviously, we have

$$Z_i(0) = \underline{Z}_i, \quad Z_i(1) = \bar{Z}_i. \tag{2}$$

Theorem 1. For any $x \in X \subseteq X^0$, for each $i = 0, 1, \dots, m$, consider the function $f_i(x)$, $f_i^l(x, X, \theta_i)$ and $f_i^u(x, X, \theta_i)$; then the following conclusions hold:

- (i) $f_i^l(x, X, \theta_i) \leq f_i(x) \leq f_i^u(x, X, \theta_i)$;
- (ii) $\|f_i^u(x, X, \theta_i) - f_i(x)\| \rightarrow 0$ and $\|f_i(x) - f_i^l(x, X, \theta_i)\| \rightarrow 0$ as $\|u - l\| \rightarrow 0$.

Proof. (i) The gradient function of the function $f_i(x) = x^T A^i x$, $i = 0, 1, \dots, m$, can be expressed as follows:

$$\frac{\partial f_i(x)}{\partial x} = 2A^i x = 2 \times \begin{pmatrix} \sum_{k=1}^n a_{1k}^i x_k \\ \sum_{k=1}^n a_{2k}^i x_k \\ \vdots \\ \sum_{k=1}^n a_{nk}^i x_k \end{pmatrix}. \tag{3}$$

Obviously, we have

$$\underline{Z}_i \leq \frac{\partial f_i(x)}{\partial x} = 2A^i x \leq \bar{Z}_i. \tag{4}$$

By the mean value theorem, for any $x \in X$, there exists a point $\xi = \alpha x + (1 - \alpha)x(\theta_i)$, where $\alpha \in [0, 1]$, such that

$$f_i(x) = f_i(x(\theta_i)) + \left(\frac{\partial f_i(\xi)}{\partial \xi}\right)^T (x - x(\theta_i)) = x(\theta_i)^T A^i x(\theta_i) + \left(\frac{\partial f_i(\xi)}{\partial \xi}\right)^T (x - x(\theta_i)). \tag{5}$$

If $\theta_i = 0$, then we have

$$\frac{\partial f_i(\xi)}{\partial \xi} \geq \underline{Z}_i = Z_i(\theta_i), \tag{6}$$

$$x - x(\theta_i) = x - l \geq 0, \quad \text{for any } x \in X.$$

If $\theta_i = 1$, it follows that

$$\frac{\partial f_i(\xi)}{\partial \xi} \leq \bar{Z}_i = Z_i(\theta_i), \tag{7}$$

$$x - x(\theta_i) = x - u \leq 0, \quad \text{for any } x \in X.$$

Therefore, we can get that

$$f_i(x) = x(\theta_i)^T A^i x(\theta_i) + \left(\frac{\partial f_i(\xi)}{\partial \xi}\right)^T (x - x(\theta_i)) \geq x(\theta_i)^T A^i x(\theta_i) + Z_i(\theta_i)^T (x - x(\theta_i)) = f_i^l(x, X, \theta_i). \tag{8}$$

Analogously, if $\theta_i = 0$, then it follows that

$$\frac{\partial f_i(\xi)}{\partial \xi} \leq \bar{Z}_i = Z_i(1 - \theta_i), \tag{9}$$

$$x - x(\theta_i) = x - l \geq 0, \quad \text{for any } x \in X.$$

If $\theta_i = 1$, the following inequalities hold:

$$\frac{\partial f_i(\xi)}{\partial \xi} \geq \underline{Z}_i = Z_i(1 - \theta_i), \tag{10}$$

$$x - x(\theta_i) = x - u \leq 0, \quad \text{for any } x \in X.$$

Hence, it follows as above that

$$f_i(x) = x(\theta_i)^T A^i x(\theta_i) + \left(\frac{\partial f_i(\xi)}{\partial \xi}\right)^T (x - x(\theta_i)) \leq x(\theta_i)^T A^i x(\theta_i) + Z_i(1 - \theta_i)^T (x - x(\theta_i)) = f_i^u(x, X, \theta_i). \tag{11}$$

The conclusion (i) is followed.

(ii) Consider $\|f_i(x) - f_i^l(x, X, \theta_i)\|$, we have

$$\begin{aligned}
& \|f_i(x) - f_i^l(x, X, \theta_i)\| \\
&= \left\| x^T A^i x - [Z_i(\theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) - Z_i(\theta_i)^T x(\theta_i)] \right\| \\
&= \left\| \left[x(\theta_i)^T A^i x(\theta_i) + \left(\frac{\partial f_i(\xi)}{\partial \xi} \right)^T (x - x(\theta_i)) \right] \right. \\
&\quad \left. - [Z_i(\theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) - Z_i(\theta_i)^T x(\theta_i)] \right\| \\
&= \left\| \left(\frac{\partial f_i(\xi)}{\partial \xi} \right)^T (x - x(\theta_i)) - [Z_i(\theta_i)^T x - Z_i(\theta_i)^T x(\theta_i)] \right\| \\
&= \left\| \left(\frac{\partial f_i(\xi)}{\partial \xi} - Z_i(\theta_i) \right)^T (x - x(\theta_i)) \right\| \\
&\leq \left\| \frac{\partial f_i(\xi)}{\partial \xi} - Z_i(\theta_i) \right\| \|x - x(\theta_i)\| \\
&\leq \|\bar{Z}_i - \underline{Z}_i\| \|x - x(\theta_i)\| \\
&\leq \|\bar{Z}_i - \underline{Z}_i\| \|u - l\|,
\end{aligned} \tag{12}$$

where $\xi = \alpha x + (1 - \alpha)x(\theta_i)$ with $\alpha \in [0, 1]$.

Since $\|\bar{Z}_i - \underline{Z}_i\| \|u - l\| \rightarrow 0$ as $\|u - l\| \rightarrow 0$, thus we can get that

$$\|f_i(x) - f_i^l(x, X, \theta_i)\| \rightarrow 0 \quad \text{as } \|u - l\| \rightarrow 0. \tag{13}$$

Similarly, we can prove that

$$\|f_i^u(x, X, \theta_i) - f_i(x)\| \rightarrow 0 \quad \text{as } \|u - l\| \rightarrow 0. \tag{14}$$

The conclusion is complete. \square

By Theorem 1, for any $x \in X$, for each $i = 0, 1, \dots, m$, we define

$$\begin{aligned}
G_i^L(x, X, \theta_i) &= Z_i(\theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) \\
&\quad - Z_i(\theta_i)^T x(\theta_i) + (d^i)^T x, \\
G_i^U(x, X, \theta_i) &= Z_i(1 - \theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) \\
&\quad - Z_i(1 - \theta_i)^T x(\theta_i) + (d^i)^T x.
\end{aligned} \tag{15}$$

Theorem 2. For any $x \in X \subseteq X^0$, for each $i = 0, 1, \dots, m$, the following conclusions hold:

- (i) $G_i^L(x, X, \theta_i) \leq G_i(x) \leq G_i^U(x, X, \theta_i)$;
(ii) $\|G_i^U(x, X, \theta_i) - G_i(x)\| \rightarrow 0$ and $\|G_i(x) - G_i^L(x, X, \theta_i)\| \rightarrow 0$ as $\|u - l\| \rightarrow 0$.

Proof. (i) By the definitions of $G_i^L(x, X, \theta_i)$, $G_i(x)$ and $G_i^U(x, X, \theta_i)$ and the conclusion (i) of Theorem 1, we have

$$\begin{aligned}
& G_i^L(x, X, \theta_i) \\
&= Z_i(\theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) - Z_i(\theta_i)^T x(\theta_i) + (d^i)^T x \\
&= f_i^l(x, X, \theta_i) + (d^i)^T x \\
&\leq f_i(x) + (d^i)^T x \\
&= G_i(x) \\
&\leq f_i^u(x, X, \theta_i) + (d^i)^T x \\
&= Z_i(1 - \theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) \\
&\quad - Z_i(1 - \theta_i)^T x(\theta_i) + (d^i)^T x \\
&= G_i^U(x, X, \theta_i).
\end{aligned} \tag{16}$$

Therefore, the conclusion (i) is followed.

(ii) By the expressions of $G_i^L(x, X, \theta_i)$, $G_i(x)$, $G_i^U(x, X, \theta_i)$, $f_i^l(x, X, \theta_i)$, $f_i(x)$, and $f_i^u(x, X, \theta_i)$, we can get that

$$\begin{aligned}
& \|G_i(x) - G_i^L(x, X, \theta_i)\| \\
&= \left\| [x^T A^i x + (d^i)^T x] - [Z_i(\theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) \right. \\
&\quad \left. - Z_i(\theta_i)^T x(\theta_i) + (d^i)^T x] \right\| \\
&= \left\| x^T A^i x - [Z_i(\theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) - Z_i(\theta_i)^T x(\theta_i)] \right\| \\
&= \|f_i(x) - f_i^l(x, X, \theta_i)\|, \\
& \|G_i^U(x, X, \theta_i) - G_i(x)\| \\
&= \left\| [Z_i(1 - \theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) - Z_i(1 - \theta_i)^T x(\theta_i) \right. \\
&\quad \left. + (d^i)^T x] - [x^T A^i x + (d^i)^T x] \right\| \\
&= \left\| [Z_i(1 - \theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) \right. \\
&\quad \left. - Z_i(1 - \theta_i)^T x(\theta_i)] - x^T A^i x \right\| \\
&= \|f_i^u(x, X, \theta_i) - f_i(x)\|.
\end{aligned} \tag{17}$$

By the proof of Theorem 1 (ii), we can get

$$\begin{aligned} & \|G_i^U(x, X, \theta_i) - G_i(x)\| \\ &= \|f_i^u(x, X, \theta_i) - f_i(x)\| \rightarrow 0 \quad \text{as } \|u - l\| \rightarrow 0, \\ & \|G_i(x) - G_i^L(x, X, \theta_i)\| \\ &= \|f_i(x) - f_i^l(x, X, \theta_i)\| \rightarrow 0 \quad \text{as } \|u - l\| \rightarrow 0. \end{aligned} \tag{18}$$

The conclusion is complete. \square

By Theorem 2, we can construct the underestimating approximation parametric linear programs relaxation problem (PLPRP) of the problem (NQCQP) over subhyperrectangle X as follows:

$$\begin{aligned} \min \quad & G_0^L(x, X, \theta_0), \\ \text{s.t.} \quad & G_i^L(x, X, \theta_i) \leq b_i, \quad i = 1, \dots, m, \quad (\text{PLPRP}) \\ & x \in X = \{x : l \leq x \leq u\} \subseteq X^0, \end{aligned}$$

where

$$\begin{aligned} G_i^L(x, X, \theta_i) = & Z_i(\theta_i)^T x + x(\theta_i)^T A^i x(\theta_i) \\ & - Z_i(\theta_i)^T x(\theta_i) + (d^i)^T x. \end{aligned} \tag{19}$$

Based on the above parametric linearizing approach, every feasible point of the problem (NQCQP) is feasible to the problem (PLPRP) over the subhyperrectangle X , and the objective function value of the problem (PLPRP) at each feasible point is less than or equal to that of the problem (NQCQP) over the subhyperrectangle X . Thus, the optimal value of the problem (PLPRP) offers a valid lower bound for the optimal value of the problem (NQCQP) over the subhyperrectangle X .

3. Cutting Down Approach

To enhance the computational speed of the investigated algorithm, based on the above parametric linear relaxation problem, a novel cutting down approach is described in the following Theorem 3. At the k -th iteration of the proposed algorithm, we will judge whether or not the subhyperrectangle X contains a global optimum point of the problem (NQCQP). The cutting down approach can be used to reject a part of the subhyperrectangle X or the whole X without deleting any global optimum point of the problem (NQCQP). For convenience, for any $x \in X = (X_j)_{n \times 1}$ with $X_j = [l_j, u_j]$ ($j = 1, \dots, n$), without loss of generality, we express the $G_i^L(x, X, \theta_i)$ in the problem (PLPRP) over the subhyperrectangle X as the following form:

$$G_i^L(x, X, \theta_i) = \sum_{j=1}^n \gamma_{ij}(\theta_i) x_j + \eta_i(\theta_i), \quad i = 0, 1, \dots, m. \tag{20}$$

Assume that UB_k is the currently known upper bound of the proposed branch and bound algorithm, and for any fixed $\theta_i, i = 0, 1, \dots, m$, let

$$PLRB_i(\theta_i) = \sum_{j=1}^n \min \{ \gamma_{ij}(\theta_i) l_j, \gamma_{ij}(\theta_i) u_j \} + \eta_i(\theta_i),$$

$$i = 0, 1, \dots, m,$$

$$\beta_p(\theta_0) = UB_k - PLRB_0(\theta_0) + \min \{ \gamma_{0p}(\theta_0) l_p, \gamma_{0p}(\theta_0) u_p \},$$

$$p = 1, \dots, n,$$

$$\lambda_{ip}(\theta_i) = b_i - PLRB_i(\theta_i) + \min \{ \gamma_{ip}(\theta_i) l_p, \gamma_{ip}(\theta_i) u_p \},$$

$$p = 1, \dots, n, \quad i = 1, \dots, m. \tag{21}$$

And define $\bar{\bar{Y}} = (\bar{\bar{Y}}_j)_{n \times 1}, \underline{Y} = (\underline{Y}_j)_{n \times 1}, \bar{Y} = (\bar{Y}_j)_{n \times 1}$, and $\hat{Y} = (\hat{Y}_j)_{n \times 1}$, where

$$\begin{aligned} \bar{Y}_j = & \begin{cases} X_j, & j \neq p, \quad j = 1, \dots, n, \\ \left(\frac{\lambda_{ip}(\theta_i)}{\gamma_{ip}(\theta_i)}, u_p \right) \cap X_p, & j = p; \end{cases} \\ \hat{Y}_j = & \begin{cases} X_j, & j \neq p, \quad j = 1, \dots, n, \\ \left[l_p, \frac{\lambda_{ip}(\theta_i)}{\gamma_{ip}(\theta_i)} \right) \cap X_p, & j = p; \end{cases} \\ \bar{\bar{Y}}_j = & \begin{cases} X_j, & j \neq p, \quad j = 1, \dots, n, \\ \left(\frac{\beta_p(\theta_0)}{\gamma_{0p}(\theta_0)}, u_p \right) \cap X_p, & j = p; \end{cases} \\ \underline{Y}_j = & \begin{cases} X_j, & j \neq p, \quad j = 1, \dots, n, \\ \left[l_p, \frac{\beta_p(\theta_0)}{\gamma_{0p}(\theta_0)} \right) \cap X_p, & j = p. \end{cases} \end{aligned} \tag{22}$$

Theorem 3. For any subhyperrectangle $X \subseteq X^0$, one has the following results.

- (i) If $PLRB_0(\theta_0) > UB_k$, then the subhyperrectangle X does not contain the global optimum point of the problem (NQCQP).
- (ii) If $PLRB_0(\theta_0) \leq UB_k$, then for any $p \in \{1, 2, \dots, n\}$, if $\gamma_{0p}(\theta_0) > 0$, then the subhyperrectangle $\bar{\bar{Y}} = (\bar{\bar{Y}}_j)_{n \times 1}$ does not contain the global optimum point of the problem (NQCQP); if $\gamma_{0p}(\theta_0) < 0$, then the subhyperrectangle $\underline{Y} = (\underline{Y}_j)_{n \times 1}$ does not contain the global optimum point of the problem (NQCQP).
- (iii) If there exists some $i \in \{1, \dots, m\}$ such that $PLRB_i(\theta_i) > b_i$, then the subhyperrectangle X does not contain the global optimum point of the problem (NQCQP).
- (iv) If $PLRB_i(\theta_i) \leq b_i$ for all $i = 1, \dots, m$, then for each $p \in \{1, 2, \dots, n\}$, if $\gamma_{ip}(\theta_i) > 0$, then the subhyperrectangle $\bar{Y} = (\bar{Y}_j)_{n \times 1}$ does not contain the global optimum point

of the problem (NQCQP); if $\gamma_{ip}(\theta_i) < 0$, then the subhyperrectangle $\hat{Y} = (\hat{Y}_j)_{n \times 1}$ does not contain the global optimum point of the problem (NQCQP).

Proof. (i) If $\text{PLRB}_0(\theta_0) > \text{UB}_k$, then for all $x \in X$, by the Theorem 2 we have

$$\begin{aligned} G_0(x) &\geq G_0^L(x, X, \theta_0) \\ &\geq \sum_{j=1}^n \min \{ \gamma_{0j}(\theta_0) l_j, \gamma_{0j}(\theta_0) u_j \} + \eta_0(\theta_0) \quad (23) \\ &= \text{PLRB}_0(\theta_0) > \text{UB}_k. \end{aligned}$$

Hence, the X does not contain the global optimum point of the problem (NQCQP).

(ii) If $\text{PLRB}_0(\theta_0) \leq \text{UB}_k$, then for any $p \in \{1, \dots, n\}$, if $\gamma_{0p}(\theta_0) > 0$, for all $x \in \bar{Y}$, we have $x_p > \beta_p(\theta_0)/\gamma_{0p}(\theta_0)$; that is, $\gamma_{0p}(\theta_0)x_p > \text{UB}_k - \text{PLRB}_0(\theta_0) + \min\{\gamma_{0p}(\theta_0)l_p, \gamma_{0p}(\theta_0)u_p\}$. Thus, we have

$$\begin{aligned} &G_0^L(x, \bar{Y}, \theta_0) \\ &= \sum_{j=1, j \neq p}^n \gamma_{0j}(\theta_0) x_j + \gamma_{0p}(\theta_0) x_p + \eta_0(\theta_0) \\ &\geq \sum_{j=1, j \neq p}^n \min \{ \gamma_{0j}(\theta_0) l_j, \gamma_{0j}(\theta_0) u_j \} + \eta_0(\theta_0) + \gamma_{0p}(\theta_0) x_p \\ &> \sum_{j=1, j \neq p}^n \min \{ \gamma_{0j}(\theta_0) l_j, \gamma_{0j}(\theta_0) u_j \} + \eta_0(\theta_0) \\ &\quad + \text{UB}_k - \text{PLRB}_0(\theta_0) + \min \{ \gamma_{0p}(\theta_0) l_p, \gamma_{0p}(\theta_0) u_p \} \\ &= \text{PLRB}_0(\theta_0) + \text{UB}_k - \text{PLRB}_0(\theta_0) \\ &= \text{UB}_k. \end{aligned} \quad (24)$$

Therefore, by the above inequality and Theorem 2, we get that

$$G_0(x) \geq G_0^L(x, \bar{Y}, \theta_0) > \text{UB}_k. \quad (25)$$

Hence, the rectangle \bar{Y} does not contain the global optimum point of the problem (NQCQP).

Similarly, if $\gamma_{0p}(\theta_0) < 0$, we can prove that the subhyperrectangle \underline{Y} does not contain the global optimum point of the problem (NQCQP).

Using similar proving method as the above, we can draw the conclusions (iii) and (iv). \square

By Theorem 3, by making use of the above cutting down approach to reject a part of the investigated subhyperrectangle which does not contain the global minimum point of the problem (NQCQP), thus we can enhance the computational speed of the proposed branch and bound algorithm.

4. Algorithm and Its Convergence

In this section, based on the former parametric linear programs relaxation problem, we present a novel optimization method for globally solving the problem (NQCQP). There are three fundamental compositions in the presented method: a branching approach, an updating upper bounds approach, and an updating lower bounds approach.

The branching approach iteratively subdivides the investigated hyperrectangle X^k into two subhyperrectangles, which produces a more refined partition for computing the global optimum point of the problem (NQCQP). In this paper we select a simple partitioning approach, which is enough to guarantee the global convergence of the presented branch and bound algorithm. For any selected subhyperrectangle $X^k = [l^k, u^k] \subseteq X^0$, the selected partitioning approach is described as follows.

(a) Let $\eta = \arg \max \{ u_i^k - l_i^k : i = 1, \dots, n \}$.

(b) Let

$$\begin{aligned} X^{k,1} &= \left\{ x \in R^n \mid l_i^k \leq x_i \leq u_i^k, i \neq \eta, l_\eta^k \leq x_\eta \leq \frac{l_\eta^k + u_\eta^k}{2} \right\}, \\ X^{k,2} &= \left\{ x \in R^n \mid l_i^k \leq x_i \leq u_i^k, i \neq \eta, \frac{l_\eta^k + u_\eta^k}{2} \leq x_\eta \leq u_\eta^k \right\}. \end{aligned} \quad (26)$$

By making use of this branching approach, the selected hyperrectangle X^k is subdivided into two subhyperrectangles $X^{k,1}$ and $X^{k,2}$.

The updating lower bounds approach needs to compute a sequence of parametric linear programs relaxation problems by using the simplex approach. The updating upper bounds approaches need to calculate the objective function value of the feasible point of the problem (NQCQP), where the feasible point can be found by solving the parametric linear programs relaxation problem and probing the feasibility of the midpoint of the investigated subhyperrectangle X^k , respectively.

4.1. Novel Optimization Algorithm. Let $\text{LB}(X^k)$ and $x^k = x(X^k)$ be the optimal value and the optimal solution of the problem (PLPRP) over subhyperrectangle X^k , respectively. Combining the former parametric linear programs relaxation problem with the cutting down approach in a branch and bound framework, a novel global optimization method for the problem (NQCQP) is described as follows.

Algorithm Steps

Step 1 (initializing). Initialize the iteration counter $k := 0$, the collection of all active node $\Omega_0 = \{X^0\}$, the feasible solution set $F = \emptyset$, the convergence error $\epsilon > 0$, and the upper bound $\text{UB}_0 = +\infty$.

Solve the problem (PLPRP) over the hyperrectangle X^0 to compute $LB_0 := LB(X^0)$ and $x^0 := x(X^0)$. For all $i = 1, \dots, m$, if $G_i(x^0) \leq b_i$, then let $F = \{x^0\}$ and $UB_0 = G_0(x^0)$.

If $UB_0 - LB_0 \leq \epsilon$, then algorithm terminates with x^0 as the global optimum point of the problem (NQCQP). Otherwise, go to Step 2.

Step 2 (partitioning hyperrectangle). Utilizing the proposed branching approach, select a branching variable x_η to partition X^k into two new subhyperrectangles, and still let the new collection of partitioned subhyperrectangles by \bar{X}^k .

Step 3 (cutting down region). For each subhyperrectangle $X \in \bar{X}^k$, for any fixed parameter vector $\theta = (\theta_0, \theta_1, \dots, \theta_m)$, compute $PLRB_i(\theta_i)$ ($i = 0, 1, \dots, m$), $\beta_p(\theta_0)$ ($p = 1, \dots, n$), and $\lambda_{ip}(\theta_i)$ ($i = 1, \dots, m, p = 1, \dots, n$).

For each $i \in \{1, \dots, m\}$, if $PLRB_i(\theta_i) > b_i$, let $X = \emptyset$; else

if $\gamma_{ip}(\theta_i) > 0$ and $\lambda_{ip}(\theta_i)/\gamma_{ip}(\theta_i) < u_p$ for some $p \in \{1, \dots, n\}$, then let $u_p = \lambda_{ip}(\theta_i)/\gamma_{ip}(\theta_i)$.

else if $\gamma_{ip}(\theta_i) < 0$ and $\lambda_{ip}(\theta_i)/\gamma_{ip}(\theta_i) > l_p$ for some $p \in \{1, \dots, n\}$, then let $l_p = \lambda_{ip}(\theta_i)/\gamma_{ip}(\theta_i)$.

If $PLRB_0(\theta_0) > UB_k$, then let $X = \emptyset$; else

if $\gamma_{0p}(\theta_0) > 0$ and $\beta_p(\theta_0)/\gamma_{0p}(\theta_0) < u_p$ for some $p \in \{1, \dots, n\}$, then let $u_p = \beta_p(\theta_0)/\gamma_{0p}(\theta_0)$,

else if $\gamma_{0p}(\theta_0) < 0$ and $\beta_p(\theta_0)/\gamma_{0p}(\theta_0) > l_p$ for some $p \in \{1, \dots, n\}$, then let $l_p = \beta_p(\theta_0)/\gamma_{0p}(\theta_0)$.

At last, still let the remaining subhyperrectangle be X , and let the remaining partitioned set be \bar{X}^k .

Step 4 (feasibility fathoming). For each new subhyperrectangle $X \in \bar{X}^k$, compute the lower bounds $LB(X)$ and $x(X)$ by solving the problem (PLPRP) over X .

If $LB(X) > UB_k$, set $\bar{X}^k := \bar{X}^k \setminus X$; otherwise, if the midpoint x^{mid} of X satisfies $G_i(x^{\text{mid}}) \leq b_i$ for all $i = 1, \dots, m$, then let $F := F \cup \{x^{\text{mid}}\}$, and if $x(X)$ satisfies $G_i(x(X)) \leq b_i$ for all $i = 1, \dots, m$, then let $F := F \cup \{x(X)\}$,

Step 5 (renewing bound). Renew the upper bound $UB_k := \min_{x \in F} G_0(x)$. If $F \neq \emptyset$, the best known feasible solution is denoted by $x^k := \arg \min_{x \in F} G_0(x)$. Let $\Theta_k := (\Theta_k \setminus X^k) \cup \bar{X}^k$, and renew the lower bound $LB_k := \inf_{x \in \Theta_k} LB(X)$.

Step 6 (convergence fathoming). If $UB_k - LB_k \leq \epsilon$, then algorithm stops, and we get that UB_k is the global ϵ -minimum value of the problem (NQCQP), and x^k is a global ϵ -optimum point. Otherwise, $k := k + 1$, and select a new subhyperrectangle X^k such that $X^k = \arg \min_{X \in \Theta_k} LB(X)$, and return to Step 2.

4.2. Global Convergence of the Algorithm. The global convergence of the presented algorithm is described as follows.

Theorem 4. *If the proposed algorithm stops finitely at the k -th iteration, then when the algorithm stops, x^k is the global optimum point of the problem (NQCQP); else it will bring about an infinite sequence $\{x^k\}$ of iteration, such that any accumulation point x^* of the sequence $\{x^k\}$ will be the global optimum point of the problem (NQCQP), and the sequence $\{UB_k\}$ is nonincreasing and the sequence $\{LB_k\}$ is nondecreasing; moreover they meet $\lim_{k \rightarrow \infty} UB_k = \lim_{k \rightarrow \infty} LB_k = v^*$, where v^* is the global minimum value of the problem (NQCQP).*

Proof. (i) If the proposed algorithm stops finitely at the k -th iteration, then when it stops, we get that $UB_k = v^* = LB_k$. Thus, by the characteristic of the proposed branch and bound algorithm, we get the global optimum point x^k of the problem (NQCQP). If the proposed algorithm is infinite, then it must bring about an infinite subhyperrectangle sequence $\{X^k\}$, since the used branching approach is exhaustive, we get that the subhyperrectangle sequence $\{X^k\}$ converges to a point. By the branch and bound characteristic of the algorithm we get that the sequence $\{UB_k\}$ is nonincreasing, and the sequence $\{LB_k\}$ is nondecreasing; therefore the sequence $\{UB_k - LB_k\}$ is a positive and nonincreasing sequence. From Theorem 2, we know that the sequence $\{UB_k - LB_k\}$ must be convergent to zero. Also $LB_k \leq v^* \leq UB_k$ for each k indicate that $\lim_{k \rightarrow \infty} UB_k = \lim_{k \rightarrow \infty} LB_k = v^*$. Since x^k is always a feasible solution of the problem (NQCQP) and the upper bound $UB_k = G_0(x^k)$, any cluster point x^* of the sequence $\{x^k\}$ must be feasible to the problem (NQCQP) with objective function value $v^* = G_0(x^*)$. Therefore, the conclusion is followed. \square

5. Numerical Experiments

To validate the performance and computational efficiency of the presented optimization method, several common test examples in literatures are put into effect on microcomputer, the algorithm program is coded in C++, the simplex approach is applied to solve a series of parametric linear programs relaxation problems, and the termination tolerance error is set to $\epsilon = 10^{-6}$. These test examples are described and their numerical results are listed as follows. In the following Tables 1, 2, and 3, number of algorithm iteration and computational time in seconds are denoted by "Iter" and "Time," respectively.

Example 1 (see [21, 32]). Consider

$$\begin{aligned}
 \min \quad & -x_1^2 + x_1 + x_2^2 - 2x_2 + x_1x_2 \\
 \text{s.t.} \quad & x_1 + x_2 \leq 6, \\
 & -2x_1^2 + x_2^2 + 2x_1 + x_2 \leq -4, \\
 & 1 \leq x_1, \quad x_2 \leq 6.
 \end{aligned} \tag{27}$$

TABLE 1: Numerical results for Examples 1–3.

Example	Refs.	$(\theta_0, \dots, \theta_m)$	Optimal solution	Optimal value	Iter	Time (s)
1	our1	(0, 0, 0)	(5.0, 1.0)	-16.000000000	3	0.00190359
	our2	(0, 0, 1)	(5.0, 1.0)	-16.000000000	2	0.000975264
	our3	(0, 1, 1)	(5.0, 1.0)	-16.000000000	2	0.000958502
	our4	(1, 1, 1)	(5.0, 1.0)	-16.000000000	3	0.00115294
	our5	(1, 0, 0)	(5.0, 1.0)	-16.000000000	4	0.0013566
	our6	(1, 0, 1)	(5.0, 1.0)	-16.000000000	3	0.00114037
	our7	(1, 1, 0)	(5.0, 1.0)	-16.000000000	4	0.00142281
	our8	(0, 1, 0)	(5.0, 1.0)	-16.000000000	3	0.00108701
	[32]		(5.0, 1.0)	-16.0	20	0.175
	[21]		(5.0, 1.0)	-16.0	5	0.00184588
2	our1	(0, 0)	(2.0, 1.666666667)	6.777778018	32	0.00662822
	our2	(0, 1)	(2.0, 1.666666667)	6.777778233	32	0.00665057
	our3	(1, 1)	(2.0, 1.666666667)	6.777778517	33	0.00698972
	our4	(1, 0)	(2.0, 1.666666667)	6.777778743	32	0.00685227
	[19]		(2.0, 1.666666667)	6.777782016	40	0.032
	[22]		(2.00003, 1.66665)	6.7780	44	0.18
	3	our1	(0, 0, 0)	(0.5, 0.5)	0.500000000	25
our2		(0, 0, 1)	(0.5, 0.5)	0.500000000	25	0.00733278
our3		(0, 1, 1)	(0.5, 0.5)	0.500000600	26	0.00686791
our4		(1, 1, 1)	(0.5, 0.5)	0.500000600	26	0.00691568
our5		(1, 0, 0)	(0.5, 0.5)	0.500000000	25	0.00735261
our6		(1, 0, 1)	(0.5, 0.5)	0.500000000	25	0.00736742
our7		(1, 1, 0)	(0.5, 0.5)	0.500000600	26	0.00769232
our8		(0, 1, 0)	(0.5, 0.5)	0.500000600	26	0.00921374
[19]			(0.5, 0.5)	0.500004627	34	0.056
[21]			(0.5, 0.5)	0.500000442	37	0.0192625
[22]			(0.5, 0.5)	0.5	91	0.85

Example 2 (see [19, 21, 22]). Consider

$$\begin{aligned}
 \min \quad & x_1^2 + x_2^2 \\
 \text{s.t.} \quad & -0.3x_1x_2 \leq -1, \\
 & 2 \leq x_1 \leq 5, \quad 1 \leq x_2 \leq 3.
 \end{aligned} \tag{28}$$

Example 3 (see [19, 21, 22]). Consider

$$\begin{aligned}
 \min \quad & x_1 \\
 \text{s.t.} \quad & 4x_2 - 4x_1^2 \leq 1, \\
 & x_1 + x_2 \geq 1, \\
 & 0.01 \leq x_1, \quad x_2 \leq 15.
 \end{aligned} \tag{29}$$

Example 4 (see [21, 23]). Consider

$$\begin{aligned}
 \min \quad & x_1x_2 - 2x_1 + x_2 + 1 \\
 \text{s.t.} \quad & 8x_2^2 - 6x_1 - 16x_2 \leq -11, \\
 & -x_2^2 + 3x_1 + 2x_2 \leq 7, \\
 & 1 \leq x_1 \leq 2.5, \quad 1 \leq x_2 \leq 2.225.
 \end{aligned} \tag{30}$$

Example 5 (see [17, 21]). Consider

$$\begin{aligned}
 \min \quad & 6x_1^2 + 4x_2^2 + 5x_1x_2 \\
 \text{s.t.} \quad & -6x_1x_2 \leq -48, \quad 0 \leq x_1, x_2 \leq 10.
 \end{aligned} \tag{31}$$

Example 6 (see [21, 25]). Consider

$$\begin{aligned}
 \min \quad & -4x_2 + (x_1 - 1)^2 + x_2^2 - 10x_3^2 \\
 \min \quad & 6x_1^2 + 4x_2^2 + 5x_1x_2 \\
 & (x_1 - 2)^2 + x_2^2 + x_3^2 \leq 2, \\
 & 2 - \sqrt{2} \leq x_1 \leq \sqrt{2}, \quad 0 \leq x_2, x_3 \leq \sqrt{2}.
 \end{aligned} \tag{32}$$

Example 7 (see [33, 34]). Consider

$$\begin{aligned}
 \min \quad & x_1 + x_2 + x_3 \\
 \text{s.t.} \quad & 833.33252x_1^{-1}x_4x_6^{-6} + 100.0x_6^{-1} \\
 & - 83333.333x_1^{-1}x_6^{-1} \leq 1, \\
 & 1250.0x_2^{-1}x_5x_7^{-1} + 1.0x_4x_7^{-1} \\
 & - 1250.0x_2^{-1}x_4x_7^{-1} \leq 1,
 \end{aligned}$$

TABLE 2: Numerical results for Examples 4–6.

Example	Refs.	$(\theta_0, \dots, \theta_m)$	Optimal solution	Optimal value	Iter	Time (s)
4	our1	(0, 0, 0)	(2.0, 1.0)	-1.000000	0	0.00071504
	our2	(0, 0, 1)	(2.0, 1.0)	-1.000000000	1	0.00165866
	our3	(0, 1, 1)	(2.0, 1.0)	-1.000000000	2	0.00169589
	our4	(1, 1, 1)	(2.0, 1.0)	-1.000000000	2	0.00156529
	our5	(1, 0, 0)	(2.0, 1.0)	-1.000000000	2	0.00153087
	our6	(1, 0, 1)	(2.0, 1.0)	-1.000000000	2	0.00159881
	our7	(1, 1, 0)	(2.0, 1.0)	-1.000000000	2	0.00158519
	our8	(0, 1, 0)	(2.0, 1.0)	-1.000000	0	0.00066607
	[21]		(2.0, 1.0)	-0.999999410	21	0.00849446
[23]		(2.0, 1.0)	-1.0	24	0.0116923	
5	our1	(0, 0)	(2.555409888, 3.130613160)	118.383672050	49	0.0271463
	our2	(0, 1)	(2.555676683, 3.130286407)	118.383672115	51	0.0287108
	our3	(1, 1)	(2.555800904, 3.130134268)	118.383672231	61	0.0350215
	our4	(1, 0)	(2.555775920, 3.130164866)	118.383671937	55	0.030462
	[17]		(2.555779370, 3.130164640)	118.383756475	210	0.78
	[21]		(2.555745855, 3.130201688)	118.383671904	59	0.0385038
6	our1	(0, 0, 0)	(1.0, 0.181818247, 0.983332154)	-11.363635387	141	0.110562
	our2	(0, 0, 1)	(1.0, 0.181818217, 0.983332160)	-11.363636364	240	0.198587
	our3	(0, 1, 1)	(1.0, 0.181818196, 0.983332163)	-11.363635785	260	0.213692
	our4	(1, 1, 1)	(1.0, 0.181818133, 0.983332175)	-11.363635682	229	0.18844
	our5	(1, 0, 0)	(1.0, 0.181783067, 0.983338664)	-11.363635889	80	0.0669022
	our6	(1, 0, 1)	(1.0, 0.181818133, 0.983332175)	-11.363635715	213	0.176459
	our7	(1, 1, 0)	(1.0, 0.181818301, 0.983332144)	-11.363635516	149	0.131391
	our8	(0, 1, 0)	(1.0, 0.181818133, 0.983332175)	-11.363636364	133	0.105004
	[21]		(1.0, 0.181818470, 0.983332113)	-11.363636364	420	0.284541
[25]		(0.998712, 0.196213, 0.979216)	-10.35	1648	0.3438	

TABLE 3: Numerical comparisons with Refs. [16, 21] for Example 9.

Refs.	Number of variable	$n = 5$	$n = 10$	$n = 20$	$n = 30$
Ours	Computational time (s)	0.0124692	0.215709	3.69194	27.4522
	Iteration Number	12	31	86	204
Ref. [16]	Computational time (s)	10.11	21.86	47.00	106.33
	Iteration Number	141	283	651	965
Ref. [21]	Computational time (s)	0.0181791	0.302157	6.01095	44.4965
	Iteration Number	12	32	88	206

$$\begin{aligned}
 &1250000.0x_3^{-1}x_8^{-1} + 1.0x_5x_8^{-1} && (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\
 &- 2500.0x_3^{-1}x_5x_8^{-1} \leq 1, && \leq (10000, 10000, 10000, 1000, \\
 &0.0025x_4 + 0.0025x_6 \leq 1, && 1000, 1000, 1000, 1000). \\
 &- 0.0025x_4 + 0.0025x_5 && \\
 &+ 0.0025x_7 \leq 1, && \\
 &0.001x_8 - 0.001x_5 \leq 1, && \\
 &(100, 1000, 1000, 10, 10, 10, 10, 10) && \\
 &\leq (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8), &&
 \end{aligned}
 \tag{33}$$

This problem is from the heat exchanger design problem; solve this problem by the proposed method, with the given convergence error $\epsilon = 10^{-6}$ and parameter vector $\theta_i = 0$, $i = 0, 1, \dots, 6$, and a global ϵ -optimal solution

$$\begin{aligned}
 (x_1, \dots, x_8)^T &= (579.307, 1359.971, 5109.971, 182.018, \\
 &295.601, 217.982, 286.417, 395.6)
 \end{aligned}
 \tag{34}$$

is found after 18377 iterations with objective function value 7049.248020529, and computational time is 11.475 s.

Using the proposed approach in Floudas et al. [33], a global optimal solution, under the given convergence tolerance $\epsilon = 10^{-4}$,

$$(x_1, \dots, x_8)^T = (579.31, 1359.97, 5109.97, 182.01, \\ 295.60, 217.98, 286.42, 395.6) \quad (35)$$

is found with global optimal value 7049.25000.

Using the proposed approach in Lin and Tsai [34], under the convergence tolerance $\epsilon = 10^{-6}$, a global optimal solution

$$(x_1, \dots, x_8)^T = (578.973, 1359.573, 5110.701, 181.99, \\ 295.572, 218.01, 286.418, 395.572) \quad (36)$$

is found with global optimization value 7049.24692.

Example 8 (see [24]). Consider

$$\begin{aligned} \min \quad & 5.3578x_3^2 + 0.8357x_1x_5 + 37.2392x_1 \\ \text{s.t.} \quad & 0.00002584x_3x_5 - 0.00006663x_2x_5 \\ & - 0.0000734x_1x_4 \leq 1, \\ & 0.000853007x_2x_5 + 0.00009395x_1x_4 \\ & - 0.00033085x_3x_5 \leq 1, \\ & 1330.3294x_2^{-1}x_5^{-1} - 0.42x_1x_5^{-1} \\ & - 0.30586x_2^{-1}x_3^2x_5^{-1} \leq 1, \\ & 0.00024186x_2x_5 + 0.00010159x_1x_2 \\ & + 0.00007379x_3^2 \leq 1, \\ & 2275.1327x_3^{-1}x_5^{-1} - 0.2668x_1x_5^{-1} \\ & - 0.40584x_4x_5^{-1} \leq 1, \\ & 0.00029955x_3x_5 + 0.00007992x_1x_3 \\ & + 0.00012157x_3x_4 \leq 1, \\ & 78.0 \leq x_1 \leq 102.0, \quad 33.0 \leq x_2 \leq 45.0, \\ & 27.0 \leq x_3 \leq 45.0, \\ & 27.0 \leq x_4 \leq 45.0, \quad 27.0 \leq x_5 \leq 45.0. \end{aligned} \quad (37)$$

This test problem has a relative high degree of difficulty, and it contains both negative and positive terms. By using the proposed algorithm in this paper, initializing the parameter $\theta_i = 1$, $i = 0, 1, \dots, 6$, with the given convergence error $\epsilon = 10^{-6}$, the global ϵ -optimal solution

$$(x_1, x_2, \dots, x_5)^T = (78.0, 33.739811424, 30.594417922, \\ 44.991210938, 35.288085938) \quad (38)$$

can be obtained after the 51 iterations.

But using the proposed algorithm in [24], with the given convergence error $\epsilon = 10^{-6}$, the global ϵ -optimal solution

$$(x_1, x_2, \dots, x_5)^T = (78.0, 32.99999946, 29.995510365, \\ 44.999998630, 36.77517397) \quad (39)$$

can be obtained after the 523 iterations.

Example 9 (see [16, 21]). Consider

$$\begin{aligned} \min \quad & -\sum_{i=1}^n x_i^2 \\ \text{s.t.} \quad & \sum_{i=1}^j x_i \leq j, \quad j \in \{1, 2, \dots, n\}, \\ & x_i \geq 0, \quad i \in \{1, 2, \dots, n\}. \end{aligned} \quad (40)$$

Using the proposed algorithm in this paper, initializing the parameter $\theta_i = 0$, $i = 0, 1, \dots, n$, the numerical results are compared with those in [16, 21] and are illustrated in Table 3.

From the numerical results for Examples 1–9, our algorithm is competitive.

6. Concluding Remarks

In this paper, a novel optimization method based on the parametric linear programs relaxation problem is proposed for globally solving the NQCQP problem. The parametric linear programs relaxation problem is constructed by underestimating each quadratic function with a parametric linear function. By making use of the currently known upper bound and the parametric linear programs relaxation of the problem (NQCQP), a cutting down approach is constructed and used to enhance the computational speed of the branch and bound algorithm. The algorithm is convergent to the global optimum point by subdividing the initial hyperrectangle and solving sequences of parametric linear programs relaxation problems. Numerical experimental results are reported to demonstrate that the presented method can be employed to effectively solve the problem (NQCQP).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] V. Visweswaran and C. A. Floudas, "A global optimization algorithm (GOP) for certain classes of nonconvex NLPs II: applications of theory and test problems," *Computers and Chemical Engineering*, vol. 14, pp. 1417–1434, 1990.
- [2] D. P. Rutenberg and T. L. Shaftef, "Product design: sub-assemblies for multiple markets," *Management Science*, vol. 18, no. 4, pp. B220–B231, 1971.
- [3] E. V. Demands and C. S. Tang, "Linear control of a Markov production system," *Operations Research*, vol. 40, pp. 259–278, 1992.
- [4] E. Phan-huy-Hao, "Quadratically constrained quadratic programming: some applications and a method for solution," *Zeitschrift für Operations Research A*, vol. 26, no. 3, pp. 105–119, 1982.
- [5] A. Weintraub and J. Vera, "A cutting plane approach for chance constrained linear programs," *Operations Research*, vol. 39, no. 5, pp. 776–785, 1991.
- [6] T. Q. Phong, P. D. Tao, and L. T. H. An, "A method for solving d.c. programming problems: application to fuel mixture non-convex optimization problem," *Journal of Global Optimization*, vol. 6, no. 1, pp. 87–105, 1995.
- [7] F. A. Al-Khayyal, C. Larsen, and T. van Voorhis, "A relaxation method for nonconvex quadratically constrained quadratic programs," *Journal of Global Optimization*, vol. 6, no. 3, pp. 215–230, 1995.
- [8] F. Al-Khayyal and T. van Voorhis, "Accelerating convergence of branch and bound algorithms for quadratically constrained optimization problems," in *State of the Art in Global Optimization: Computational Methods and Applications*, C. A. Floudas, Ed., vol. 7, Kluwer Academic, London, UK, 1996.
- [9] J. A. Filar and T. A. Schultz, "Bilinear programming and structured stochastic games," *Journal of Optimization Theory and Applications*, vol. 53, no. 1, pp. 85–104, 1987.
- [10] H. D. Sherali and C. H. Tuncbilek, "A reformulation-convexification approach for solving nonconvex quadratic programming problems," *Journal of Global Optimization*, vol. 7, no. 1, pp. 1–31, 1995.
- [11] T. Van Voorhis, "A global optimization algorithm using Lagrangian underestimates and the interval Newton method," *Journal of Global Optimization*, vol. 24, no. 3, pp. 349–370, 2002.
- [12] J. Linderoth, "A simplicial branch-and-bound algorithm for solving quadratically constrained quadratic programs," *Mathematical Programming B*, vol. 103, no. 2, pp. 251–282, 2005.
- [13] S. Burer and D. Vandenbussche, "A finite branch-and-bound algorithm for nonconvex quadratic programming via semidefinite relaxations," *Mathematical Programming A*, vol. 113, no. 2, pp. 259–282, 2008.
- [14] X. J. Zheng, X. L. Sun, and D. Li, "Convex relaxations for nonconvex quadratically constrained quadratic programming: matrix cone decomposition and polyhedral approximation," *Mathematical Programming B*, vol. 129, no. 2, pp. 301–329, 2011.
- [15] N. V. Thoai, "Duality bound method for the general quadratic programming problem with quadratic constraints," *Journal of Optimization Theory and Applications*, vol. 107, no. 2, pp. 331–354, 2000.
- [16] Y. Gao, H. Xue, and P. Shen, "A new rectangle branch-and-reduce approach for solving nonconvex quadratic programming problems," *Applied Mathematics and Computation*, vol. 168, no. 2, pp. 1409–1418, 2005.
- [17] Y. Gao, Y. Shang, and L. Zhang, "A branch and reduce approach for solving nonconvex quadratic programming problems with quadratic constraints," *OR Transactions*, vol. 9, no. 2, pp. 9–20, 2005.
- [18] P. Shen, Y. Duan, and Y. Ma, "A robust solution approach for nonconvex quadratic programs with additional multiplicative constraints," *Applied Mathematics and Computation*, vol. 201, no. 1-2, pp. 514–526, 2008.
- [19] P. P. Shen and L. M. Liu, "A global optimization approach to quadratic programming problems with nonconvex quadratic constraints," *Chinese Journal of Engineering Mathematics*, vol. 25, no. 5, pp. 923–926, 2008.
- [20] S.-J. Qu, K.-C. Zhang, and Y. Ji, "A global optimization algorithm using parametric linearization relaxation," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 763–771, 2007.
- [21] H. Jiao and Y. Chen, "A global optimization algorithm for generalized quadratic programming," *Journal of Applied Mathematics*, vol. 2013, Article ID 215312, 9 pages, 2013.
- [22] Y. Wang and Z. Liang, "A deterministic global optimization algorithm for generalized geometric programming," *Applied Mathematics and Computation*, vol. 168, no. 1, pp. 722–737, 2005.
- [23] P. Shen and H. Jiao, "A new rectangle branch-and-pruning approach for generalized geometric programming," *Applied Mathematics and Computation*, vol. 183, no. 2, pp. 1027–1038, 2006.
- [24] P. Shen, "Linearization method of global optimization for generalized geometric programming," *Applied Mathematics and Computation*, vol. 162, no. 1, pp. 353–370, 2005.
- [25] P. Shen and X. Li, "Branch-reduction-bound algorithm for generalized geometric programming," *Journal of Global Optimization*, vol. 56, no. 3, pp. 1123–1142, 2013.
- [26] X. P. Hou, P. P. Shen, and Y. Q. Chen, "A global optimization algorithm for signomial geometric programming problem," *Abstract and Applied Analysis*. In press.
- [27] J.-F. Tsai, M.-H. Lin, and Y.-C. Hu, "On generalized geometric programming problems with non-positive variables," *European Journal of Operational Research*, vol. 178, no. 1, pp. 10–19, 2007.
- [28] A. Nazemi and E. Sharifi, "Solving a class of geometric programming problems by an efficient dynamic model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 3, pp. 692–709, 2013.
- [29] P.-P. Shen and X.-D. Bai, "Global optimization for generalized geometric programming problems with discrete variables," *Optimization*, vol. 62, no. 7, pp. 895–917, 2013.
- [30] K. Lange and H. Zhou, "MM algorithms for geometric and signomial programming," *Mathematical Programming*, vol. 143, no. 1-2, pp. 339–356, 2014.
- [31] C. A. Floudas and C. E. Gounaris, "A review of recent advances in global optimization," *Journal of Global Optimization*, vol. 45, no. 1, pp. 3–38, 2009.
- [32] P. P. Shen and L. M. Liu, "A linearization method for the global optimal solution of a quadratic programming problem with nonconvex quadratic constraints," *Numerical Mathematics*, vol. 30, no. 3, pp. 261–267, 2008.
- [33] C. A. Floudas, P. M. Pardalos, C. S. Adjiman et al., *Handbook of Test Problems in Local and Global Optimization*, vol. 33, Kluwer Academic, Boston, Mass, USA, 1999.
- [34] M.-H. Lin and J.-F. Tsai, "Range reduction techniques for improving computational efficiency in global optimization of signomial geometric programming problems," *European Journal of Operational Research*, vol. 216, no. 1, pp. 17–25, 2012.